Title: Modular structure of Type IIB superstrings in the low energy expansion

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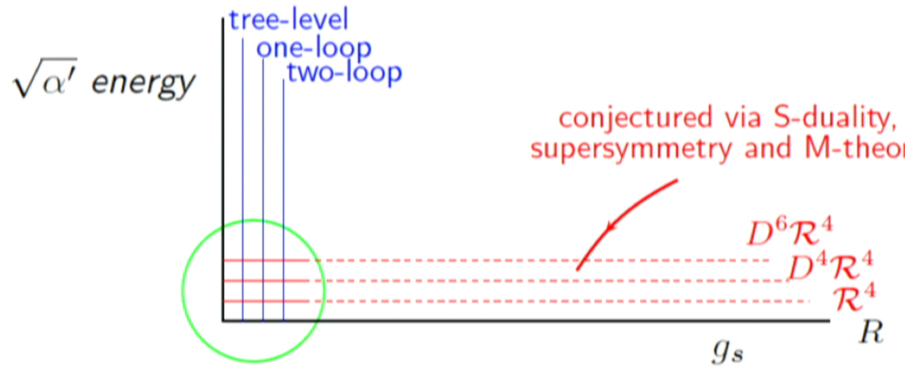
Abstract:

Expansions of Type IIB Superstring Theory



- **Superstring Perturbation Theory in powers of** g_s
  - holds for weak coupling g_s
  - but for all energies
- **Classical supergravity** R
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$
  - Conjectured for all couplings via S-duality, supersmmetry and M-theory

D-instantons and Eisenstein series

- **Full $\mathcal{R}^4$ effective interaction conjectured from D-instanton** [Green Gutperle 1997]

$$(T_2)^{\frac{1}{2}} E_{\frac{3}{2}}(T)\,\mathcal{R}^4 \qquad T = T_1 + iT_2 \qquad T_2 = \frac{1}{g_s}$$

- **The "non-holomorphic" Eisenstein series is defined by,**

$$E_s(T) = \sum_{(m,n)\neq(0,0)} \frac{(T_2)^s}{\pi^s |mT+n|^{2s}}$$

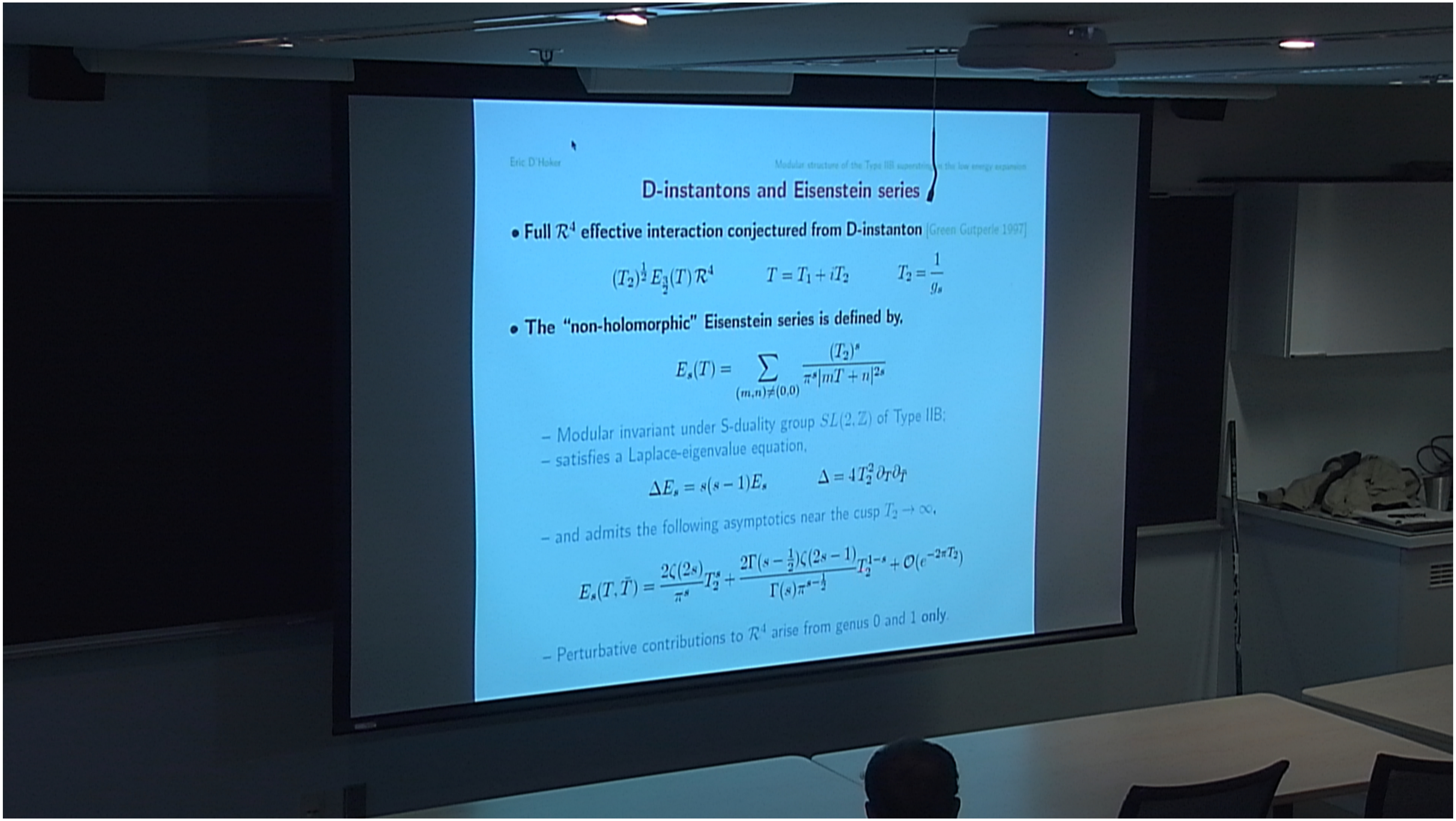
  - Modular invariant under S-duality group $SL(2,\mathbb{Z})$ of Type IIB;
  - satisfies a Laplace-eigenvalue equation,

$$\Delta E_s = s(s-1)E_s \qquad \Delta = 4\,T_2^2\,\partial_T \partial_{\bar{T}}$$

  - and admits the following asymptotics near the cusp $T_2 \to \infty$,

$$E_s(T,\bar{T}) = \frac{2\zeta(2s)}{\pi^s} T_2^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}} T_2^{1-s} + \mathcal{O}(e^{-2\pi T_2})$$

  - Perturbative contributions to $\mathcal{R}^4$ arise from genus 0 and 1 **only**.


Eric D'Hoker
Modular structure of the Type IIB superstring in the low energy expansion
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Supersymmetry and S-duality

- **Laplace-eigenvalue eq from space-time supersymmetry** [Green, Sethi, 1998]
  - Eisenstein series = unique modular solution with polynomial growth at cusp

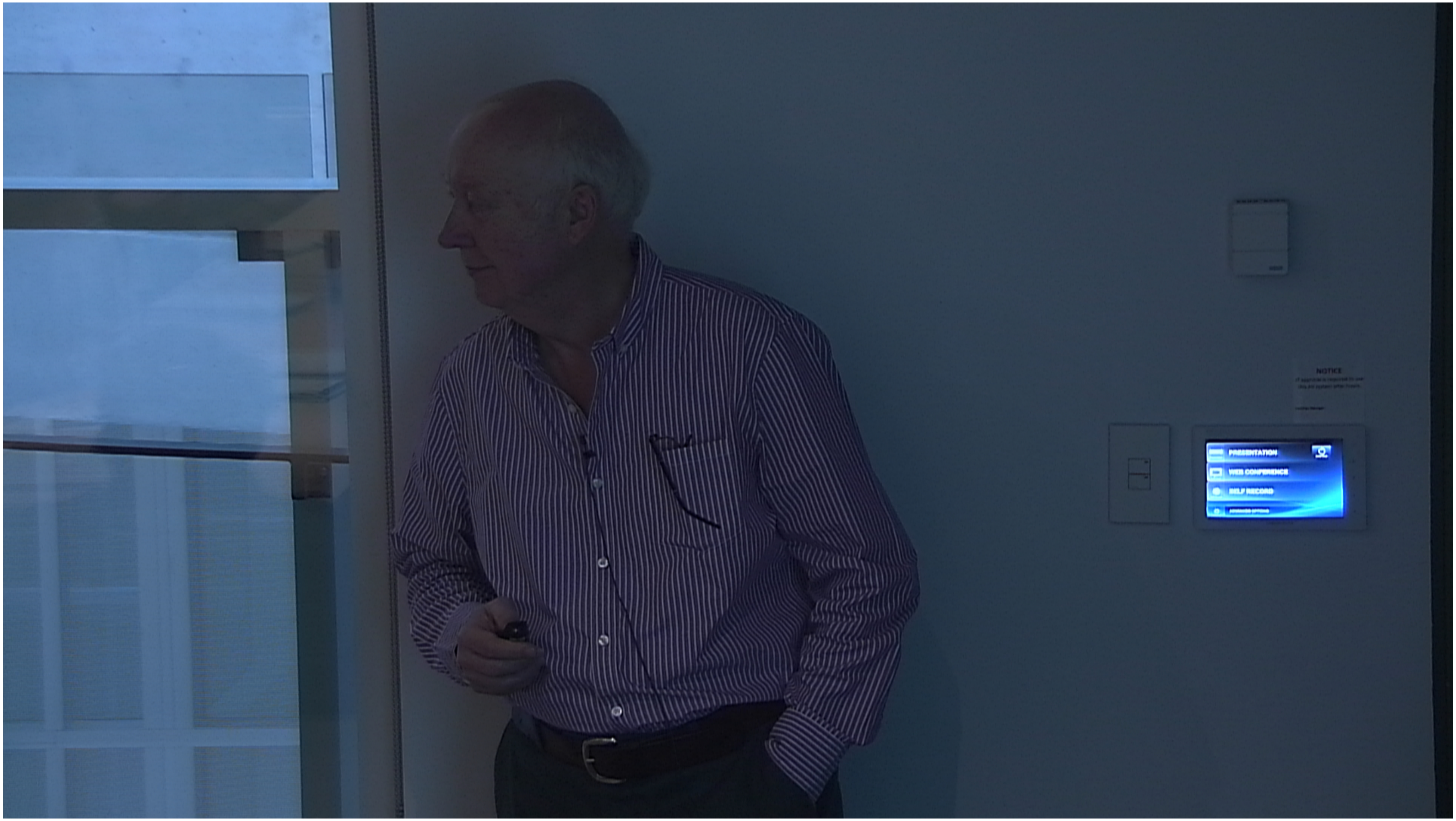
- **Predicts vanishing contributions for high enough loop order,**

$\mathcal{R}^4$	1/2 BPS	$h \geq 2$	$E_{\frac{3}{2}}$
$D^4\mathcal{R}^4$	1/4 BPS	$h \geq 3$	$E_{\frac{5}{2}}$
$D^6\mathcal{R}^4$	1/8 BPS	$h \geq 4$	$(\Delta - 12)\mathcal{E}_{D^6\mathcal{R}^4} = (E_{\frac{3}{2}})^2$

[Green, Gutperle, Vanhove 1997; Green, Vanhove 2005]

- **Predicts relations between non-vanishing contributions** (e.g. with tree-level),

$\mathcal{R}^4$	$h = 1$	[Green, Gutperle 1997]
$D^4\mathcal{R}^4$	$h = 2$	[ED, Gutperle, Phong 2005]
$D^6\mathcal{R}^4$	$h = 2$	[ED, Green, Pioline, Russo 2014]
	$h = 3$	[Gomez, Mafra 2013]


PRESENTATION
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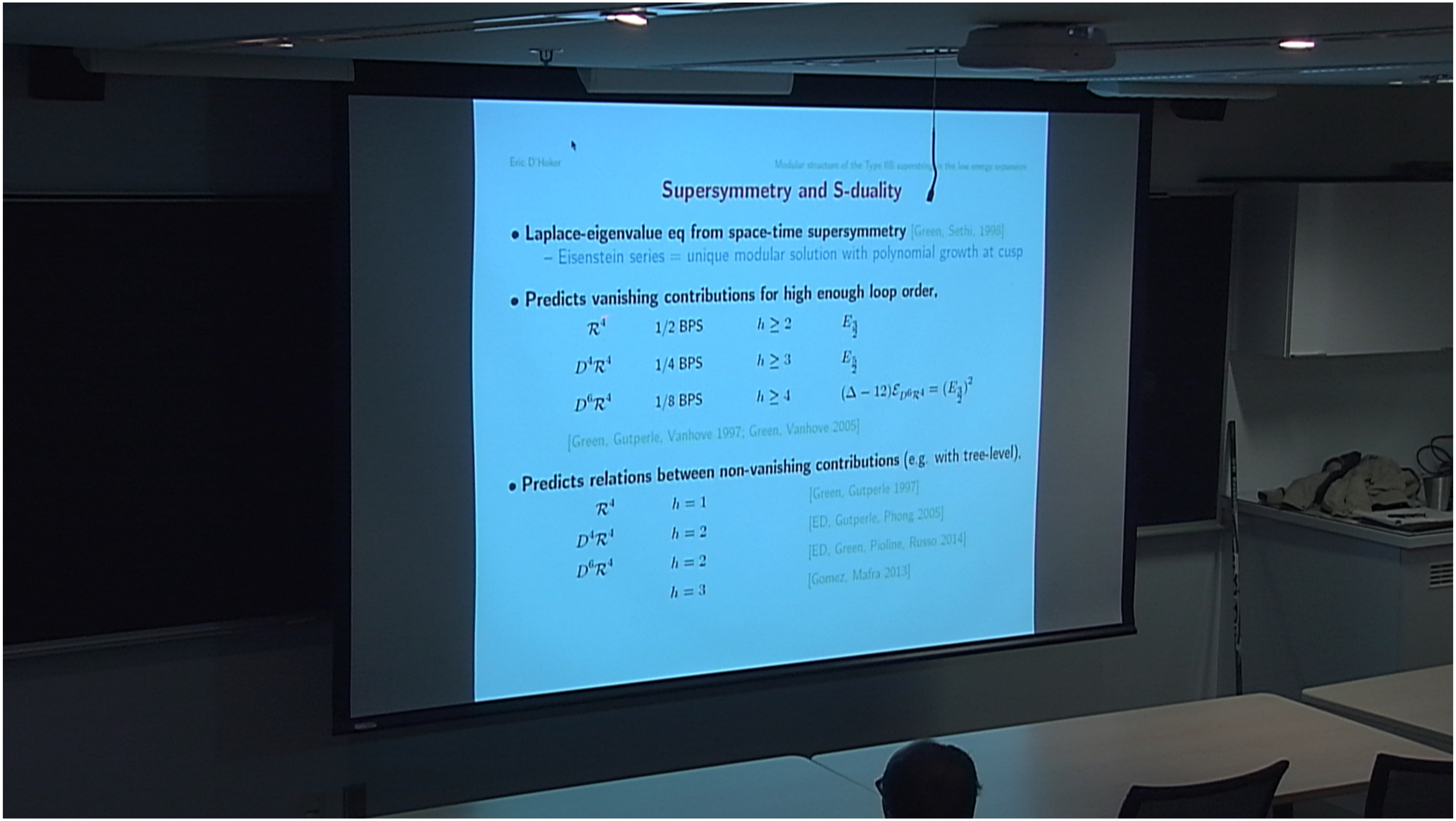
SELF RECORD



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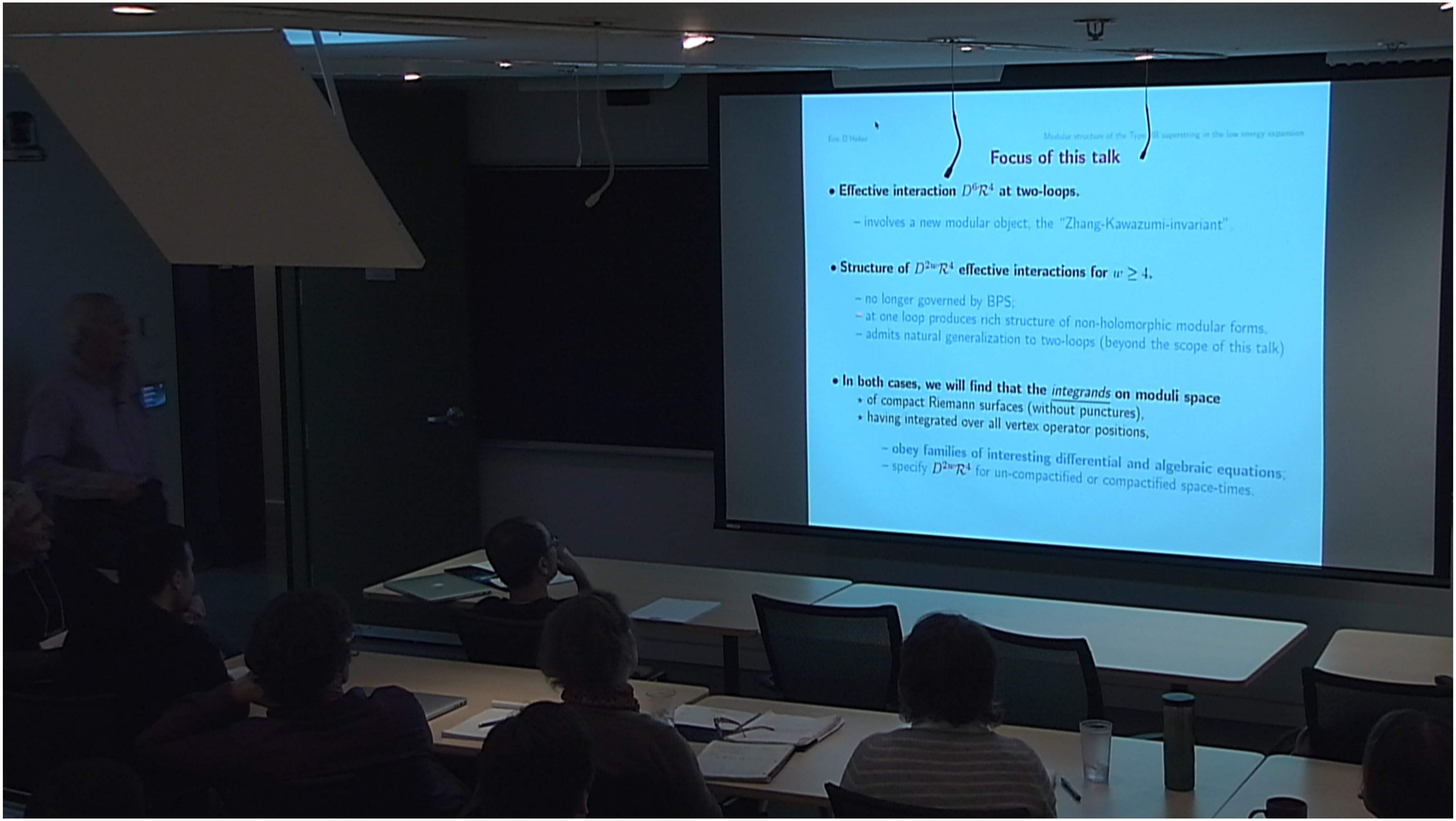
Focus of this talk
• Effective interaction $D^6\mathcal{R}^4$ at two-loops.
– involves a new modular object, the "Zhang-Kawazumi-invariant".
• Structure of $D^{2w}\mathcal{R}^4$ effective interactions for $w \geq 4$.
– no longer governed by BPS;
– at one loop produces rich structure of non-holomorphic modular forms.
– admits natural generalization to two-loops (beyond the scope of this talk)
• In both cases, we will find that the integrands on moduli space
⋆ of compact Riemann surfaces (without punctures),
⋆ having integrated over all vertex operator positions,
– obey families of interesting differential and algebraic equations;
– specify $D^{2w}\mathcal{R}^4$ for un-compactified or compactified space-times.

The effective interaction $D^6\mathcal{R}^4$ at genus-two

- **Start with Type II four-graviton amplitude at genus 2,** [ED, Phong 2005]

$$\mathcal{A}^{(2)} = \frac{\pi}{64}\kappa^2\mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}^{(2)}(s,t,u|\Omega)$$

$$\mathcal{B}^{(2)} = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} G(i,j)$$

  - $\mathcal{M}_2$ is the moduli space with Siegel volume form $d\mu_2$;
  - $G(i,j)$ is the scalar Green function;
  - $\mathcal{Y} = (s-t)\Delta(1,3) \wedge \Delta(4,2) + 2$ permutations;
  - $\Delta(i,j)$ is a holomorphic $(1,0)_i \otimes (1,0)_j$ form independent of s,t,u.

- Contributions produced to local effective interactions
  - $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s=t=u=0$;
  - $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
  - $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}^{(2)} = 32(s^2+t^2+u^2) + 192\, stu\, \varphi(\Omega) + \mathcal{O}(s^4, \cdots, u^4)$$

  - $\varphi(\Omega)$ coincides with the Zhang -Kawazumi invariant [ED, Green 2013].

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The Zhang-Kawazumi invariant for genus-two

- **Definition of the ZK-invariant**
  - Let A_I, B_I be canonical homology basis for $H_1(\Sigma, \mathbb{Z})$,
  - ω_I dual holomorphic (1,0) forms normalized via,

$$\oint_{A_I} \omega_J = \delta_{IJ} \qquad \oint_{B_I} \omega_J = \Omega_{IJ} = X_{IJ} + iY_{IJ}$$

then the ZK-invariant takes the following form,

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2Y^{-1}_{IL} Y^{-1}_{JK}\right) \int_{\Sigma^2} G(x,y)\omega_I(x)\overline{\omega_J(x)}\omega_K(y)\overline{\omega_L(y)}$$

  - invariant under the modular group $Sp(4, \mathbb{Z})$
  - equivalent to definition via Arakelov geometry [Zhang 2007, Kawazumi 2008]
  - related to the genus-two Faltings invariant [De Jong 2010]

- **Direct evaluation of $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$ appeared out of reach ... until ...**

Evidence

- **Initial indications from $D^6\mathcal{R}^4$ interaction for compactification on $\mathbb{T}^d$,**

$$\mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = \pi \int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega) \, \Gamma_{d,d,2}(\rho_d|\Omega)$$

  - $\Gamma_{d,d,2}$ is the torus partition function
    - ⋆ dependent on $\rho_d = G + B \in SO(d,d,\mathbb{R})/SO(d,\mathbb{R}) \times SO(d,\mathbb{R})$
    - ⋆ satisfies $\left(2\Delta - \Delta_{SO(d,d)} + 3d - d^2\right) \Gamma_{d,d,2} = 0$;
  - Susy & duality conjectured relation with genus-one $\mathcal{E}^{(1)}_{\mathcal{R}^4}$ (for $d \neq 2$)

$$\left(\Delta_{SO(d,d)} - (d+2)(5-d)\right)\mathcal{E}^{(2)}_{D^6\mathcal{R}^4} = -\left(\mathcal{E}^{(1)}_{\mathcal{R}^4}\right)^2$$

  - Elimination of $\Delta_{SO(d,d)}$ gives,

$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)(\Delta - 5)\Gamma_{d,d,2}(\rho_d, \Omega) = -\frac{\pi}{2}\left(\int_{\mathcal{M}_1} d\mu_1 \Gamma_{d,d,1}(\rho, \tau)\right)^2$$

integration by parts, and notice no d-dependence in eigenvalue !

- **Further evidence from asymptotics of φ** [via De Jong 2012, Wentworth 1991]

Proof via deformations of complex structures

- **Laplacian Δ on genus-two moduli space $\mathcal{M}_2$**
  - = Laplace-Beltrami operator for the Siegel metric on upper half space
  - – In terms of the period matrix $\Omega_{IJ} = X_{IJ} + iY_{IJ}$, with $I, J = 1, 2$

$$\Delta = \sum_{I \leq J} \sum_{K \leq L} Y_{IK} Y_{JL} \frac{\partial}{\partial \bar\Omega_{IJ}} \frac{\partial}{\partial \Omega_{KL}}$$

- **Variations in Ω_{IJ} result from variation by Beltrami differential μ**

$$\delta_\mu \phi = \frac{1}{2\pi} \int_\Sigma d^2 w \, \mu_{\bar w}{}^w \, \delta_{ww} \phi$$

  - – $\delta_{ww}\phi$ is obtained by variation of $\bar\partial$ or insertion of the stress tensor T_{ww}

$$\begin{aligned} \delta_{ww} \Omega_{IJ} &= 2\pi i \, \omega_I(w) \omega_J(w) \\ \delta_{ww} \omega_I(x) &= \omega_I(w) \partial_x \partial_w \ln E(x,w) \\ \delta_{ww} G(x,y) &= -\partial_w G(w,x) \partial_w G(w,y) + \cdots \end{aligned}$$

- **Careful calculation of mixed derivatives proves $(\Delta - 5)\varphi = 0$ inside $\mathcal{M}_2$**
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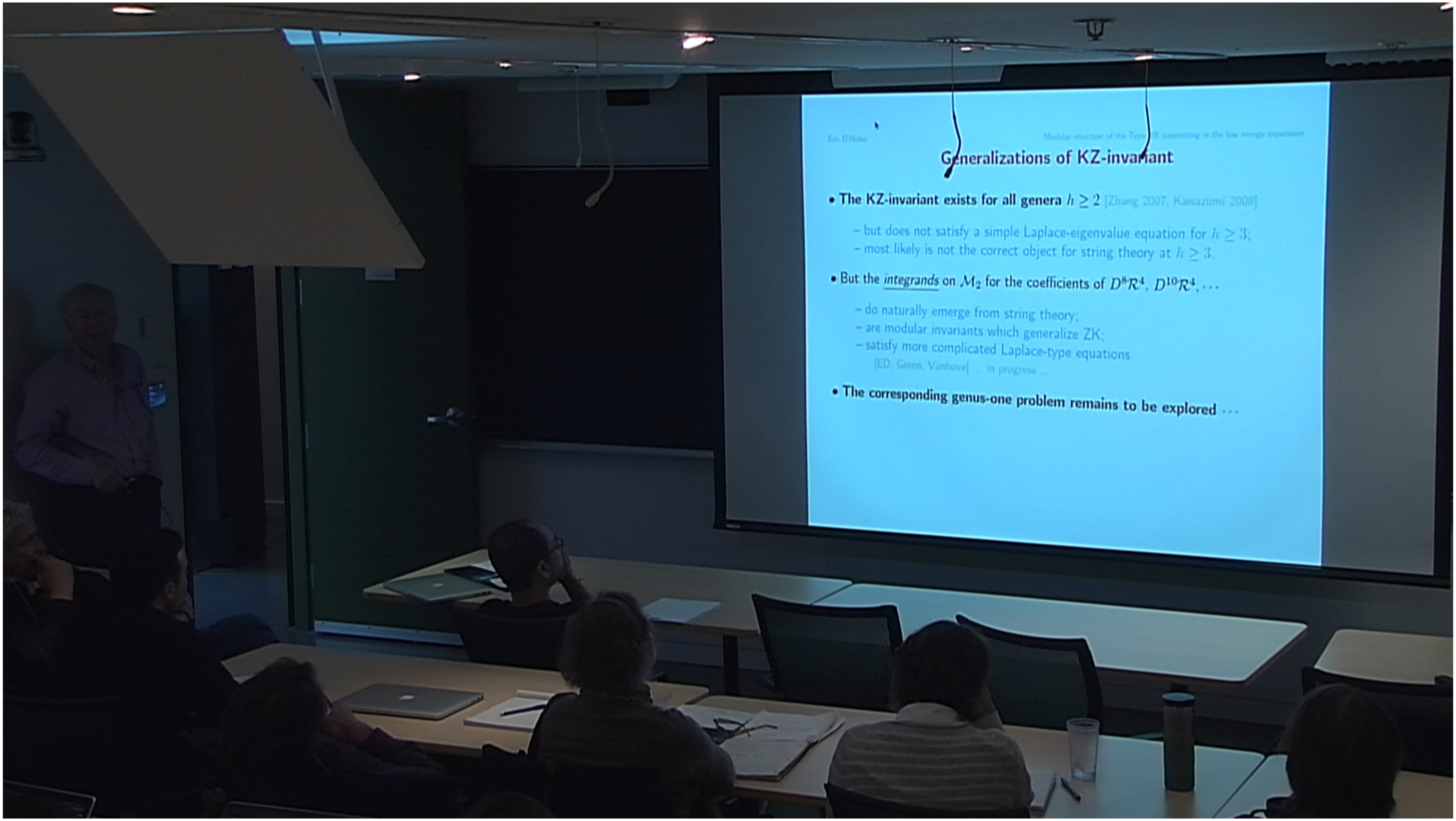
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Generalizations of KZ-invariant
• The KZ-invariant exists for all genera $h \geq 2$ [Zhang 2007, Kawazumi 2008]
– but does not satisfy a simple Laplace-eigenvalue equation for $h \geq 3$;
– most likely is not the correct object for string theory at $h \geq 3$.
• But the *integrands* on $\mathcal{M}_2$ for the coefficients of $D^8\mathcal{R}^4$, $D^{10}\mathcal{R}^4$, $\cdots$
– do naturally emerge from string theory;
– are modular invariants which generalize ZK;
– satisfy more complicated Laplace-type equations
[ED, Green, Vanhove] ... in progress ...
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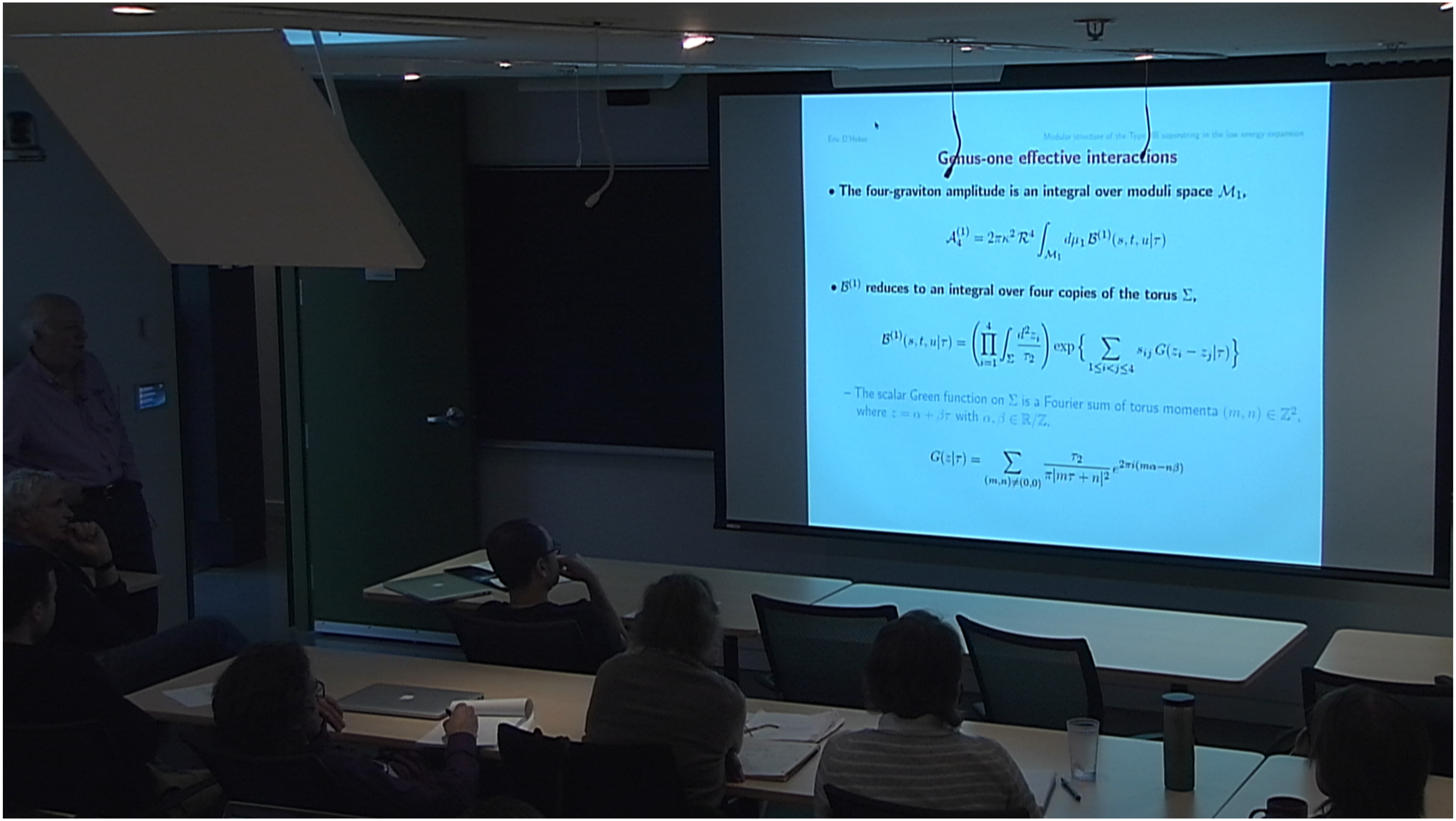
Genus-one effective interactions
• The four-graviton amplitude is an integral over moduli space $\mathcal{M}_1$,
$\mathcal{A}_4^{(1)} = 2\pi\kappa^2 \mathcal{R}^4 \int_{\mathcal{M}_1} d\mu_1 \mathcal{B}^{(1)}(s,t,u|\tau)$
• $\mathcal{B}^{(1)}$ reduces to an integral over four copies of the torus Σ,
$\mathcal{B}^{(1)}(s,t,u|\tau) = \left(\prod_{i=1}^{4} \int_\Sigma \frac{d^2 z_i}{\tau_2} \right) \exp\left\{ \sum_{1 \leq i < j \leq 4} s_{ij}\, G(z_i - z_j|\tau) \right\}$
– The scalar Green function on Σ is a Fourier sum of torus momenta $(m,n) \in \mathbb{Z}^2$, where $z = \alpha + \beta\tau$ with $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$.
$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi |m\tau + n|^2} e^{2\pi i(m\alpha - n\beta)}$

Worldsheet Feynman diagrams

- **Expansion in powers of s_{ij} organized in "worldsheet Feynman diagrams"**
  - Each integration point z_i on Σ is represented by a vertex;
  - Each Green function $G(z_i - z_j|\tau)$ by a line —— between z_i and z_j;
  - Diagrams with a single G ending in a point vanish by $\int_\Sigma d^2z\, G(z|\tau) = 0$
  - A diagram with w lines of G,
    - ⋆ has *weight* w;
    - ⋆ contributes to $D^{2w}\mathcal{R}^4$.


Worldsheet Feynman diagrams (connected)
$D^4\mathcal{R}^4$
$D^6\mathcal{R}^4$
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$D^{10}\mathcal{R}^4$
Eisenstein
$C_{a,b,c}$

Kronecker-Eisenstein series

- **One-loop worldsheet Feynman diagrams generate Eisenstein series.**
  - for example to order $s^2 + t^2 + u^2$

$$\int_\Sigma \frac{d^2 z}{\tau_2} G(z|\tau)^2 = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^2}{\pi^2 |m\tau + n|^4} = E_2(\tau)$$

- **Two-loop Feynman diagrams generate "Kronecker-Eisenstein series".**

$$C_{a_1,a_2,a_3}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2} \right)^{a_r}$$

- The total worldsheet momenta $m = m_1 + m_2 + m_3$, $n = n_1 + n_2 + n_3$ vanish;
- the *weight* is $w = a_1 + a_2 + a_3$;

- For our diagrams we have $a_r \geq 1$ and the sums converge;
- $C_{a_1,a_2,a_3}(\tau)$ is a modular function under $SL(2, \mathbb{Z})$.

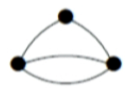
Examples at low weight w

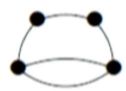
- **We find inhomogeneous Laplace-eigenvalue equations,**

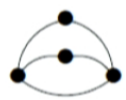
$w = 3$ $\quad C_{1,1,1} =$  $\quad \Delta C_{1,1,1} = 6E_3$

- Use $\Delta E_3 = 6E_3$ to get $\Delta(C_{1,1,1} - E_3) = 0$;
- constant determined from asymptotics $C_{1,1,1} = E_3 + \zeta(3)$

  (obtained earlier by Zagier using direct calculation of sums)

$w = 4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $\quad C_{2,2,1} =$  $\quad \Delta C_{2,2,1} = 8E_5$

- Note eigenvalues of the form $s(s-1)$ for $s = 1, 2, 3$;

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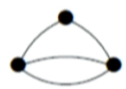
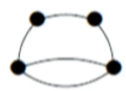
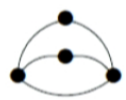
$w = 4$ $C_{2,1,1} =$  $(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w = 5$ $C_{3,1,1} =$  $(\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w = 5$ $C_{2,2,1} =$  $\Delta C_{2,2,1} = 8E_5$

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Structure Theorem for $C_{a,b,c}$ modular functions

- $C_{a,b,c}(\tau)$ are linear combinations of modular functions $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ which satisfy

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}, \zeta(s'')\big)$$

  - an inhomogeneous eigenvalue equation of weight $w = a + b + c$;
  - $\mathfrak{F}$ is a polynomial of degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;
  - depends on $\zeta(s'')$ for s'' an odd integer $3 \leq s'' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w = 3$	$s = 1$	$0^{(1)}$
$w = 4$	$s = 2$	$2^{(1)}$
$w = 5$	$s = 1, 3$	$0^{(1)} \oplus 6^{(1)}$
$w = 6$	$s = 2, 4$	$2^{(1)} \oplus 12^{(2)}$
$w = 7$	$s = 1, 3, 5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$
$w = 8$	$s = 2, 4, 6$	$2^{(1)} \oplus 12^{(2)} \oplus 30^{(2)}$

The generating function

- **There is a natural generating function,**

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{a,b,c=1}^{\infty} t_1^{a-1} t_2^{b-1} t_3^{c-1} C_{a,b,c}(\tau)$$

Summing gives the sunset diagram for three scalars with masses $M_r^2 = -t_r \tau_2$,

$$\mathcal{W}(t_1, t_2, t_2|\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\, \delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi |m_r \tau + n_r|^2 - t_r \tau_2} \right)$$

- **Algebraic representation of Laplacian induces differential action on $\mathcal{W}$,**

$$\Delta \mathcal{W} - \mathfrak{L}^2 \mathcal{W} = \mathfrak{R}$$

$$\mathfrak{D} = t_1 \partial_1 + t_2 \partial_2 + t_3 \partial_3$$

$$\mathfrak{L}^2 = \mathfrak{D}^2 + \mathfrak{D} + (t_1^2 + t_2^2 + t_3^2 - 2t_1 t_2 - 2t_2 t_3 - 2t_3 t_1)(\partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1)$$

$\mathfrak{R}$ = quadratic polynomial in the Eisenstein series E_s

Proof via generating function

- **Permutations of (a,b,c) induces permutations of (t_1,t_2,t_3)**
  - $\mathfrak{S}_3$ adapted coordinates,

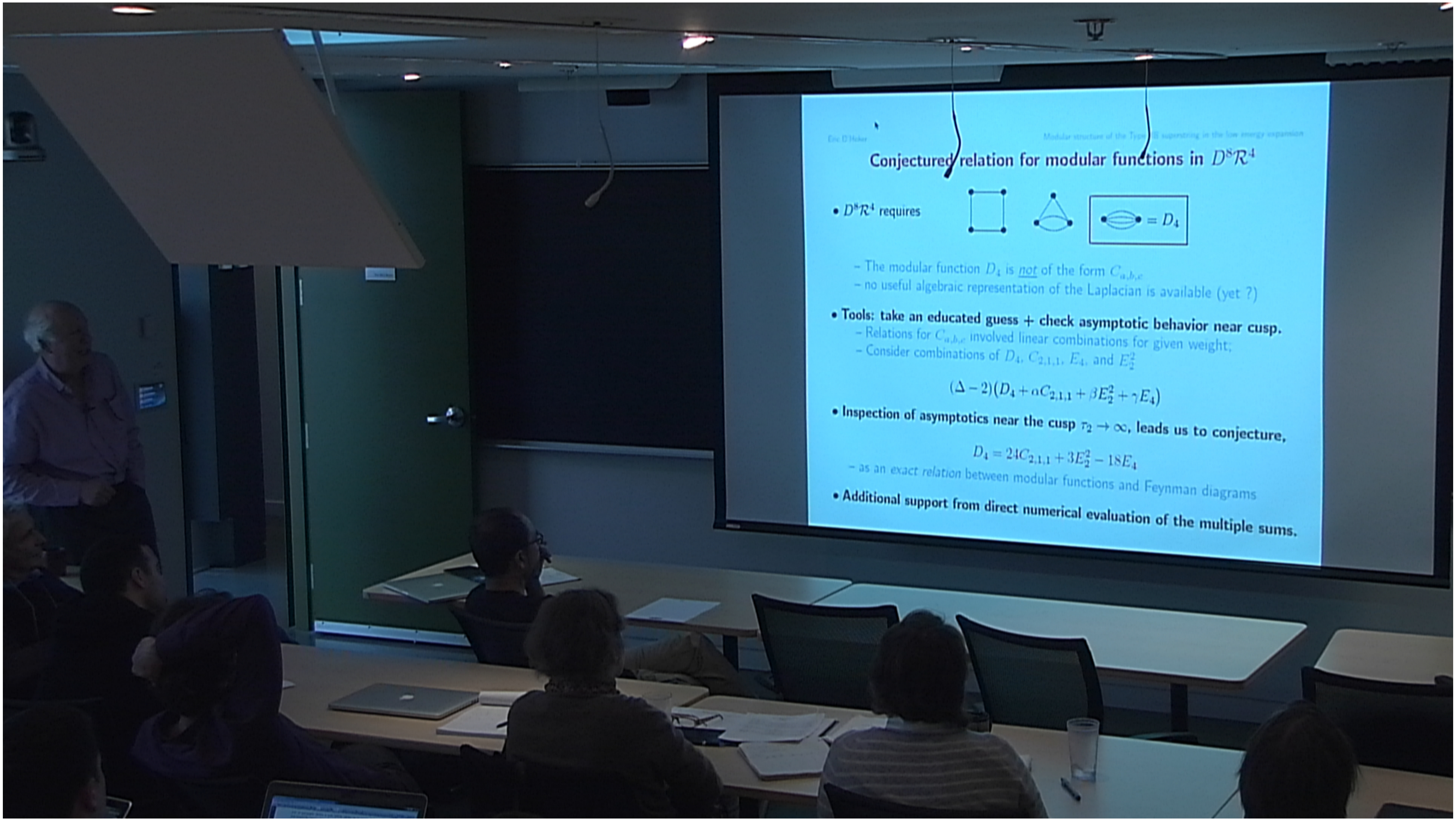
$$
\begin{aligned}
u &= t_1+t_2+t_3 & \varepsilon &= e^{2\pi i/3} \\
v/\sqrt{2} &= t_1+\varepsilon t_2+\varepsilon^2 t_3 & (t_1,t_3,t_2)(u,v,\bar v) &= (u,\bar v,v) \\
\bar v/\sqrt{2} &= t_1+\varepsilon^2 t_2+\varepsilon t_3 & (t_2,t_3,t_1)(u,v,\bar v) &= (u,\varepsilon^2 v,\varepsilon \bar v)
\end{aligned}
$$

  - $\mathfrak{L}^2=\mathfrak{L}_0^2-\mathfrak{L}_1^2-\mathfrak{L}_2^2$ Casimir of $SO(1,2)$ generated by $\mathfrak{L}_0,\mathfrak{L}_1,\mathfrak{L}_2$;
  - Simultaneously diagonalize the $\mathfrak{S}_3$-invariant operators $\mathfrak{D}$, $\mathfrak{L}_0^2$, and $\mathfrak{L}^2$

$$
\begin{aligned}
\mathfrak{D}\mathcal{W}_{w;s;\mathfrak{p}} &= w\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{D} &= t_1\partial_1+t_2\partial_2+t_3\partial_3 \\
\mathfrak{L}^2\mathcal{W}_{w;s;\mathfrak{p}} &= s(s-1)\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}^2 &= -(u^2-2v\bar v)(\partial_u^2-2\partial_v\partial_{\bar v}) \\
\mathfrak{L}_0^2\mathcal{W}_{w;s;\mathfrak{p}} &= -9\mathfrak{p}^2\mathcal{W}_{w;s;\mathfrak{p}} & \mathfrak{L}_0 &= iv\partial_v-i\bar v\partial_{\bar v}
\end{aligned}
$$

  - $\mathfrak{S}_3$-invariance of eigenfunctions requires $\mathfrak{p}$ to be integer;
  - which explains multiplicities $[(s-1)/3]$.

$\Longrightarrow$ constructive proof of Structure Theorem.



Conjectured relation for modular functions in $D^8\mathcal{R}^4$
• $D^8\mathcal{R}^4$ requires
$= D_4$
– The modular function D_4 is *not* of the form $C_{a,b,c}$
– no useful algebraic representation of the Laplacian is available (yet ?)
• Tools: take an educated guess + check asymptotic behavior near cusp.
– Relations for $C_{a,b,c}$ involved linear combinations for given weight;
– Consider combinations of D_4, $C_{2,1,1}$, E_4, and E_2^2
$(\Delta - 2)(D_4 + \alpha C_{2,1,1} + \beta E_2^2 + \gamma E_4)$
• Inspection of asymptotics near the cusp $\tau_2 \to \infty$, leads us to conjecture,
$D_4 = 24 C_{2,1,1} + 3E_2^2 - 18 E_4$
– as an *exact relation* between modular functions and Feynman diagrams
• Additional support from direct numerical evaluation of the multiple sums.

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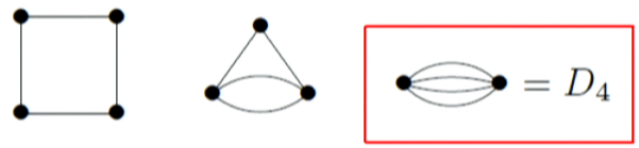

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Structure of the asymptotics near the cusp

- The expansion near the cusp $\tau_2 \to \infty$ takes the following form,

$$D_4(\tau) = \sum_{k,\bar{k}=0}^{\infty} \mathcal{D}_4^{(k,\bar{k})}(\pi\tau_2)\, q^k \bar{q}^{\bar{k}} \qquad\qquad q = e^{2\pi i \tau}$$

- We checked the following asymptotics (similarly for $C_{2,1,1}$, E_4, E_2^2)

$$\begin{aligned}
\mathcal{D}_4^{(0,0)}(y) &= \frac{y^4}{945} + \frac{2\zeta(3)y}{3} + \frac{10\zeta(5)}{y} - \frac{3\zeta(3)^2}{y^2} + \frac{9\zeta(7)}{4y^3} \\
\mathcal{D}_4^{(0,1)}(y) &= \frac{4y^2}{15} + \frac{2y}{3} + 2 + \frac{4}{y} + \frac{12\zeta(3)}{y} - \frac{6\zeta(3)}{y^2} + \frac{9}{2y^2} + \frac{9}{4y^3} \\
\mathcal{D}_4^{(1,0)}(y) &= \mathcal{D}_4^{(0,1)}(y)
\end{aligned}$$

How could the conjecture fail ?

- Consider the difference $F = D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4$
  - the conjecture states $F = 0$
- If the conjecture were to fail, then $F \neq 0$ and its properties are,
  - modular function under $SL(2,\mathbb{Z})$;
  - its pure power part in the expansion near the cusp vanishes;
    - $\Longrightarrow$ F is a *cuspidal function*
  - Vanishing of leading exponential restricts it further.

Progress towards a full proof

- **Inspired by a calculation of Zagier for C_{111}, we first perform n-sums**

$$D_4(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{4} \frac{\tau_2}{\pi|m_r\tau + n_r|^2}$$

  - partition sum according to the number of vanishing m_r;
  - solve for n_4; decompose in partial fractions in n_3; sum over n_3,
  - for $m_3 \neq 0$, sum using the formula (and its derivatives in z)

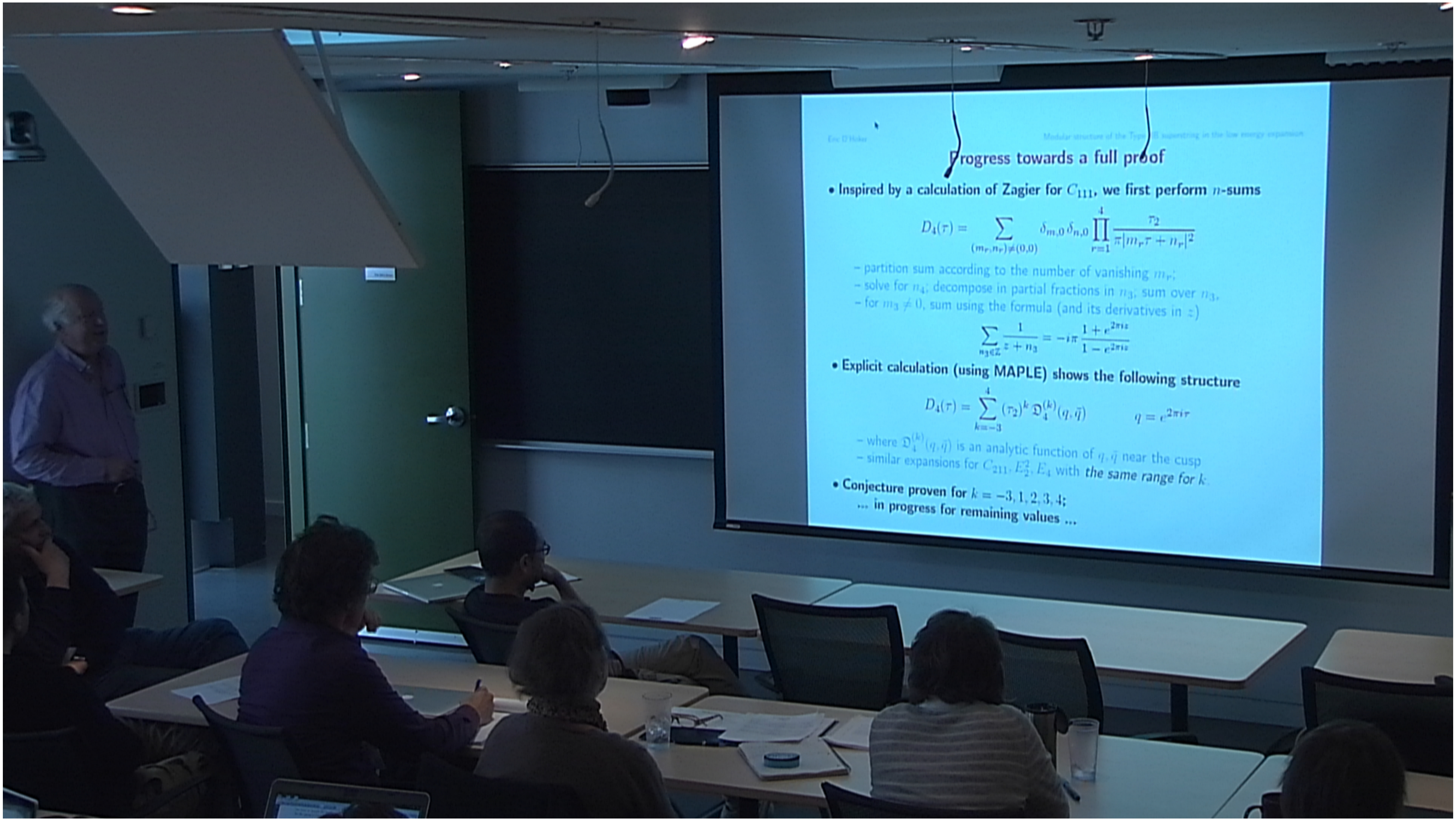
$$\sum_{n_3\in\mathbb{Z}} \frac{1}{z+n_3} = -i\pi \frac{1+e^{2\pi i z}}{1-e^{2\pi i z}}$$

- **Explicit calculation (using MAPLE) shows the following structure**

$$D_4(\tau) = \sum_{k=-3}^{4} (\tau_2)^k\, \mathfrak{D}_4^{(k)}(q,\bar q) \qquad\qquad q = e^{2\pi i \tau}$$

  - where $\mathfrak{D}_4^{(k)}(q,\bar q)$ is an analytic function of $q,\bar q$ near the cusp
  - similar expansions for C_{211}, E_2^2, E_4 with *the same range for k.*

- **Conjecture proven for $k = -3, 1, 2, 3, 4$;**
  ... in progress for remaining values ...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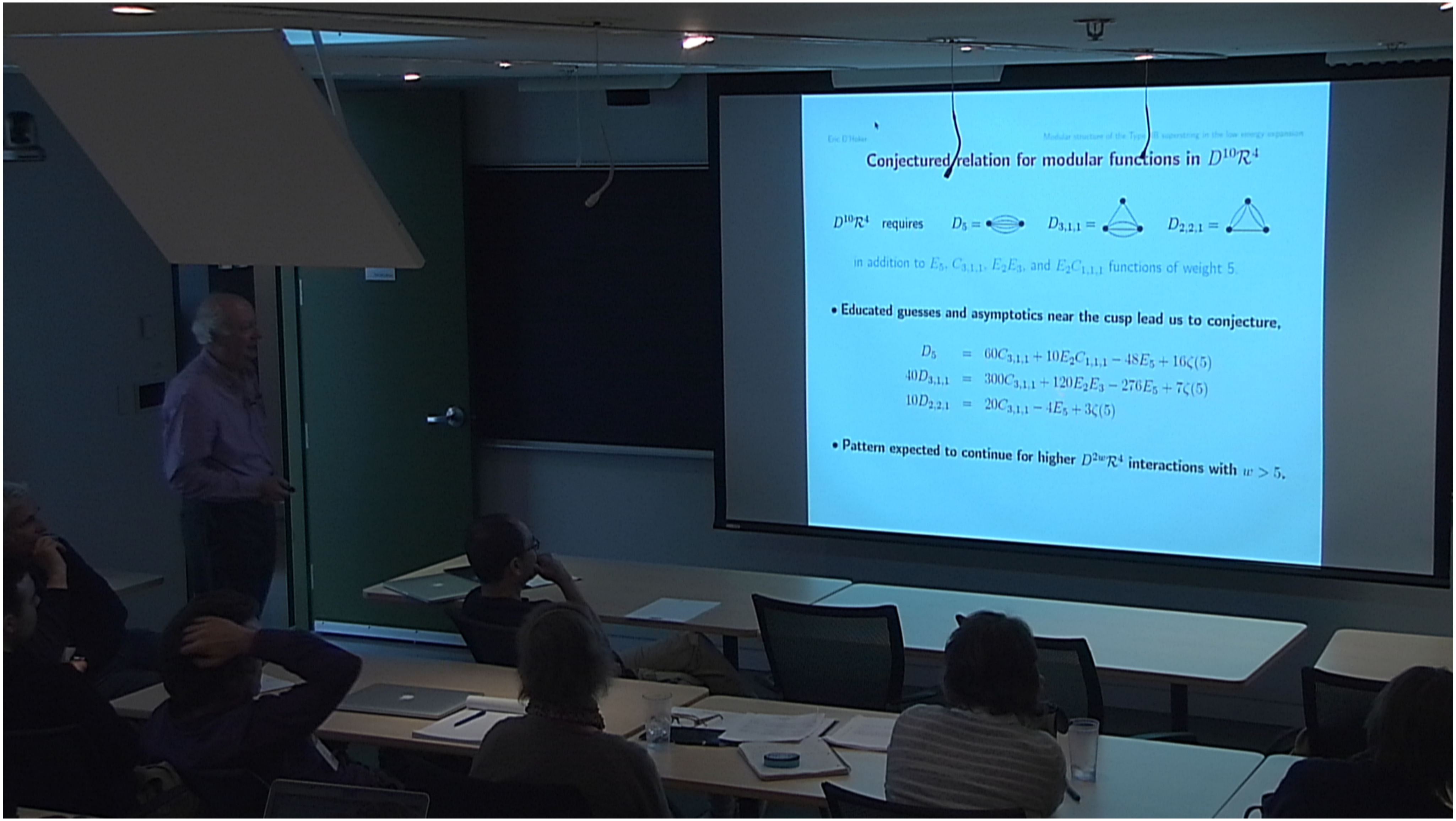
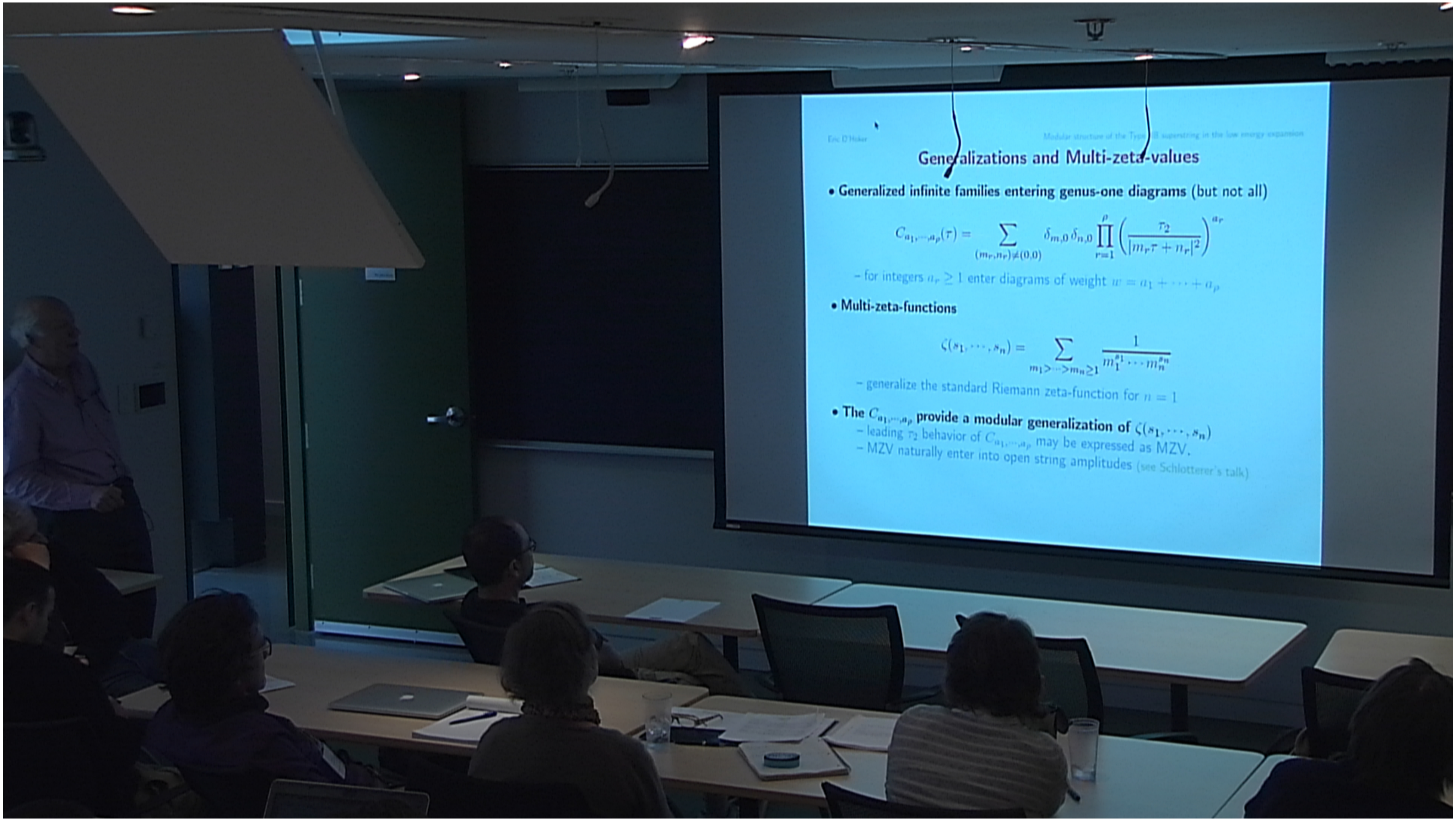

Eric D'Hoker
Conjectured relation for modular functions in $D^{10}\mathcal{R}^4$
$D^{10}\mathcal{R}^4$ requires $D_5 =$ $D_{3,1,1} =$ $D_{2,2,1} =$
in addition to E_5, $C_{3,1,1}$, E_2E_3, and $E_2C_{1,1,1}$ functions of weight 5.
• Educated guesses and asymptotics near the cusp lead us to conjecture,
$D_5 = 60C_{3,1,1} + 10E_2C_{1,1,1} - 48E_5 + 16\zeta(5)$
$40D_{3,1,1} = 300C_{3,1,1} + 120E_2E_3 - 276E_5 + 7\zeta(5)$
$10D_{2,2,1} = 20C_{3,1,1} - 4E_5 + 3\zeta(5)$
• Pattern expected to continue for higher $D^{2w}\mathcal{R}^4$ interactions with $w > 5$.


Eric D'Hoker
Modular structure of the Type IIB superstring in the low energy expansion
Generalizations and Multi-zeta-values
• Generalized infinite families entering genus-one diagrams (but not all)
$$C_{a_1,\cdots,a_p}(\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{p} \left(\frac{\tau_2}{|m_r\tau + n_r|^2} \right)^{a_r}$$
– for integers $a_r \geq 1$ enter diagrams of weight $w = a_1 + \cdots + a_p$
• Multi-zeta-functions
$$\zeta(s_1, \cdots, s_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}}$$
– generalize the standard Riemann zeta-function for $n = 1$
• The $C_{a_1,\cdots,a_p}$ provide a modular generalization of $\zeta(s_1, \cdots, s_n)$
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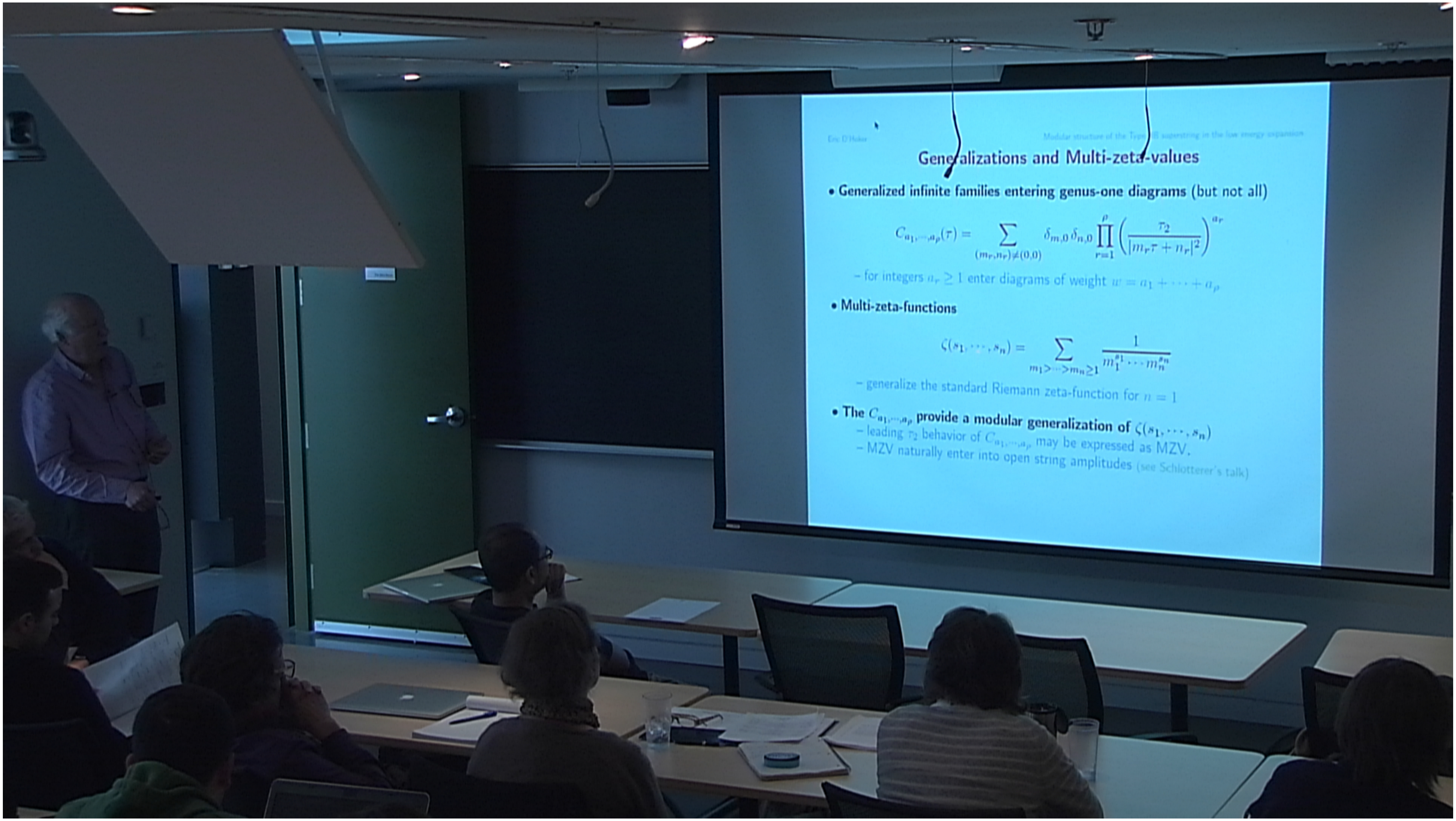
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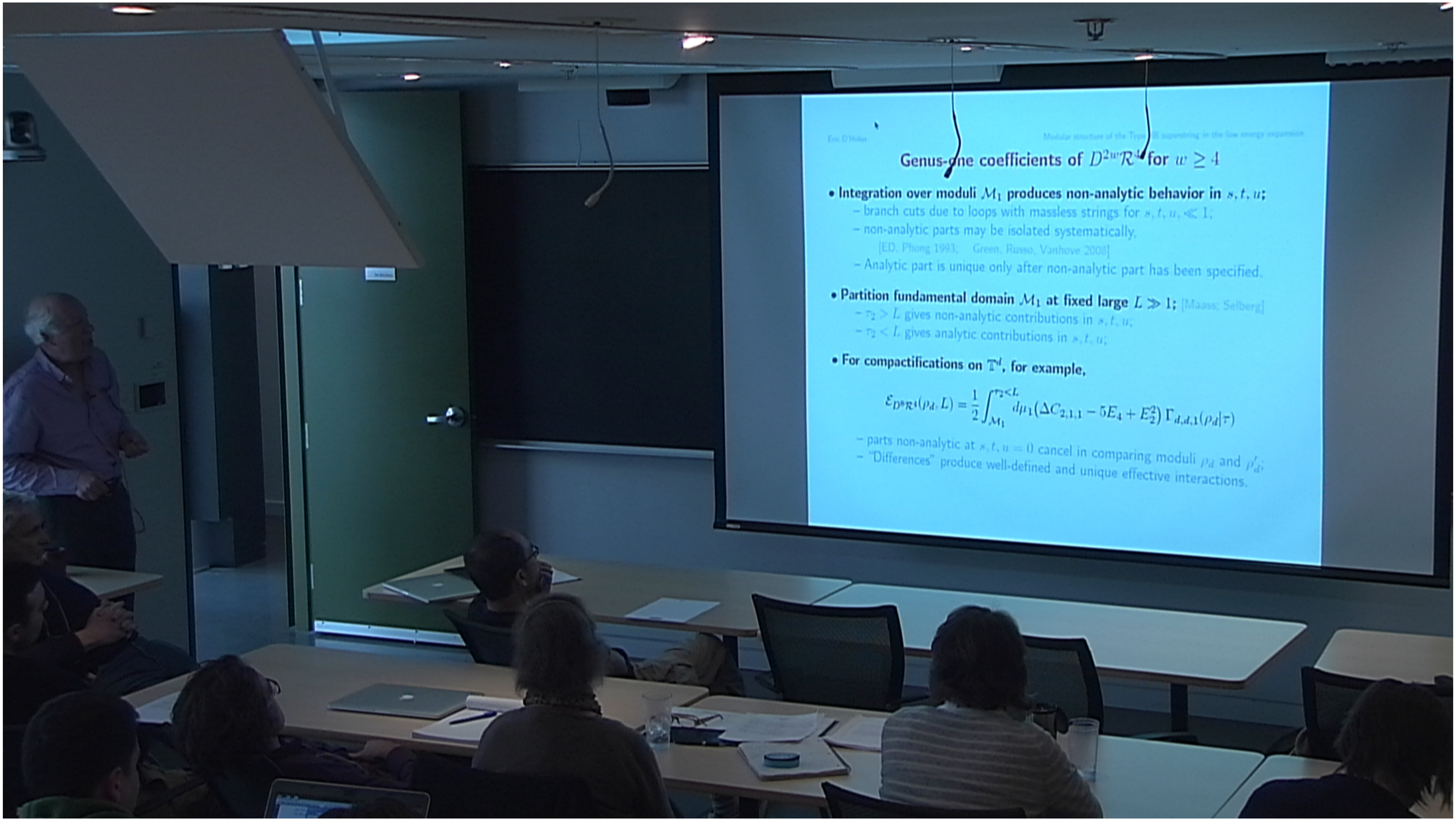
Genus-one coefficients of $D^{2w}\mathcal{R}^4$ for $w \geq 4$
• Integration over moduli $\mathcal{M}_1$ produces non-analytic behavior in s, t, u;
– branch cuts due to loops with massless strings for $s, t, u, \ll 1$;
– non-analytic parts may be isolated systematically.
[ED, Phong 1993; Green, Russo, Vanhove 2008]
– Analytic part is unique only after non-analytic part has been specified.
• Partition fundamental domain $\mathcal{M}_1$ at fixed large $L \gg 1$; [Maass; Selberg]
– $\tau_2 > L$ gives non-analytic contributions in s, t, u;
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• For compactifications on $\mathbb{T}^d$, for example,
$\mathcal{E}_{D^8\mathcal{R}^4}(\rho_d, L) = \frac{1}{2}\int_{\mathcal{M}_1}^{\tau_2<L} d\mu_1 (\Delta C_{2,1,1} - 5E_4 + E_2^2)\, \Gamma_{d,d,1}(\rho_d|\tau)$
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  - non-holomorphic Kronecker-Eisenstein series on genus-one Riemann surfaces;
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  - differential and algebraic interrelations;
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- **Extensions at genus-one**
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  - Identify structure of the ring of all such non-holomorphic modular forms.
  - Equations obeyed by entire string integrand ? [ED, Green] ... in progress ...
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