Title: Superstring amplitudes in the operator formalism

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Abstract: The operator formalism provides a constructive approach to superstring amplitudes that has two main virtues: it makes it possible to study explicitly all the degeneration limits and it is flexible enough to work in different setups. I will illustrate these two features in a concrete example: the twisted open string partition function at two loops, which describes the interactions between three D-branes in type II theories.

Superstring amplitudes in the operator formalism

Based on:
Lorenzo Magnea, Sam Playle, R.R., Stefano Sciuto 1503.05182
Lorenzo Magnea, Sam Playle, R.R., Stefano Sciuto 1305.6631
Sam Playle, R.R., Stefano Sciuto work in progress

See also:
R.R. and Stefano Sciuto hep-th/0701292
R.R. and Stefano Sciuto hep-th/0312205
R.R. and Stefano Sciuto hep-th/0306129

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Perimeter, 23/04/2015

Plan

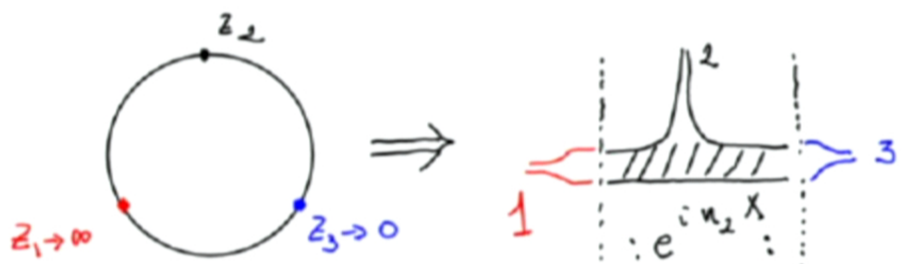
- The amplitudes building blocks in the operator formalism:
  - The 3-string vertex
  - The propagator
- Building higher genus Riemann surfaces through the sewing procedure: the Schottky parametrization
- How to deal with monodromies along the a-cycles
- A particularly interesting example: the three body problem with Dp-branes
- Super Riemann surfaces in the Schottky parametrization
- Degeneration limits and field theory integrands

The 3-string couplings

- Traditionally the couplings between three on-shell states are given by a correlators of composite operators in the 2D worldsheet CFT
- An example: one tachyon and two gluons in open bosonic theory

$$\langle c\epsilon_1 \partial X e^{ik_1 X}|_{z_1} c e^{ik_2 X}|_{z_2} c\epsilon_3 \partial X e^{ik_3 X}|_{z_3}\rangle \sim \epsilon_1\epsilon_3 - 2\alpha'(\epsilon_1 k_3)(\epsilon_3 k_1)$$

- A single vertex operator is a generating functional of a whole tower of 3-string couplings



- Can we have three arbitrary strings?

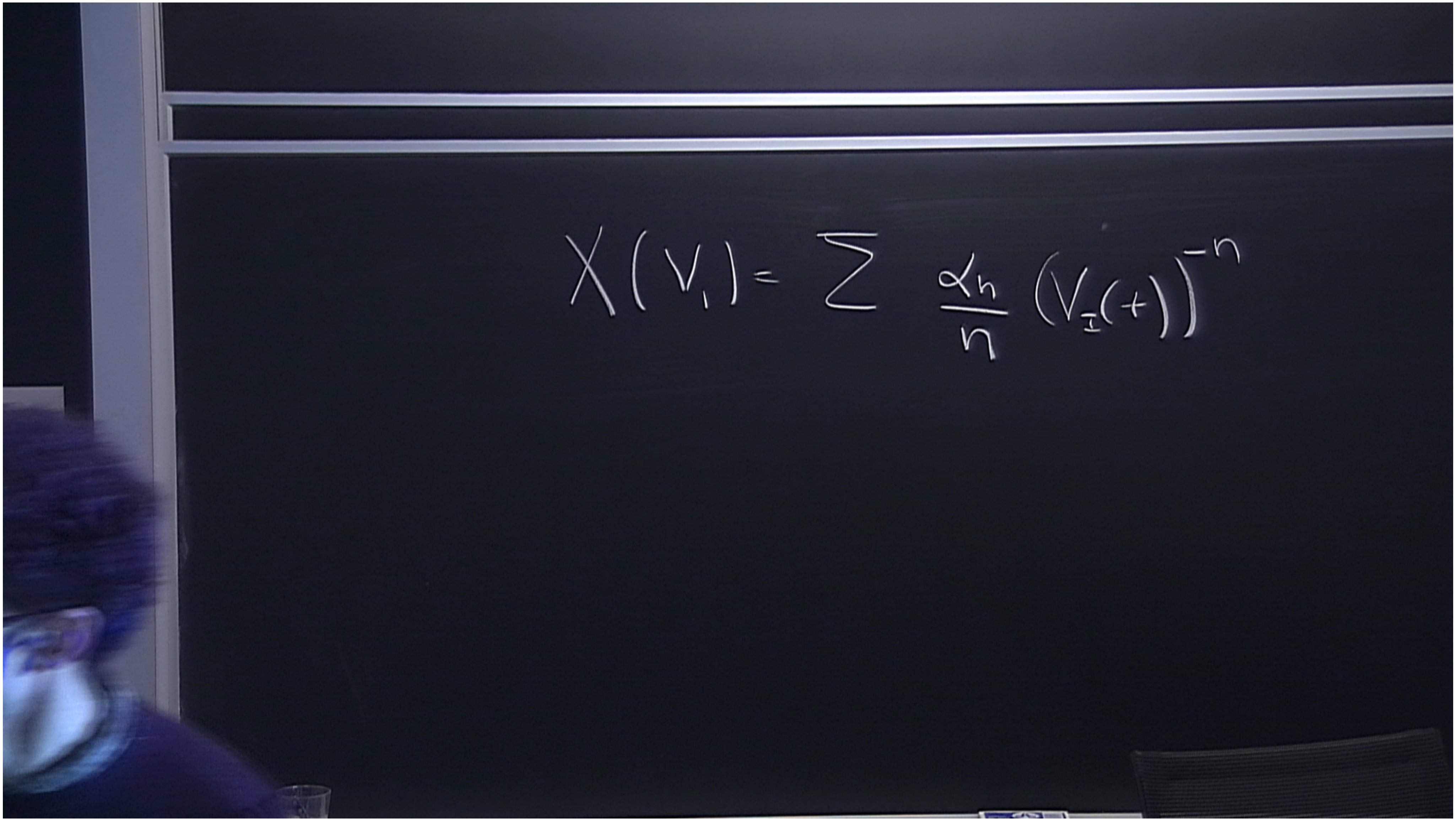
The 3-string vertex

- Introduce a new Hilbert space (Sciuto 1969)

$$\langle \mathcal{V}_I^X | = {}_I\langle 0, x_0 = 0| : \exp \left\{ \oint_0 dz \, (-X(V_I(z))\partial_z X_I(z)) \right\} : ,$$

- $V_I(z)$ is a projective transformation providing local coordinates around the puncture, $V_I(0) = z_I$
- For each on-shell state $|s_I\rangle$, $\langle \mathcal{V}_I^X | s_I \rangle$ is the usual vertex operator.
- In terms of modes $\langle \mathcal{V}_I^X |$ is (with $\Gamma(z) = 1/z$; Dirichlet b.c.)

$${}_I\langle 0, x_0 = 0| \exp \left[\sum_{n,m=1}^{\infty} a_n^\dagger D_{nm}(V_I) a_m^I \right] \exp \left[\sum_{n,m=1}^{\infty} a_n D_{nm}(\Gamma V_I) a_m^I \right]$$


$X(V_1) = \sum \frac{\alpha_n}{n} (V_I(t))^{-n}$

- D_{nm} are a representation of the projective group

$$D_{nm} = \frac{\sqrt{m}}{\sqrt{n}} \frac{1}{m!} \partial_z^m [V(z)]^n \Big|_{z=0}$$

- In the Neumann case there are zero-modes

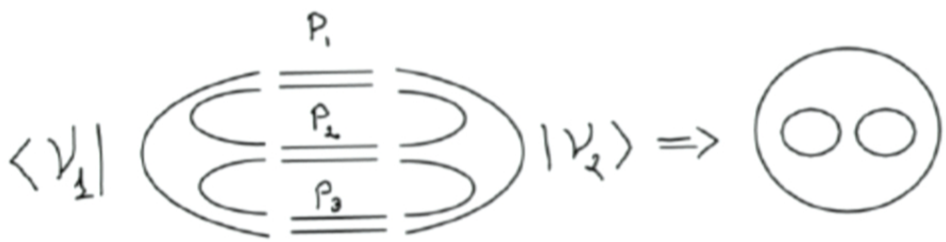
$$D_{00} = \frac{1}{2} \ln V'(0)\,,\ D_{n0} = \frac{1}{\sqrt{n}} [V(0)]^n\,,\ D_{0m} = \frac{\sqrt{m}}{2m!} \partial_z^m \ln V'(0) \Big|_{z=0}$$

- D_{nm} with $n, m \geq 0$ form a representation when used in the 3-string vertex and the propagator $P(\gamma)$

$$P(\gamma) =: \exp \left[\sum_{n,m=0}^{\infty} a_n^\dagger D_{nm}(\gamma) a_m - \sum_{n=1}^{\infty} a_n^\dagger a_n \right] :$$

Sewing

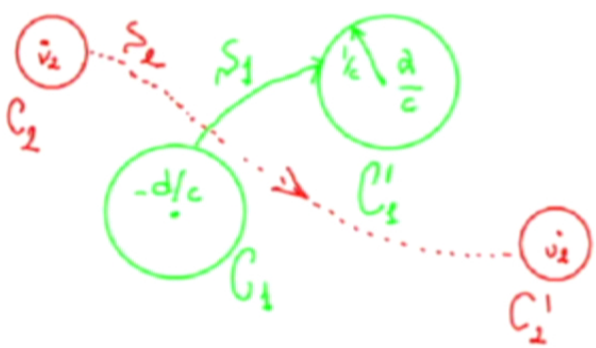
- It is natural to use the group property to build a generic amplitude by sewing vertices and propagators



- This programme was pursued in the 70ies (dual models) and again in the 80ies when the ghost sector was understood.
- For the bosonic and NS cases general expressions were derived: all classical objects in the theory of Riemann surfaces (Abelian differentials, Prime form, etc.) arise in the so-called Schottky parametrization
- The approach was generalised to Super Riemann surfaces (see later)

The Schottky parametrization

- Start from the sphere mapped to $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.
- A handle is represented by a projective transformation $S \in SL(2, \mathbf{C})$ (generator) with two distinct eigenevectors
- This defines two isometric circles $\mathcal{C}$ and $\mathcal{C}'$ identified under S
- A genus g surface is described by g generators S_i with non-overlapping isometric circles



$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

Eigenvectors	Eigenvalue
u	$\sqrt{k}$
v	$1/\sqrt{k}$

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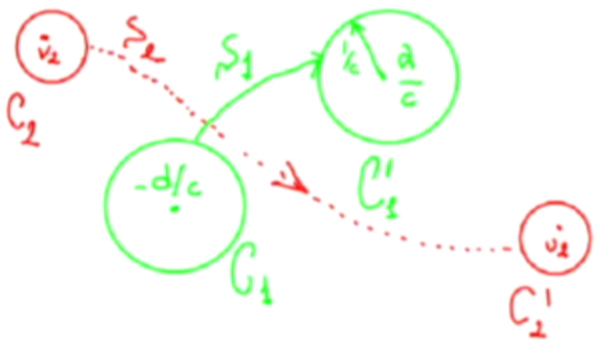

$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

Eigenvectors	Eigenvalue
u	$\sqrt{\kappa}$
v	$1/\sqrt{\kappa}$

- There is a built-in "Hamiltonian" choice of a and b-cycles
- The Schottky group is the set of all (finite) products of S_i and their inverses. Each element T_α is characterised by u_α, v_α and k_α.
- For instance, in this parametrization the period matrix is

$$\tau_{ij} = \frac{1}{2\pi \mathrm{i}} \left[\delta_{ij} \log k_i - {}^{(j)}\sum_{\alpha}{}^{(i)} \log \frac{\langle u_j | T_\alpha | v_i \rangle \langle v_j | T_\alpha | u_i \rangle}{\langle u_j | T_\alpha | u_i \rangle \langle v_j | T_\alpha | v_i \rangle} \right] .$$

where $\langle u| = (u_2, -u_1)$, $|u\rangle = (u_1, u_2)^{\mathrm{t}}$

- We can represent S as follows

$$S = \mathbb{1} + \frac{1}{\langle v|u\rangle} \left[\left(1 + k^{\frac{1}{2}}\right) |v\rangle\langle u| - \left(1 + k^{-\frac{1}{2}}\right) |u\rangle\langle v| \right]$$

- Oriented surfaces with borders can be obtained by starting from $\overline{\mathbf{C}}^{+}$ and using $SL(2, \mathbf{R})$ transformations to add borders

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What's it good for?

- The generators of the Schottky group are combinations of the local coordinates and the propagators
- The multiplier (eigenvalue) of each generator is a gluing parameter
- In the degeneration limit $k \to 0$ the Poincaré series over the Schottky group converge rapidly
- The order of a group element is directly related to the (mass) level of the states propagating in the "vacuum" diagram
- A one-to-one correspondence between the field theory diagrams and the different degenerations of the string amplitudes?
  - Worldsheet moduli $\longleftrightarrow$ Schwinger proper times
  - Spacetime vev's $\longleftrightarrow$ field theory (bare) parameters
- This is fixed at tree level, then there are no free parameters

Drawbacks and a workaround

- The modular properties are only partially manifest. Redefinitions mixing '*a* and *b*-cycles are non-analytic in Schottky's variables
- It is possible to consider monodromies along the *b*-cycles, but almost all string applications requires also monodromies along the *a*-cycles (R spin structures, external fields, ...)
- A workaround: identify Schottky results with objects that have known modular properties. An example: Jacobi's theta function

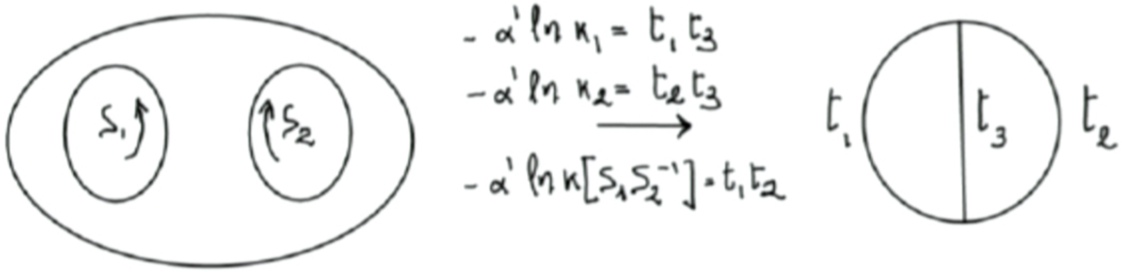
$$\theta(\epsilon|\tau) = \prod_{n=1}^{\infty}(1-k^n)(1+k^{n-\frac{1}{2}}\mathrm{e}^{2\pi\mathrm{i}\epsilon})(1+k^{n-\frac{1}{2}}\mathrm{e}^{-2\pi\mathrm{i}\epsilon})$$

$$= \sum_{n\in\mathbf{Z}} \exp\left[\mathrm{i}\pi n^2\tau + 2\pi\mathrm{i}\epsilon n\right] , \quad \text{with } k = \mathrm{e}^{2\pi\mathrm{i}\tau}$$

- It's a consequence/proof of bosonization for $\lambda = 1/2$ at 1 loop

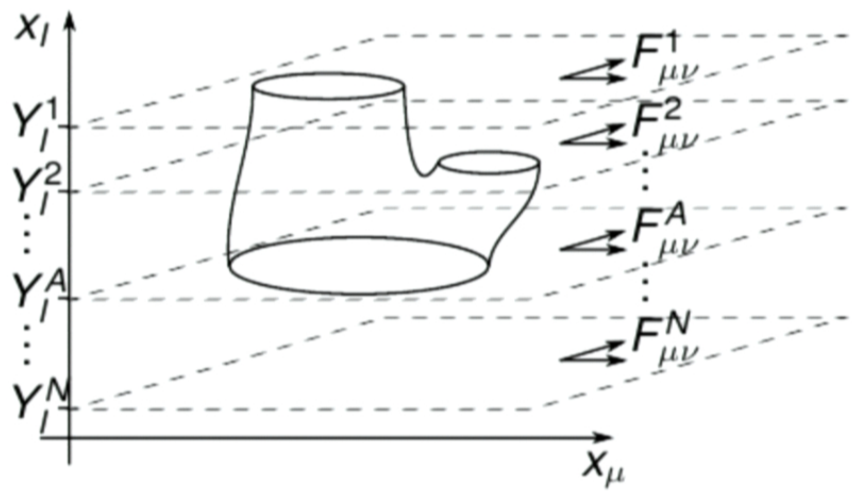
D-brane interaction: a three body problem

- I focus on the interaction among Dp-branes in type II theories
- It vanishes for identical Dp-branes in a supersymmetric setup
- It's non-trivial in presence of relative boosts/rotations or (T-dual setup) when electromagnetic fields strengths are present on some Dp-branes
- It's a Gaussian problem when the above parameters are constant. They translate in monodromies along the a-cycles (open channel) or b-cycles (closed channel)



- $\Rightarrow$ M(atrix) model, Yukawa couplings, topological string ...

- For concreteness I focus on the case of magnetized Dp-branes
- A spacetime picture of the 3-body problem (we take $p \geq 2$)



- $F^A_{\mu\nu} = gB_A(\eta_{\mu 1}\eta_{2\nu} - \eta_{\mu 2}\eta_{\nu 1})$ and $2\pi\alpha' g(B_{A_0} - B_{A_i}) = \tan \pi\epsilon_i$
- The goal is to derive the partition function $Z(\vec{\epsilon}, Y, p)$

The 2-body problem

- The one loop case is by now a classical result (Tseytlin and Bachas, Porrati). In the bosonic case $Z \sim B \int \frac{d\tau}{\tau} \frac{1}{k} f_{26}(\epsilon, Y; \tau)$ where

$$f_d = \frac{\mathrm{e}^{\pi \mathrm{i} \epsilon^2 \tau}}{\sin \pi \epsilon \tau} \frac{\mathrm{e}^{\mathrm{i} \frac{Y^2 \tau}{2\pi\alpha'}}}{(\mathrm{Im}\tau)^{\frac{p-1}{2}}} \prod_{n=1}^{\infty} [(1-k^n)^{d-4}(1-k^{n+\epsilon})(1-k^{n-\epsilon})]^{-1}$$

- For the superstring $Z \sim B \int \frac{d\tau}{\tau} \frac{1}{\sqrt{k}} f_{10}(\epsilon, Y; \tau) \sum g^3(a,b,0) g(a,b,\epsilon)$

$$g(a,b,\epsilon) = k^{\frac{a^2}{8}} \prod_{n=1}^{\infty} (1 + \mathrm{e}^{\mathrm{i}\pi b} k^{n - \frac{1-a}{2} + \epsilon})(1 + \mathrm{e}^{\mathrm{i}\pi b} k^{n - \frac{1+a}{2} - \epsilon})$$

  where the sum is on the even spin structures $(a,b) \neq (1,1)$.

- This can be expressed just in terms of theta functions... but it is a peculiarity of the one loop case!

A new ingredient

- The 2-loop case involves the period matrix τ_ϵ of the $g-1$ twisted (Prym) differentials Ω
- A novelty of $g=2$. The sewing approach provides an explicit expressions for Ω's in closed channel (twists along the *b*-cycles)
- Bosonization identities can be used to rewrite the result in terms of "geometric" objects (Δ , E ...) and to perform the modular transformation to the open channel (twists along the *a*-cycles)
- The determinant of τ_ϵ generalizes the combination $\sin \pi\tau\epsilon$ of the one loop case
- A generalization of Riemann bilinear's identity (Antoniadis, Narain and Taylor) yields a computationally convenient form for $det(\tau_\epsilon)$
- The same approach can be used in the superstring NS sector

The superstring case (NS sector)

- Super Schottky formalism: start from the projective superspace $z|\zeta$. It is convenient to use $\langle \mathbf{z}| = (z_2, -z_1, \hat{\zeta})$, $|\mathbf{z}\rangle = (z_1, z_2, \hat{\zeta})^{\mathrm{t}}$
- The super projective transformation can be written in terms of the eigenvectors $u|\theta$ and $v|\phi$

$$\mathbf{S} = \mathbb{1} + \frac{1}{\langle \mathbf{v}|\mathbf{u}\rangle}\left[\left(1 + \mathrm{e}^{\mathrm{i}\pi\varsigma} k^{\frac{1}{2}}\right) |\mathbf{v}\rangle\langle \mathbf{u}| - \left(1 + \mathrm{e}^{-\mathrm{i}\pi\varsigma} k^{-\frac{1}{2}}\right) |\mathbf{u}\rangle\langle \mathbf{v}|\right]$$

- The multipliers (eigenvalues) remain bosonic: there are $3(g-1)|2(g-1)$ worldsheet supermoduli
- There are two different quartic bosonic superinvariants (y and u)
- We have explicit Schottky expressions for all ingredients below

$$\mathbf{Z}_2 = \mathrm{e}^{\mathrm{i}\pi(\varsigma_1+\varsigma_2)} \int \frac{dk_1}{k_1^{3/2}} \frac{dk_2}{k_2^{3/2}} \frac{du}{y}\; d\theta\, d\phi\; \mathbf{F}_{\mathrm{gh}}(\boldsymbol{\mu})\, \mathbf{F}_{\parallel}^{(\vec{\epsilon})}(\boldsymbol{\mu})\, \mathbf{F}_{\perp}(\boldsymbol{\mu})\, \mathbf{F}_{\mathrm{scal}}^{(\vec{\mathrm{d}}\,)}(\boldsymbol{\mu})$$

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Degeneration limits & Yang-Mills diagrams

- We kept separate the CFT origin of the various terms in the $\mathbf{Z}_2$ integrand. So not only we can trace the level of each contribution, but also the its spacetime meaning (gluons, scalar, ghost)
- Different (open strings) degeneration limits

The symmetric (Fig. a), incomplete (Fig. b), and separating (Fig. c)

- Care is needed with Berezin's integration (Witten)
- A complete list of 2-loop 1PI diagrams (bosonic part of $N = 4$)
  - Fig. (a) corresponds to $p_i \to 0$, with $k_i = p_i p_3$, $k[S_1 S_2^{-1}] = p_1 p_2$
  - Fib. (b) corresponds to $k_{1,2} \to 0$, with u (or y) arbitrary between $p_3 = \epsilon$ and $u = \epsilon$

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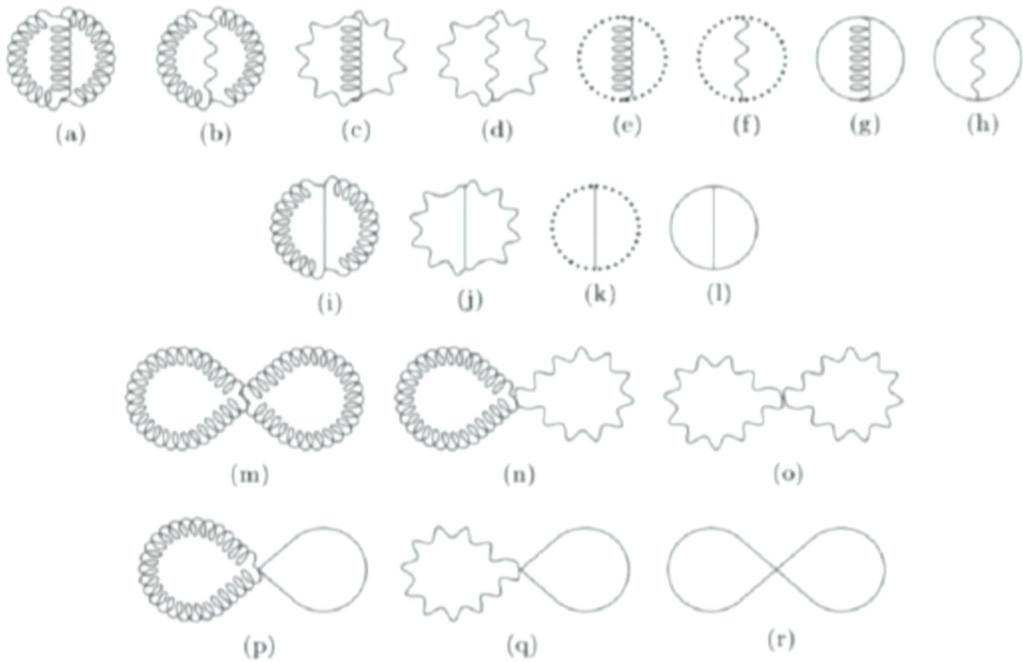
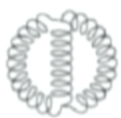


Figure 1: Two-loop 1PI vacuum Feynman graphs in Yang-Mills with adjoint scalars with VEVs. The dotted edges signify Faddeev-Popov ghosts, and the plain edges symbolize scalars, the helical edges denote gluons polarized *parallel* to the plane of the background magnetic field and the wavy edges indicate gluons polarized *perpendicular* to the background magnetic field.

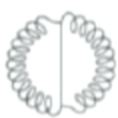
- There is perfect a one-to-one correspondence if the Yang-Mills diagrams are calculated in the background field approach with dimensional reduction and the Gervais-Neveu $\gamma = 1$ gauge (NPB 1972)

$$\mathcal{G}(\mathcal{A}, \mathcal{Q}) = \mathfrak{D}_M \mathcal{Q}^M + \mathrm{i}\,\gamma\, g\, \mathcal{Q}_M \mathcal{Q}^M = 0$$

- Some examples ($\Delta_B = \dfrac{\cosh\left[g(B_1 t_1 - B_2 t_2 - B_3 t_3)\right]}{2g^2 B_2 B_3} +$ cycl. perm.)



$$= -\frac{g^2}{(4\pi)^d}\int_0^\infty \left(\frac{\prod_{i=1}^3 dt_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_B}\right)\frac{1}{\Delta_B}\left[\frac{\sinh(gB_3 t_3)}{gB_3}\ (2 + (1-\gamma^2)\cosh(2gB_1t_1 - 2gB_2t_2)) \times \cosh(2gB_3t_3 - gB_2t_2 - gB_1t_1) + \text{cyclic permutations}\right]$$



$$= \frac{g^2}{(4\pi)^d}\int_0^\infty \left(\frac{\prod_{i=1}^3 dt_i\, \mathrm{e}^{-t_i m_i^2}}{\Delta_0^{d/2-1}\Delta_B}\right)\left[((1+\gamma^2)m_3^2 - 2(m_1^2 + m_2^2)) \times \cosh(2gB_1t_1 - 2gB_2t_2) + \text{ cyclic permutations}\right]$$

$$= -\frac{g^2}{(4\pi)^d}\int_0^\infty \left(\prod_{i=1}^2 \frac{dt_i\, \mathrm{e}^{-t_i m_i^2}\, gB_i}{t_i^{d/2-1}\sinh(gB_i t_i)}\right)\Big\{2\cosh(2gB_1t_1 + 2gB_2t_2) - \frac{1-\gamma^2}{2}\Big(2\cosh(2gB_1t_1 - 2gB_2t_2) + 4\cosh(2gB_1t_1)\cosh(2gB_2t_2)\Big)\Big\} + \text{cyclic permutations}$$

Extension to the R sector (Work in progress)

- If we simply send $\vec{\varsigma} \to \vec{b} \pm \vec{a}\tau$ the Poincaré series over the Schottky group do not converge

$$\prod_{\alpha}{}' \prod_{n=1}^{\infty} \left(1 + e^{i\pi\vec{\varsigma}\cdot\vec{N}_\alpha}\, k_\alpha^{n-1/2}\right)^2$$

- But if we rewrite this in terms of a Riemann's theta then extension to the R case is straightforward
- How to extend the super period matrix? We can start from the split surface Σ_0 (i.e. $(\theta, \phi) = (0, 0)$) and see the super surface as a complex structure deformation (D'Hoker and Phong).
- The super-Abelian differentials $\omega_i = \alpha_i(z) + \zeta b_i(z)$ satisfy

$$\alpha_i^{(1)} = -\frac{1}{2\pi} \int_{\Sigma_0} S\begin{bmatrix} a \\ b \end{bmatrix}(z, w) \chi_{\bar{w}}^{\zeta}\, b_i^{(0)}(w) d^2 w$$

- $\chi_{\bar{z}}^{\zeta} = -\partial_{\bar{z}} c^{\zeta}$ is part of the super Beltrami. $S(z, w)$ is the Szego kernel with (a, b) spin structures
- There is a large redundancy in the definition of c^{ζ}, but its monodromies are fixed. In Schottky's case they can be read from the generators (which represent the non-trivial transition functions). A natural gauge is

$$c^{\zeta} = -\phi \frac{z-u}{u-v}(1 + k^{\frac{1}{2}})\delta_{\mathcal{C}_2} + \theta \frac{z-v}{u-v}(1 + k^{\frac{1}{2}})\delta_{\mathcal{C}'_2}$$

- It matches the result from the sewing approach on the NS case!
- The R and twisted cases should follow by changing the monodromies of the propagator.
- Check against the field theory diagrams with spin 1/2 particles

Conclusions and outlook

- The analysis of the degeneration limit of string amplitudes can be performed in detail also for amplitudes with a rich structure
- It can shed light on properties valid at full string level
- In the setup discussed there are various interesting open issues
  - As mentioned, work out the R spin structures
  - Study the closed string degenerations
  - Small ϵ expansion; the $p = 0$ case relevant for M(atrix) theory
- Study amplitudes with external states. Does the map between string and field theory integrands hold off-shell? (see Rudra's and Sen's talks)
- The twisted building blocks discussed are relevant for string amplitudes in other context as well such as orbifolds
- Use an improved knowledge of string amplitudes to derive interesting new spacetime information