Title: Explorations in String Theory -7

Date: Apr 14, 2015 11:30 AM

URL: http://pirsa.org/15040146

Abstract:

Simplest Example:

1 cut solution

$$E = \frac{4\lambda n^2 \alpha(1-\alpha)\pi^2}{L}$$

$\frac{N}{L}$

from $u \simeq 0$ also

lytics }

$$\frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K}(u-u_j^*)}$$

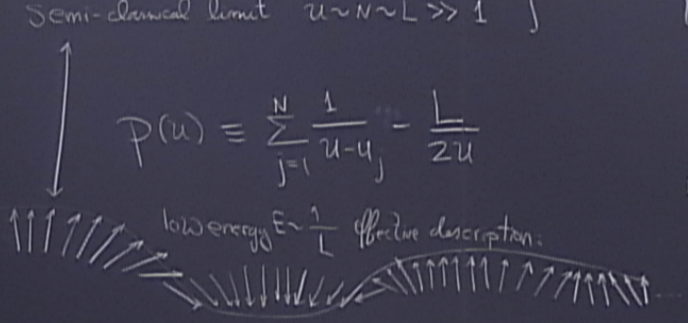
$\sim$ rational fn

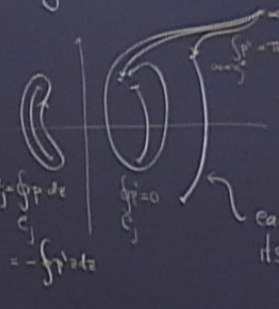
$2K+3$ parameters

{ Solutions to Bethe Equations in the Semi-classical limit $u \sim N \sim L \gg 1$ } $\longleftrightarrow$ { Algebraic Curves w/ prescribed analytics }

$$P(u) \equiv \sum_{j=1}^{N} \frac{1}{u-u_j} - \frac{L}{2u}$$

low energy $E \sim \frac{1}{L}$ effective description:





$$(P^1)^2 = \frac{\left(\sum_{j=0}^{K} C_j x^j\right)^2}{u^4 \prod_{j=1}^{2K}(u-u_j^*)}$$

$\sim$ rational fn w/ $3K$ parameters $+1$

$= 3K-1 + 2$

$3K-1$: picture; $+2$: $u \simeq 0$

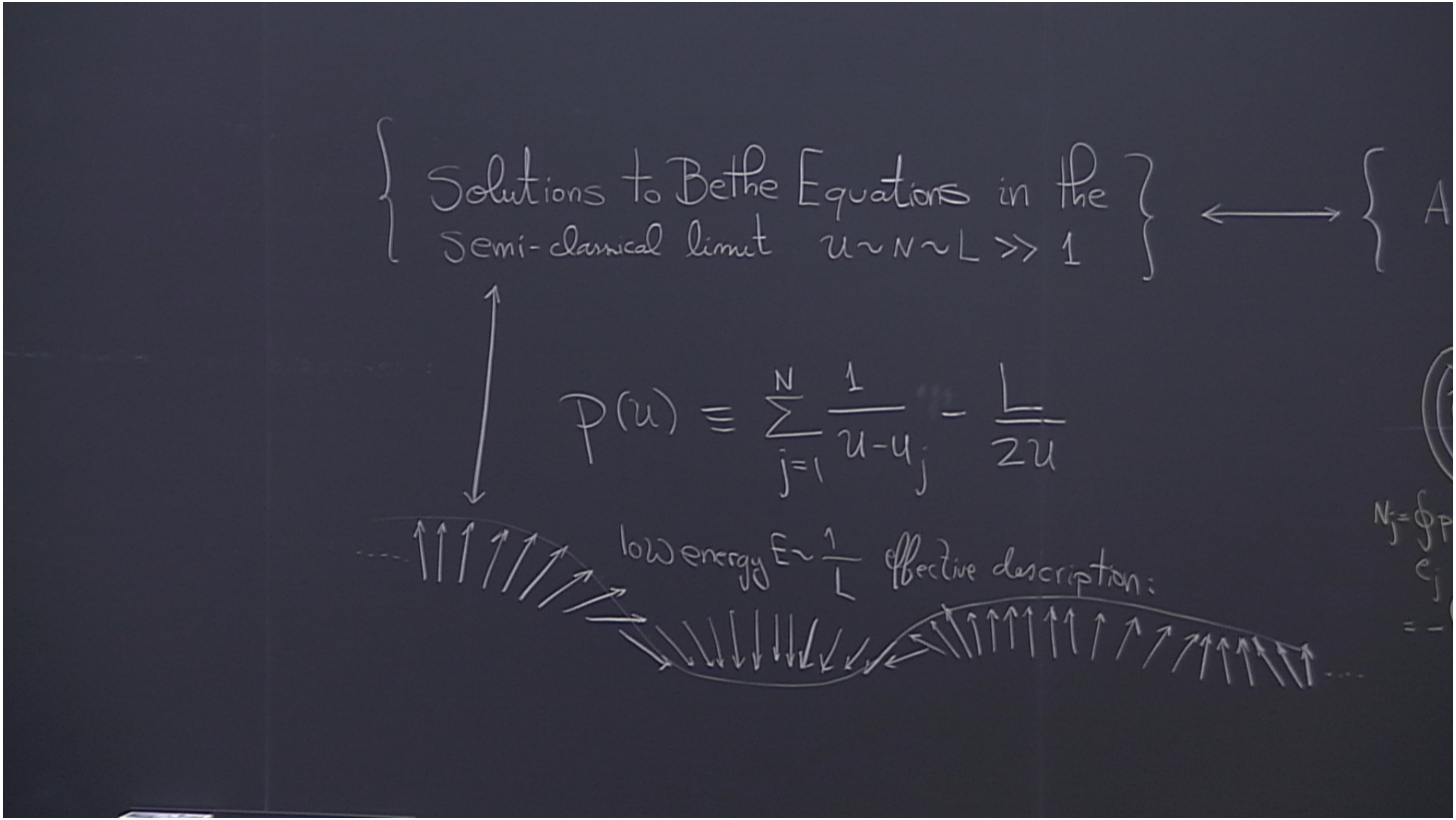
Simplest Example

1 cut solution

$$E = \frac{4\lambda n^2 \alpha(1-\alpha)\pi^2}{L}$$

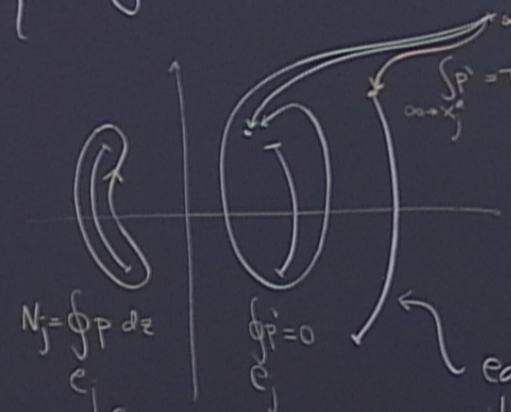
$\alpha = \frac{N}{L}$

from $u \simeq 0$ also


{ Solutions to Bethe Equations in the semi-classical limit $u \sim N \sim L \gg 1$ } ⟷ { A
$p(u) \equiv \sum_{j=1}^{N} \frac{1}{u-u_j} - \frac{L}{2u}$
low energy $E \sim \frac{1}{L}$ effective description:
$N_j = \oint_{C_j} p$

$$E = \sum \frac{1 \times \lambda}{u_j^2 + \frac{1}{4}}$$

$$\left\{ \right\} \longleftrightarrow \left\{ \text{Algebraic Curves w/ prescribed analytics} \right\}$$



each cut has its mode number n_j

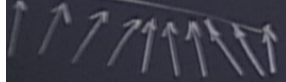
$$(p')^2 = \frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K} (u - u_j^*)}$$

$\sim$ rational w/

para

$= 3$

tion:



$$E = \sum \frac{1 \times \lambda}{u_j^2 + \frac{1}{4}}$$

Simplest Example.

1 cut solution

$$E = \frac{4\lambda n^2 \alpha(1-\alpha)\pi^2}{L}$$

$\frac{N}{L}$

from $u \simeq 0$ also

prescribed analytics

$$(P')^2 = \frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K} (u - u_j^*)}$$

$\sim$ rational fn

$2K+1$ parameters

$= 3K-1+2$

(picture) (from $u \simeq 0$)

has

number n_j

$$E = \sum \frac{1 \times \lambda}{u_j^2 + \frac{1}{4}}$$

$$\updownarrow$$

$$H = -\lambda \sum \vec{S}_n \vec{S}_{n+1}$$

prescribed analytics

Simplest Example: $\frac{N}{L}$

1 cut solution

$$E = \frac{4\lambda n^2 \alpha(1-\alpha)\pi^2}{L}$$

from $u \simeq 0$ also

$$(p')^2 = \frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K} (u - u_j^*)}$$

$\sim$ rational fn

$2K + 1$

parameters

$= 3K - 1 + 2$

(picture) ($u \simeq 0$)

t has

e number n_j

{ Effective Low Energy FT for such Spin-Chains } ← today

↝ tomorrow

$E = \sum \frac{1 \times \lambda}{u_j^2 + \frac{1}{4}}$

$H = -\lambda \sum \vec{S}_n \vec{S}$

{ Solutions to Bethe Equations in the semi-classical limit $u \sim N \sim L \gg 1$ } ⟷ { Algebraic Curves w/ prescribed analytics }

$$P(u) \equiv \sum_{j=1}^{N} \frac{1}{u - u_j} - \frac{L}{2u}$$

$N_j = \oint_{C_j} p \, du = -\oint_{C_j} p' \, dz$

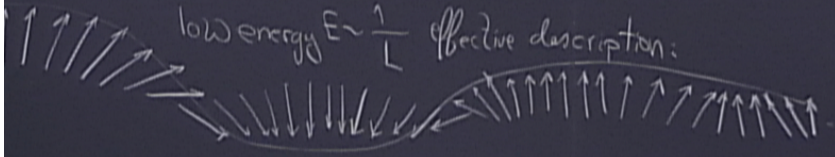
$\oint_{C_j} p' = 0$

$\int_{u_j^*}^{\infty} p' = \pi n_j$

each cut has its mode number n_j

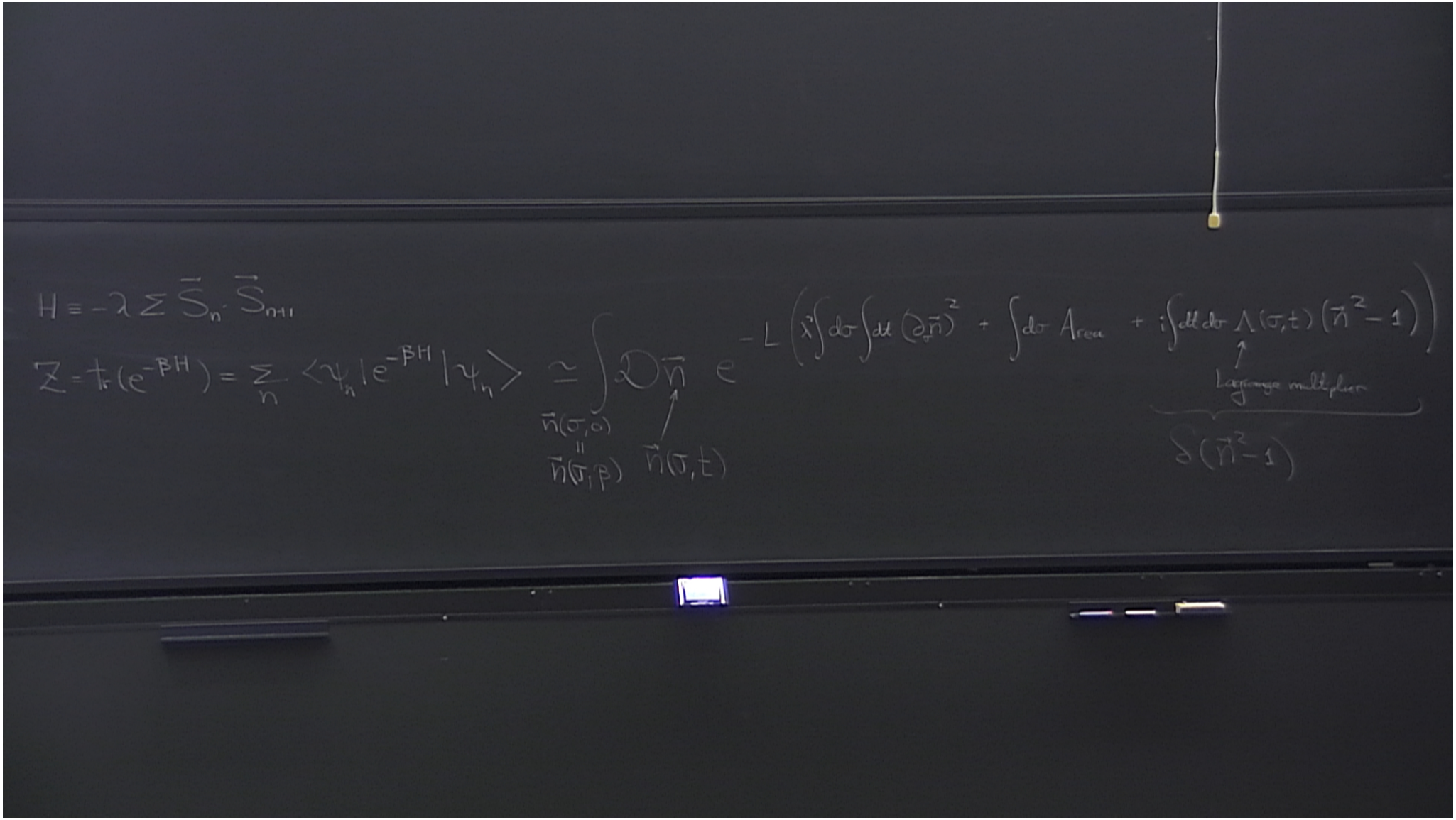
$$(p')^2 = \frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K} (u - u_j^*)}$$

low energy $E \sim \frac{1}{L}$ effective description:



$$H \equiv -\lambda \sum_n \vec{S}_n \cdot \vec{S}_{n+1}$$

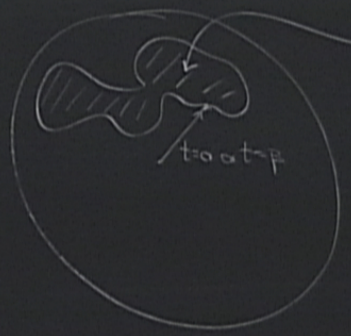
$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_n \langle \psi_n | e^{-\beta H} | \psi_n \rangle \simeq$$


$H \equiv -\lambda \sum \vec{S}_n \cdot \vec{S}_{n+1}$
$Z = \mathrm{Tr}(e^{-\beta H}) = \sum_n \langle \psi_n | e^{-\beta H} | \psi_n \rangle \simeq \int_{\vec{n}(\sigma,0) = \vec{n}(\sigma,\beta)} \mathcal{D}\vec{n}\; e^{-L\left(\lambda \int d\sigma \int dt\, (\partial_\mu \vec{n})^2 + \int d\sigma\, A_{reu} + i \int dt\, d\sigma\, \Lambda(\sigma,t)(\vec{n}^2 - 1)\right)}$
$\vec{n}(\sigma,t)$
Lagrange multiplier
$\delta(\vec{n}^2 - 1)$

also $\int \mathcal{D}\Lambda(\sigma,t)$ implicit (or just $\int \mathcal{D}\vec{n}\,\delta(1-\vec{n}^2)$)

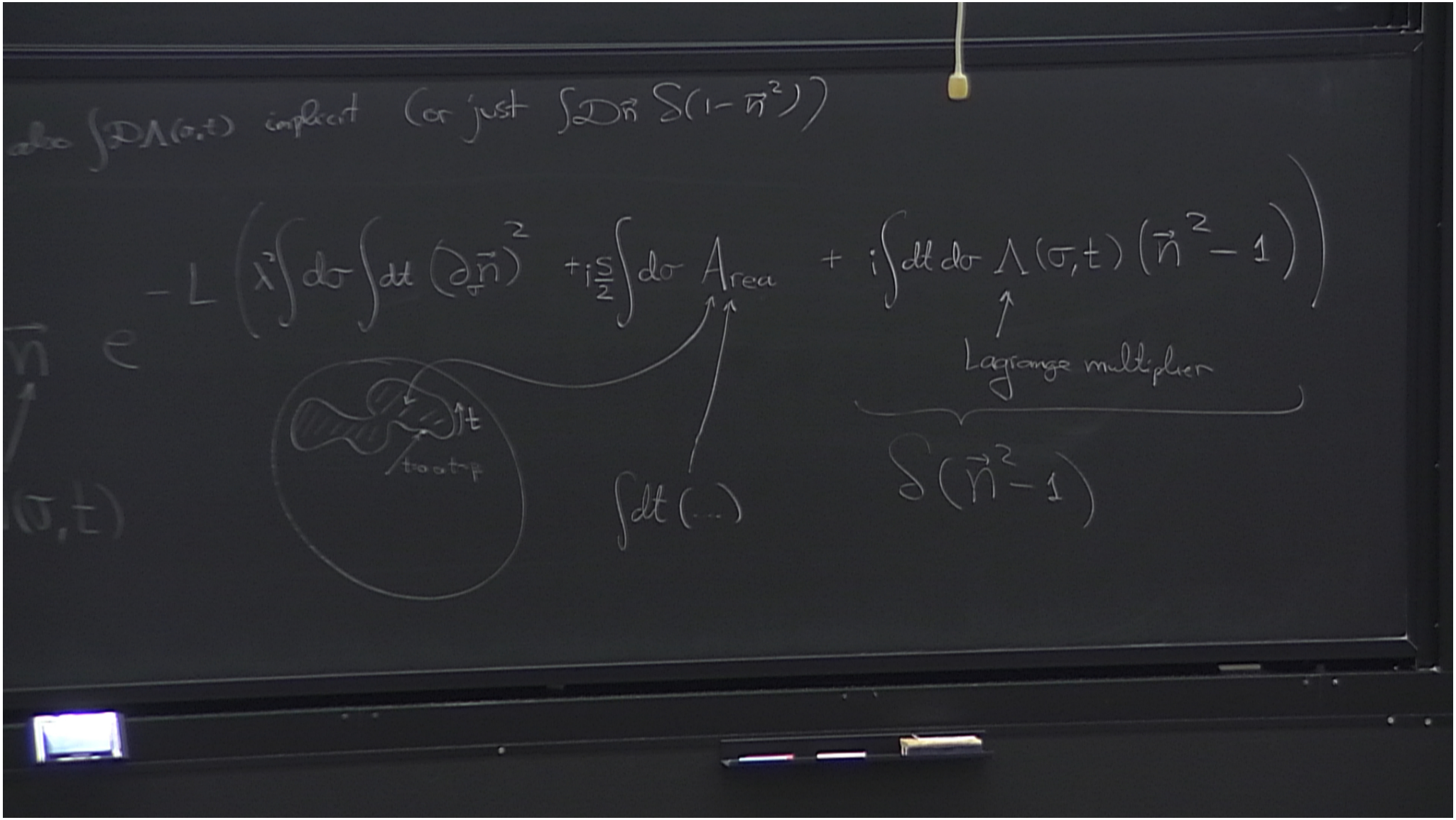
$$|\psi_n\rangle \simeq \int_{\vec{n}(\sigma,0)=\vec{n}(\sigma,\beta)} \mathcal{D}\vec{n}\; e^{-L\left(\lambda^2\int d\sigma\int dt\,(\partial_\sigma \vec{n})^2 + \int d\sigma\, A_{\text{area}} + i\int dt\,d\sigma\,\Lambda(\sigma,t)\right)}$$

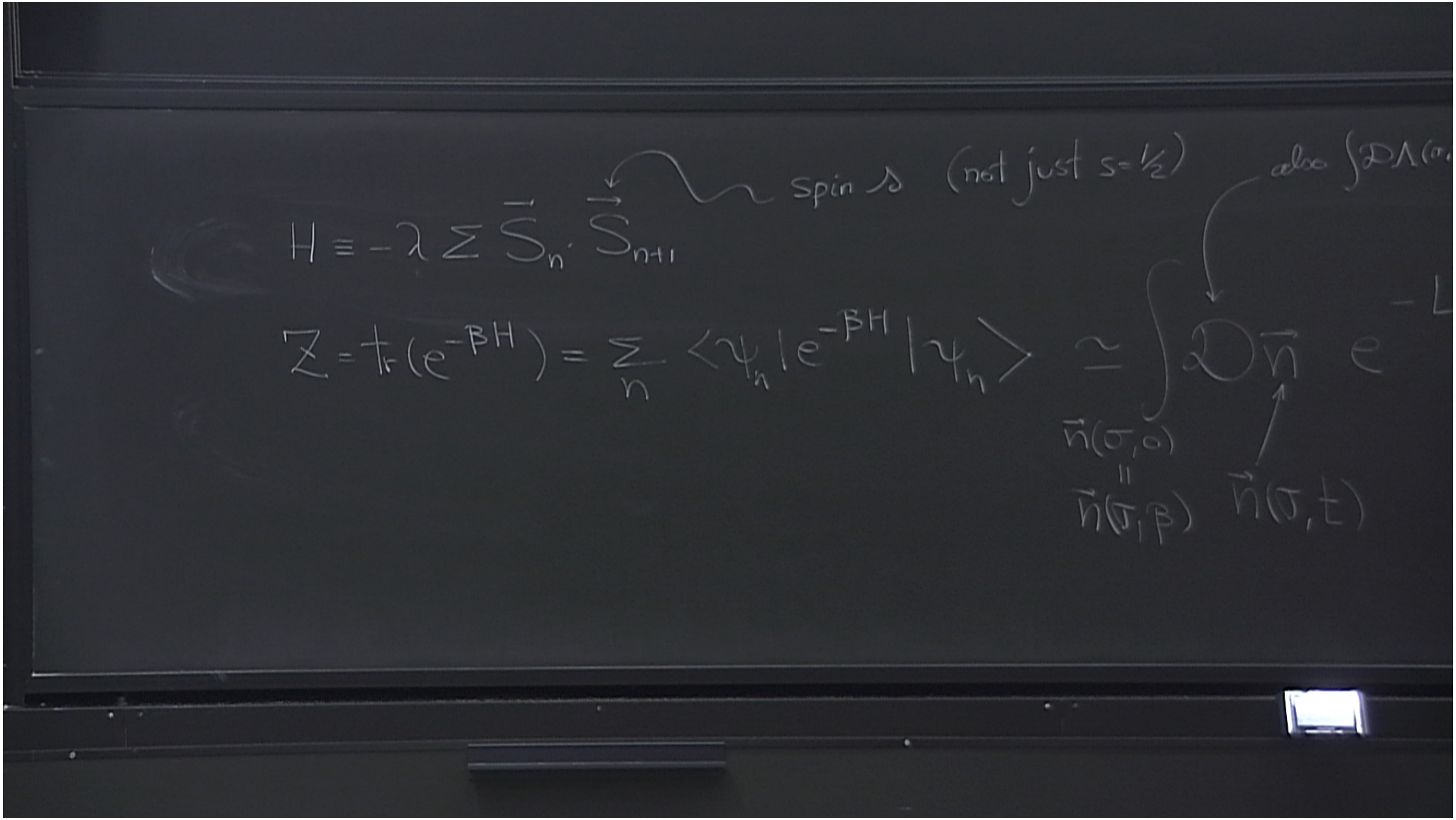
$\vec{n}(\sigma,t)$



Lagrange mul

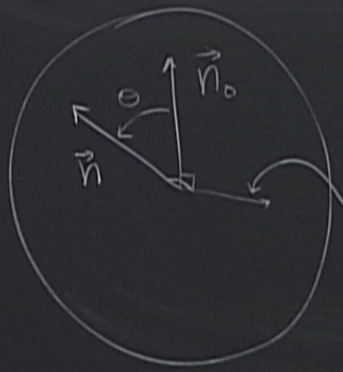
$\delta(\vec{n}^2-1)$


also $\int \mathcal{D}\Lambda(\sigma,t)$ implicit (or just $\int \mathcal{D}\vec{n}\,\delta(1-\vec{n}^2)$)
$\vec{n}$
$e^{-L\left(\lambda\int d\sigma\int dt\,(\partial_\sigma \vec{n})^2 + i\frac{S}{2}\int d\sigma\, A_{area} + i\int dt\,d\sigma\,\Lambda(\sigma,t)(\vec{n}^2-1)\right)}$
Lagrange multiplier
(σ,t)
t
$\int dt\,(\ldots)$
$\delta(\vec{n}^2-1)$


spin s (not just s = 1/2)
also ∫ 𝒟 Λ (σ
H ≡ − λ Σ S⃗ₙ · S⃗ₙ₊₁
Z = Tr(e^{−βH}) = Σₙ ⟨ψₙ| e^{−βH} |ψₙ⟩ ≃ ∫ 𝒟 n⃗ e^{−L
n⃗(σ, 0) = n⃗(σ, β)
n⃗(σ, t)

$$|\vec{n}\rangle = U(\vec{n}_0 \to \vec{n})\,|\vec{n}_0\rangle$$

$$|\uparrow\rangle \,,\quad \vec{n}_0 = (0,0,1)$$



$$\theta \cdot \cos\theta = \vec{n}\cdot\vec{n}_0$$

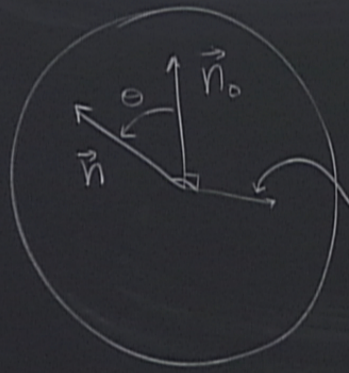
$$\frac{\vec{n}_0 \times \vec{n}}{\sin\theta}$$

$$U(\vec{n}_0 \to \vec{n}) = e^{\,i\theta\, \frac{\vec{n}_0 \times \vec{n}}{\sin\theta}\cdot \vec{S}}$$

$$|\vec{n}\rangle = U(\vec{n}_0 \to \vec{n})\,|\vec{n}_0\rangle = e^{i\theta \frac{\vec{n}_0 \times \vec{n}}{\sin\theta}\cdot\vec{S}}\,|\uparrow\rangle$$

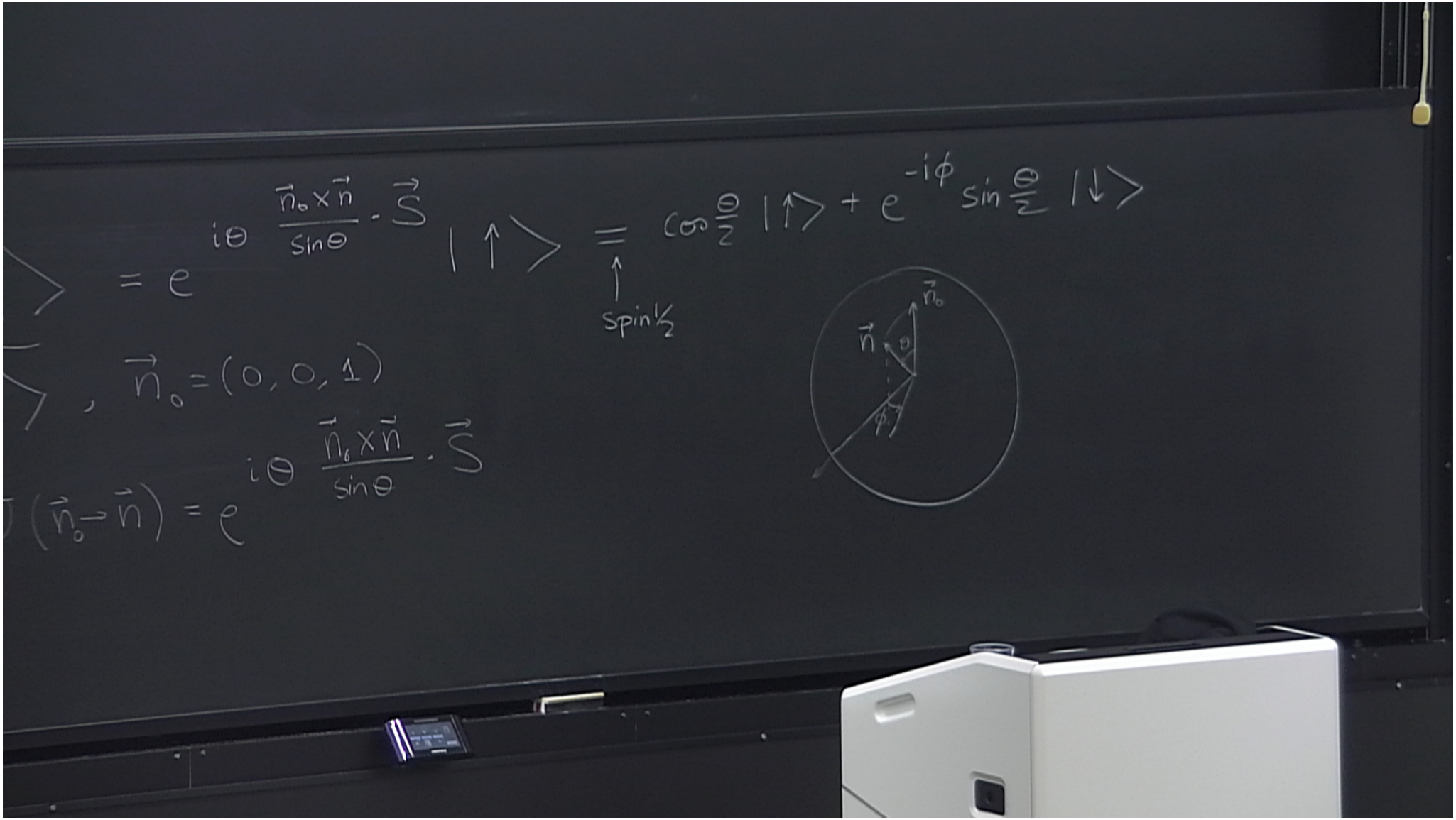
$$|\uparrow\rangle \,,\ \vec{n}_0 = (0,0,1)$$

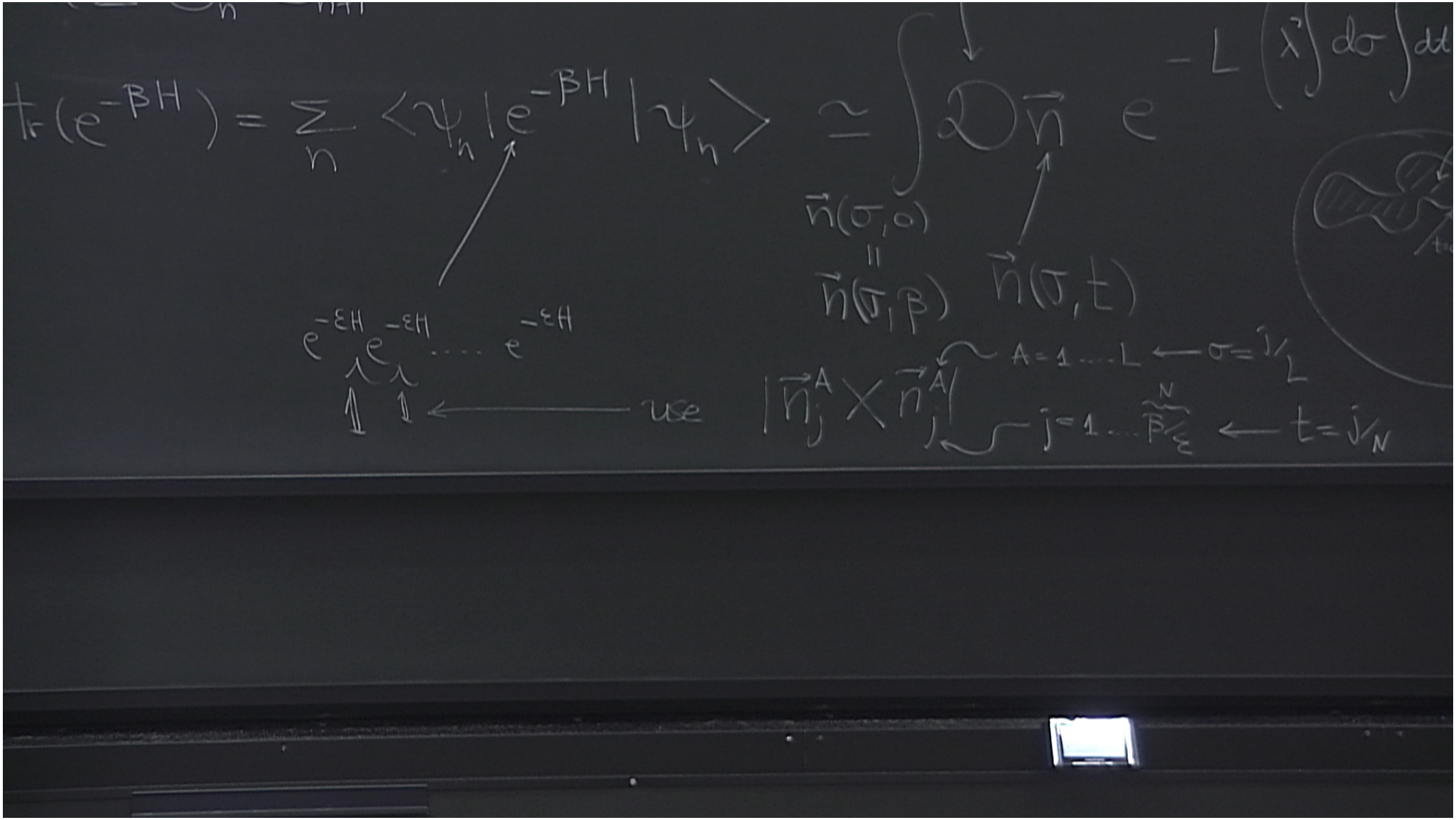
$$U(\vec{n}_0 \to \vec{n}) = e^{i\theta \frac{\vec{n}_0 \times \vec{n}}{\sin\theta}\cdot\vec{S}}$$

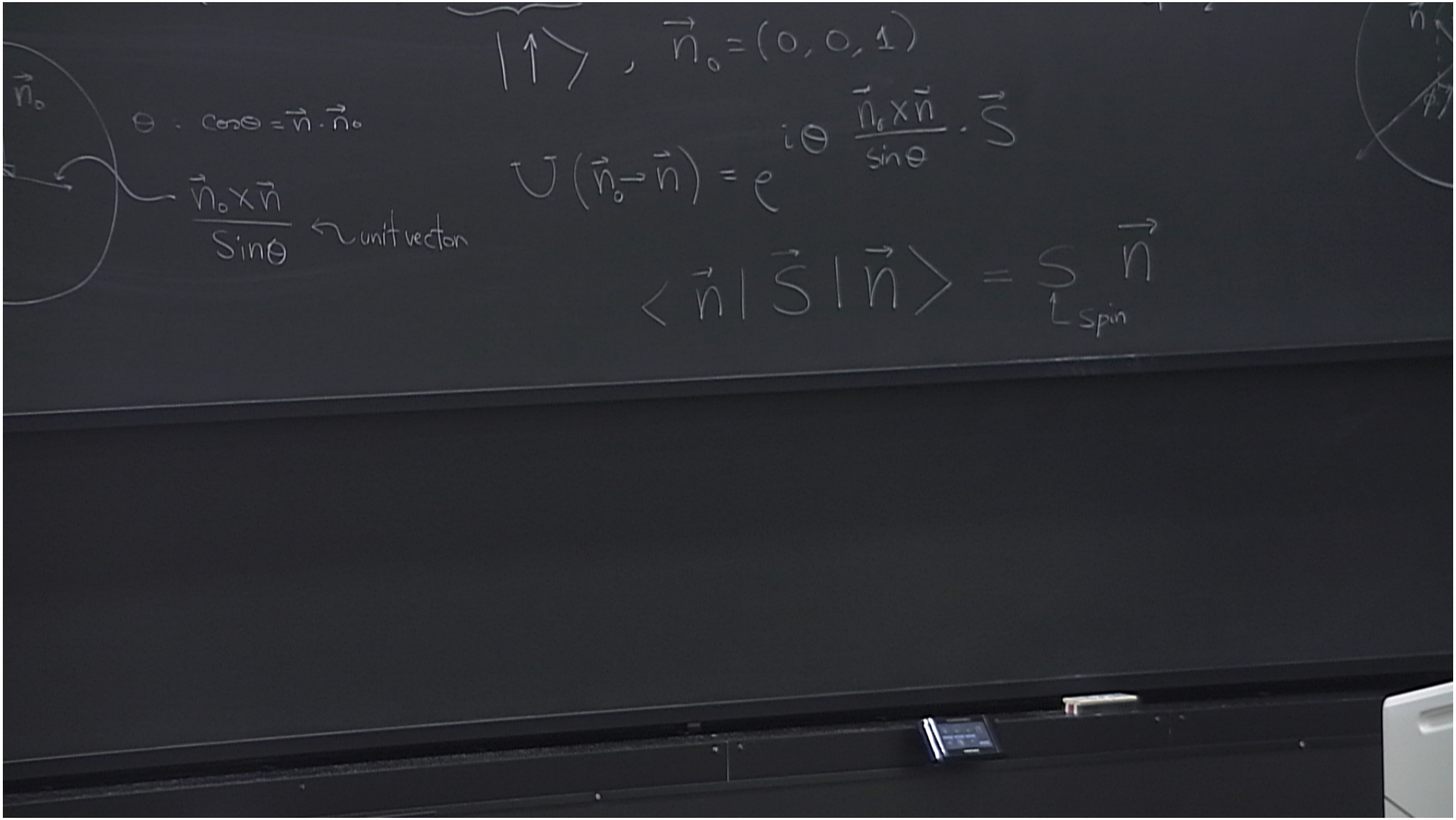


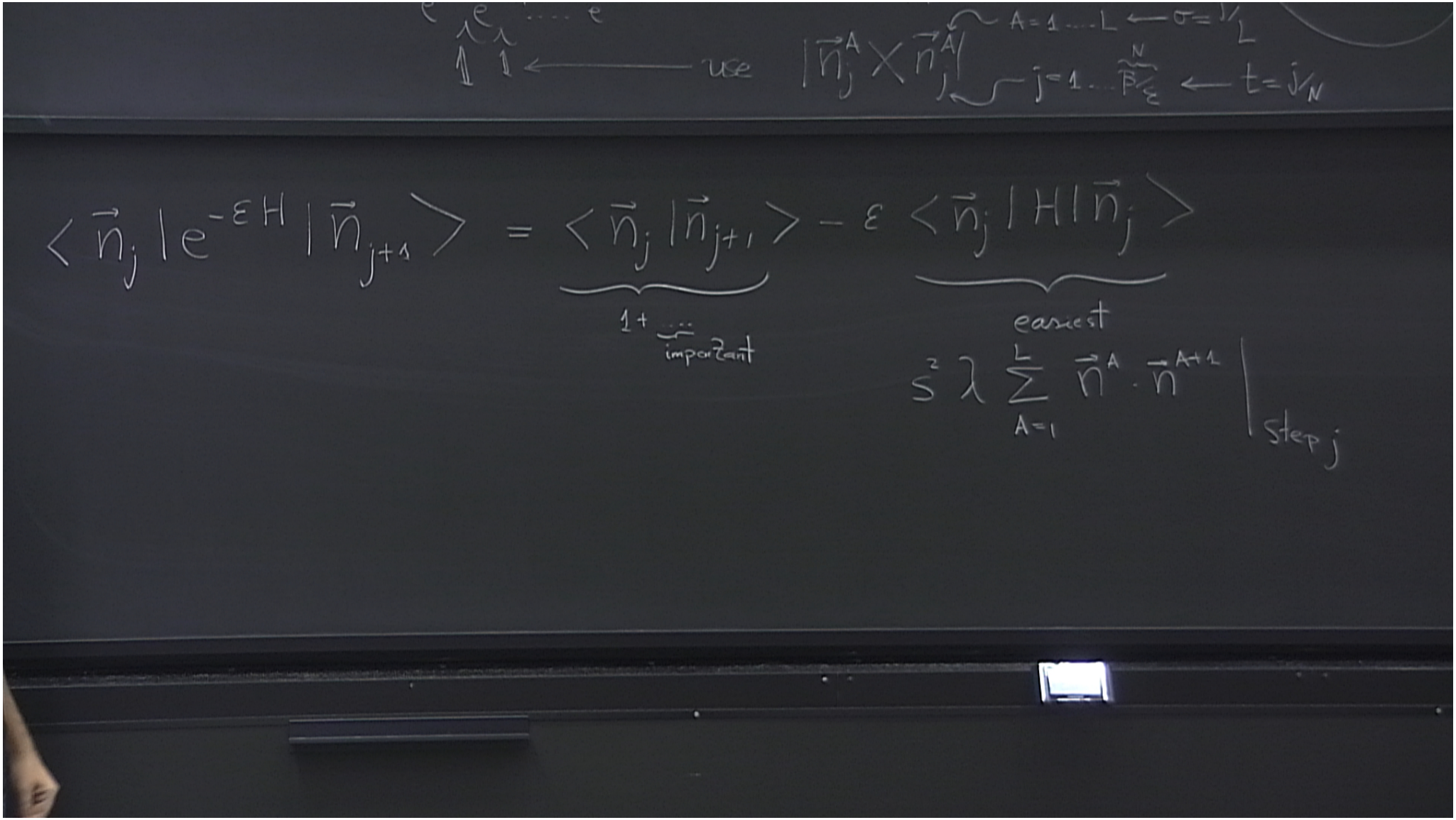
$\theta \cdot \cos\theta = \vec{n} \cdot \vec{n}_0$

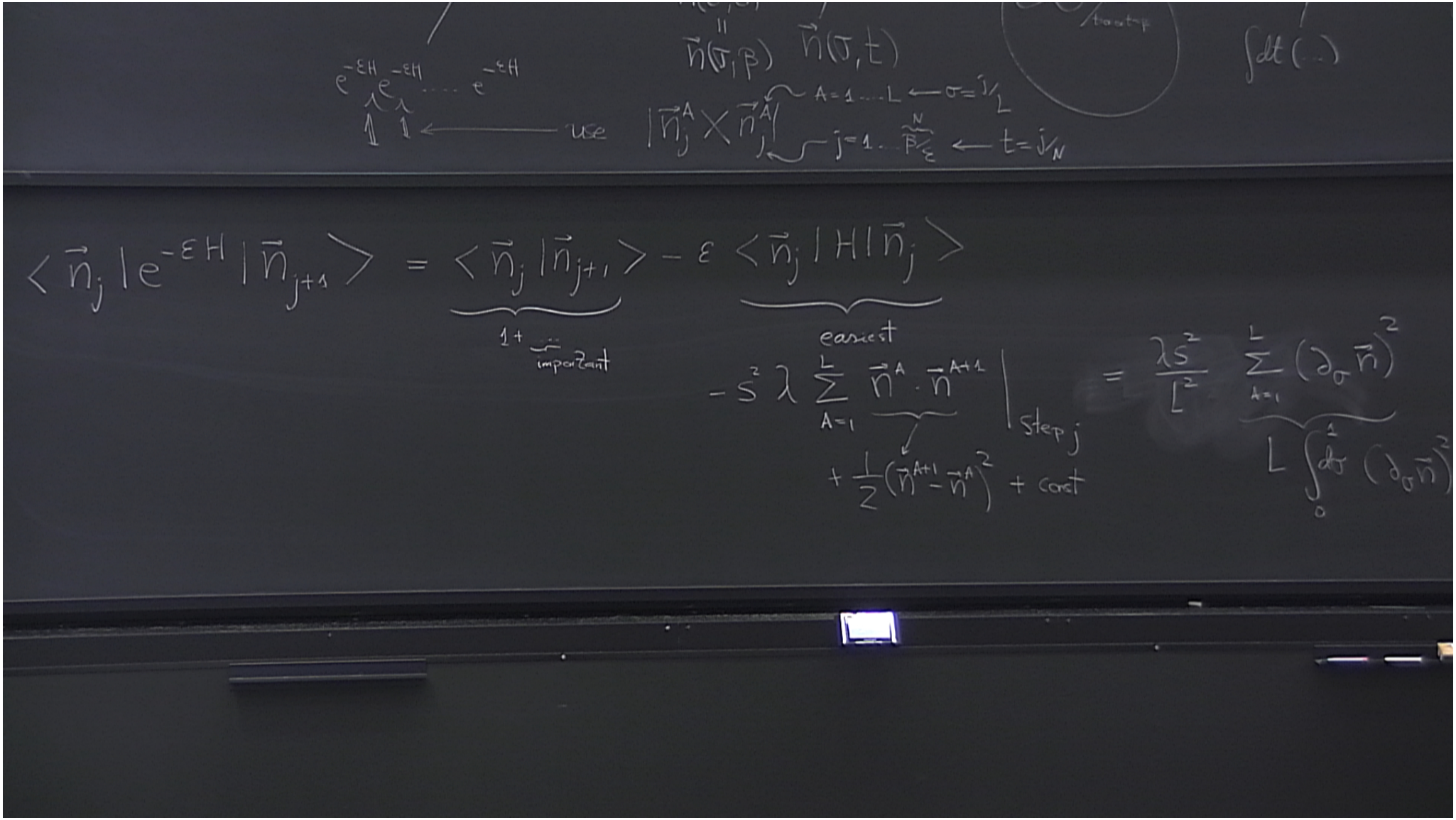
$\frac{\vec{n}_0 \times \vec{n}}{\sin\theta}$ ← unit vector


$= e^{i\theta \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \cdot \vec{S}} |\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{-i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$
Spin ½
$\vec{n}_0 = (0,0,1)$
$(\vec{n}_0 \to \vec{n}) = e^{i\theta \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \cdot \vec{S}}$
$\vec{n}_0$
$\vec{n}$
θ
ϕ


$\mathrm{Tr}(e^{-\beta H}) = \sum_n \langle \psi_n | e^{-\beta H} | \psi_n \rangle \simeq \int \mathcal{D}\vec{n}\, e^{-L\left(\dot{x}^j\right) d\sigma \int dt}$
$\vec{n}(\sigma,0) = \vec{n}(\sigma,\beta)$
$\vec{n}(\sigma,t)$
$e^{-\varepsilon H} e^{-\varepsilon H} \cdots e^{-\varepsilon H}$
use
$|\vec{n}_j^A \times \vec{n}_{j'}^{A'}|$
$A = 1 \ldots L \leftarrow \sigma = j/L$
$j = 1 \ldots N/\varepsilon \leftarrow t = j/N$

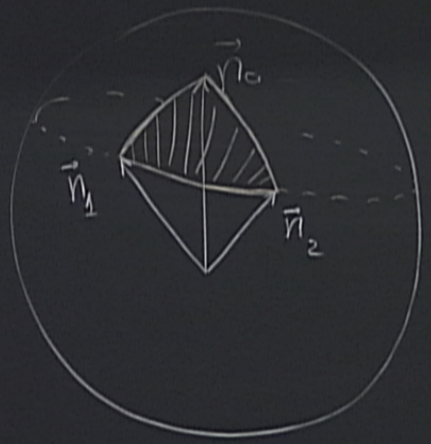

$|\uparrow\rangle$, $\vec{n}_0 = (0,0,1)$
$\theta \cdot \cos\theta = \vec{n} \cdot \vec{n}_0$
$U(\vec{n}_0 \to \vec{n}) = e^{i\theta \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \cdot \vec{S}}$
$\frac{\vec{n}_0 \times \vec{n}}{\sin\theta}$ ↜ unit vector
$\langle \vec{n} | \vec{S} | \vec{n} \rangle = \frac{S}{2}_{spin} \vec{n}$


use
A=1 ... L
j=1 ... N
$\langle \vec{n}_j | e^{-\varepsilon H} | \vec{n}_{j+1} \rangle = \langle \vec{n}_j | \vec{n}_{j+1} \rangle - \varepsilon \langle \vec{n}_j | H | \vec{n}_j \rangle$
1 + ... important
easiest
$s^2 \lambda \sum_{A=1}^{L} \vec{n}^A \cdot \vec{n}^{A+1} \Big|_{\text{step } j}$

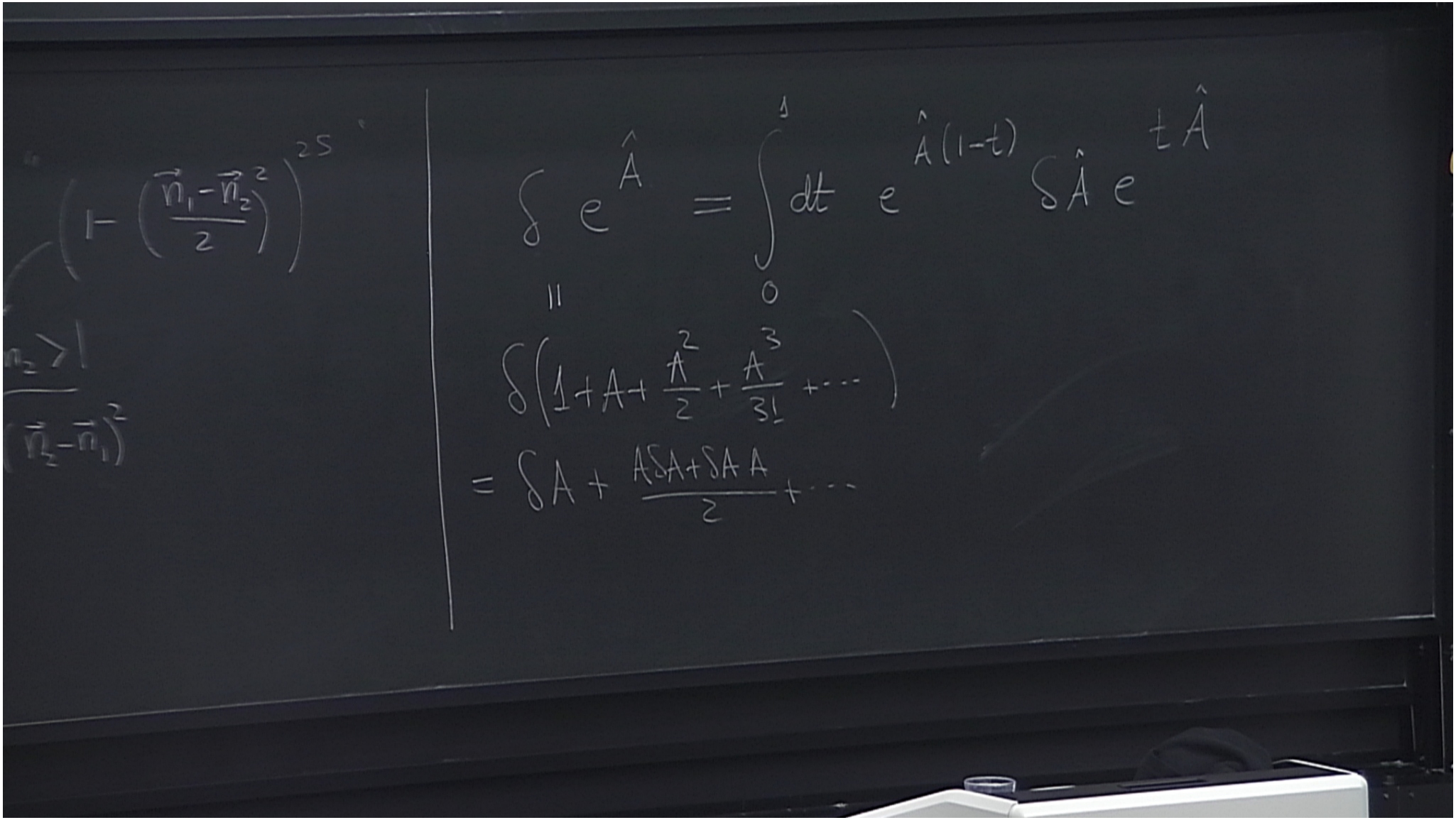

$\vec{n}(\sigma,\beta)$ $\vec{n}(\sigma,t)$
$\int dt(\ldots)$
$e^{-\varepsilon H}$ $e^{-\varepsilon H}$ $e^{-\varepsilon H}$
use
$|\vec{n}_j^A \times \vec{n}_j^{A'}|$
$A=1 \ldots L \leftarrow \sigma = j/L$
$j=1 \ldots N/\varepsilon \leftarrow t = j/N$
$\langle \vec{n}_j | e^{-\varepsilon H} | \vec{n}_{j+1} \rangle = \langle \vec{n}_j | \vec{n}_{j+1} \rangle - \varepsilon \langle \vec{n}_j | H | \vec{n}_j \rangle$
$1 +$ ... important
easiest
$-s^2 \lambda \sum_{A=1}^{L} \vec{n}^A \cdot \vec{n}^{A+1} \Big|_{\text{step } j} = \frac{\lambda s^2}{L^2} \sum_{A=1}^{L} (\partial_\sigma \vec{n})^2$
$+ \frac{1}{2}(\vec{n}^{A+1} - \vec{n}^A)^2 + \text{const}$
$L \int_0^1 d\sigma \, (\partial_\sigma \vec{n})^2$

$\sin\theta$

$\langle \vec{n} | \vec{S} | \vec{n} \rangle$ =



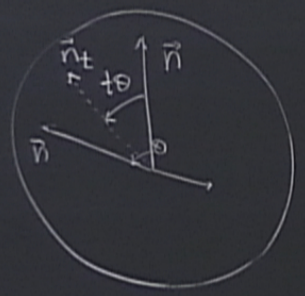
$\langle \vec{n}_1 | \vec{n}_2 \rangle = e^{iS\,\text{Area}_\Delta}$ $\underbrace{|\langle n_1 | n_2 \rangle|}_{1 + \mathcal{O}(\vec{n}_2 - \vec{n}_1)^2}$


$\left(1-\left(\frac{\vec{n}_1-\vec{n}_2}{2}\right)^2\right)^{25}$
$\delta e^{\hat{A}} = \int_0^1 dt\, e^{\hat{A}(1-t)}\,\delta\hat{A}\,e^{t\hat{A}}$
$\delta\left(1+A+\frac{A^2}{2}+\frac{A^3}{3!}+\cdots\right)$
$= \delta A + \frac{A\delta A+\delta A A}{2}+\cdots$

$$+ \frac{1}{2}\left(\vec{n}^{A+1} - \vec{n}^{A}\right)^2 + \text{const}$$

$$\int_0 \left(\delta_\sigma \vec{n}\right)$$

$$\langle \uparrow | e^{-i\theta t \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \vec{S}} \left[i\theta \frac{\vec{n}_0 \times \delta\vec{n}}{\sin\theta} \right] \cdot \vec{S}\, e^{i\theta t \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \vec{S}} | \uparrow \rangle$$



$$= i\theta s \frac{\vec{n}_0 \times \delta\vec{n}}{\sin\theta} \cdot \int_0^1 dt\ \vec{n}_t$$

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\delta\vec{n} = \delta\phi \frac{\partial\vec{n}}{\partial\phi} + \delta\theta \frac{\partial\vec{n}}{\partial\theta}$$

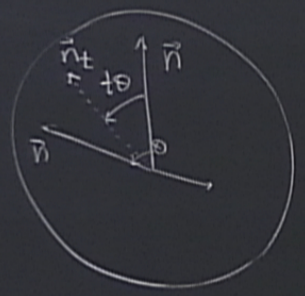
$$\vec{n}_0 = (0, 0, 1)$$

$$\vec{n}_t = \vec{n}\big|_{\theta \to \theta t}$$

$$+\frac{1}{2}\left(\vec{n}^{A+1}-\vec{n}^{A}\right)^2+\text{const}$$

$$\int_0^{\theta}\left(\delta_0 n\right)$$

$$\langle\uparrow|\, e^{-i\theta t\,\frac{\vec{n}_0\times\vec{n}}{\sin\theta}\,\vec{S}}\left[i\theta\,\frac{\vec{n}_0\times\delta\vec{n}}{\sin\theta}\right]\cdot\vec{S}\; e^{i\theta t\,\frac{\vec{n}_0\times\vec{n}}{\sin\theta}\,\vec{S}}\,|\uparrow\rangle$$



$$= i\theta s\,\frac{\vec{n}_0\times\delta\vec{n}}{\sin\theta}\cdot\int_0^1 dt\;\vec{n}_t$$

$$= i s\,\delta\phi\,(1-\cos\theta)$$

$$\vec{n}=(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\delta\vec{n}=\delta\phi\,\frac{\partial\vec{n}}{\partial\phi}+\delta\theta\,\frac{\partial\vec{n}}{\partial\theta}$$

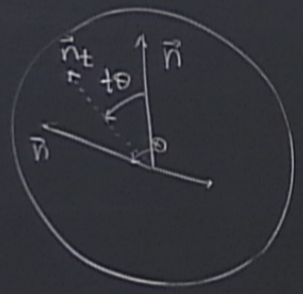
$$\vec{n}_0=(0,0,1)$$

$$\vec{n}_t=\vec{n}\big|_{\theta\to\theta t}$$

$$+\frac{1}{2}\left(\vec{n}^{A+1}-\vec{n}^{A}\right)^2+\text{const}$$

$$\int_0^{\sigma}\left(\delta_\sigma \vec{n}\right)$$

$$\langle\uparrow| e^{-i\theta t\,\frac{\vec{n}_0\times\vec{n}}{\sin\theta}\,\vec{S}}\left[i\theta\,\frac{\vec{n}_0\times\delta\vec{n}}{\sin\theta}\right]\cdot\vec{S}\; e^{i\theta t\,\frac{\vec{n}_0\times\vec{n}}{\sin\theta}\,\vec{S}}|\uparrow\rangle$$



$$= i\theta s\,\frac{\vec{n}_0\times\delta\vec{n}}{\sin\theta}\cdot\int_0^1 dt\;\vec{n}_t$$

$$= is\,\delta\phi\,(1-\cos\theta) = -is\,\delta\phi\int_0^{\theta}\sin\theta\,d\theta$$

$$\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

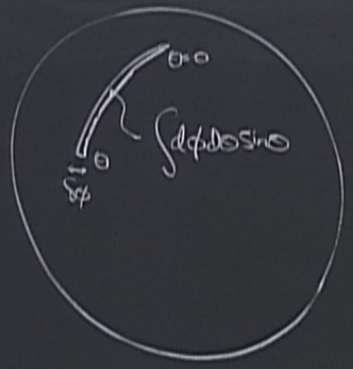
$$\delta\vec{n} = \delta\phi\,\frac{\partial\vec{n}}{\partial\phi} + \delta\theta\,\frac{\partial\vec{n}}{\partial\theta}$$

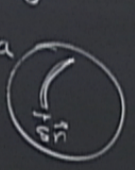
$$\vec{n}_0 = (0,0,1)$$

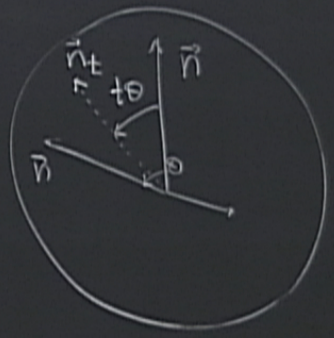
$$\vec{n}_t = \vec{n}\big|_{\theta\to\theta t}$$

$$\langle \vec{n} \mid \vec{n} + \delta\vec{n} \rangle = \int_0^1 dt \; \langle \uparrow \mid e^{-i\theta t \, \frac{\vec{n}_0 \times \vec{n}}{\sin\theta} \cdot \vec{S}} \left[i\theta \, \frac{\vec{n}_0 \times \delta\vec{n}}{\sin\theta} \right] \cdot \vec{S}$$

$$= 1 + i\,\text{Area}$$

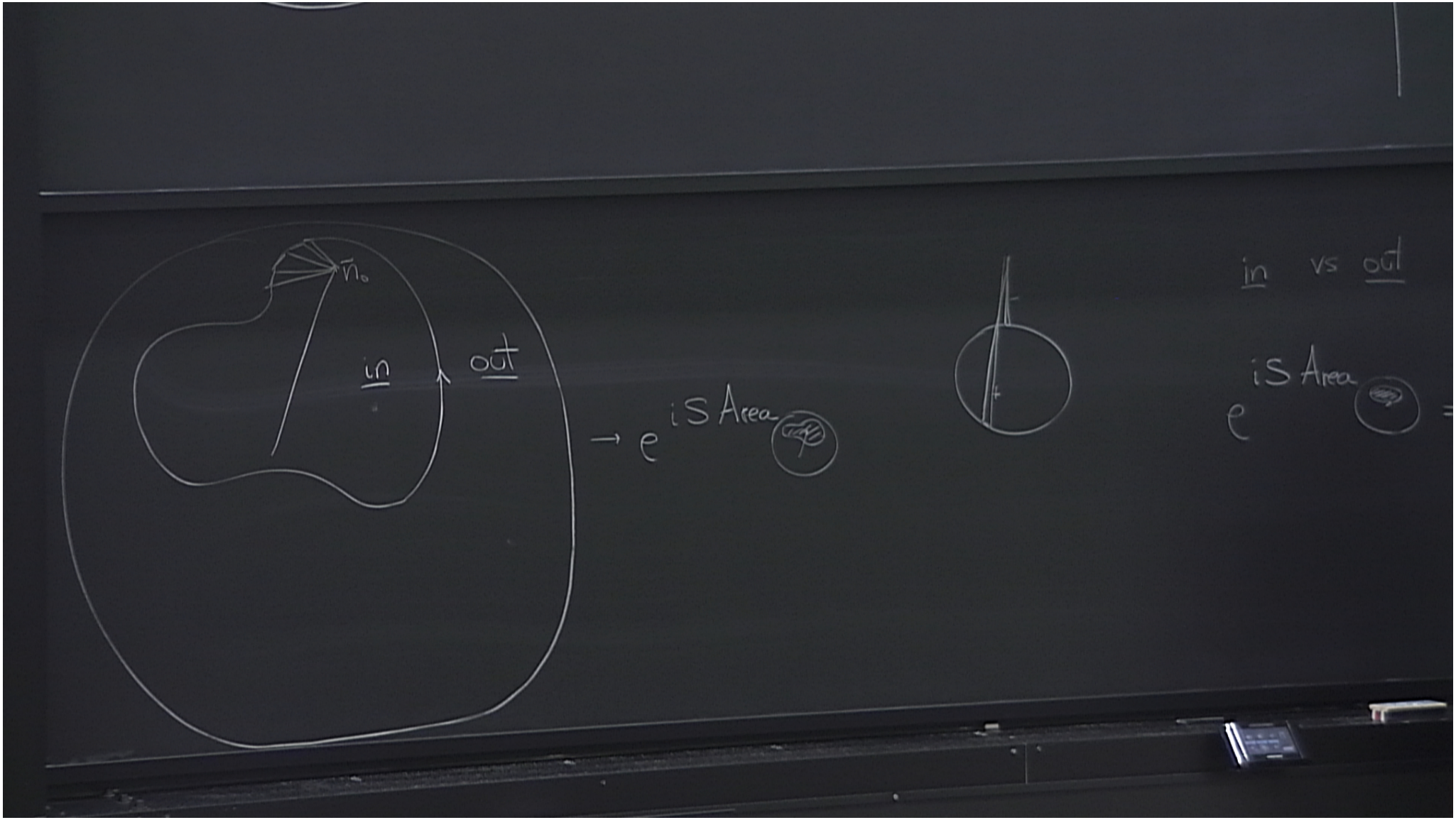


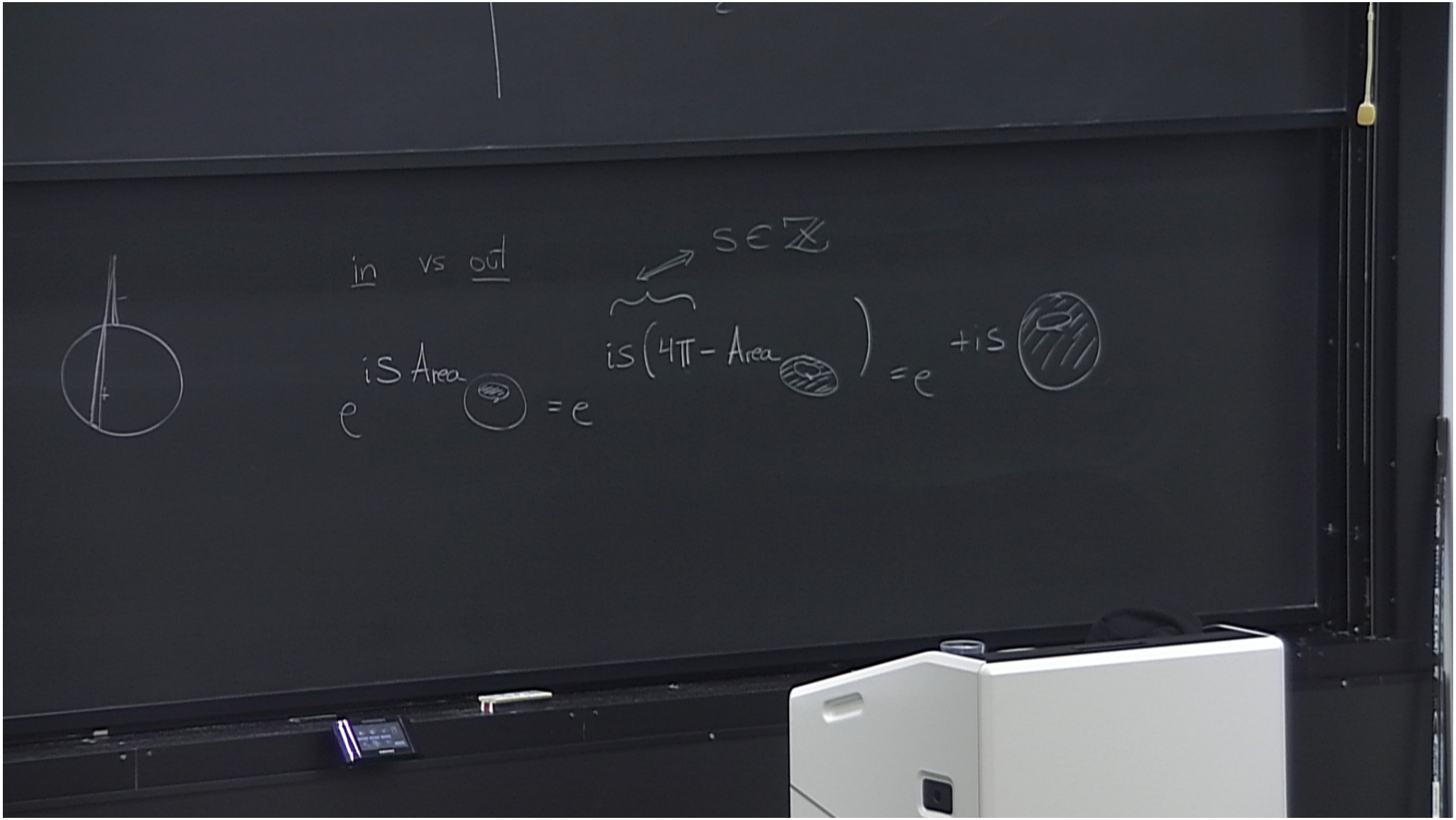


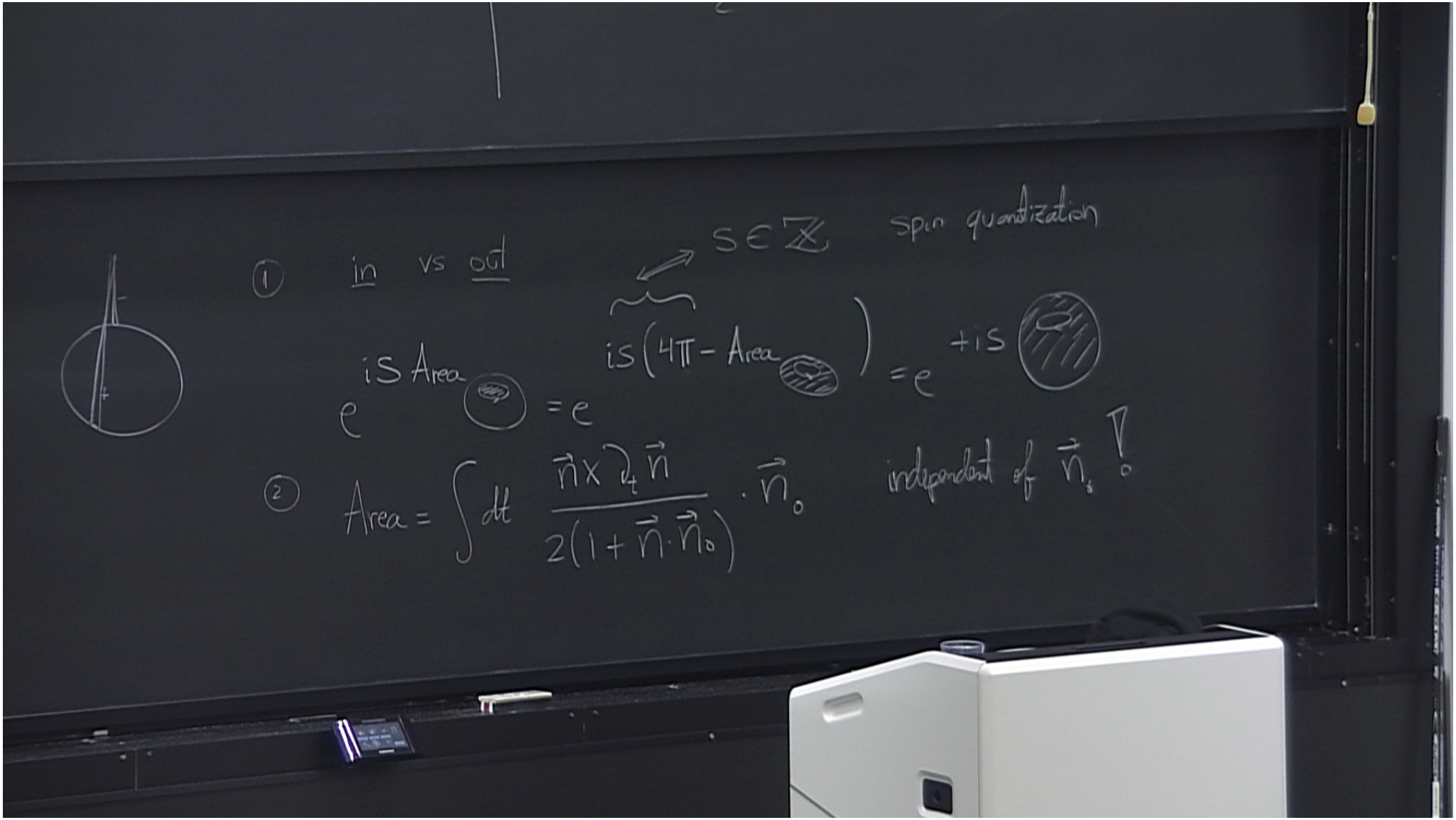


$$= i\theta s \, \frac{\vec{n}_0 \times \delta\vec{n}}{\sin\theta} \cdot \int_0^1 dt \; \vec{n}$$

$$= i S \, \delta\phi \, (1 - \cos\theta) = -iS$$


in
out
$\rightarrow e^{iS\,\mathrm{Area}}$
in vs out
$e^{iS\,\mathrm{Area}}$


in vs out
$s \in \mathbb{Z}$
$e^{is\,\text{Area}} = e^{is(4\pi - \text{Area})} = e^{+is}$

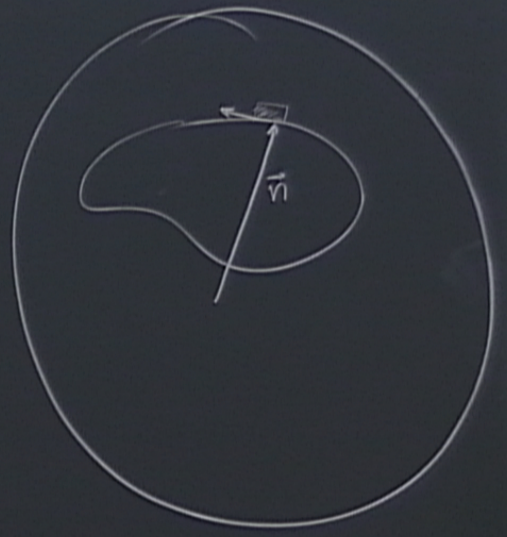

① in vs out
$s \in \mathbb{Z}$ spin quantization
$e^{is\,\text{Area}} = e^{is(4\pi - \text{Area})} = e^{+is}$
② $\text{Area} = \int dt \, \frac{\vec{n} \times \partial_t \vec{n}}{2(1+\vec{n}\cdot\vec{n}_0)} \cdot \vec{n}_0$ independent of $\vec{n}_0$!

Classical Limit.

EOM for Landau-Lifshitz σ-model

$$\Lambda \vec{n} + \lambda \partial_\sigma^2 \vec{n} + \underbrace{\text{contribution from Area}}_{\vec{n} \times \partial_t \vec{n}}$$

$$\delta \text{Area} = \delta \vec{n} \cdot (\vec{n} \times \partial_t \vec{n})$$



$$\vec{n} \times [\text{eom}]$$

$$\vec{n} \times \vec{n} = 0$$

$$\vec{n} \times (\vec{n} \times \partial_t \vec{n}) = (\vec{n} \cdot \vec{n}) \partial_t \vec{n} - (\vec{n} \cdot \partial_t \vec{n}) \vec{n}$$

$\vec{n} \times [\text{e.om}] \Rightarrow \boxed{\partial_t \vec{n} - \frac{\lambda}{4L^2} \vec{n} \times \partial_\sigma^2 \vec{n} = 0}$ (Spin ½)

$\boxed{\text{Energy} = \frac{\lambda}{L} \int d\sigma (\partial_\sigma \vec{n})^2}$

$\vec{n} \times \vec{n} = 0$

$\vec{n} \times (\vec{n} \times \partial_t \vec{n}) = \underbrace{(\vec{n} \cdot \vec{n})}_{1} \partial_t \vec{n} - \underbrace{(\vec{n} \cdot \partial_t \vec{n})}_{= 0 = \partial_t (1 = \vec{n} \cdot \vec{n})} \vec{n}$

$\vec{n} \times [\text{e.om}] \Rightarrow \boxed{\partial_t \vec{n} - \frac{\lambda}{2L^2} \vec{n} \times \partial_\sigma^2 \vec{n} = 0}$ (Spin ½)

$\boxed{\text{Energy} = \frac{J}{4L} \int d\sigma (\partial_\sigma \vec{n})^2}$

$\vec{n} \times \vec{n} = 0$

$\vec{n} \times (\vec{n} \times \partial_t \vec{n}) = \underbrace{(\vec{n} \cdot \vec{n})}_{1} \partial_t \vec{n} - \underbrace{(\vec{n} \cdot \partial_t \vec{n})}_{= 0 = \partial_t (1 = \vec{n} \cdot \vec{n})} \vec{n}$

{ Effective Low Energy FT for such Spin-Chains } ← today

~ tomorrow

{ Sol of BOX }

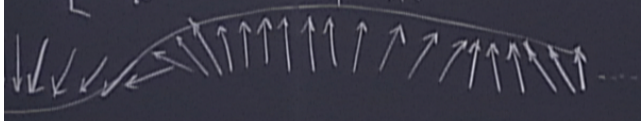
$E = \sum \frac{1 \times \lambda}{u_j^2 + \frac{1}{4}}$

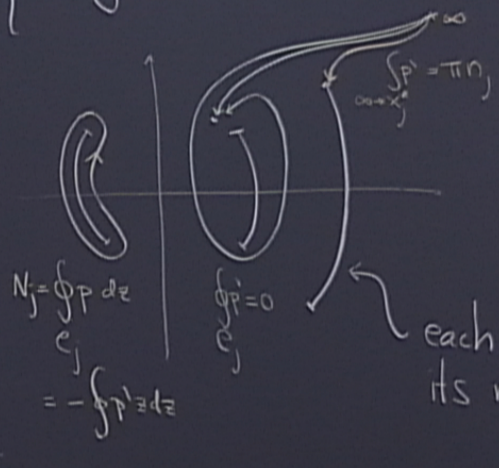
$H = -2 \sum \vec{S}_n \vec{S}_{n+1}$

{ Equations in the $u \sim N \sim L \gg 1$ } ⟷ { Algebraic Curves w/ prescribed analytics }

$\frac{N}{2} \frac{1}{u - u_j} - \frac{L}{2u}$

$\sim \frac{1}{L}$ effective description:





each cut has its mode number n_j

$(p')^2 = \frac{\left(\sum_{j=0}^{K} c_j x^j\right)^2}{u^4 \prod_{j=1}^{2K} (u - u_j^*)}$ ~ r