Title: On holomorphic vertex operator algebras of central charge 24

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Abstract: I will talk about the recent progress on the classification of (strongly regular) holomorphic vertex operator algebras of central charge 24. In particular, I will discuss a construction of certain holomorphic vertex operator algebras of central charge 24 using orbifold construction associated to inner automorphisms. This talk is based on a joint work with Hiroki Shimakura.

On classification of holomorphic vertex operator algebras of central charge 24

Ching Hung Lam

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Based on joint works with Hiroki Shimakura (Tokoku University)

April 15, 2015

Definition

A VOA V is said to be rational if all admissible V-modules are completely reducible.

Remark: If V is rational, then it has only finitely inequivalent irreducible modules. [DLM]

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A simple rational VOA V is said to be holomorphic if it has only one irreducible, i.e., V itself. The most famous example is $V^\natural$.

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A VOA V is of CFT type if $V_n = 0$ for all $n < 0$ and $\dim V_0 = 1$.

Introduction (Problem)

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Try to classify (C_2-cofinite) holomorphic vertex operator algebras (VOAs) (of CFT type) of central charge $c = 24$.

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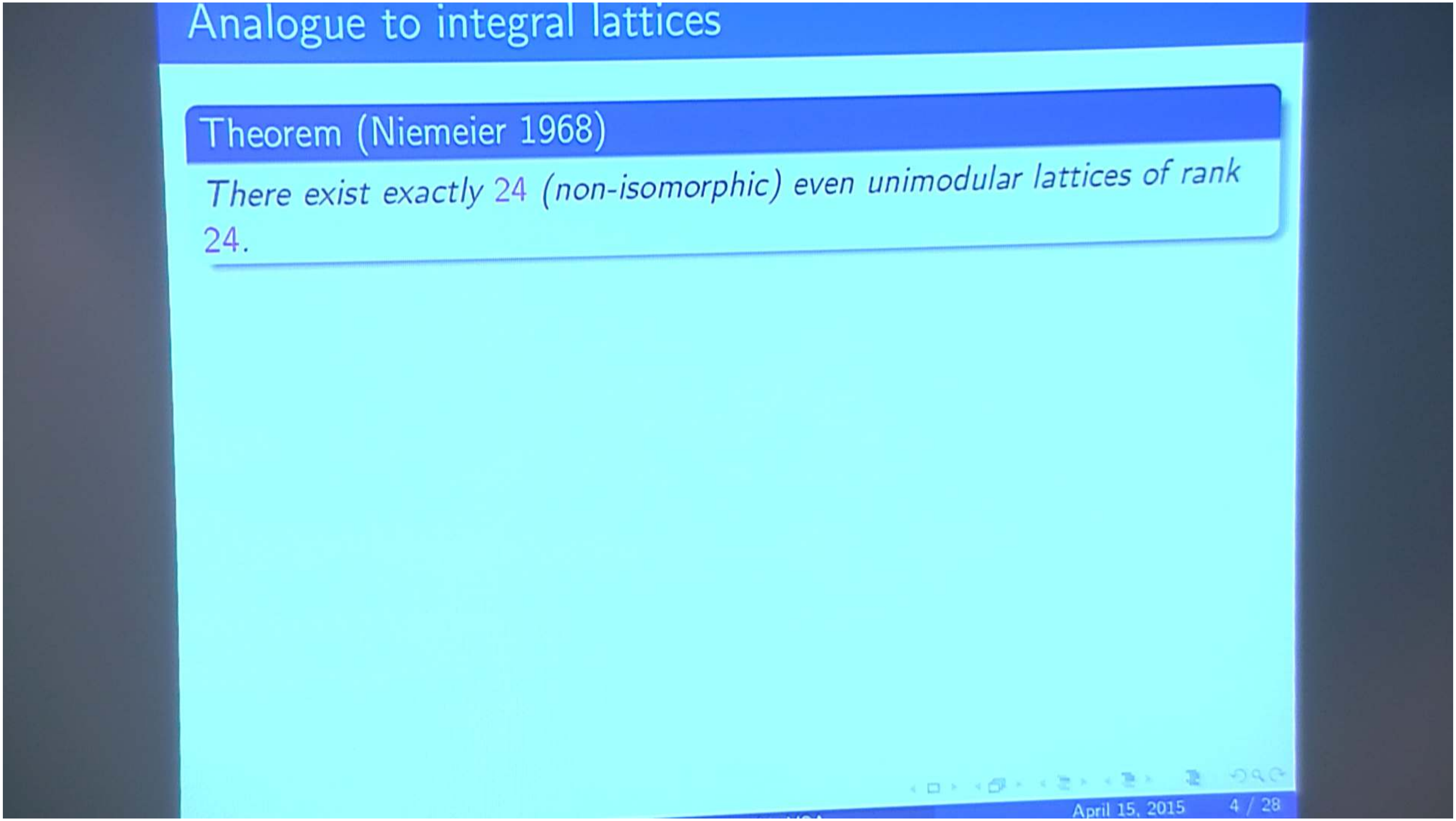
Some Known results

- ([Zhu 1996]) The central charge of any holomorphic VOA is in $8\mathbb{Z}_{>0}$.
- ([Dong-Mason 2004]) If $c \leq 16$ then any holomorphic VOA is isomorphic to a lattice VOA.

Therefore, $c = 24$ is the first nontrivial case.


Analogue to integral lattices
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- The isometry type is determined by the root system.
- There is a unique even unimodular lattices of rank 24 whose root system is trivial.(Leech lattice)

Question: Is it possible to classify holomorphic VOA of central charge 24 using vectors of minimal weight? For VOA of CFT type

$$V = \bigoplus_{n=0}^{\infty} V_n, \quad V_0 = \mathbb{C}1.$$

Structure of V_1

Assume V is of CFT type, i.e., $V_n = 0$ for $n < 0$ and $\dim V_0 = 1$.

Theorem (Borcherds)

$(V_1, [\ ,\ ])$ *is a Lie algebra with* $[a, b] = a_0 b$.

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$(V_1, [\ ,\ ])$ is a Lie algebra with $[a, b] = a_0 b$. Moreover, $(\ .\ )$ defined by $(a, b) \cdot 1 = a_1 b$ for $a, b \in V_1$ is a symmetric invariant form. The affine Lie algebra of V_1 acts on V.

Theorem (Dong-Mason)

If V is rational and C_2, then V_1 is reductive and V is an integrable representation of affine Lie algebra of V_1.

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Theorem (Dong-Mason)

If V is rational and C_2, then V_1 is reductive and V is an integrable representation of affine Lie algebra of V_1, i.e., the level is integral.

Theorem (Dong-Mason 2004)

Let V be a holomorphic and C_2-cofinite VOA of central charge 24. Then V_1 is 0, abelian, or semisimple.
If $V_1 \neq 0$ and is abelian, then V is isomorphic to the Leech lattice VOA.

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Suppose V_1 is semisimple and let V_1 be a direct sum

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \mathfrak{g}_{2,k_2} \oplus \cdots \oplus \mathfrak{g}_{n,k_n}$$

of simple Lie algebra $\mathfrak{g}_i$ whose affine Lie algebra has level k_i on V.

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Theorem (Dong-Mason)

Let $h_i^\vee$ be the dual Coxeter number of $\mathfrak{g}_i$. Then

$$\frac{h_i^\vee}{k_i} = \frac{(\dim V_1 - 24)}{24}.$$

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Known results

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However only 39 holomorphic VOAs of $c = 24$ have been known at that time.

Proposition (FLM 1988, Dong 1993, Dolan-Goddard-Montague 1996)

- *There are 24 holomorphic lattice VOAs V_L of $c = 24$.*
- *There are 15 holomorphic VOAs of $c = 24$ obtained by the $\mathbb{Z}_2$-orbifolds $\tilde{V}_L$ of the lattice VOAs V_L, which are not isomorphic to the lattice VOAs.*

Main philosophy (Still a conjecture):
The VOA structure of V should be uniquely determined by the Lie algebra structure of V_1.

Special case: When $V_1 = 0$, it is conjectured that $V \cong V^\natural$, the Moonshine VOA.This conjecture is considered to be one of the most difficult problems in VOA theory.

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Problem

1. Construct all 71 theories.
2. Establish the uniqueness for each case.
3. Verify Schellekens' theorem rigorously.

In the last 20 years, there are many attempts to construct the 71 theories of Schellekens.

For example, Montague (1994) proposed a construction of 70 of 71 theories using $\mathbb{Z}_2$ and $\mathbb{Z}_3$-orbifoldings.
His method is not rigorous and there are also mistakes.
However, his idea is very inspiring.

One can obtain many new VOAs if the orbifold construction can be established, mathematically.

What is an orbifold construction?

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Let $g \in Aut(V)$ be an automorphism of order p (Assume p is prime).
Then V^g is a subVOA. Moreover, for each $1 \leq i \leq p-1$,
there is a unique g^i-twisted module $V^T(g^i)$ for V, whose weights are graded by $\frac{1}{p^2}\mathbb{Z}$.
Suppose the weights of the twisted modules are in $\frac{1}{p}\mathbb{Z}$.
Let $[V^T(g^i)]^{g^i}$ be the submodule of V^g which has integral weights.

Conjecture

$\tilde{V}(g) = V^g \oplus \left(\oplus_{i=0}^{p-1} [V^T(g^i)]^{g^i} \right)$ is a holomorphic VOA.

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1. If V is C_2 and rational , then so is V^g for any $g \in Aut(V)$ of finite order.

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1. If V is C_2 and rational , then so is V^g for any $g \in Aut(V)$ of finite order.
2. The fusion ring of V^g is isomorphic to the group ring $\mathbb{Z}[A]$, where A is an abelian group of order p^2.

Other difficulties

- We don't know the full automorphism groups for holomorphic VOAs.

Some recent progress

- Shimakura- L studied a special class of vertex operator algebras, called **framed VOA** — a simple VOA containing a full subVOA isomorphic to a tensor product of Virasoro VOA $L(\frac{1}{2},0)^{\otimes r}$.

$L(\frac{1}{2},0),\quad L(\frac{1}{2},\frac{1}{2}),\quad L(\frac{1}{2},\frac{1}{16}).$

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  — We can reduce the classification of holomorphic framed VOA to some combinatorial problem
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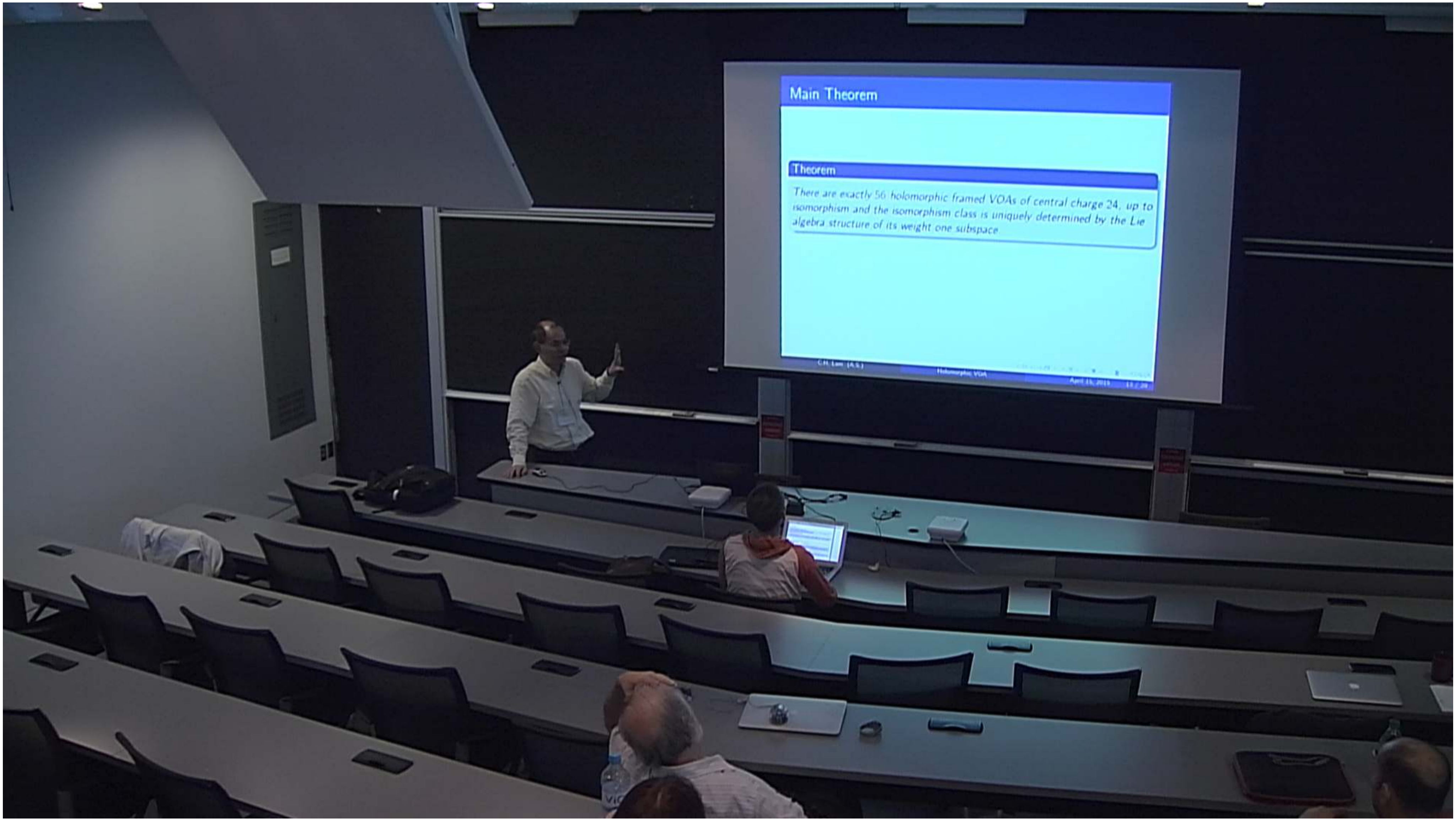
- One can easily define some involutions in framed VOA [Miyamoto].
  — We can established the $\mathbb{Z}_2$-orbifold construction rigorously for framed VOAs.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10	48	$D_{5,8} A_{1,2}$
13	60	$D_{4,4}(A_{2,2})^4$
18	72	$A_{7,4}(A_{1,1})^3$
19	72	$D_{5,4} C_{3,2}(A_{1,1})^2$
22	84	$C_{4,2}(A_{4,2})^2$
26	96	$(A_{5,2})^2 C_{2,1}(A_{2,1})^2$
33	120	$A_{7,2}(C_{3,1})^2 A_{3,1}$
35	120	$C_{7,2} A_{3,1}$
36	132	$A_{8,2} F_{4,2}$
40	144	$A_{9,2} A_{4,1} B_{3,1}$
44	168	$E_{6,2} C_{5,1} A_{5,1}$
48	192	$(C_{6,1})^2 B_{4,1}$
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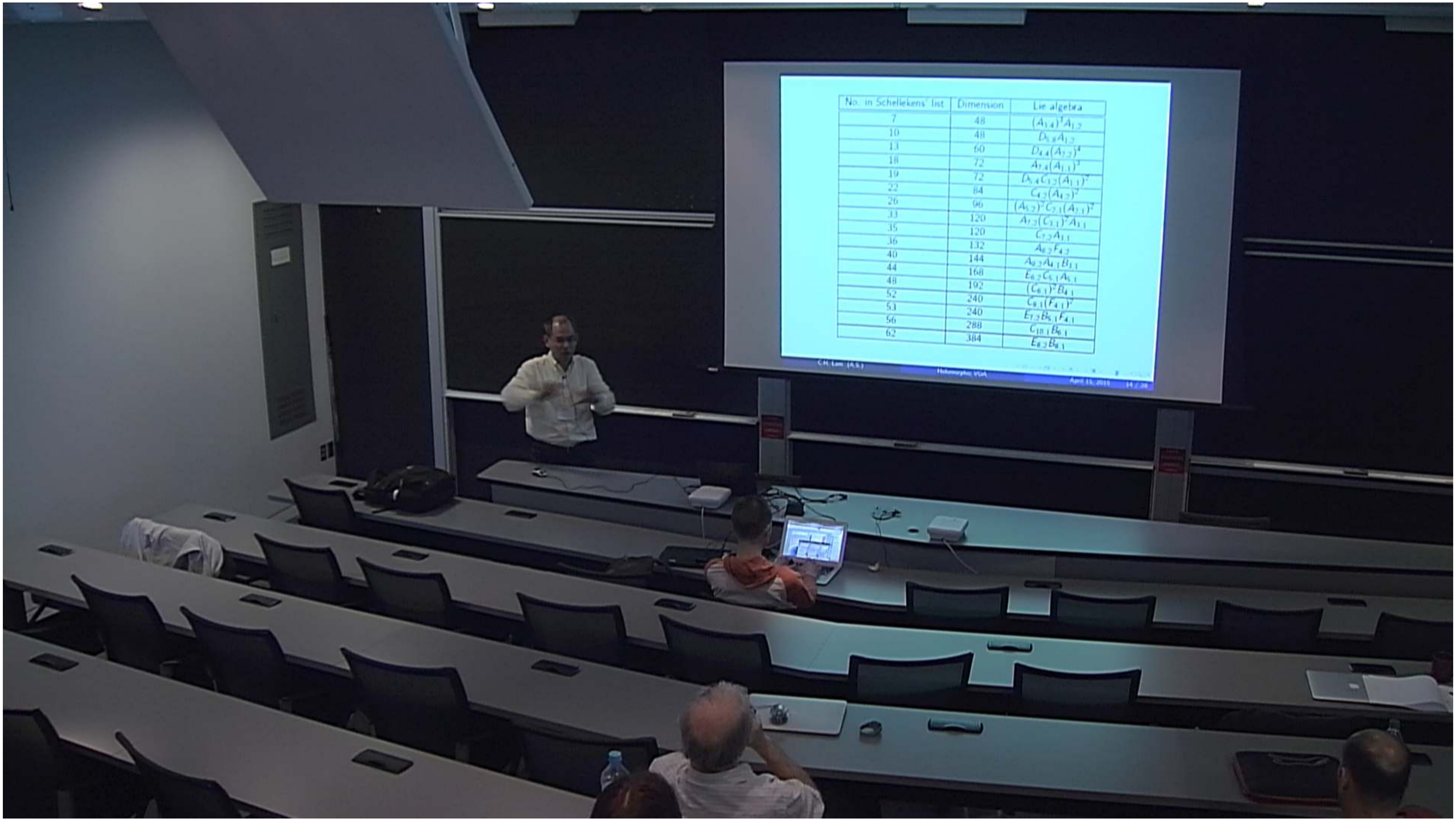
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$\mathbb{Z}_3$-orbifold construction

- Recently, Miyamoto(2013) established the $\mathbb{Z}_3$-orbifold construction for lattice VOA. He constructed a new VOA with the Lie algebra $E_{6,3}G_{2,1}^3$.
- Using Miyamoto's construction, Sagaki and Shimakura constructed another two VOAs with Lie algebra $A_{2,3}^6$ and $A_{1,1}^3A_{5,3}D_{4,3}$.
- They also showed that the holomorphic VOA that can be obtained by Miyamoto $\mathbb{Z}_3$-orbifold construction must be a lattice VOA or has the Lie algebra isomorphic to one of the three cases: $E_{6,3}G_{2,1}^3$, $A_{2,3}^6$ and $A_{1,1}^3A_{5,3}D_{4,3}$.

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General Settings

Let V be a vertex operator algebra of CFT type and let $h \in V_1$.

Suppose that $h_{(0)}$ acts semisimply on V and that there exists a positive integer $T \in \mathbb{Z}_{>0}$ such that $\mathrm{Spec}\, h_{(0)} < \frac{1}{T}\mathbb{Z}$.

Then $\sigma_h = \exp(-2\pi\sqrt{-1}h_{(0)})$ is an automorphism of V with $\sigma_h^T = 1$.

Define

$$\Delta(h, z) = z^{h_{(0)}} \exp\left(\sum_{n=1}^{\infty} \frac{h_{(n)}}{-n}(-z)^{-n}\right).$$

Proposition ([Li])

Let $h \in V_1$ be as above and let (M, Y_M) be a V-module.
Define $(M^{(h)}, Y_{M^{(h)}}(\cdot, z))$ as follows:

$$M^{(h)} = M \quad \text{as a vector space;}$$
$$Y_{M^{(h)}}(a, z) = Y_M(\Delta(h, z)a, z) \quad \text{for any } a \in V.$$

Then $(M^{(h)}, Y_{M^{(h)}}(\cdot, z))$ is a σ_h-twisted V-module.

Furthermore, if M is irreducible, then so is $M^{(h)}$.

Lemma

Let M be a V-module whose $L(0)$-weights are half-integral.
Let $h \in V_1$ such that $\langle h|h\rangle \in \mathbb{Z}$. Assume that the spectra of $h_{(0)}$ on M are half-integral.
Then $L^{(h)}(0)$-weights of the σ_h-twisted V-module $M^{(h)}$ are also half-integral.

Proposition ([DLM])

Suppose V is holomorphic. Let $h \in V_1$ such that $\langle h|h\rangle \in \mathbb{Z}$ and the spectra of $h_{(0)}$ on V are half-integral.
Then the V^{σ_h}-module $\tilde{V} = V^{\sigma_h} \oplus (V^{(h)})_{\mathbb{Z}}$ has a VOA structure as a simple current extension of V^{σ_h} graded by $\mathbb{Z}_2$.
Furthermore, $\tilde{V}$ is holomorphic, rational and C_2-cofinite.

Proposition

Let V, $\tilde{V}$, $\mathfrak{H}$, h_i and k_i be defined as before.
If $-\sum_{i=1}^{t} k_i h_i$ is not a weight for $\mathfrak{H}$ on V, i.e.,

$$\{v \in V \mid x_{(0)}v = (-\sum_{i=1}^{t} k_i h_i | x)v\} = \{0\},$$

then $\mathfrak{H}$ is a Cartan subalgebra of the (semisimple) Lie algebra $\tilde{V}_1$.

Theorem

Let $g = \exp(-2\pi\sqrt{-1}h_{(0)})$ be an inner automorphism of order 2 and $\langle h|h\rangle \in \mathbb{Z}$. Then we have

$$(1)\ \dim V(g)_{1/2} = \frac{\dim V_2^g - 98580}{2^{11}};$$

$$(2)\ \dim V_1 + \dim \tilde{V}_1 = 3 \dim V_1^g + 24(1 - \dim V(g)_{1/2}).$$

New examples

- From $E_{6,3}G_{2,1}^3$ to $D_{7,3}A_{3,1}G_{2,1}$.
  Use $h = \frac{1}{2}(\Lambda_1 - \Lambda_6, \Lambda_2, \Lambda_2, 0)$ and $\langle h|h\rangle = 2$.
- From $A_{3,1}D_{7,3}G_{2,1}$ to $A_{5,1}E_{7,3}$.
  $h = \frac{1}{2}(2\Lambda_1, \Lambda_6 - \Lambda_7, \Lambda_2) \in \bigoplus_{i=1}^3(\mathfrak{g}_i \cap \mathfrak{H})$ and $\langle h|h\rangle = 2$.
- From $A_{5,1}E_{7,3}$ to $A_{2,1}^2A_{8,3}$,
  $h = \frac{1}{2}(\Lambda_2, \Lambda_3)$ and $\langle h|h\rangle = 3$.

Assuming the existence of VOA with Lie algebras $A_{4,5}^2$ and $C_{5,3}G_{2,2}A_{1,1}$.

- From $A_{4,5}^2$ to $A_{1,1}^2D_{6,5}$.
  $h = \frac{1}{2}(\Lambda_1, \Lambda_1)$ and $\langle h|h\rangle = 2$.

Shimakura and I also studied the following cases.

- $N(E_6^4)$, order 6 element to $A_{1,1}C_{5,3}G_{2,2}$;
- $N(A_4^6)$, order 10 element to $C_{4,10}$;
- $N(D_4^6)$, Dih_6 to $A_{2,6}D_{4,12}$;

Combining all results and claims,
we have constructions of 69 holomorphic VOAs.

- For $A_{2,2}F_{4,6}$, Shimakura and I are working on a construction using an inner automorphism of order 3 from $A_{2,6}D_{4,12}$.

  The fixed point Lie subalgebra V_1^σ has dim $= 18$ and has the shape $A_{2,6}A_{2,12}U(1)^2$.

- The most difficult case seems to be the VOA with $V_1 = A_{6,7}$. It probably cannot be obtained by orbifold construction from a holomorphic VOA of $c = 24$.