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Abstract:

Modular properties of superstring scattering amplitudes

Michael B. Green
University of Cambridge

Workshop on "Superstring Perturbation Theory"
Perimeter Institute, April 23, 2015

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Consider narrowly-focused aspects of the low energy expansion of closed string theory obtained from maximally supersymmetric closed string scattering amplitudes.

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- Features of Closed String Perturbation theory (see Eric D'Hoker's talk)
  Modular invariants of higher-genus Riemann surfaces
- Non-Perturbative features - Duality:
  Connects perturbative with non-perturbative effects
  Constraints imposed by SUSY, Duality, Unitarity
  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; ....

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  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; ….
- **Connections with maximal supergravity.**
  Comments on ultraviolet divergences of maximal supergravity

With: Eric D'Hoker; Pierre Vanhove; Stephen Miller;
Carlos Mafra; Oliver Schlotterer;
Boris Pioline; Jorge Russo; Rudolfo Russo;
Don Zagier; ………..

The Low Energy Expansion of String Theory

- Lowest order term reproduces the results of classical supergravity

Einstein-Hilbert

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$ Interactions of other supergravity fields

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metric – $G_{\mu\nu}$

scalar field - dilaton

$e^{-\phi} = \frac{1}{g_s}$ ← string coupling constant

compactify space-time to dimensions D < 10

- Higher order terms:

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Coefficient depends on moduli (scalar fields)

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- Expansion in powers of $\alpha' R, \quad \alpha' D^2, \quad \ldots$

THE LOW ENERGY EXPANSION OF (TYPE IIB) STRING THEORY

HIGHER DERIVATIVE CORRECTIONS to Einstein theory

Mass dimension

R^4 ⟶ Δ

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$\longrightarrow \quad \Delta$

$R^4 \quad d^2R^4 \quad d^4R^4 \quad d^6R^4 \quad d^8R^4$

The Low Energy Expansion of (Type IIB) String Theory

Higher derivative corrections to Einstein theory

$R^6 \quad d^2 R^6 \quad d^4 R^6$

$R^5 \quad d^2 R^5 \quad d^4 R^5 \quad d^6 R^5$

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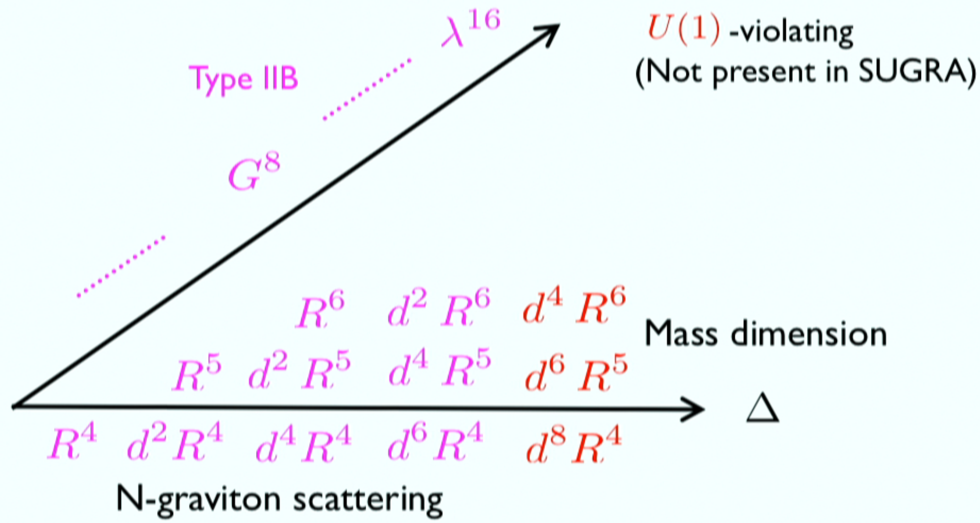

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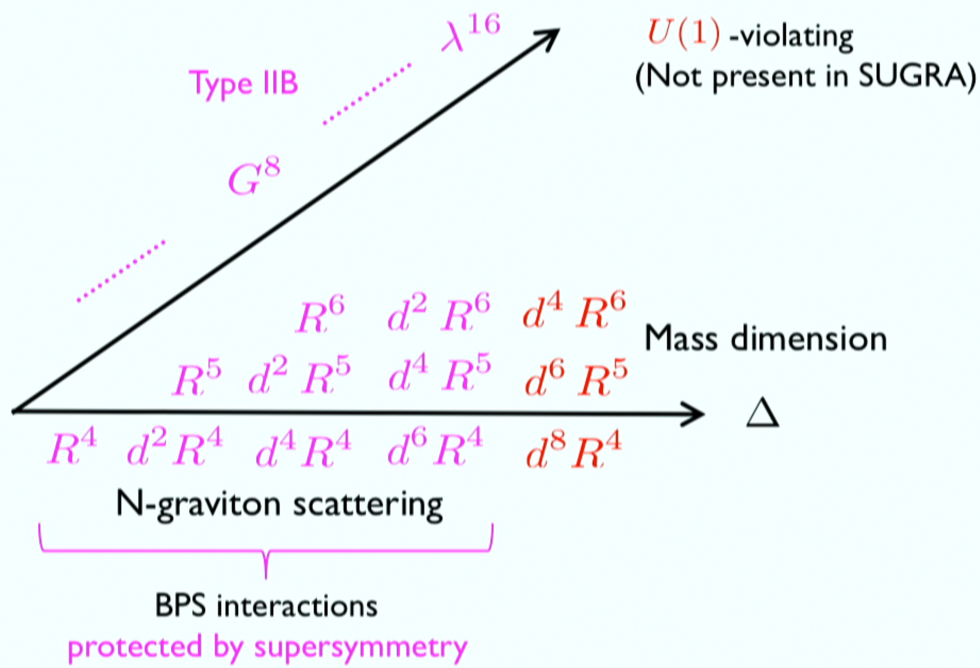

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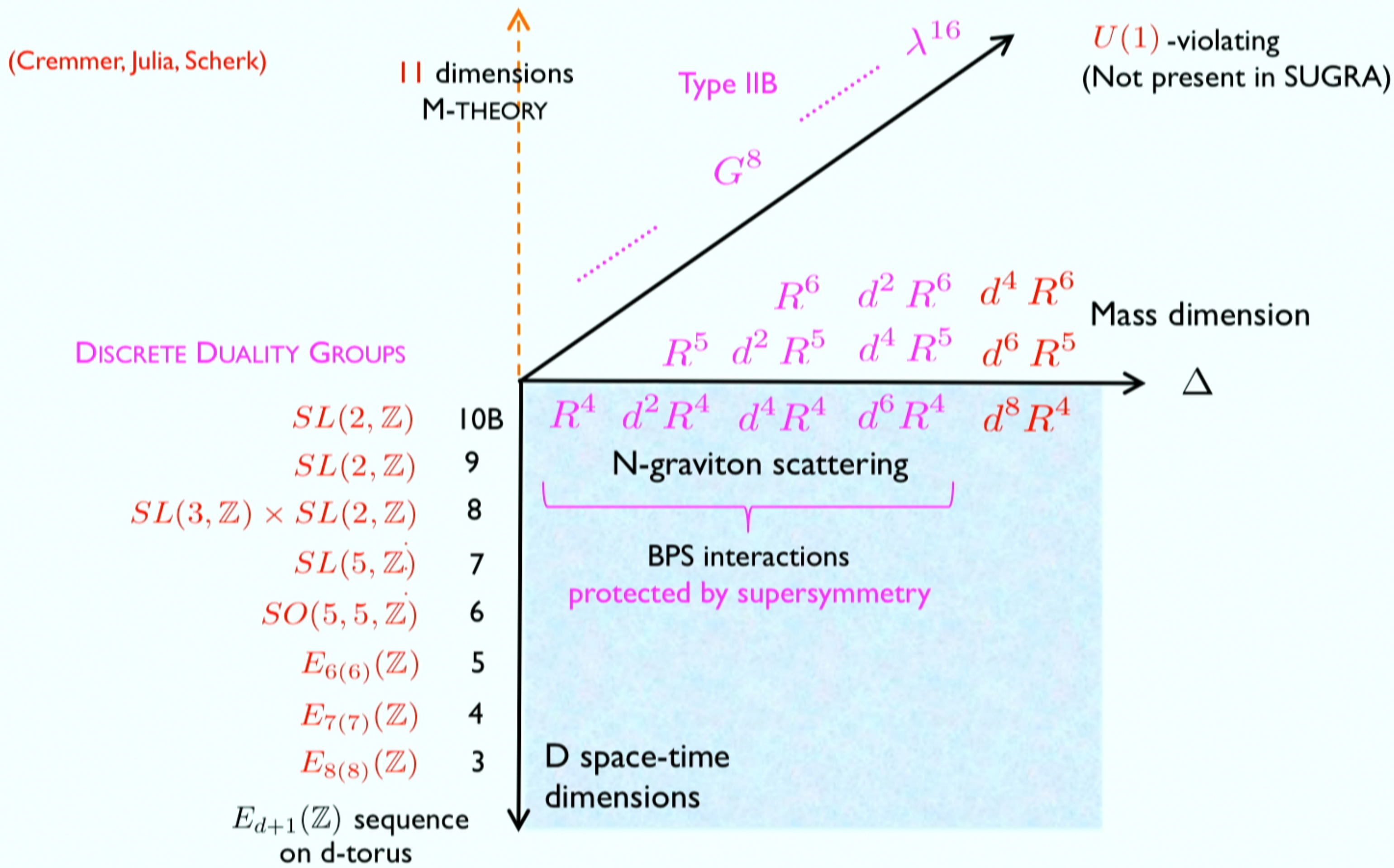

SCALAR FIELDS (MODULI) AND S-DUALITY

(Cremmer, Julia)

SUPERGRAVITY (low energy limit of string theory):

Scalar fields parameterize a symmetric space $G(\mathbb{R})/K(\mathbb{R})$

groups in E_n series (real split forms)

Maximal compact subgroup

STRING THEORY:

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

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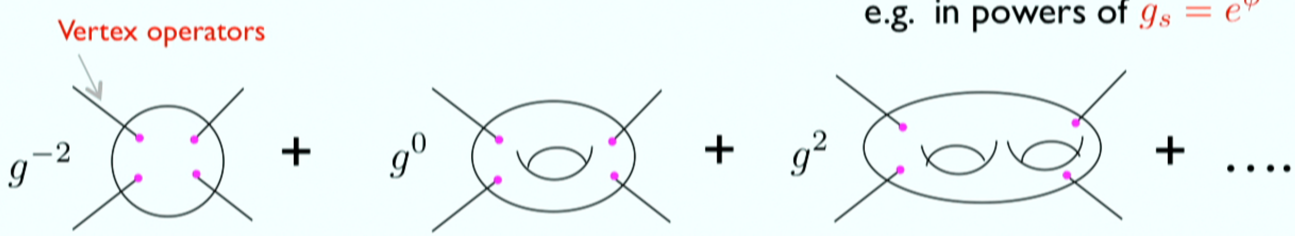
Only a discrete arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory

String perturbation theory: Expansion around boundary of moduli space

e.g. in powers of $g_s = e^{\phi}$



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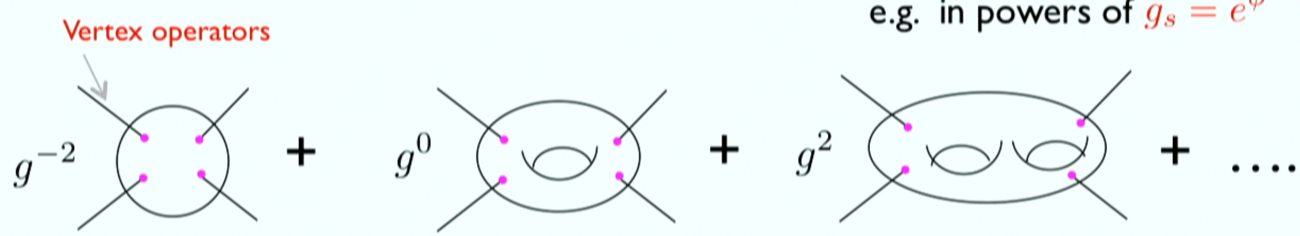

acts on complex structure of torus $\longrightarrow$ $SL(2,\mathbb{Z})$ $\quad Sp(4,\mathbb{Z}) \quad$ "$Sp(2h,\mathbb{Z})$"

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad a,\, b,\, c,\, d \in \mathbb{Z} \quad ad - bc = 1$$

WORLD-SHEET automorphic symmetries

HOW POWERFUL ARE THE CONSTRAINTS IMPOSED BY (MAXIMAL) SUSY AND DUALITY ??

Investigate the exact moduli dependence of low lying terms in the low energy expansion.

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Consider simple example:

10-dimensional Type IIB - maximal supersymmetry

One complex modulus $\Omega = \Omega_1 + i\Omega_2$

inverse string coupling constant $\longrightarrow$ $\Omega_2 = \frac{1}{g} = e^{-\phi}$

Duality group $SL(2,\mathbb{Z})$ $\Omega \to \frac{a\,\Omega + b}{c\Omega + d}$ $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$

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e.g.

Four-Graviton Scattering in Type II String Theory

$$A^{(4)}(\epsilon_r, k_r; \Omega) = \mathcal{R}^4\, T(s,t,u;\Omega)$$

$s = -2\, k_1 \cdot k_2$
$t = -2\, k_1 \cdot k_4$
$u = -2\, k_1 \cdot k_3$

$\mathcal{R}$ linearized curvature $\sim k_\mu\, k_\nu\, \epsilon_{\rho\sigma}$

Symmetric function of Mandelstam invariants s, t, u (with $s + t + u = 0$).
Has an expansion in power series of $\sigma_2 = s^2 + t^2 + u^2$ and $\sigma_3 = s^3 + t^3 + u^3$

(non-analytic pieces are essential, but will be ignored for now)

$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q \sim s^{2p+3q} + \dots$$

Coefficients are $SL(2,\mathbb{Z})$ -invariant functions of scalar fields (moduli, or coupling constants).

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FOUR-GRAVITON SCATTERING IN TYPE II STRING THEORY

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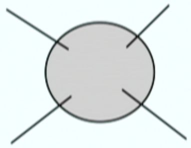
TO WHAT EXTENT CAN WE DETERMINE THESE COEFFICIENTS?

BOUNDARY DATA: STRING PERTURBATION THEORY $\Omega_2 \to \infty \quad (g \to 0)$

TREE-LEVEL (VIRASORO AMPLITUDE)



$$A_0^{(4)}(\epsilon_r, k_r) = g_s^{-2}\,\mathcal{R}^4\,T_0(s,t,u)$$

$$T_0^{(4)} = \frac{4}{stu}\,\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

$$s^k R^4 \sim d^{2k} R^4$$

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,\alpha'^3 + \zeta(5)\,\alpha'^5\sigma_2 + \frac{2\,\zeta(3)^2}{3}\,\alpha'^6\,\sigma_3 + \frac{\zeta(7)}{2}\,\alpha'^7\,\sigma_2^2$$
$$+ \frac{2\,\zeta(3)\,\zeta(5)}{3}\,\alpha'^8\,\sigma_2\,\sigma_3 + \frac{\zeta(9)}{4}\,\alpha'^8\,\sigma_2^3 + \frac{2}{27}(2\,\zeta(3)^2 + \zeta(9))\,\alpha'^9\,\sigma_3^2 + \dots$$

$\sigma_2 = s^2 + t^2 + u^2$

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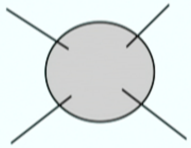
Tree-level (Virasoro Amplitude)



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GENUS ONE

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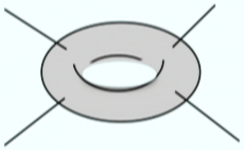


$$\mathcal{A}_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16}\,\mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{y^2}\,\mathcal{B}_1(s,t,u;\tau)$$

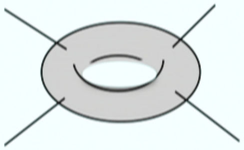
Integral over complex structure $\tau = x + iy$

$$\mathcal{B}_1(s,t,u;\tau) = \frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2z\,\exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\,G(z_i,z_j)\right)$$

Vertex operator Corr. function

Green function

GENUS ONE



GENUS ONE AMPLITUDE

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Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

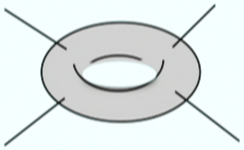
(MBG, D'Hoker, Russo, Vanhove)

Expanding in a power series in momenta gives (with $\alpha' = 4$)

$$\frac{1}{w!}\frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2z_i \left(\sum_{0<i<j\le 4} s_{ij} G(z_i - z_j)\right)^w = \sum_i \sigma_2^{p_i}\,\sigma_3^{q_i}\, j^{(p_i,q_i)}(\tau)$$

$$\sum_i (2p_i + 3q_i) = w$$

Genus one



Genus one amplitude

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Coefficients of higher derivative interactions

Modular invariants for surface

Feynman diagrams on toroidal world-sheet

Coefficients of higher derivative interactions:

(genus-one generalisation of the tree-level ζ values)

$$\Xi^{(p,q)} = \int_{\mathcal{M}_1}\frac{d^2\tau}{y^2}\, j^{(p,q)}(\tau)$$

GENUS-ONE EXPANSION (after much work!):

Integrating over τ (and dealing with divergences) gives the one-loop expansion:

$$A_1^{(4)} = \frac{\pi}{3}\left(1 + 0\,\sigma_2 + \frac{\pi\zeta(3)}{9}\,\sigma_3 + 0\,\sigma_2^2 + \frac{29}{180}\,\zeta(5)\,\sigma_2\sigma_3 + \dots\right)\mathcal{R}^4$$

$\mathcal{R}^4$ $\quad d^4\,\mathcal{R}^4$ $\quad d^6\,\mathcal{R}^4$ $\quad d^8\,\mathcal{R}^4$ $\quad d^{10}\,\mathcal{R}^4$

These coefficients are analogous to the tree-level coefficients:

WHAT IS THE CONNECTION BETWEEN THEM??

D=10 genus-one threshold terms: $A_1^{(4)\,non-an} \sim s\log s + \cdots + s^4\,\log s + \dots$

arising from the degeneration limit

SUGRA threshold

1st stringy threshold

GENUS TWO

Amplitude is explicit but little studied. Low energy limit

(D'Hoker, Gutperle, Phong) (D'Hoker, MBG, Pioline, R. Russo)

Result:

$$A_2^{(4)} = g_s^2\Big(\frac{4}{3}\zeta(4)\,\sigma_2 R^4 \; + 4\zeta(4)\sigma_3 R^4 + \dots\Big)$$

$d^4\,R^4$ $d^6\,R^4$

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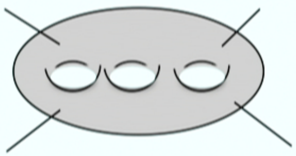


Technical difficulties analysing 3-loops. Gomez and Mafra evaluated the leading low energy behaviour using PURE SPINOR FORMALISM, giving

$$A_3^{(4)} = g_s^4\left(\frac{4}{27}\,\zeta(6)\,\sigma_3 + \dots\right)\mathcal{R}^4$$

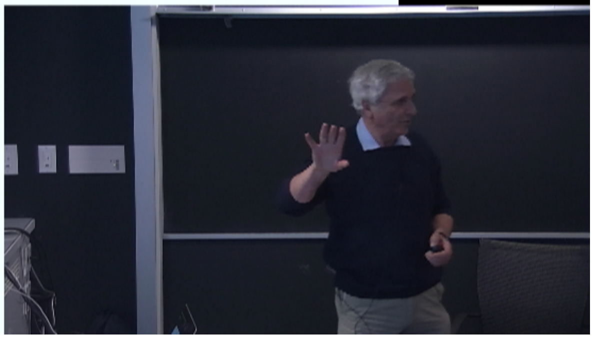
$d^6\,R^4$

EXTENSION TO N-PARTICLE AMPLITUDES

Very brief summary

- OPEN-STRING TREES: For $N > 4$ coefficients of higher derivative interactions
  (Yang-Mills) (Mafra, Schlotterer, Stieberger)

  non-trivial multi-zeta values (MZV's) $$\zeta(n_!, \ldots, n_r) = \sum_{0<k_1<\ldots k_r} \prod_{\ell=1}^{r} k_l^{-n_l}$$



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  First case is $\zeta(5,3) + \ldots$ weight $w = 8$

- CLOSED-STRING TREES: For $N > 4$ non-trivial MZV's with odd weights arise
  (gravity)

  First case is $\zeta(5,3,3) + \ldots$ weight $w = 11$

Special values of (single-valued) multiple polylogarithms – also arise in dimensional regularisation of renormalisable quantum field theories. Zagier, Brown, Goncharov, …..

HOW DOES THIS GENERALIZE TO HIGHER GENUS ??

Extension to N-particle amplitudes

Very brief summary

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How does this generalize to higher genus ??

NON-PERTURBATIVE EXTENSION

Focus here on the simplest nontrivial duality group $SL(2,\mathbb{Z})$

$$A^{(4)}(\epsilon_r, k_r; \Omega) = \mathcal{R}^4\, T(s,t,u;\Omega)$$

$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q \longrightarrow \sim s^{2p+3q} + \ldots$$

$SL(2,\mathbb{Z})$ invariant functions

Combine information from:

MAXIMAL (NONLINEAR) SUPERSYMMETRY

DUALITY BETWEEN M-THEORY ON T^2 AND STRING THEORY

Consequences of Supersymmetry

Consider $SL(2,\mathbb{Z})$ -invariant local interaction (Einstein frame)

$$\alpha'^n\, S^{(n)} = (\alpha')^{n-4} \int d^{10}x\, e\, \mathcal{F}_n^{(u)}(\Omega)\, \mathcal{P}_{2n+2}^{(-u)}$$

e.g.	$U(1)$ charge
$\mathcal{R}^4$	$u=0$
λ^{16} dilatino	$u=-24$

Coefficient function is a modular form

$$\mathcal{F}_n^{(u)}(\Omega) \to \left(\frac{c\bar{\Omega}+d}{c\Omega+d}\right)^{\frac{u}{2}} \mathcal{F}_n^{(u)}(\Omega)$$

$$\Omega \underset{SL(2,\mathbb{Z})}{\rightarrow} \frac{a\Omega+b}{c\Omega+d}$$

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$U(1)$ charge: $u=0$ ($\mathcal{R}^4$), $u=-24$ (λ^{16})

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$$\Omega \underset{SL(2,\mathbb{Z})}{\longrightarrow} \frac{a\Omega+b}{c\Omega+d}$$

Invariance of action

$$\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0$$

i.e, $(\delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots)(S^{(0)} + \alpha'^3 S^{(3)} + \ldots) = 0$

Classical SUSY ($\delta^{(0)}$) — Classical SUGRA ($S^{(0)}$)

On-shell algebra

$$[\delta\,,\delta]\,\Phi \equiv [\delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots, \delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots]\,\Phi$$
$$\approx a\cdot P\,\Phi$$

Consequences of Supersymmetry

Consider $SL(2,\mathbb{Z})$ -invariant local interaction (Einstein frame)

$$\alpha'^n\, S^{(n)} = (\alpha')^{n-4} \int d^{10}x\, e\, \mathcal{F}_n^{(u)}(\Omega)\, \mathcal{P}_{2n+2}^{(-u)}$$

e.g. $\mathcal{R}^4$; λ^{16} dilatino

$U(1)$ charge: $u=0$; $u=-24$

Coefficient function is a modular form

$$\mathcal{F}_n^{(u)}(\Omega) \to \left(\frac{c\bar{\Omega}+d}{c\Omega+d}\right)^{\frac{u}{2}} \mathcal{F}_n^{(u)}(\Omega)$$

$$\Omega \underset{SL(2,\mathbb{Z})}{\longrightarrow} \frac{a\Omega+b}{c\Omega+d}$$

Invariance of action

$$\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0$$ i.e, $$(\delta^{(0)} + \alpha'^3\delta^{(3)} + \dots)(S^{(0)} + \alpha'^3 S^{(3)} + \dots) = 0$$

Classical SUSY ($\delta^{(0)}$) — Classical SUGRA ($S^{(0)}$)

On-shell algebra

$$[\delta\,,\delta]\,\Phi \equiv [\delta^{(0)} + \alpha'^3\delta^{(3)} + \dots, \delta^{(0)} + \alpha'^3\delta^{(3)} + \dots]\,\Phi$$
$$\approx a\cdot P\,\Phi$$

Strongly constrains the form of $S^{(n)}$ and $\delta^{(n)}$

- Difficult to implement in general in absence of off-shell superspace formalism.

- Low orders

$O(\alpha'^3)$ $\qquad \delta^{(0)}\, S^{(3)} + \delta^{(3)}\, S^{(0)} = 0$ $\qquad$ MBG, Sethi

$O(\alpha'^6)$ $\qquad \delta^{(0)}\, S^{(6)} + \delta^{(3)}\, S^{(3)} + \delta^{(6)}\, S^{(0)} = 0$

- Simple examples

MBG, Sethi

$$\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\,\mathcal{F}_n^{(u+2)} \qquad \bar{\mathcal{D}}\,\mathcal{F}_n^{(u+2)} = \bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$$

Modular covariant derivative

$$\mathcal{D} = i\Omega_2\,\frac{\partial}{\partial\Omega} - \frac{u}{4}$$

- First order differential equations relating term of charge u to one of charge $u+2$

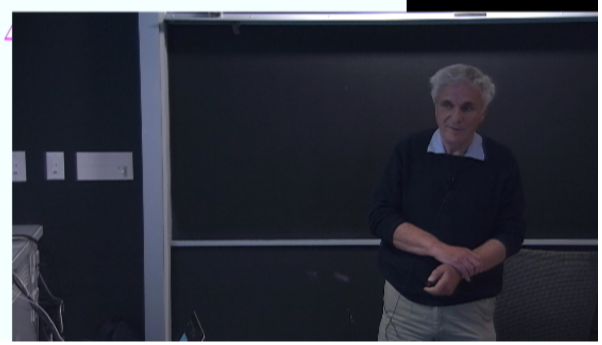
Implies LAPLACE EIGENVALUE EQUATION :

$$\bar{\mathcal{D}}\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$$

Consider $u = 0$ $\qquad \Delta_\Omega\,\mathcal{F}_n^{(0)} = s(s-1)\,\mathcal{F}_n^{(0)}$

e.g. $\quad \mathcal{R}^4 \quad n = 2s = 3 \quad \Delta_\Omega\,\mathcal{E}_{(0,0)}(\Omega) = \frac{3}{4}\,\mathcal{E}_{(0,0)}(\Omega)$

$d^4\,\mathcal{R}^4 \quad n = 2s = 5 \quad \Delta_\Omega\,\mathcal{E}_{(1,0)}(\Omega) = \frac{15}{4}\,\mathcal{E}_{(0,0)}(\Omega)$



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- Simple examples MBG, Sethi $\quad \mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\,\mathcal{F}_n^{(u+2)} \qquad \bar{\mathcal{D}}\,\mathcal{F}_n^{(u+2)} = \bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$

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$\Delta_\Omega = \Omega_2^2(\partial_{\Omega_1}^2 + \partial_{\Omega_2}^2)$

SOLUTIONS: NON-HOLOMORPHIC EISENSTEIN SERIES

(with power growth as $\Omega_2 \to \infty$)

NON-HOLOMORPHIC EISENSTEIN SERIES

$$E_s(\Omega) = \sum_{\gcd(p,q)=1} \frac{\Omega_2^s}{|p+q\Omega|^{2s}} = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma\Omega)^s$$

Poincare series – manifest $SL(2,\mathbb{Z})$

Parabolic subgroup $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

- $SL(2,\mathbb{Z})$ invariant (generalises to higher rank duality groups)
- Solution of LAPLACE EIGENVALUE EQN.

$$\Delta_\Omega \, E_s(\Omega) = s(s-1)\, E_s(\Omega)$$

- Fourier series $E_s(\Omega) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\Omega_2)\cos(2\pi i k \Omega_1)\,.$
- ZERO MODE $k=0$ - TWO POWER-BEHAVED TERMS (perturbative) :

$$\mathcal{F}_0 = \Omega_2^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)}\Omega_2^{1-s}$$

Non-holomorphic Eisenstein series

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- Solution of Laplace eigenvalue eqn.

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- Non-zero modes $k>0$ - D-instanton sum

$$\mathcal{F}_k = \frac{2\pi^s}{\zeta(2s)\Gamma(s)}|k|^{s-\frac{1}{2}}\,\sigma_{2s-1}(k)\,\Omega_2^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi|k|\Omega_2)$$

$$\sim \frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)}|k|^{s-1}\,\sigma_{2s-1}(k)\,e^{-2\pi|k|\Omega_2}\,(1+O(\Omega_2^{-1}))$$

divisor sum $\sigma_n(k) = \sum_{p|k} p^n$

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Low order interaction coefficients

α'^3 $\mathcal{E}_{(0,0)} = 2\zeta(3)\, E_{\frac{3}{2}}(\Omega)$

MBG, Gutperle, Vanhove

$$2\zeta(3)\, g_s^{-\frac{1}{2}}\, E_{\frac{3}{2}}(\Omega) = 2\zeta(3)\, g_s^{-2} + 4\zeta(2)\, g_s^{0} + \sum_{k\neq 0}(\dots)\, \sigma_2(k)\, e^{-2\pi|k|\Omega_2} e^{2\pi i k \Omega_1}$$

Perturbative terms: tree-level genus-two D-instantons

Non-renormalisation beyond 1-loop for R^4 $\frac{1}{2} - BPS$

α'^5 $\mathcal{E}_{(1,0)} = \zeta(5)\, E_{\frac{5}{2}}(\Omega)$

$$\zeta(5)\, g_s^{\frac{1}{2}}\, E_{\frac{5}{2}}(\Omega) = \zeta(5)\, g_s^{-2} + \frac{4}{3}\zeta(4)\, g_s^{2} + \sum_{k\neq 0}(\dots)\, \sigma_4(k)\, e^{-2\pi|k|\Omega_2} e^{2\pi i k \Omega_1}$$

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NO genus-one term (no ¼-BPS states)

Non-renormalisation beyond 2 loops for $d^4\,\mathcal{R}^4$ $\frac{1}{4} - BPS$

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NON-RENORMALISATION BEYOND 2 LOOPS FOR $d^4\, \mathcal{R}^4$ $\frac{1}{4} - BPS$

These coefficients can be obtained by consideration of duality between one- and two-loop eleven-dimensional supergravity compactified on a two-torus and type IIB compactified on a circle.

NEXT ORDER $O(\alpha'^6\, d^6\, \mathcal{R}^4)$ $\quad g_s\, \mathcal{E}_{(0,1)}(\Omega)\, \sigma_3\, \mathcal{R}^4$

$\mathcal{E}_{(0,1)}(\Omega)$ NOT Eisenstein series but satisfies INHOMOGENEOUS Laplace equation

MBG, Vanhove

$$(\Delta_\Omega - 12)\, \mathcal{E}_{(0,1)}(\Omega) = -\left(2\zeta(3)\, E_{\frac{3}{2}}(\Omega)\right)^2$$

→ The square of the coefficient of R^4

This equation was conjectured by consideration of duality between two-loop eleven-dimensional supergravity compactified on a two-torus and type IIB compactified on a circle.

Closely related to the maximal degeneration limit of the ZHANG-KAWAZUMI INVARIANT (defined in Eric D'Hoker's talk)

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Structure also motivated by (but not directly derived from) supersymmetry

THE SOLUTION OF THIS EQUATION HAS SOME WEIRD AND WONDERFUL (PUZZLING) FEATURES.

Solution of the Inhomogeneous Laplace Equation

MBG, Miller, Vanhove

$$(\Delta_\Omega - 12)\, f(\Omega) = -\Big(2\zeta(3)\, E_{\frac{3}{2}}(\Omega)\Big)^2$$

Fourier series: $$f(\Omega) \;=\; \sum_n \widehat{f}_n(\Omega_2)\, e^{2\pi i n\Omega_1}\,.$$

Equation for Fourier modes : $$(\Omega_2^2\,\partial^2_{\Omega_2} - 12 - 4\pi^2 n^2\Omega_2^2)\,\widehat{f}(\Omega_2) = S_n(\Omega_2)$$ Fourier mode of source

Boundary conditions : $$\widehat{f}_n(\Omega_2) = O(\Omega_2^3)\,, \qquad \Omega_2 \to \infty$$ Weak coupling $\Omega_2 = g_s^{-1}$

Weak coupling (tree level) power behaviour

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Weak coupling (tree level) power behaviour

$$\widehat{f}_n(\Omega_2) = O(\Omega_2^{-2})\,, \qquad \Omega_2 \to 0$$ Strong coupling

Subtle implication of $SL(2,\mathbb{Z})$ invariance

SOLUTION OF THE INHOMOGENEOUS LAPLACE EQUATION

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SUBTLE implication of $SL(2,\mathbb{Z})$ invariance

These b.c.'s determine a unique solution by fixing the coefficient of the solution of the homogeneous equation, $\alpha_n \sqrt{y}\, K_{\frac{7}{2}}(2\pi|n|y)$, for each value of n.

ZERO MODE - four power-behaved terms :

$$\widehat{f}_0(\Omega_2) = \frac{2\,\zeta(3)^2}{3}\,\Omega_2^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\,\Omega_2 + 4\zeta(4)\,\Omega_2^{-1} + \frac{4\,\zeta(6)}{27}\,\Omega_2^{-3} + \sum_{m\neq 0} \widehat{f}_0^{m}(\Omega_2)$$

GENUS	0	1	2	3	Non-Perturbative

- ALL PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT STRING PERT. THEORY CALCULATIONS

- NON-PERTURBATIVE TERMS

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_0^{m}(\Omega_2) = \frac{32\,\pi\,\sigma_2(|m|)^2}{315\,|m|^3} \sum_{i,j=0,1} r^{i,j}(\pi|m|\Omega_2)\,K_i(2\pi|m|\Omega_2)\,K_j(2\pi|m|\Omega_2)$$

Bilinear in $K_0\,,\ K_1$

$\sim e^{-4\pi m\Omega_2}$

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Bilinear in $K_0\,,\;K_1$

$$\Omega_2 \to \infty \qquad \sim e^{-4\pi|m|\Omega_2}\left(\frac{\sigma_2(|m|)^2}{|m|^5\,\Omega_2^2} + O(\Omega_2^{-3})\right) \qquad \sim e^{-4\pi m\Omega_2}$$

Behaviour suggestive of charge-zero INSTANTON / ANTI-INSTANTON pairs.

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Genus $\quad 0 \quad 1 \quad 2 \quad 3 \quad$ Non-Perturbative

- All perturbative contributions agree with explicit string pert. theory calculations

- Non-perturbative terms

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Behaviour suggestive of charge-zero Instanton / Anti-Instanton pairs.

$$\Omega_2 \to 0 \qquad \sim \frac{945\,\zeta(3)^2\,\zeta(5)}{4\,\pi^5}\,\frac{1}{\Omega_2^2} + O(\log\Omega_2)$$

cancellation of Ω_2^{-3} term by infinite number of "instantons".

NON-ZERO MODES:

$i.j = 0, 1$

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_n(\Omega_2) = \alpha_n \sqrt{\Omega_2}\, K_{\frac{7}{2}}(2\pi|n|\Omega_2) + \sum_{\substack{n_1+n_2=n \\ (n_1,n_2)\neq(0,0)}} M^{ij}_{n_1,n_2}(\pi|n|\Omega_2)\, K_i(2\pi|n_1|\Omega_2)\, K_j(2\pi|n_2|\Omega_2)$$

Constant α_n determined by cancellation of the Ω_2^{-3} term in the $\Omega_2 \to 0$ limit.

$$\underset{\Omega_2 >> 1}{\sim} e^{-2\pi(|n_1|+|n_2|)\Omega_2}$$

"BPS INSTANTON PAIR" if $|n_1| + |n_2| = |n| = |n_1 + n_2|$ $\quad(\text{sign}\, n_1 = \text{sign}\, n_2)$

charge = action

"INSTANTON / ANTI-INSTANTON" pair if $|n_1| + |n_2| < |n|$ $\quad(\text{sign}\, n_1 = -\text{sign}\, n_2)$

charge < action

$$\widehat{f}_n(\Omega_2) \underset{\Omega_2 >> 1}{\sim} e^{-2\pi|n|\Omega_2}\left(8\,\frac{\sigma_2(|n|)}{|n|^{5/2}}\,\zeta(3)\,\Omega_2^{1/2} + O(1)\right) + c\, e^{-2\pi(|n|+1)\Omega_2}(\ldots) + \ldots$$

- Solution can be expressed as a Poincare series:

$$f(\Omega) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \Phi(\gamma\Omega)$$

where $\quad \Phi(\Omega) = a_0(\Omega_2) + \sum_{n\neq 0} a_n(\Omega_2)\, e^{2\pi i n \Omega_1}$ $\quad$ ($a_n(\Omega_2)$ is linear in K_0 , K_1)

Compactification on a d-torus to D=10-d dimensions

MBG, Miller, Russo, Vanhove

Duality symmetry –Higher Rank duality groups

$$\mathcal{E}^{(D)}_{(p,q)}(\gamma\cdot\varphi) = \mathcal{E}^{(D)}_{(p,q)}(\varphi); \qquad \gamma \in E_{d+1}(\mathbb{Z})$$

Dimensional reduction of Laplace equations

$\mathcal{R}^4$ $$\left(\Delta^{(D)} - \frac{3\,(11-D)\,(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = 6\pi\,\delta_{D-8,0}$$

$d^4\,\mathcal{R}^4$ $$\left(\Delta^{(D)} - \frac{5\,(12-D)\,(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = 7\mathcal{E}_{(0,0)}\,\delta_{D-6,0} + 40\zeta(2)\,\delta_{D-7,0}$$

$d^6\,\mathcal{R}^4$ $$\left(\Delta^{(D)} - \frac{6\,(14-D)\,(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} = -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 40\zeta(3)\,\delta_{D-6,0}$$

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- Note Kronecker delta terms contribute in "critical" dimensions

($D = D_C = 4 + 6/L$ where maximal supergravity has L-loop log UV divergences)

SOLUTIONS: Maximal Parabolic LANGLANDS EISENSTEIN SERIES

(for $\mathcal{R}^4$, $d^4\,\mathcal{R}^4$)

Generalisations of SL(2) Eisenstein series to higher rank duality groups in

For a group G associated with a maximal parabolic subgroup labeled by a simple root, β , with elements

$$P_\beta = L_\beta \times U_\beta \subset E_{d+1}$$

"LEVI subgroup"
(block diagonal)ß

"Unipotent radical"
(upper triangular)

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$$E^G_{\beta;s} \; := \sum_{\gamma \in P_\beta(\mathbb{Z})\backslash G(\mathbb{Z})} e^{2\,s\,\langle \omega_\beta\,, H(\gamma g)\rangle} \; ,$$

Cartan elements

fundamental weight vector ω_β for a simple root β.

c.f $SL(2)$case

$$E_s = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma\Omega)^s$$

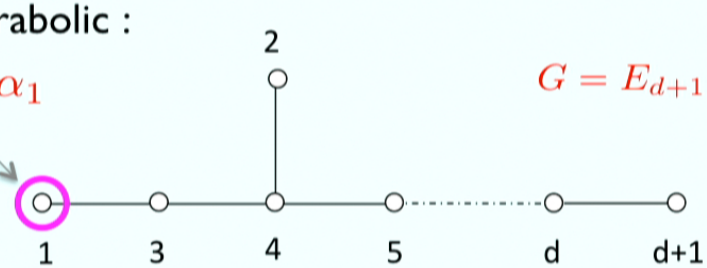
Choice of maximal parabolic :

B.c..'s require $\beta = \alpha_1$

With Levi sub-group

$L_{\alpha_1} = Spin(d,d)$

$G = E_{d+1}$ Dynkin diagram



Striking simplifications when $s = \frac{3}{2}, \quad s = \frac{5}{2}$ (for $\mathcal{R}^4$, $d^4\,\mathcal{R}^4$)

$E^G_{\alpha_1;\frac{3}{2}}\,\mathcal{R}^4$ $\quad E^G_{\alpha_1;\frac{5}{2}}\,d^4\mathcal{R}^4$ $\quad \mathcal{E}^G_{(0,1)}\,d^6\mathcal{R}^4$ Satisfies inhomogeneous Laplace equation for G

- Encodes STRING PERTURBATION RESULTS in compactified theories.
- D-INSTANTON contributions correspond to wrapped euclidean p-branes that fill out expected fractional BPS orbits – minimal (1/2-BPS), next-to-minimal (1/4-BPS)

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  Thresholds change their singularity structure as $r_d \to \infty$, due to condensation of Kaluza-Klein modes.

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