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Abstract:

Analytic Bootstrap Bounds

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to be published, with Hyungrok Kim and Petr Kravchuk

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When I accepted Shamit's kind invitation to speak,
I was hoping that our project would lead to a result of relevance to the Moonshine by the time of the conference.

Unfortunately, we have not gotten there yet
— as usual, our project has made unexpected turns, but I hope that the results I will present today would still be of interest to you and that they would eventually become relevant to the Moonshine, too.

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Tohru Eguchi and Anne Taormina,

- ☆ *Unitary Representations of N=4 Superconformal Algebra*
  Phys. Lett. B196 (1987) 75.
- ☆ *Character Formulas for the N=4 Superconformal Algebra*
  Phys. Lett. B200 (1988) 315.
- ☆ *On the Unitary Representations of N=2 and N=4 Superconformal Algebras,* Phys. Lett. B210 (1988) 125.

Volume 210, number 1,2 PHYSICS LETTERS B 18 August 1988

$$\mathrm{ch}_0^{\mathrm{R}}(l=0;\ -1/\tau)=\mathrm{ch}_0^{\mathrm{NS}'}(l=1/2;\ \tau)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{2\cosh\pi\alpha}\,\mathrm{ch}^{\mathrm{NS}'}(h=\alpha^2/2-1/8;\ \tau)\ ,$$

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$$\mathrm{ch}_0^{\mathrm{R}}(k=1,\ l=0;\ z)=\sum_m q^{m^2/2+m/2+1/4}z^{m+1/2}\frac{1}{1+zq^m}f^{\mathrm{R}}(z)\ ,$$

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Freeman Dyson at the **Ramanujan Centenary Conference** in 1987:

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. ... My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions...

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Tohru Eguchi, Anne Taormina, Sung-Kil Yang + H.O.,

☆ *Extended Superconformal Algebras and String Compactifications*

Nucle. Phys. B315 (1989) 193.

$$\Phi(\hat{A}) = 2\frac{\left(\vartheta_2^4 - \vartheta_4^4\right)}{\eta^4}\left(\frac{\vartheta_3}{\eta}\right)^2$$

$$= -q^{-1/4}\left(2 - 40q^{1/2} - 124q + \dots\right), \qquad (3.3)$$

eq. (3.3) may also be derived directly from (3.1); we note that the boundary condition of $\Phi(\hat{A})$ is invariant under the transformations S and T^2 (T: $\tau \to \tau + 1$) and thus $\Phi(\hat{A})$ is a modular form invariant under Γ_2, the level-2 principal congruence subgroup. This fact uniquely determines $\Phi(\hat{A})$ up to an overall constant

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My PhD thesis, Int. J. Mod. Phys. A4 (1989) 4303:

$$F(\tau) \equiv \sum_h (N_{h,1} - 2N_{h,0}) q^h = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots$$

If q^h in $F(\tau)$ had a negative coefficient for some value of h, the corresponding $N_{h,0}$ would be non-vanishing. If this were the case, there should be a holomorphic field of dimension h, and the symmetry of the system could not be just the $N = 4$ symmetry but always larger than that. I have computed the expansion coefficients of $F(\tau)$ to the order of q^{50} and found that they are all positive and exponentially increasing. One can also examine the asymptotic behaviour of $F(\tau)$ as $\tau \to 0$, i.e. $q \to 1$. Using the modular transformation property of the ϑ-functions and the Mordell's formula[23] for $h_3(\tau)$, My PhD thesis, Int. J. Mod. Phys. A4 (1989) 4303:

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$$\begin{aligned} F(-1/\tau) &= \sum_h (N_{h,1} - 2N_{h,0}) \cdot \tilde{q}^h \\ &\longrightarrow 2\sqrt{-i\tau} q^{-1/8} - 12 + \cdots . \end{aligned} \tag{24}$$

This observation seems to imply that the q-expansion coefficients of $F(\tau)$ are all positive and the symmetry of the generic non-linear σ-model is just the $N = 4$ superconformal symmetry, though I have no rigorous proof for it.

It took us 22 years to divide these coefficients by 2.

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Tohru Eguchi, Yuji Tachikawa + H.O.,

☆ *Notes on the K3 Surface and the Mathieu Group M24*

Experimental Mathematics 20 (2011) 91.

n	1	2	3	4	5
A_n	45	231	770	2277	5796

Conformal Bootstrap

To understand these 1/4 BPS states beyond their degeneracies, it would be interesting to study their operator product expansions.

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$\Rightarrow$ **Conformal Bootstrap**

Currently, we are trying to construct N=4 conformal blocks for 4-point functions on the 2-sphere.

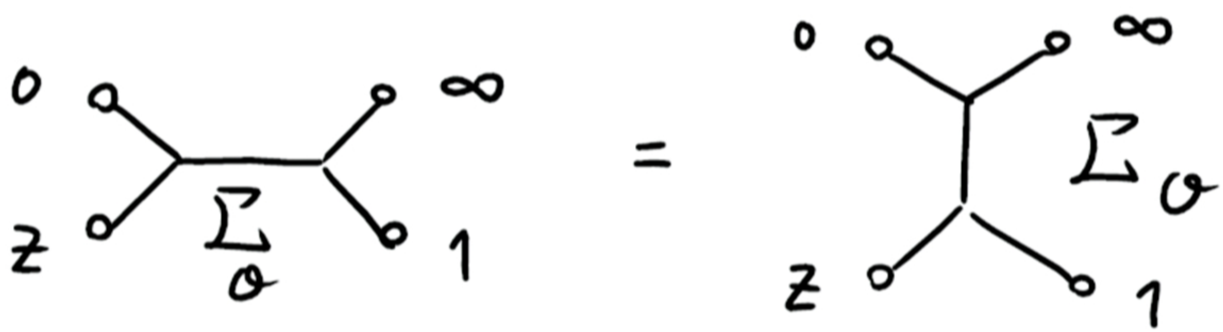
In meanwhile, we found that there are things we can do **without full conformal blocks**.

$$G(z) = \langle \phi(0)\, \phi(z)\, \phi(1)\, \phi(\infty) \rangle$$

scalar 4-point in CFT_d

In any dimensions, we can use conformal symmetry to locate 4 points on a 2-plane and at $0, z, 1, \infty$.

Crossing Symmetry : $G(1-z) = G(z)$



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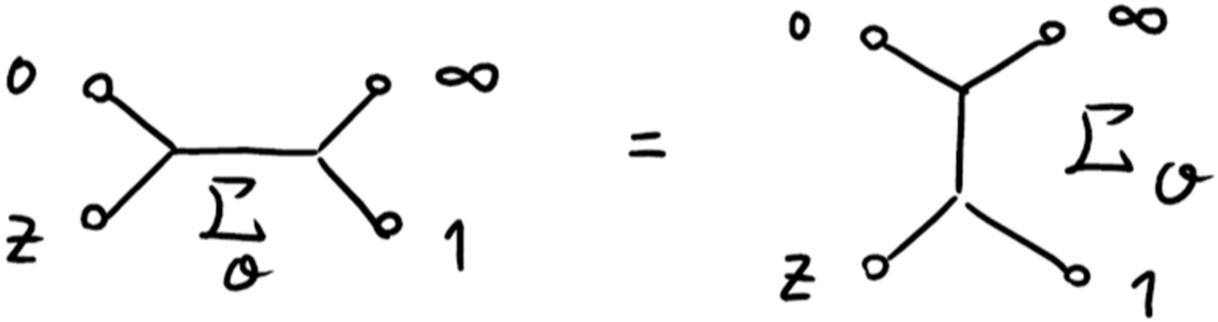

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$$G(z) = \langle \phi(0)\,\phi(z)\,\phi(1)\,\phi(\infty)\rangle$$

scalar 4-point in CFT_d

$$= \begin{cases} \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2 \, z^{-2\Delta_0} \, z^{\Delta_{\mathcal{O}}} & (1) \\ \sum_{\mathcal{O}_{p(\text{rimary})}} C_{\phi\phi\mathcal{O}_p}^2 \, z^{-2\Delta_0} \, F_{\Delta_{\mathcal{O}_p};\, l_{\mathcal{O}_p}}(z) & (2) \end{cases}$$

scaling blocks

conformal blocks

We will compare bootstrap results for (1) and (2) for any d.

$$G(z) = \langle \phi(0)\, \phi(z)\, \phi(1)\, \phi(\infty) \rangle$$

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scaling blocks

Treat conformal descendants separately.

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scaling blocks

Treat conformal descendants separately.

Consider $z = 1/2$ (general z, later).

- By definition, $\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1$

- Crossing Symmetry for z : real

$$\Leftrightarrow \quad \int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0 ,$$

where $[x]^n \equiv x(x-1)\cdots(x-n+1)$

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$$G(z) = \int_0^\infty z^{\Delta - 2\Delta_0} S(\Delta)\, d\Delta$$

We want to use the crossing symmetry,

$$G(1-z) = G(z),$$

to find bounds on

$$B_z(\Delta) \equiv \frac{1}{G(z)} z^{\Delta - 2\Delta_0} S(\Delta)$$

"Branching Ratio"

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Duality of Linear Optimization

$$\max(\vec{c}\cdot\vec{x}),$$

subject to $A\vec{x} = \vec{b}\ ,\quad \vec{x} \geqq \vec{0}$

$$\Updownarrow$$

$$\min(\vec{b}\cdot\vec{y})$$

subject to $A^T\vec{y} \geqq \vec{c}$

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$$\max \left(\int_{\Delta-\varepsilon}^{\Delta+\varepsilon} B_{1/2}(\Delta')\, d\Delta' \right)$$

$\leftarrow \max(\vec{c}\cdot\vec{x})$

subject to

$$\begin{cases} \bullet \ \displaystyle\int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0 \\ \bullet \ \displaystyle\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1, \end{cases}$$

$\leftarrow A\vec{x} = \vec{b}$

and $B_{1/2}(\Delta) \geq 0$ $\quad \leftarrow \vec{x} \geq \vec{0}$

Dual Problem :

$$P(\Delta) = 1 + \sum_k [\Delta - 2\Delta_0]^{2k+1} \lambda_k$$

$$\min \left(\frac{1}{P(\Delta)} \right),$$

subject to $P(\Delta') \geqq 0$,

$\Delta' \in (0, \infty)$

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18/46

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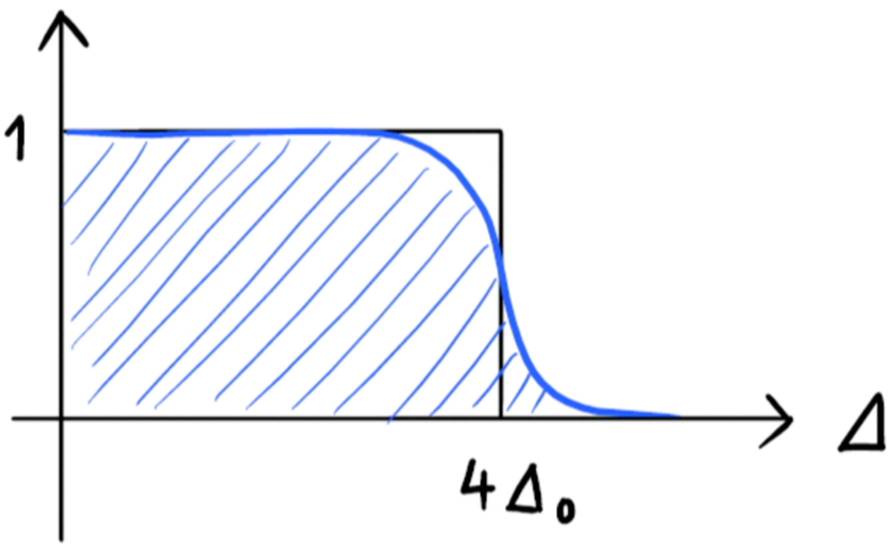
Optimization subject to

$$B_z(\Delta) \equiv \frac{1}{G(z)} z^{\Delta-2\Delta_0} S(\Delta)$$

$$\int_0^\infty [\Delta - 2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta = 0$$

$$\int_0^\infty B_{1/2}(\Delta)\, d\Delta = 1 , \quad B_{1/2}(\Delta) \geq 0$$

$$\int_\Delta^\infty B_{1/2}(\Delta')\, d\Delta'$$



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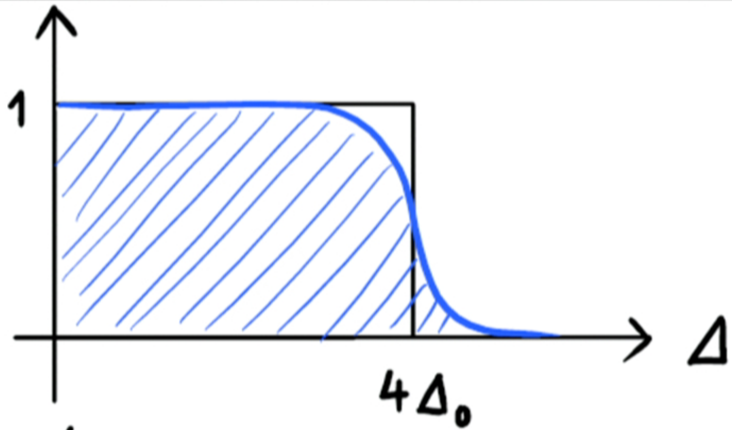
$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta'$$



To understand why, note :

$$\int_0^{\infty} (\Delta - 2\Delta_0)^{2k+1} B_{1/2}(\Delta)\, d\Delta \simeq 0$$

$$k = 1, 2, \cdots, n \ll \sqrt{\Delta_0}$$

$$\Rightarrow \quad B_{1/2}(\Delta) \sim B_{1/2}(4\Delta_0 - \Delta) = 0 \quad \text{for } 4\Delta_0 < \Delta$$

when smeared over $\delta\Delta/\Delta \sim 1/n$

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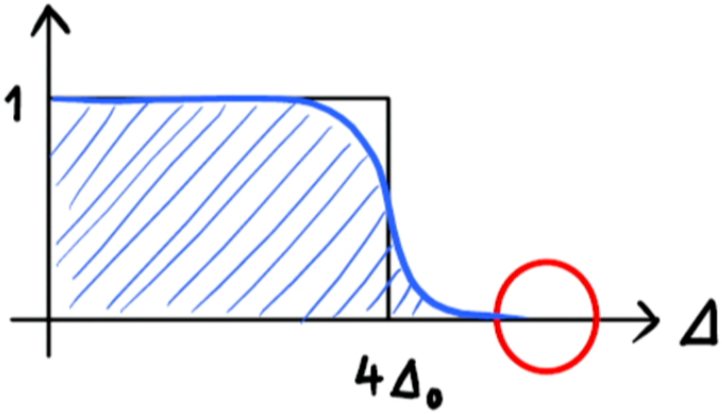


Analytical bound on the exponential tail

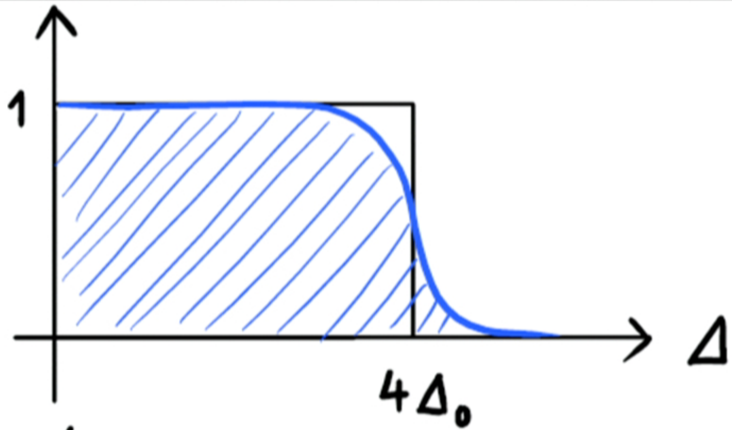
$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta' \leqq \frac{1}{1+\frac{\Gamma(\Delta-2\Delta_0+1)\Gamma(2\Delta_0)}{\Gamma\left(\frac{\Delta+3}{2}\right)\Gamma\left(\frac{\Delta-1}{2}\right)}}$$

$$\int_0^{\infty} [\Delta-2\Delta_0]^{2k+1} B_{1/2}(\Delta)\, d\Delta \simeq 0$$

Solution to the original problem w/ansatz appropriate for large Δ.

$$\sim 2^{-\Delta}\, \Delta^{2\Delta_0-1/2}\, \frac{\sqrt{2\pi}}{\Gamma(2\Delta_0)}$$

$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\,d\Delta'$$



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when smeared over $\delta\Delta/\Delta \sim 1/n$

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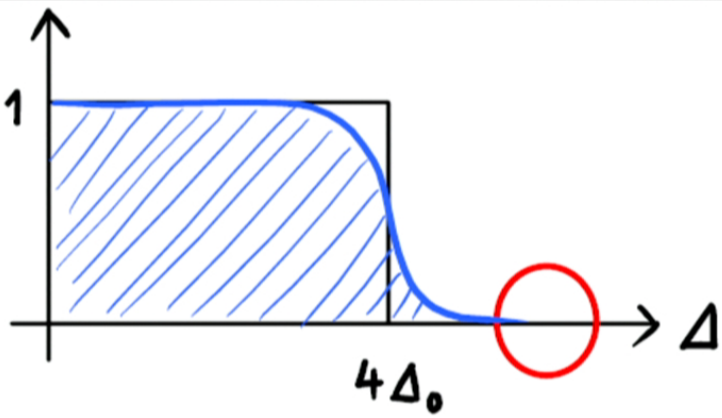


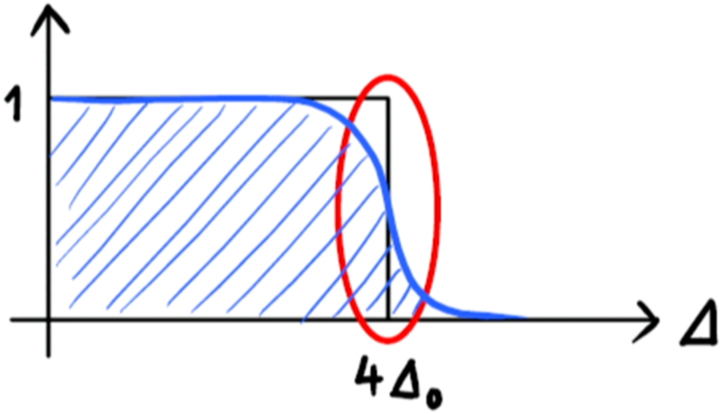
Analytical bound on the exponential tail

$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta' \leqq \frac{1}{1+\dfrac{\Gamma(\Delta-2\Delta_0+1)\,\Gamma(2\Delta_0)}{\Gamma\left(\frac{\Delta+3}{2}\right)\Gamma\left(\frac{\Delta-1}{2}\right)}}$$

For $\Delta_0 \gg 1$,

$$\sim 2^{-\Delta}\,\Delta^{2\Delta_0-1/2}\,\frac{\sqrt{2\pi}}{\Gamma(2\Delta_0)}$$

it is related to 1208.6449 by Pappadopulo, Rychkov, Espin, Rattazzi



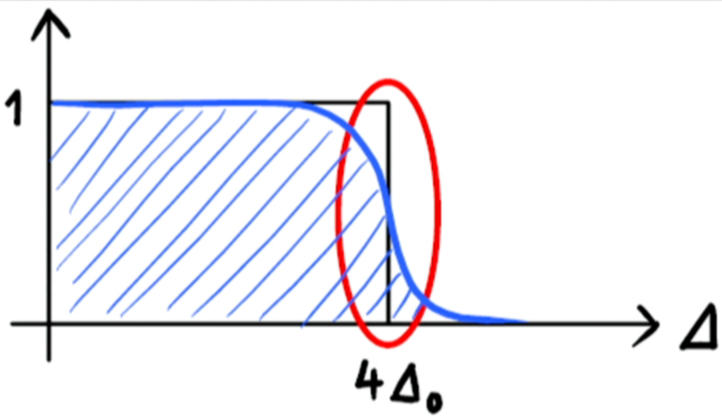
$$\int_\Delta^\infty B_{1/2}(\Delta')\, d\Delta' \le \frac{2}{1 + T_{2m+1}\left(\frac{\Delta}{2\Delta_0} - 1\right)}$$

To understand why, look at the dual problem :

Minimize $1/P(\Delta)$ for

$$P(\Delta) = 1 + \sum_k [\Delta - 2\Delta_0]^{2k+1} \lambda_k$$

subject to $P(\Delta') \geqq 0$ for $\Delta' \in (0, \infty)$.



$$\int_{\Delta}^{\infty} B_{1/2}(\Delta')\, d\Delta' \le \frac{2}{1 + T_{2n+1}\left(\frac{\Delta}{2\Delta_0} - 1\right)}$$

Approximate this by the truncated problem:

Minimize $1/P(\Delta)$ for $P(\Delta) = 1 + Q_n(\Delta)$.

$$\begin{cases} \bullet\ Q_n(\Delta) = -Q_n(4\Delta_0 - \Delta) \\ \bullet\ \text{degree } 2n+1 \ll \sqrt{\Delta_0} \\ \bullet\ P(\Delta') = 1 + Q_n(\Delta') \geq 0 \end{cases}$$

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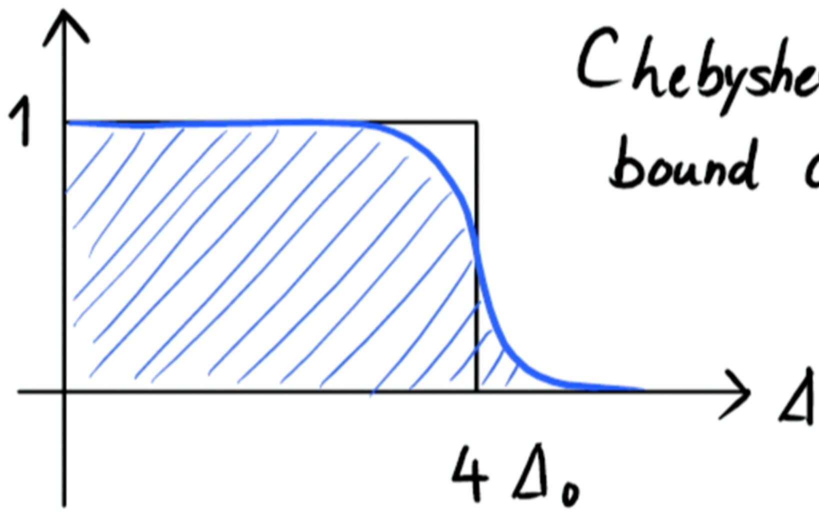
Crossing symmetry at $z = 1/2$

$$B_{1/2}(\Delta) \sim B_{1/2}(4\Delta_0 - \Delta)$$

$= 0$ for $\Delta > 4\Delta_0$

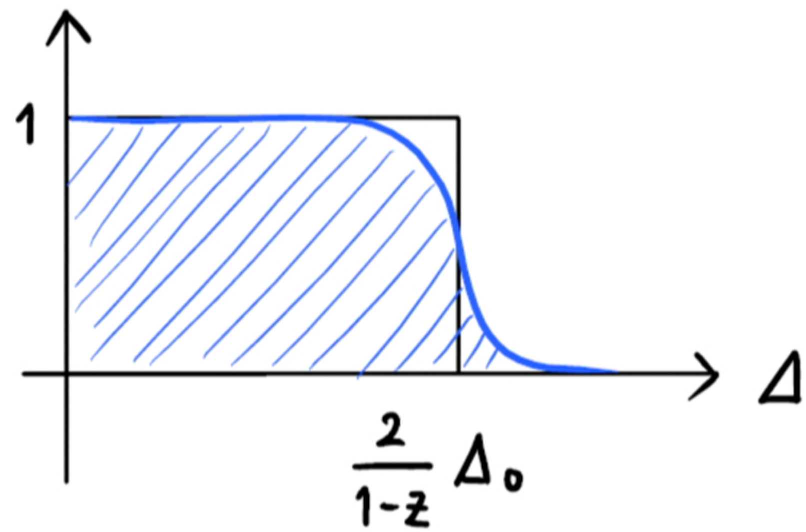
Chebyshev polynomials bound corrections.



For general z : real,

$$\int_{\Delta}^{\infty} B_z(\Delta')\, d\Delta' \leq \frac{2}{1 + T_{2m+1}\left((1-z)\frac{\Delta}{\Delta_0} - 1\right)}$$

$$\int_{\Delta}^{\infty} B_z(\Delta')\, d\Delta'$$



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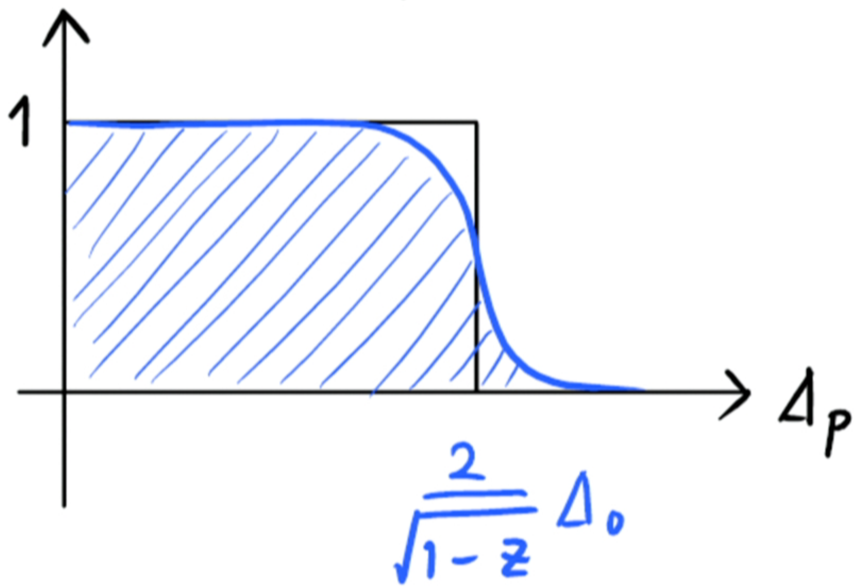
Crossing symmetry

$$G(z) = G(1-z) \Rightarrow B_z(\Delta) \sim B_{1-z}(\tilde{\Delta}),$$

$$\Delta_p = \frac{2}{\sqrt{1-z}}\Delta_0 - \sqrt{\frac{z}{1-z}}\,\tilde{\Delta}_p \lesssim \frac{2}{\sqrt{1-z}}\Delta_0$$



Chebyshev polynomials bound corrections.

This follows from the large Δ behavior of conformal blocks :

$$F_{\Delta_p, \ell} \sim \left(1 - \frac{\rho^2}{16}\right)^{-\frac{d}{2}} \rho^{\Delta_p}\left[1 + O\left(\frac{1}{\Delta_p}\right)\right]$$

$$\text{where} \quad \rho = \frac{4z}{(1+\sqrt{1-z})^2}$$

This amounts to

$$\Delta \longrightarrow \Delta_p = \sqrt{1-z}\,\Delta$$

$$\mathcal{B}_z(\Delta_p) \sim \mathcal{B}_{1-z}\left(\frac{2}{\sqrt{1-z}}\Delta_0 - \sqrt{\frac{z}{1-z}}\,\Delta_p\right)$$

When $z = 1/2$,

$$\mathcal{B}_{1/2}(\Delta_p) \sim \mathcal{B}_{1/2}(2\sqrt{2}\,\Delta_0 - \Delta_p)$$

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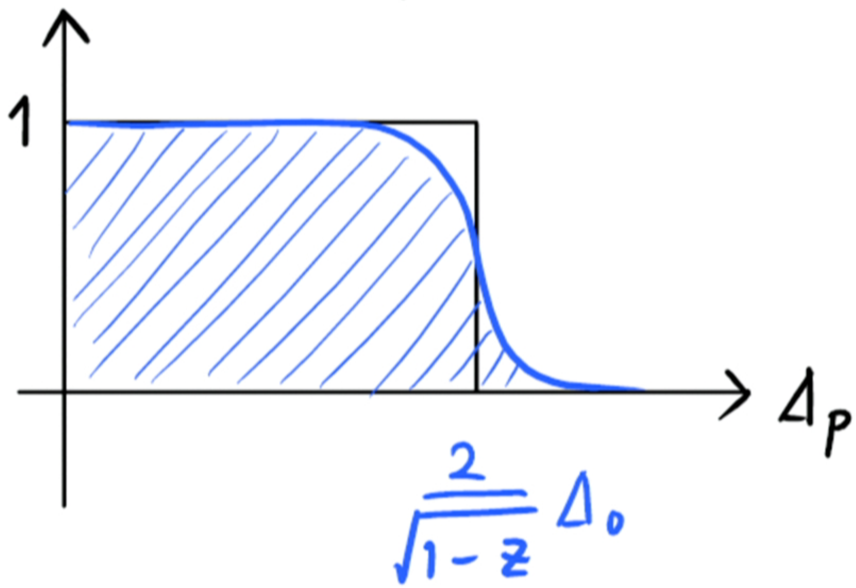
Crossing symmetry

$$G(z) = G(1-z) \Rightarrow B_z(\Delta) \sim B_{1-z}(\tilde{\Delta}),$$

$$\Delta_p = \frac{2}{\sqrt{1-z}}\Delta_0 - \sqrt{\frac{z}{1-z}}\,\tilde{\Delta}_p \lesssim \frac{2}{\sqrt{1-z}}\Delta_0$$



Chebyshev polynomials bound corrections.

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When $z = 1/2$,

$$\mathcal{B}_{1/2}(\Delta_p) \sim \mathcal{B}_{1/2}(2\sqrt{2}\,\Delta_0 - \Delta_p)$$

$$\phi_{\Delta_0}(\tfrac{1}{2})\,\phi_{\Delta_0}(0) \;\to\; \phi_{\Delta_p}(0)$$

$\sim 50\%$: $0 \le \Delta_p \le \sqrt{2}\,\Delta_0$

$\sim 50\%$: $\sqrt{2}\,\Delta_0 \le \Delta_p \le 2\sqrt{2}\,\Delta_0$

Holographic interpretation?

$$S \sim \Delta^{\alpha}, \quad \frac{2}{3} \le \alpha \le 2$$

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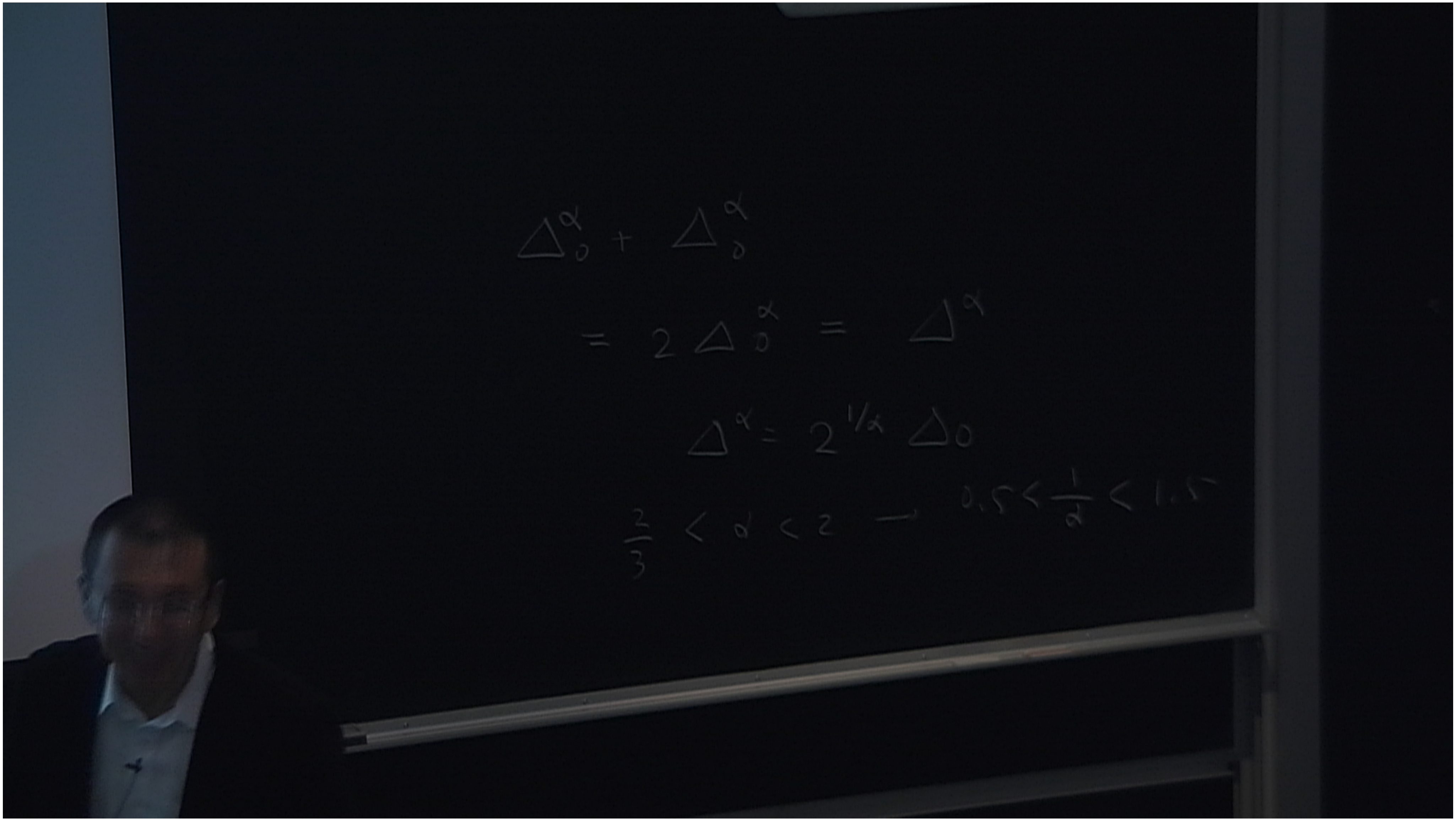
$\Delta_0^\alpha + \Delta_0^\alpha$
$= 2\Delta_0^\alpha = \Delta^\alpha$
$\Delta^\alpha = 2^{1/\alpha}\Delta_0$
$\frac{2}{3} < \alpha < 2 \longrightarrow 0.5 < \frac{1}{\alpha} < 1.5$

When $z = 1/2$,

$$\mathcal{B}_{1/2}(\Delta_p) \sim \mathcal{B}_{1/2}(2\sqrt{2}\,\Delta_0 - \Delta_p)$$

$$\phi_{\Delta_0}(\tfrac{1}{2})\,\phi_{\Delta_0}(0) \rightarrow \phi_{\Delta_p}(0)$$

$$\sim 50\% : \quad 0 \le \Delta_p \le \sqrt{2}\,\Delta_0$$

$$\sim 50\% : \quad \sqrt{2}\,\Delta_0 \le \Delta_p \le 2\sqrt{2}\,\Delta_0$$

Holographic interpretation ?

$$S \sim \Delta^{\alpha} , \quad \frac{2}{3} \le \alpha \le 2$$

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Summary

The linear optimization of the bootstrap constraints shows that the main contributions to

$$\phi_{\Delta_0}(z)\,\phi_{\Delta_0}(0) \longrightarrow \phi_{\Delta}(0)$$

are from $\Delta < \frac{2}{\sqrt{1-z}}\,\Delta_0$,

with the analytic bounds on the tails .

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$$F(\tau) \equiv \sum_h (N_{h,1} - 2N_{h,0})q^h = 90q + 462q^2 + 1540q^3 + 4554q^4 + 11592q^5 + 27830q^6 + 61686q^7 + 131100q^8 + \cdots$$

Future Directions:

☆ Apply conformal bootstrap to 1/4 BPS states in K3 sigma-model using scaling blocks.

☆ Construct N=4 superconformal blocks and refine conformal bootstrap analysis.

☆ Study solvable examples such as orbifolds and Gepner models.

☆ Understand operator product expansions of 1/4 BPS states and learn more about M24 structure.