Title: Umbral Moonshine and K3 Surfaces

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Abstract: Recently, 23 cases of umbral moonshine, relating mock modular forms and finite groups, have been discovered in the context of the 23 even unimodular Niemeier lattices. One of the 23 cases in fact coincides with the so-called Mathieu moonshine, discovered in the context of K3 non-linear sigma models. Here we establish a uniform relation between all 23 cases of umbral moonshine and K3 sigma models, and thereby take a first step in placing umbral moonshine into a geometric and physical context. This is achieved by relating the ADE root systems of the Niemeier lattices to the ADE du Val singularities that a K3 surface can develop, and the configuration of smooth rational curves in their resolutions. A geometric interpretation of our results is given in terms of the marking of K3 surfaces by Niemeier lattices.

- We are all here for some reason or another because of the observation of EOT relating the group M_{24} and K3 elliptic genus
- This has led to a variety of fascinating progress in the area of moonshine, physics, and mathematics, including the still-mysterious umbral moonshine, of which M_{24} moonshine is only the first case
- What about the other cases of umbral moonshine? Do they have some relation to K3? Can this help us find a unifying structure?

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Elliptic genus of singularities

- Consider singularities of the form $\mathbb{C}^2/G$ with G a subgroup of $SU_2(\mathbb{C})$
- Local description of singularities of K3 surfaces
- ADE classification: Intersection matrix of curves in resolution yields an ADE Dynkin diagram
- In terms of hypersurfaces, it is given by $W^0_\Phi = 0$ with

$$
\begin{aligned}
W^0_{A_{m-1}} &= x_1^2 + x_2^2 + x_3^m \\
W^0_{D_{m/2+1}} &= x_1^2 + x_2^2 x_3 + x_3^{m/2} \\
W^0_{E_6} &= x_1^2 + x_2^3 + x_3^4 \\
W^0_{E_7} &= x_1^2 + x_2^3 + x_2 x_3^3 \\
W^0_{E_8} &= x_1^2 + x_2^3 + x_3^5.
\end{aligned}
$$

Elliptic genus of singularities

The 2d conformal field theory description of these (isolated) singularities was proposed by Ooguri and Vafa to be:

$$\text{Minkowski } \mathbb{R}^{5,1} \otimes \left(\mathcal{N} = 2 \text{ minimal } \otimes \mathcal{N} = 2 \ \frac{SL(2,\mathbb{R})}{U(1)} \right) / (\mathbb{Z}/m\mathbb{Z}). \tag{1.1}$$

where m is the coxeter number of the corresponding simply laced root system. For A_{m-1}, this describes near-horizon geometry of m NS5-branes. Interesting work by Harvey and Murthy where one replaces $\mathbb{R}^{5,1}$ with $K3 \times \mathbb{R}^{1,1}$.

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One can resolve the singularity by considering $W^0_\Phi = \mu$.
It was proposed that the sigma model with the non-compact target space $W^0_\Phi = \mu$ has an alternative description as the Landau–Ginsburg model with superpotential

$$\tilde{W}_\Phi = -\mu x_0^{-m} + W^0_\Phi,$$

where x_0 is an additional chiral superfield and m is again given by the Coxeter number of Φ.
In order to compute the elliptic genus, we need to understand both the contribution from the $\mathcal{N} = 2$ minimal model and the noncompact coset model.

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The $\mathcal{N} = 2$ minimal models

The $\mathcal{N} = 2$ minimal models are known to have an ADE classification based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra. These correspond to a $2m \times 2m$ matrix Ω^{Φ} for each ADE root system Φ.
The central charge is given by the coxeter number of the root system:

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{1.2}$$

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Elliptic genus of singularities

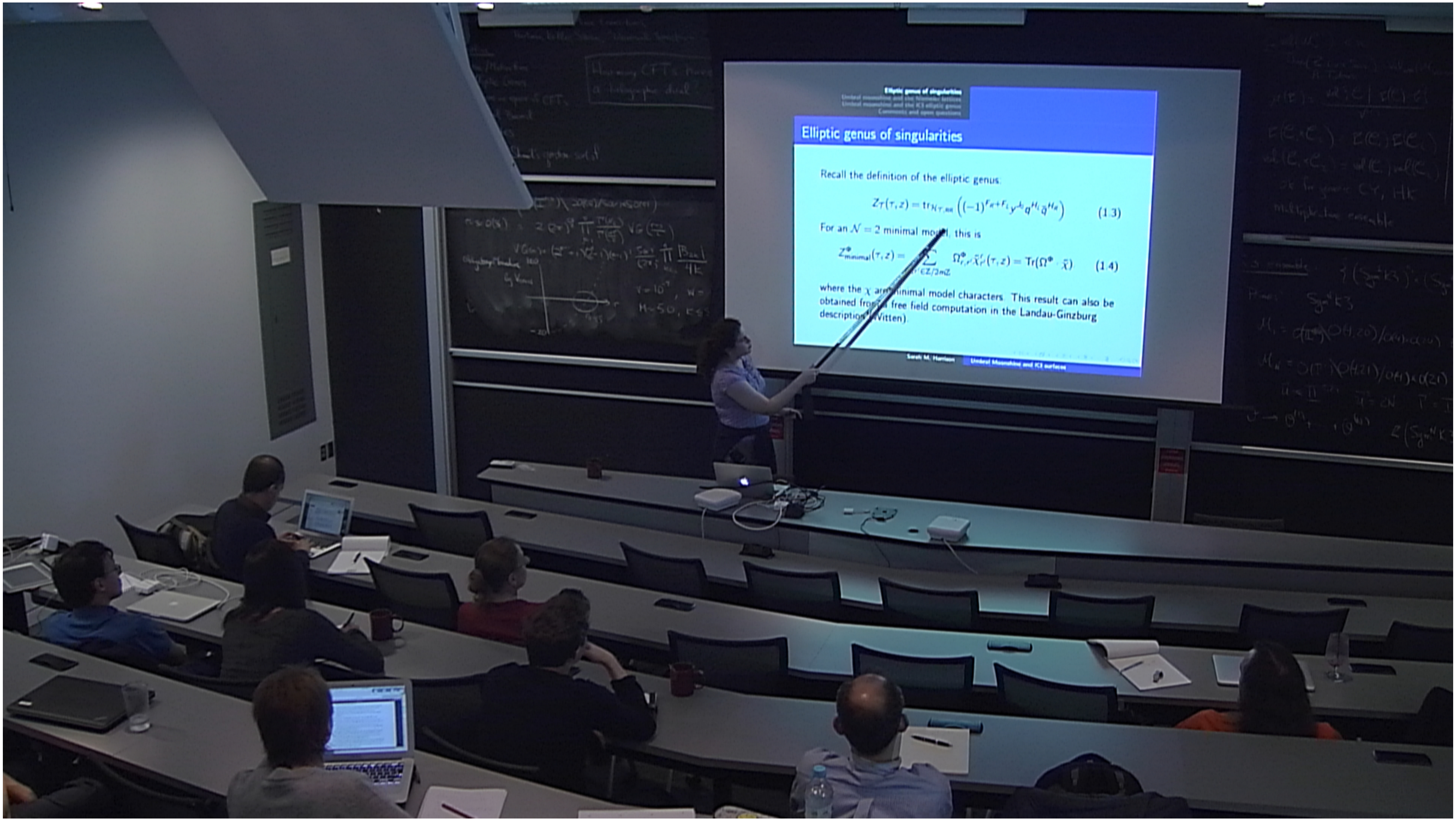
Recall the definition of the elliptic genus:

$$Z_{\mathcal{T}}(\tau, z) = \mathrm{tr}_{\mathcal{H}_{\mathcal{T},\mathrm{RR}}}\left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R}\right) \tag{1.3}$$

For an $\mathcal{N}=2$ minimal model, this is

$$Z^{\Phi}_{\text{minimal}}(\tau, z) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^{\Phi}_{r,r'} \tilde{\chi}^{r}_{r'}(\tau, z) = \mathrm{Tr}(\Omega^{\Phi} \cdot \tilde{\chi}). \tag{1.4}$$

where the χ are minimal model characters. This result can also be obtained from a free field computation in the Landau-Ginzburg description (Witten).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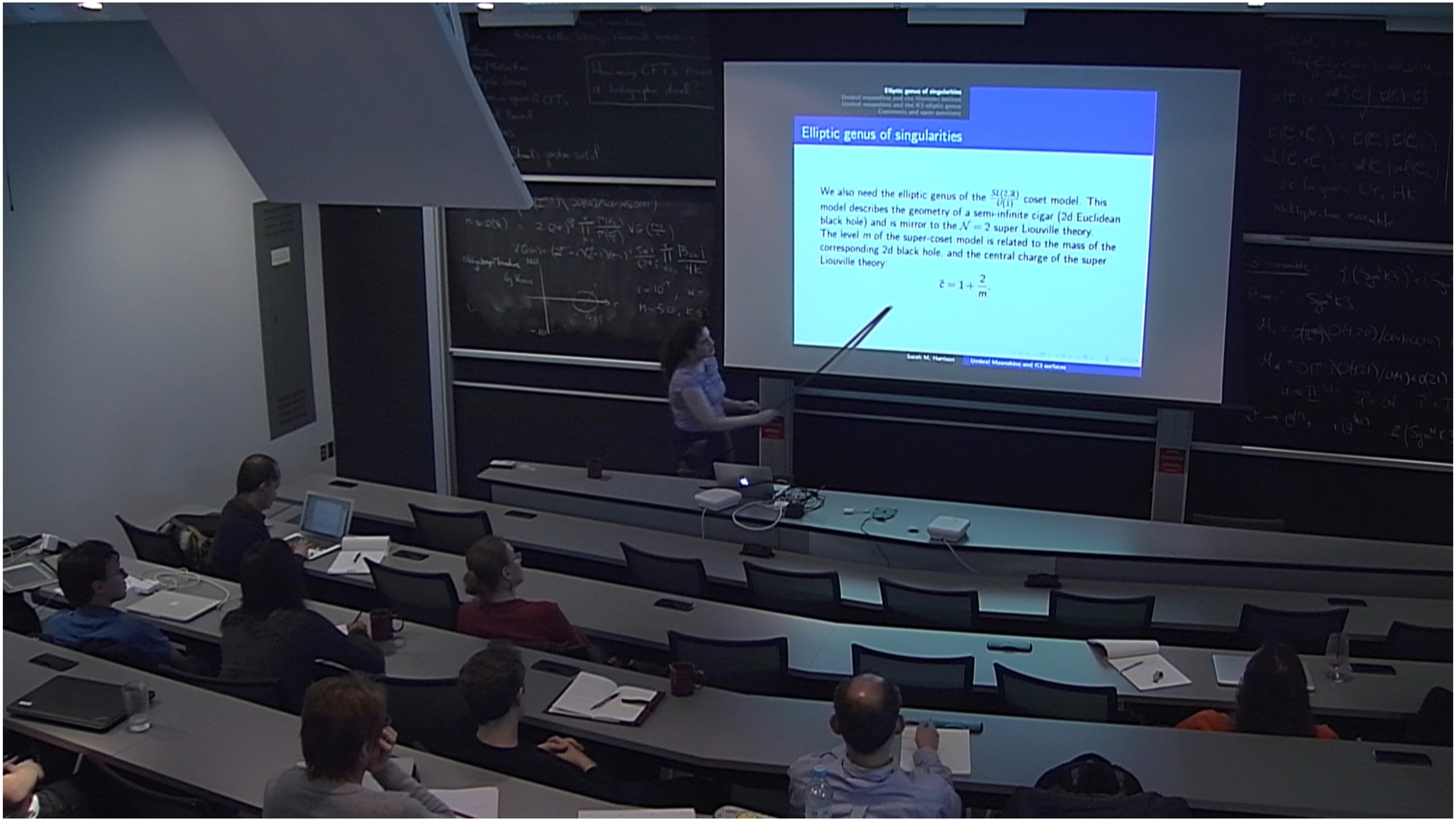
Elliptic genus of singularities

We also need the elliptic genus of the $\frac{SL(2,\mathbb{R})}{U(1)}$ coset model. This model describes the geometry of a semi-infinite cigar (2d Euclidean black hole) and is mirror to the $\mathcal{N}=2$ super Liouville theory.
The level m of the super-coset model is related to the mass of the corresponding 2d black hole, and the central charge of the super Liouville theory:

$$\hat{c} = 1 + \frac{2}{m}.$$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Elliptic genus of singularities

This model is noncompact $\rightarrow$ spectrum contains both discrete and continuous states. (DVV) Geometrically, the discrete states are localized at the tip of the cigar, and the continuous ones are those states whose wave-functions spread into the infinitely long half-cylinder and are only present above a "mass gap" $\frac{1}{4m}$ on the conformal weight.

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Elliptic genus of singularities

The elliptic genus computed in the Hamiltonian formalism contains only contributions from the discrete part of the spectrum.
The building block of the elliptic genus is the Ramond character graded by $(-1)^F$

$$\mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; s) = \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{k\in\mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{\frac{s-1}{m}}}{1 - yq^{mk}}$$

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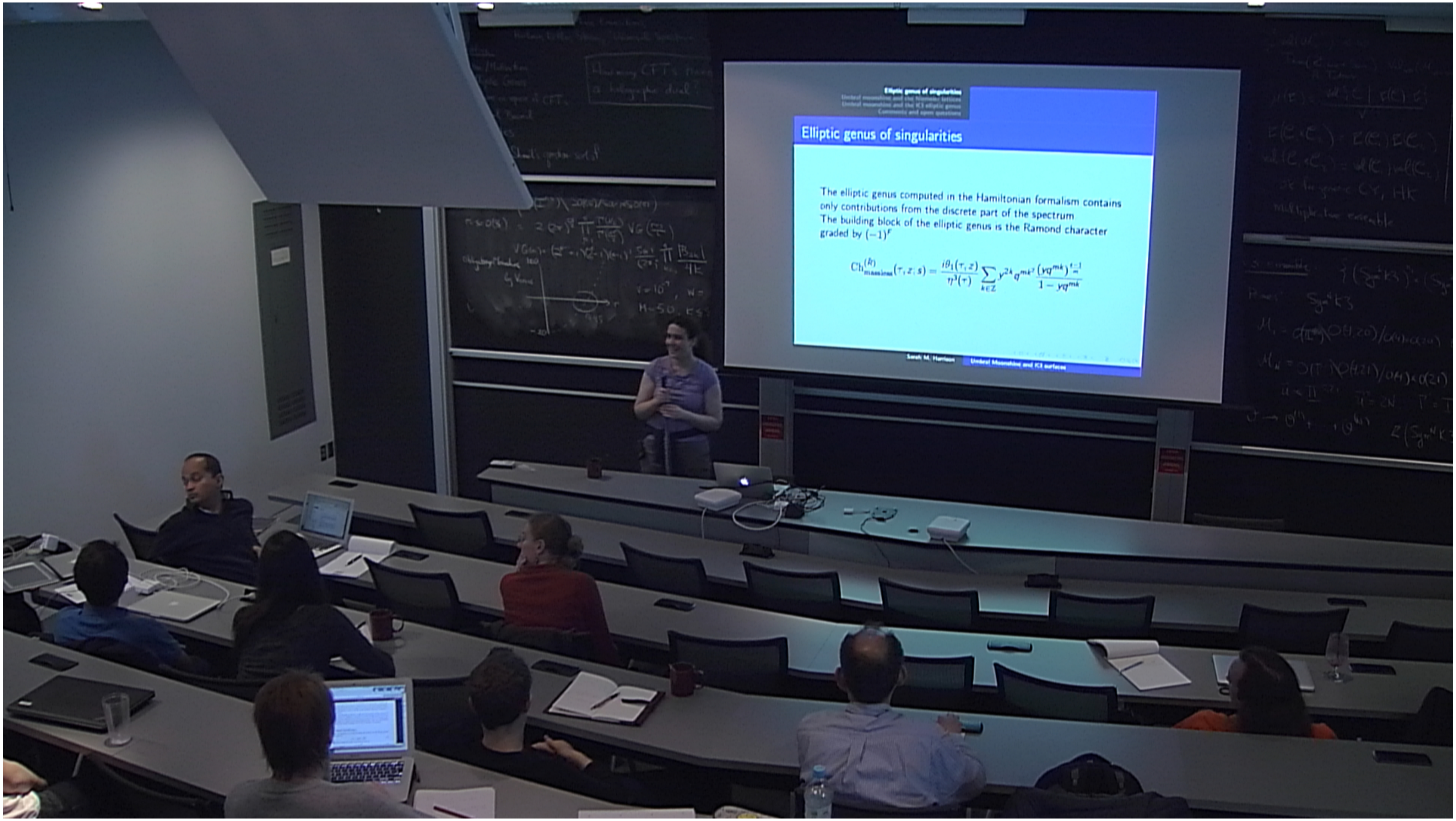
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Elliptic genus of singularities

Putting them together, from the spectrum of the super-coset model it is straightforward to work out the elliptic genus of the theory

$$Z_{L_m}(\tau,z) = \frac{1}{2}\sum_{s=1}^{m} \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,z;m+2-s) + \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,z;s)$$

$$= \frac{1}{2}\mu_{m,0}\left(\tau,\frac{z}{m}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3},$$

where we have used the (specialised) Appell–Lerch sum

$$\mu_{m,0}(\tau,z) = -\sum_{k\in\mathbb{Z}} q^{mk^2}y^{2km}\frac{1+yq^k}{1-yq^k}.$$

[Troost, Ashok-Troost, Eguchi-Sugawara]

Elliptic genus of singularities

One could also consider a path integral approach to the computation of the elliptic genus, which should naturally yield a modular object:

$$\hat{Z}^{\Phi,S}(\tau,z) =$$
$$\frac{1}{2m}\frac{i\theta_1(\tau,z)}{\eta^3(\tau)}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}}(-1)^{a+b}q^{a^2/2}y^a\, Z^{\Phi}_{\mathrm{minimal}}(\tau,z+a\tau+b)\;\hat{\mu}_{m,0}(\tau,\frac{z+a\tau+b}{m})$$

The Niemeier lattices

Here we review the main components of umbral moonshine.

- Even, unimodular, positive-definite lattices of rank 24
- More broadly relevant because their theta functions are modular invariant
- 24 such lattices, classified by Niemeier: Leech lattice + 23 others which have ADE classification
- Uniquely determined by their root systems $\Delta(L)$, that are all unions of the simply-laced root systems

The Niemeier lattices

Two conditions: all of the irreducible components have the same Coxeter numbers; total rank is 24:

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

The Niemeier lattices

Umbral groups come from the automorphism group of the lattice mod the Weyl group generated by reflections of the roots:

$$G^X = \mathrm{Aut}(L^X)/\mathrm{Weyl}(X).$$

The mock modular forms

We will use these to define special mock modular forms. Mock modular property: require H^X to be a weight 1/2 vector-valued mock modular form whose shadow is given by S^X:

$$\hat{H}^X_r(\tau) = H^X_r(\tau) + \mathrm{e}(-\tfrac{1}{8}) \frac{1}{\sqrt{2m}} \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-\frac{1}{2}} \, \overline{S^X_r(-\bar{\tau}')} \, \mathrm{d}\tau',$$

then

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \hat{H}^X_r(\tau) \, \theta_{m,r}(\tau, z)$$

transforms as a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

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The mock modular forms

Analyticity condition: require its growth near the cusp to be

$$q^{1/4m} H_r^X(\tau) = O(1) \quad \text{as } \tau \to i\infty \tag{2.1}$$

for every element $r \in \mathbb{Z}/2m\mathbb{Z}$.
The above two conditions turn out to be sufficient to determine H^X uniquely (up to a rescaling), as shown. We also fix the scaling by requiring $q^{1/4m} H_1^X(\tau) = -2 + O(q)$.

The mock modular forms

Modular properties?

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X \, \theta_{m,r}$$

is a mock Jacobi form which is the finite part of a meromorphic Jacobi form with simple poles at m-torsion points.

The mock modular forms

To fix modular properties of the twining functions H^X_g we need

- The shadow $S^X_g = \Omega^X_g S_m = S^X_{g,r}$
- The multiplier system

Given this info, we can define:

$$\hat{H}^X_{g,r}(\tau) = H^X_{g,r}(\tau) + \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S^X_{g,r}(-\bar{\tau}')}\,\mathrm{d}\tau',$$

which appears in the theta decomposition of a weight 1 index m Jacobi form under a modular subgroup:

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{H}^X_{g,r}(\tau)\,\theta_{m,r}(\tau,z).$$

The mock modular forms

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H^X_{g,r}\,\theta_{m,r} \tag{2.2}$$

is a mock Jacobi form of weight 1 and index m under a subgroup The functions H^X_g also satisfy a similar growth condition as in the case of the identity. These are defined for every $[g]\subset G^X$ and for all 23 Niemeier lattices L^X.

Umbral moonshine and the K3 elliptic genus

Now let's see how we can relate all of these mock modular forms to the $K3$ elliptic genus.
Recall the $\mathcal{N}=4$ decomposition of the K3 elliptic genus

$$\begin{aligned}\mathbf{EG}(\tau,z;K3) &= 20\,\mathrm{ch}_{2;\frac{1}{4},0} - 2\,\mathrm{ch}_{2;\frac{1}{4},\frac{1}{2}} + \big(90\,\mathrm{ch}_{2;\frac{5}{4},\frac{1}{2}} + 462\,\mathrm{ch}_{2;\frac{9}{4},\frac{1}{2}} \\ &\quad + 1540\,\mathrm{ch}_{2;\frac{13}{4},\frac{1}{2}} + \dots\big) \\ &= \frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\Big\{24\,\mu_{2,0}(\tau,z) + (\theta_{2,-1}(\tau,z) - \theta_{2,1}(\tau,z)) \\ &\quad \times\big(-2q^{-1/8} + 90q^{7/8} + 462q^{15/8} + 1540q^{23/8} + \dots\big)\Big\}\end{aligned}$$

Umbral moonshine and the K3 elliptic genus

We can view the two contributions to $\mathbf{EG}(\tau, z; K3)$, given by

$$24\,\mu_{2,0}(\tau, z)$$

and

$$-\sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{X=A_1^{24}}(\tau)\theta_{2,r}(\tau, z),$$

as contributions from the BPS and non-BPS $\mathcal{N}=4$ multiplets respectively (up to the polar term in $H_r^{X=A_1^{24}}(\tau)$).

Umbral moonshine and the K3 elliptic genus

There is an alternative interpretation due to the identity between the short $\mathcal{N}=4$ characters and the elliptic genus of an $\Phi = A_1$ singularity:

$$Z^{A_1,S}(\tau,z) = \mathrm{ch}_{2;\frac{1}{4},0}(\tau,z),$$

In other words, we can re-express the elliptic genus of $K3$ as

$$\mathbf{EG}(\tau,z;K3) = 24Z^{A_1,S}(\tau,z) - \frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)} \sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{A_1^{24}}\theta_{2,r}(\tau,z).$$

This provides a geometric interpretation of this decomposition.

Umbral moonshine and the K3 elliptic genus

We can rewrite the former expression as

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi^X \Big(\tau, \frac{z + a\tau + b}{m}\Big)$$

for $X = A_1^{24}$, where

$$\phi^X = \frac{i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau)\, \theta_{m,r}(\tau, z)$$

encodes the umbral moonshine mock modular form.

Umbral moonshine and the K3 elliptic genus

Where, for

$$X = A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E},$$

we write

$$Z^{X,S} = d_A Z^{A_{m-1}} + d_D Z^{D_{m/2+1}} + d_E Z^{E^{(m)}}.$$

corresponding to a collection of no n-interacting ADE theories with the total Hilbert space given by the direct sum of the Hilbert spaces of the component theories.
So we can offer an alternate interpretation of this decomposition as a contribution from 24 copies of A_1-type surface singularities and an "umbral moonshine" contribution given by the umbral moonshine mock modular forms H^X with $X = A_1^{24}$

Umbral moonshine and the K3 elliptic genus

In fact, this type of decomposition holds for all 23 cases of umbral moonshine!

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi^X\left(\tau, \frac{z + a\tau + b}{m}\right)$$

In other words: for the 23 Niemeier lattices L^X we have 23 different ways of separating $\mathbf{EG}(K3)$ into two parts.

- Replace the Niemeier root system X with the corresponding configuration of singularities to obtain a contribution to the $K3$ elliptic genus by the singularities.
- Use the umbral moonshine construction for the mock Jacobi form ϕ^X associated to each L^X to get the rest of $\mathbf{EG}(K3)$ after a summation procedure reminiscent of the "orbifoldization" formula for the elliptic genus of orbifold SCFTs

Umbral moonshine and the K3 elliptic genus

We can also define a twisted version of this relation:

$$Z_g^X(\tau,z) = Z_g^{X,S}(\tau,z) + \frac{1}{2m} \sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\, \phi_g^X\big(\tau, \frac{z+a\tau+b}{m}\big).$$

Umbral moonshine and the K3 elliptic genus

Consequences of this relation:
Consider [g] which obey a geometric condition, $g \in G^X$ has at least 5 orbits and one fixed point on the 24-dimensional representation. For these elements,

- Whenever $g_1 \in G^{X_1}$ and $g_2 \in G^{X_2}$ both satisfy the geometric condition and have the same 24-dimensional cycle shape $\Pi_{g_1}^{X_1} = \Pi_{g_2}^{X_2}$, we obtain
$$Z_{g_1}^{X_1} = Z_{g_2}^{X_2} \tag{3.1}$$
- The result coincides with the geometrically twined elliptic genus for a $K3$ admitting $\langle g \rangle$-symmetry
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Umbral moonshine and the K3 elliptic genus

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$4A$	4^6	H_{4C}
$2B$	$1^8 2^8$	H_{2A}
$2C$	2^{12}	H_{2B}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$2^3 6^3$	$\tilde{T}_{6A}^X = 3\Lambda_2 + 2\Lambda_3 - \Lambda_4 - 3\Lambda_6 + \Lambda_{12}$
$3B$	3^8	$\tilde{T}_{3B}^X = 2(-4\Lambda_3 + \Lambda_9 - (1)^6/(3)^2)$
$6B$	6^4	$\tilde{T}_{6B}^X = 2\frac{(1)^5(3)}{(2)(6)}$
$4B$	$2^4 4^4$	H_{4A}
$4C$	$1^4 2^2 4^4$	H_{4B}
$5A$	$1^4 5^4$	H_{5A}
$10A$	$2^2 10^2$	H_{10A}
$12A$	12^2	$\tilde{T}_{12A}^X = 2\frac{(1)(2)^5(3)}{(4)^2(6)}$
$6C$	$1^2 2^2 3^2 6^2$	H_{6A}
...	...	...

Geometric interpretation

Main results of Nikulin:

- Every $K3$ surface admits a marking by (at least) one of the 23 Niemeier lattices
- For every L^X with the exception of $X = A_{24}$ and $X = A_{12}^2$, there exists a $K3$ surface that can only be marked using L^X and not by any other Niemeier lattice (the exceptions are a conjecture)
- For any L^X, any primitive sublattice of L^X which can be primitively embedded into $\Gamma_{3,19}(-1)$ arises from the Picard lattice $\mathrm{Pic}(M)$ of a certain $K3$ surface M

Geometric interpretation

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Geometric interpretation

Applications:

- For the generic cases, a $K3$ surface M that can be marked by L^X has the configuration of all smooth rational curves given by $X \cap S_M$. In particular, if one thinks of the rational curves as arising from the minimal resolutions of the $\mathbb{C}^2/G$ singularities, then the singularities have to be given by a sub-diagram of the Dynkin diagram corresponding to X.
- If M is a generic $K3$ surface admits a marking by L^X, then the finite symplectic automorphism group G_M of M is a subgroup of G^X.

Comments

We find some evidence for the relation between groups for index m umbral moonshine and $\mathbb{Z}_m$ orbifold K3 sigma models. [WIP with Francesca Ferrari and Natalie Paquette] In particular, we find in a $\mathbb{Z}_3$ orbifold model an order 11 symmetry which comes from the $m = 3$ decomposition of $\mathbf{EG}(\tau, z; K3)$, and is a different function from the order 11 twining of $\mathbf{EG}(\tau, z; K3)$ coming from $M_{24}(m = 2)$.

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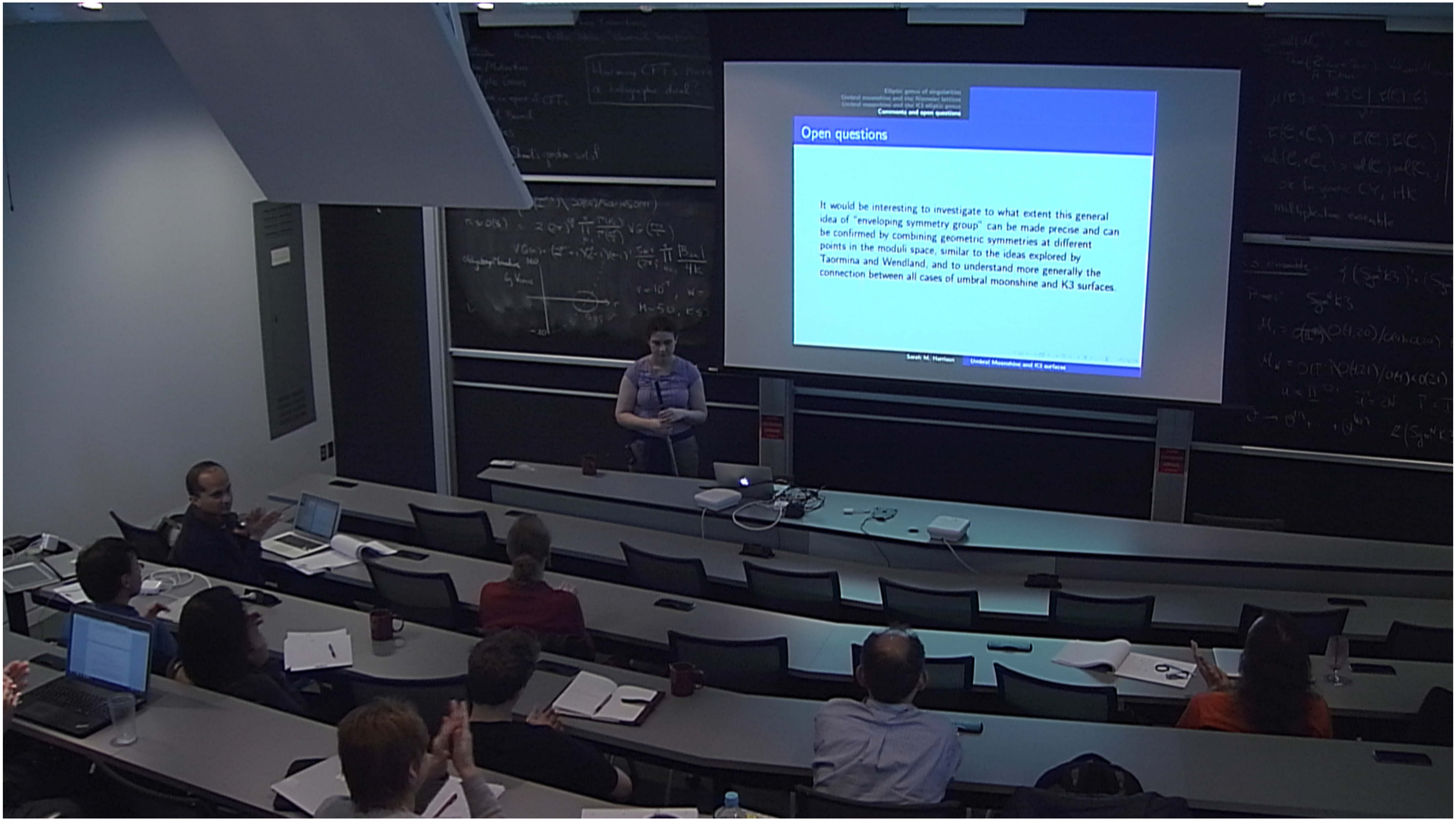
There is another rewriting of the formula

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\, \phi^X(\tau, \frac{z + a\tau + b}{m})$$

which is mathematically equivalent:

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \phi_{-2,1}(\tau, z) \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau)\, S_{m,r}(\tau)$$

Notably, the piece coming from the umbral mock modular forms, appears in other contexts, e.g. the spacetime BPS index χ_2 computed by Harvey and Murthy coming from NS5-branes wrapped on K3.



Comments and open questions
Open questions
It would be interesting to investigate to what extent this general idea of "enveloping symmetry group" can be made precise and can be confirmed by combining geometric symmetries at different points in the moduli space, similar to the ideas explored by Taormina and Wendland, and to understand more generally the connection between all cases of umbral moonshine and K3 surfaces.