Title: ADE Little string theories, Mock modular forms, and Umbral moonshine

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Abstract:

ADE fivebranes, mock modular forms, and Umbral moonshine

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Mock modularity,
moonshine and string theory
Perimeter Institute, Apr 17, 2015

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"We were looking to invite interesting participants for the moonshine conference."

We invited the rhinoceri, Washington, and Lincoln.

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Based on:

J. Harvey, S. M., arXiv:1307.7717

S. M., arXiv:1311.0918

J. Harvey, C. Nazaroglu, S. M., arXiv:1410.6174

And the interesting and impressive work of many people (many of whom are in the audience) on the themes of the conference.

The original Mathieu moonshine observation remains mysterious

(Eguchi, Ooguri, Tachikawa '10)

$$h^{(2)}(\tau) = \sum_{n=0}^{\infty} a_n \, q^{n-1/8}$$

$$= q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots)$$

M24-module $V = \bigoplus_{n=0}^{\infty} V_n$ with $a_n = \dim V_n$?

$$h^{(2)}(\tau) = \mathrm{Tr}\, q^{L_0}$$

The K3 elliptic genus seems to be a natural starting point

(Eguchi, Ooguri, Taormina, Yang '89)

$$\chi_{\rm ell}(M;\tau,z) = \mathrm{Tr}_{\mathcal{H}(M)}\,(-1)^F\, q^{L_0}\,\zeta^{J_0}\,,$$

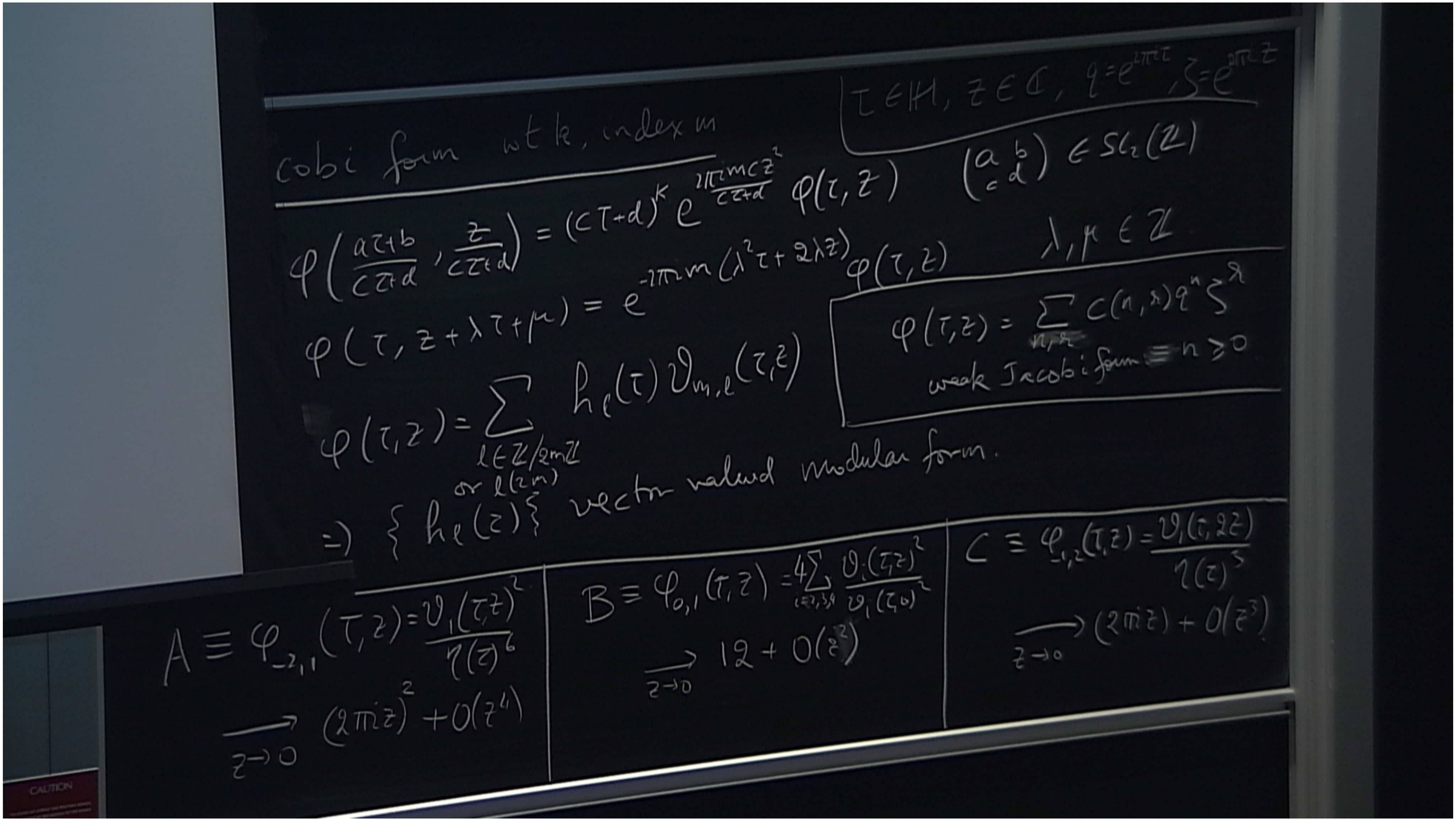
$$q := e^{2\pi i\tau},\ \zeta := e^{2\pi i z}\,.$$

$$\chi_{\rm ell}(K3;\tau,z) = 8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}\right)\,.$$

$$= (2\zeta + 20 + 2\zeta^{-1})$$

$$+(20\zeta^2 - 128\zeta + 216 - 128\zeta^{-1} + 20\zeta^{-2})\,q + \cdots$$

$$= 2\,\varphi_{0,1}(\tau,z)$$


cobi form wt k, index m
$\tau \in \mathbb{H},\ z \in \mathbb{C},\ q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}$
$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \varphi(\tau, z)$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z)$
$\lambda, \mu \in \mathbb{Z}$
$\varphi(\tau,z) = \sum_{n,r} c(n,r) q^n \zeta^r$
weak Jacobi form $\equiv n \geq 0$
$\varphi(\tau,z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\vartheta_{m,\ell}(\tau,z)$
or $\ell(2m)$
$\Rightarrow \{h_\ell(\tau)\}$ vector valued modular form.
$A \equiv \varphi_{-2,1}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}$
$\xrightarrow[z\to 0]{} (2\pi i z)^2 + O(z^4)$
$B \equiv \varphi_{0,1}(\tau,z) = 4\sum_{i=2,3,4} \frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2}$
$\xrightarrow[z\to 0]{} 12 + O(z^2)$
$C \equiv \varphi_{-1,2}(\tau,z) = \frac{\vartheta_1(\tau,2z)}{\eta(\tau)^3}$
$\xrightarrow[z\to 0]{} (2\pi i z) + O(z^3)$

Decomposing the K3 elliptic genus gives the Mathieu moonshine function

$$\chi_{\mathrm{ell}}(K3;\tau,z) \;=\; 20\,\mathrm{ch}_{\frac{1}{4},0}(\tau,z) + 2a_0\,\mathrm{ch}_{\frac{1}{4},\frac{1}{2}}(\tau,z) + 2\sum_{n=1}^{\infty} a_n\,\mathrm{ch}_{n+\frac{1}{4},\frac{1}{2}}(\tau,z)$$

$$h^{(2)}(\tau) = \sum_{n=0}^{\infty} a_n\, q^{n-1/8}$$

$$= q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots)$$

Mock modular form generating the Mathieu reps

A new ingredient is *Mock* Modular Forms

Mock modular form of weight k

$$\widehat{h}(\tau) = h(\tau) + g^*(\tau) \quad ,$$

Completion: transforms as a wt k holomorphic modular form

For $k = \frac{1}{2}$, $\quad g(\tau) = \sum_{n>0} b_n \, q^n$ cusp

$$g^*(\tau) = \sum_{n>0} \frac{\bar{b}_n}{\sqrt{n}} \, \mathrm{erfc}(4\pi n \tau_2) \, q^{-n}$$

Shadow (a modular form of weight 2-k)

Holomorphic anomaly $\quad (4\pi\tau_2)^k \frac{\partial}{\partial \bar{\tau}} \widehat{h}(\tau) = \overline{g(\tau)}$

It is convenient to organize vector valued MMFs in terms of Mock Jacobi Forms

A mock Jacobi form of weight k, index m

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\, \vartheta_{m,\ell}(\tau, z)$$

MMF of weight k-1/2

Completion of mock Jacobi form

$$\widehat{\varphi}(\tau, z) = \varphi(\tau, z) + \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} g_\ell^*(\tau)\, \vartheta_{m,\ell}(\tau, z)$$

transforms as a holomorphic Jacobi form wt k, index m, and

$$\tau_2^{k-1/2} \frac{\partial}{\partial \overline{\tau}} \widehat{\varphi}(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} \overline{g_\ell(\tau)}\, \vartheta_{m,\ell}(\tau, z)$$

One can produce *mock* Jacobi forms by starting from *meromorphic* Jacobi forms

(A. Dabholkar, S.M., D.Zagier, '12, following Zwegers, '02)

Let k=1, and assume simple poles at $z = 0 + \mathbb{Z}\tau + \mathbb{Z}$ with residue one.

$$\varphi^{\mathrm{mero}}(\tau, z) = \mathcal{A}_{1,m}(\tau, z) + \varphi^{\mathrm{F}}(\tau, z)$$

Mock Jacobi form

Appell-Lerch sum

$$\mathcal{A}_{1,m}(\tau, z) \; = \; -\frac{1}{2} \sum_{s \in \mathbb{Z}} q^{ms^2} \, \zeta^{2ms} \, \frac{1 + q^s \zeta}{1 - q^s \zeta}$$

Example (m=2)

Choose mero Jacobi form to be: $\Psi_{1,2} = \frac{C}{A}\frac{B}{12}$

Apply decomposition theorem:

$$\Psi_{1,2}(\tau, z) = \Psi^F_{1,2}(\tau, z) + \mathcal{A}_{1,2}(\tau, z)$$

One gets the Mock Jacobi form:

$$\Psi^F_{1,2}(\tau, z) = h^{(2)}(\tau)\left(\vartheta_{2,1}(\tau, z) - \vartheta_{2,-1}(\tau, z)\right)$$

$$\equiv h^{(2,1)}(\tau)\left(\vartheta_{2,1}(\tau, z) - \vartheta_{2,-1}(\tau, z)\right)$$

More examples (m>2)

Choose: $\Psi_{1,m} = \frac{C}{A} \varphi_{0,m-1}$

where $\varphi_{0,m-1} = \frac{B^{m-1}}{12^{m-1}} + A(\cdots)$

Apply decomposition theorem, one gets mock Jacobi forms of wt 1, index m, and shadow:

$$\sum_{\ell} \overline{S_{m,\ell}(\tau)}\, \vartheta_{m,\ell}(\tau, z)$$

Umbral moonshine

(Cheng, Duncan, Harvey '12, '13)

	Meromorphic Jacobi form	Mock Jacobi form (v.v. MMF)		Finite group
m=2	$\frac{C}{A}B$	$h^{(2)}(\tau) \equiv h^{(2,1)}(\tau)$	⟷	M24
m=3	$\frac{C}{A}(B^2 - E_4 A^2)$	$\left(h^{(3,1)}(\tau)\,,\ h^{(3,2)}(\tau)\right)$	⟷	M12
	$\ddots$	$\ddots$		$\ddots$
	$\frac{C}{A}\ \varphi_{0,m}^{\text{opt}}$	$\left\{h^{(m,r)}(\tau)\right\}_{r=1,\cdots,m-1}$	⟷	$G^{(m,r)}$

↓

(Niemeier lattices)

The choice of the input seed is completely fixed algorithmically

The input Jacobi forms $\varphi_{0,m}^{\rm opt}$ are closely related to Jacobi forms that arise in the study of black hole state counting.

("Optimal forms", DMZ, '12)

They are chosen so that the optimal property is satisfied.

(c.f. talk of M. Cheng)

Seems to be no analog of K3 for $m > 2$!
(Identity operator missing.)

(Cheng, Duncan, Harvey '12, '13)

Can a MMF be a partition function of a physical system?

Is there a physical system for which the function $h^{(2)}(\tau)$ is an observable, e.g. (BPS) partition function?

Does not transform nicely under modular transformations!

We expect that the observables of a CFT/string theory should transform nicely under symmetry transformations of the torus.

Can a *completed* MJF be a supersymmetric partition function of a physical system?

$$\chi_{\rm ell}(M;\tau,z) = {\rm Tr}_{\mathcal{H}(M)}\,(-1)^{F+\overline{F}}\,q^{L_0}\,\overline{q}^{\overline{L_0}}\,\zeta^{J_0}$$

- Only right-moving ground states contribute for a SCFT with a discrete spectrum.
- For a SCFT with a continuous spectrum:

$$\mathrm{Tr}\, q^{L_0}\,\overline{q}^{\overline{L_0}} = \sum_{n,\overline{n}=0}^{\infty} a(n,\overline{n})\, q^n \overline{q}^{\overline{n}} + \int dp\, \rho(p)\, (q\overline{q})^{p^2}$$

and $\rho_{\rm bos} \neq \rho_{\rm fer}$ in general.

Yes it can

(Troost, Eguchi-Sugawara, Ashok-Troost; '10-'13)

- N=(2,2) "cigar" SCFT $\dfrac{SL(2,\mathbb{R})_m}{U(1)}$

- Free-field representation of SL(2) current algebra, then integral over U(1) gauge field $u = a\tau + b, \quad a, b \in [0,1]$

(Gawedzki, '89)

$$\chi_{\rm ell}({\rm cig}/\mathbb{Z}_m; \tau, z) \;=\; \frac{\vartheta(\tau,z)}{\eta(\tau)^3}\,\widehat{\mathcal{A}}_{1,m}(\tau,z)$$

We can understand the mock nature in fairly simple terms

(S.M. '13, Ashok-Doroud-Troost '13)

- This result can be derived using free field theory from a GLSM written down in (Hori-Kapustin '01) .
- The momentum and winding states of a compact boson do not decouple from the Q-cohomology.
- The holomorphic anomaly is related to the U(1) chiral anomaly in 2d.

NS5-branes wrapped on K3

(J. Harvey, S. M. '13, + C. Nazaroglu, '14)

Worldsheet SCFT of type II string theory on:

$$\mathbb{R}^t \times S^1 \times K3 \times \left(\frac{SL(2)_m}{U(1)} \times \frac{SU(2)_m}{U(1)} \right) \Big/ \mathbb{Z}_m$$

2d *spacetime* SCFT

Non-compact
“cigar” SCFT

ADE classification

Near-horizon geometry of NS 5-branes on Coulomb branch.

(Giveon, Kutasov '99; Ooguri, Vafa '95)

Spacetime BPS index of string states

Compute the 2nd helicity supertrace in spacetime:

$$\widehat{\chi}_2(\tau) = \mathrm{Tr}\,(J_{\mathrm{sp}})^2\,(-1)^{F_s}\,q^{L_0-c/24}\,\overline{q}^{\widetilde{L}_0-\widetilde{c}/24}$$

For a holomorphic $\chi_2(\tau) = \sum_N a(N)\,q^N$,

a(N) = # of states with N=nw (momentum n, winding w).

Technical comments

- Involves sum over spin structures (R,NS), $(1,(-1)^{\mathrm{F}})$ on the string worldsheet.

- Need an explicit SCFT description of K3.

- Final answer only depends on the elliptic genus of K3.

- No N=4 decomposition, no dropping of states.

Partition function of strings near 5-branes

$$\widehat{\chi}_2(\tau) = \frac{1}{2}\int_{T^2(\tau)} \frac{du_1 du_2}{\tau_2}\,(2\tau_2)^{1/2}\, e^{-2\pi u_2^2/\tau_2}\,\frac{\eta(\tau)^6}{\vartheta_1(\tau,u)}\,\overline{\vartheta_1(\tau,u)}\,\chi^{\rm ell}(K3;\tau,u)$$

(D. Gaiotto, D. Zagier)

(J. Harvey, S. M. '13)

m=2 $\widehat{\chi}_2(\tau) = -\eta(\tau)^3\,\widehat{h}^{(2)}(\tau)$

M24 reps

$$\widehat{h}^{(2)}(\tau) = h^{(2)}(\tau) + 12\sum_{k\in\mathbb{Z}} \mathrm{sgn}(4k+1)\,q^{-(4k+1)^2/8}\left(-1 + \mathrm{Erf}\left[\frac{4k+1}{2}\sqrt{2\pi\tau_2}\right]\right)$$

5-brane partition function modular

But not holomorphic!

ADE fivebranes

(J. Harvey, S. M., C. Nazaroglu, '14)
(S.M., D. Zagier, in progress)

- For a fivebrane Y=A/D/E, we find a formula for the second helicity supertrace in terms of "mock Eisenstein series".

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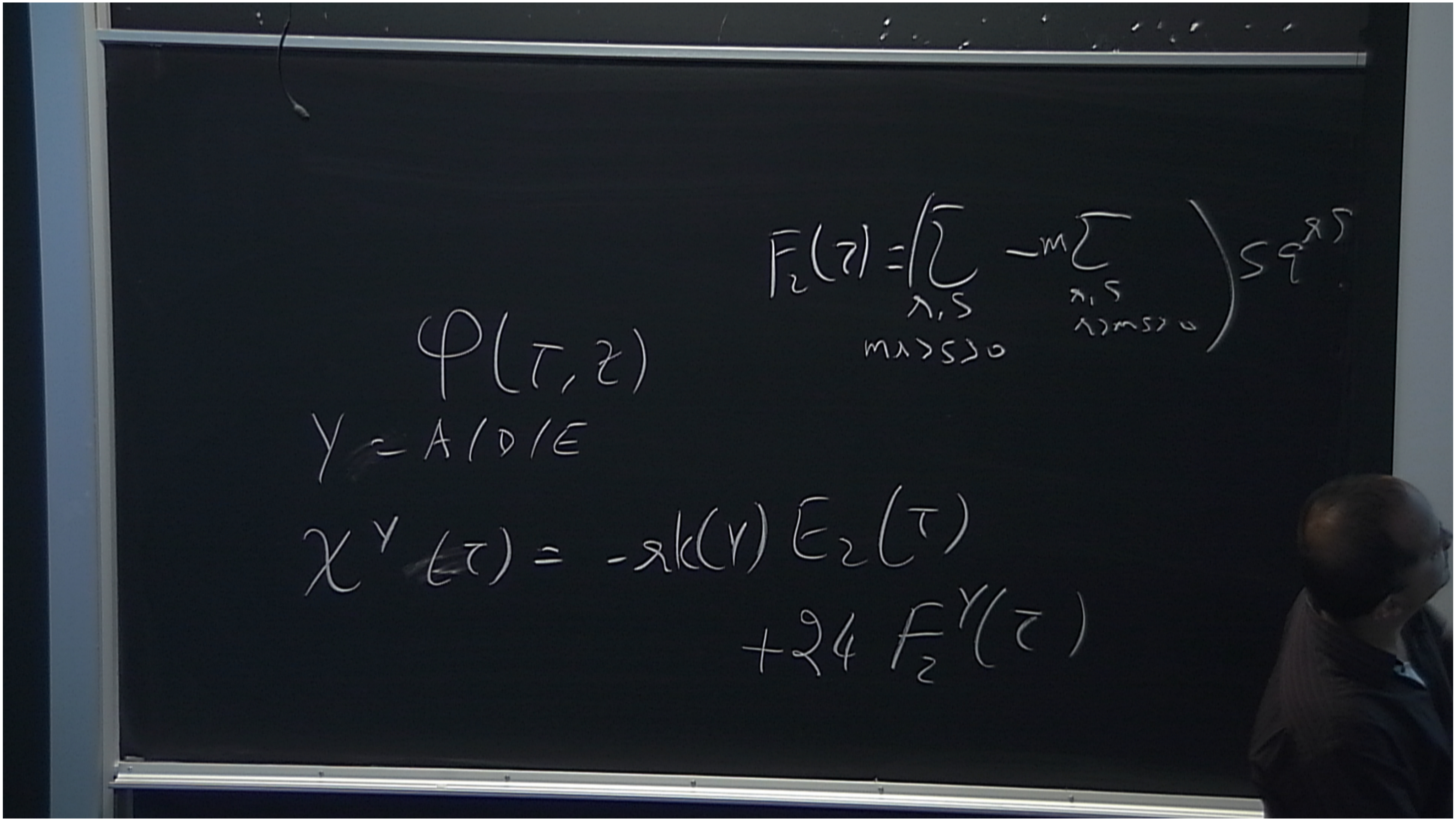
- For a fivebrane Y=A/D/E, we find a formula for the second helicity supertrace in terms of "mock Eisenstein series".

- When rank(Y) | 24, then there is a factorization:

$$\chi_2^{Y=A_3} = 3 - 96\,q - 288\,q^2 - 384\,q^3 - 576\,q^4 - 360\,q^5 + \cdots$$
$$= 3\left(1 - 32\,q - 96\,q^2 - 126\,q^3 - 192\,q^4 - 120\,q^5 + \cdots\right)$$


$F_2(\tau) = \Big(\sum_{\substack{\lambda, s \\ m\lambda > s \geq 0}} - m \sum_{\substack{\lambda, s \\ \lambda \geq ms > 0}} \Big) s q^{\lambda s}$
$\varphi(\tau, z)$
$Y = A/D/E$
$\chi^Y(\tau) = -rk(Y) E_2(\tau) + 24 F_2^Y(\tau)$

ADE fivebranes

- When rank(Y) | 24, second helicity supertrace of fivebrane system agrees with special linear combinations of the completed Umbral mock Jacobi forms.

$$\widehat{\chi}_2^{(m)}(\tau) = (m-1)\sum_{r=1}^{m-1}\widehat{h}^{(m,r)}(\tau)\,S_{m,r}(\tau)$$

$Y = A_{m-1}$ Rank

- Can full Umbral mock Jacobi forms be reconstructed from knowledge of a finite number of such linear combinations?

Summary and future directions

- Connections with Mathieu and Umbral mock Jacobi forms: moonshine or mock modularity?
- Mathematical connection with black hole counting functions — any physics?
- 5-branes in string theory naturally complete the holomorphic partition function into a modular one.
- Home in CFT/string theory for mock modular forms.