Title: U-duality, exotic instantons and automorphic forms on Kac-Moody groups

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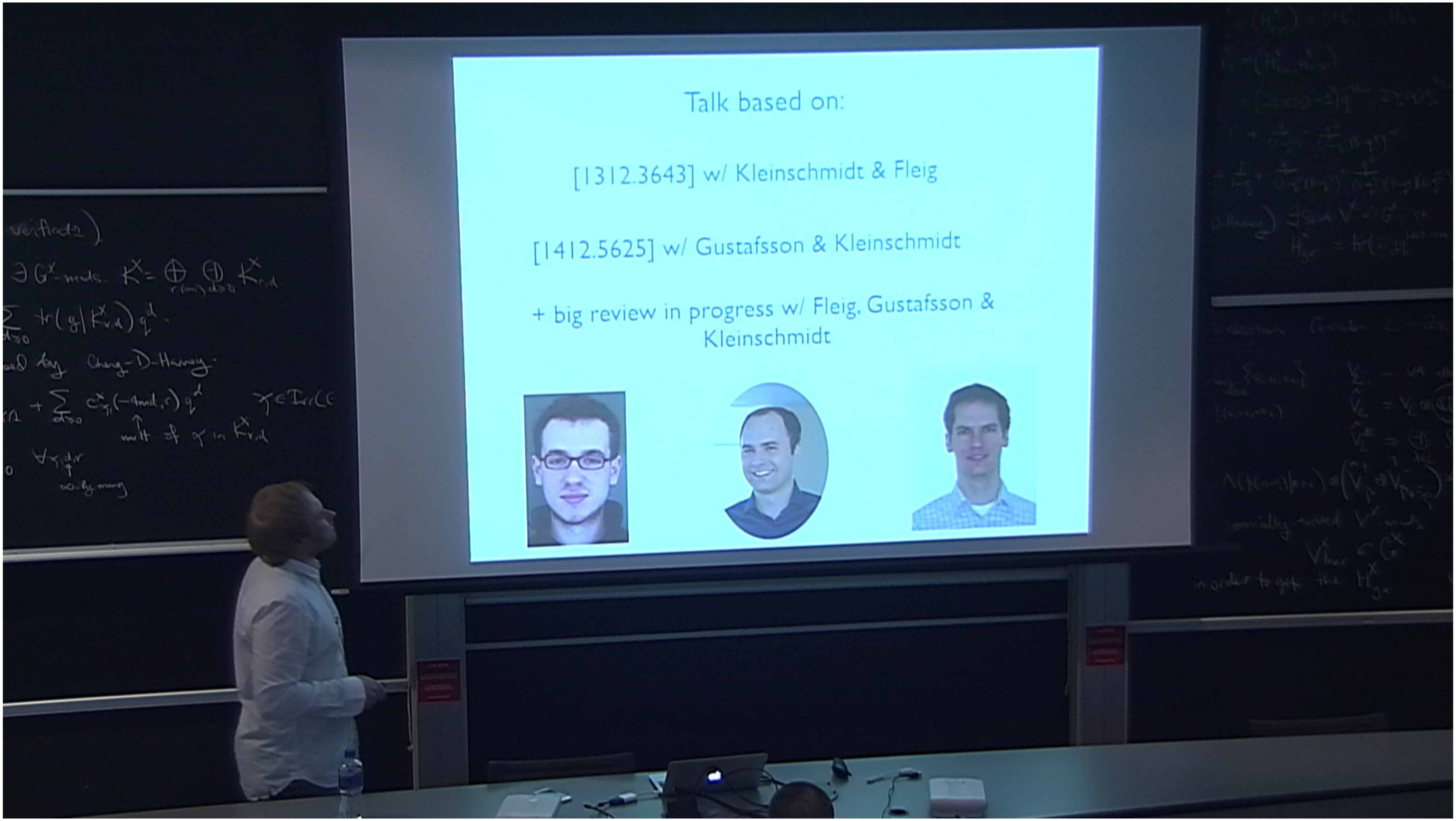
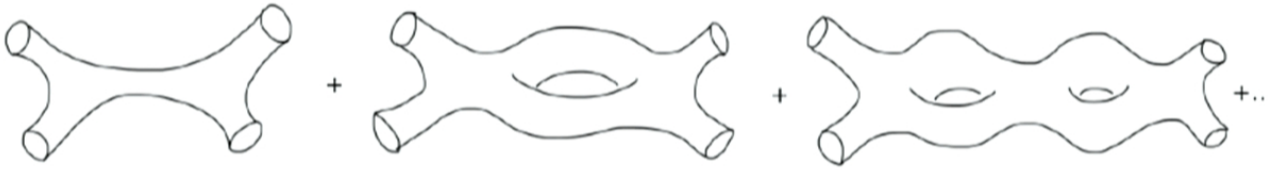
Abstract:


Talk based on:
[1312.3643] w/ Kleinschmidt & Fleig
[1412.5625] w/ Gustafsson & Kleinschmidt
+ big review in progress w/ Fleig, Gustafsson & Kleinschmidt

Motivation

Understand the structure of **string interactions**



Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

→ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

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Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics

Toroidal compactifications yield the famous chain of **U-duality groups**

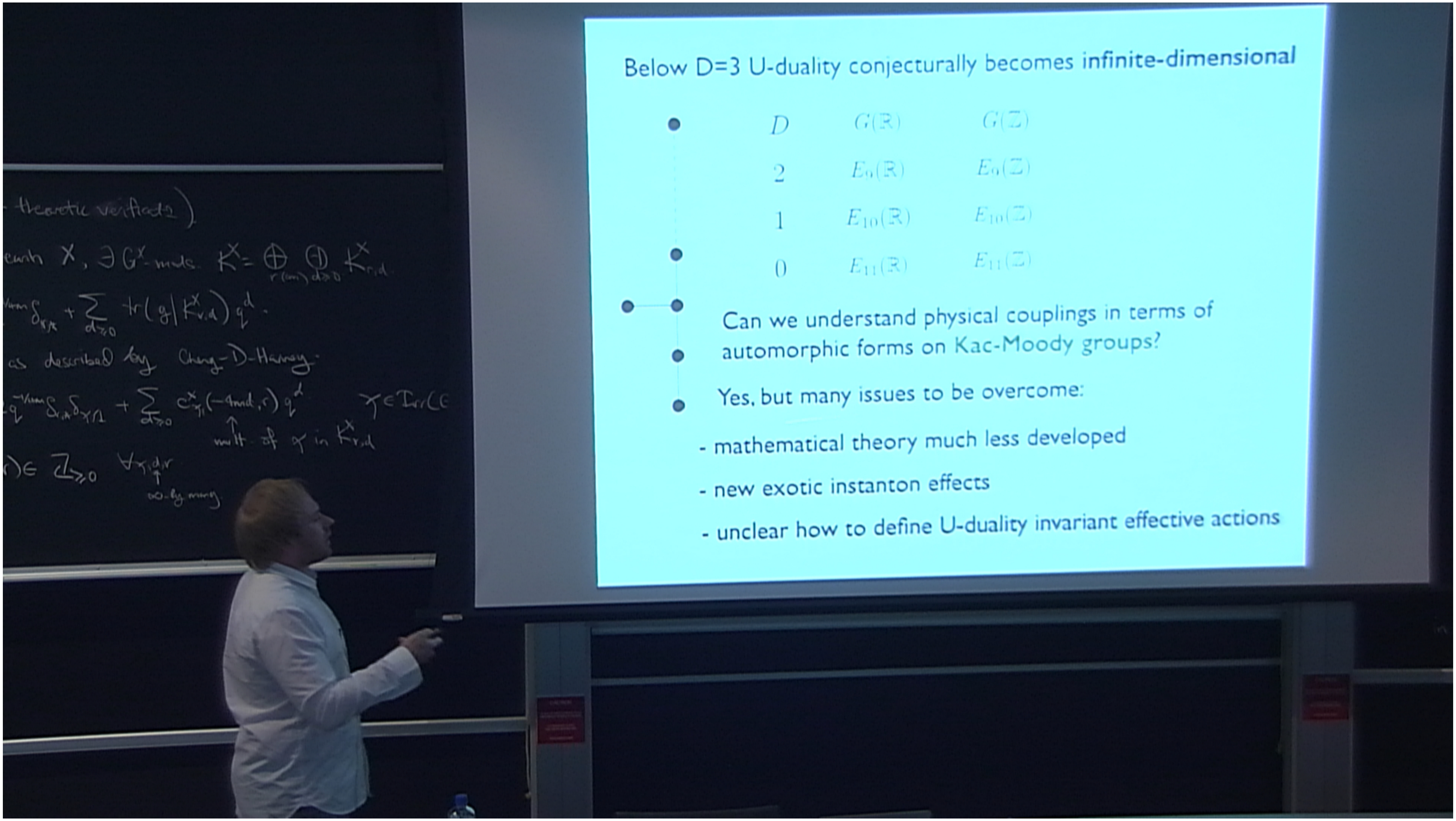
[Cremmer, Julia][Hull, Townsend]

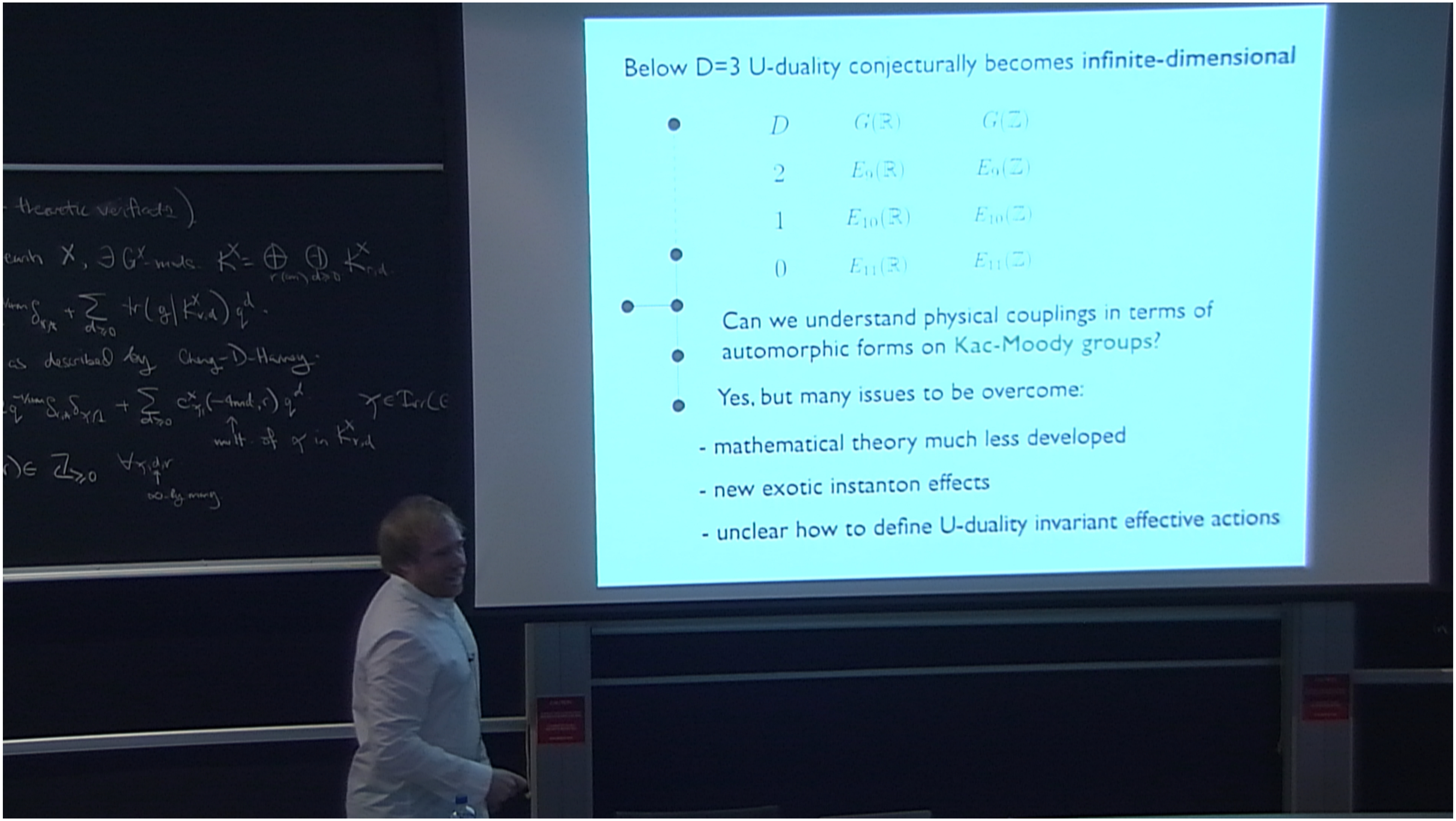
D	G	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5,\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

Physical couplings are given by **automorphic forms** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Green, Gutperle, Sethi, Vanhove, Kiritsis, Pioline, Obers, Kazhdan, Waldron, Basu, Russo, Cederwall, Bao, Nilsson, D.P., Lambert, West, Gubay, Miller, Fleig, Kleinschmidt,…


Below D=3 U-duality conjecturally becomes infinite-dimensional
D G(ℝ) G(ℤ)
2 E9(ℝ) E9(ℤ)
1 E10(ℝ) E10(ℤ)
0 E11(ℝ) E11(ℤ)
Can we understand physical couplings in terms of automorphic forms on Kac-Moody groups?
Yes, but many issues to be overcome:
- mathematical theory much less developed
- new exotic instanton effects
- unclear how to define U-duality invariant effective actions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- mathematical theory much less developed
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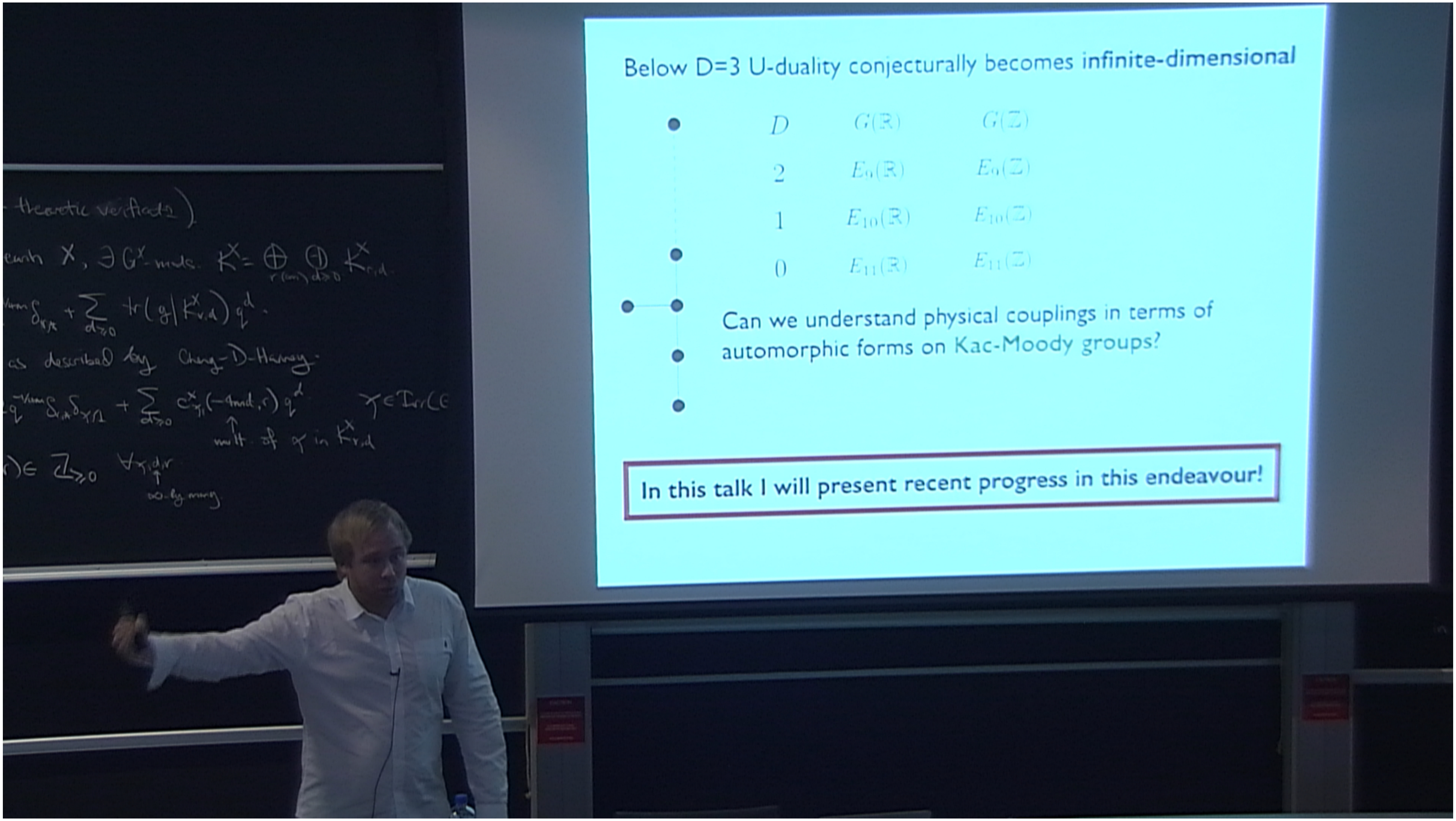
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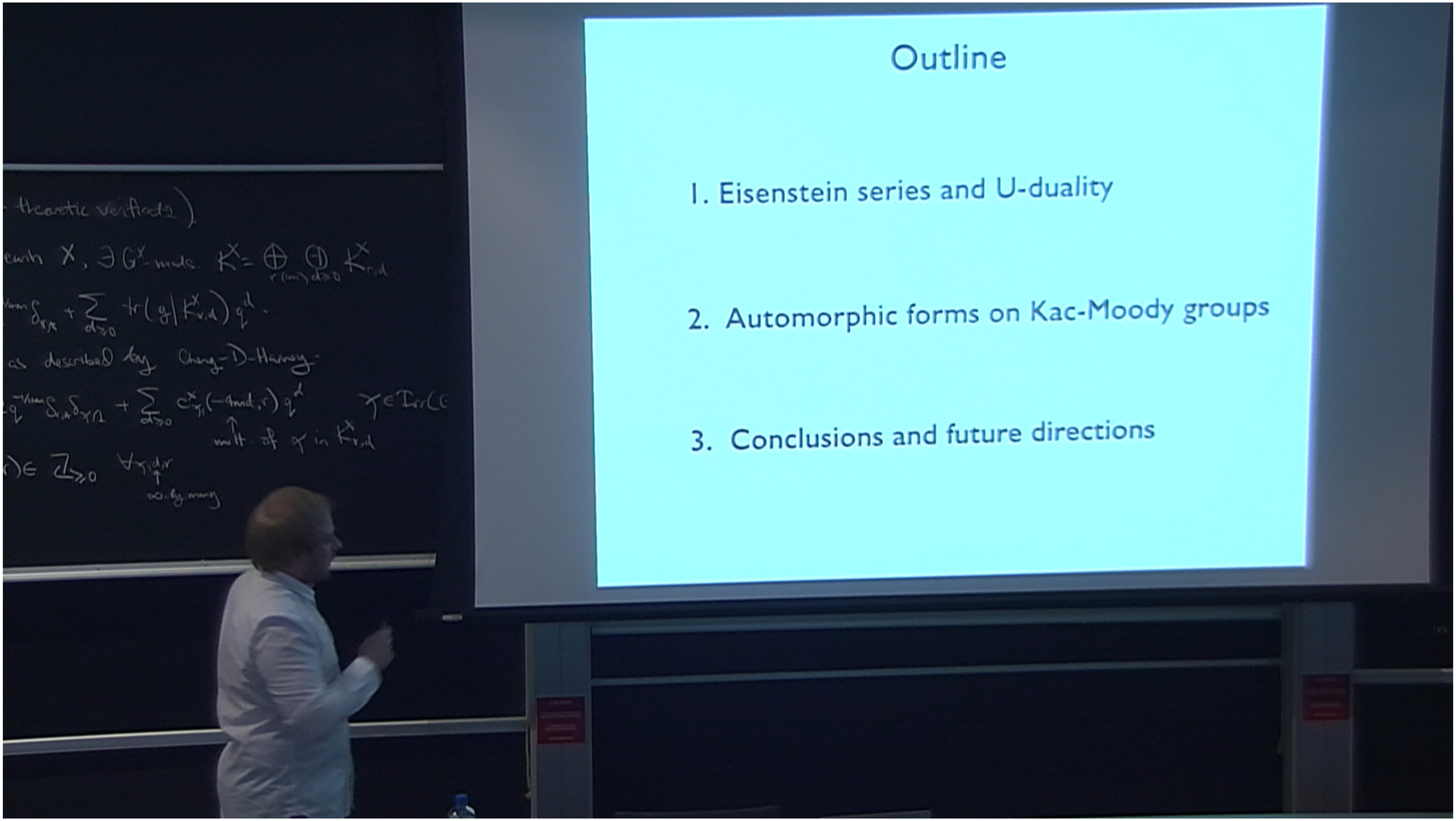
In this talk I will present recent progress in this endeavour!


Outline
1. Eisenstein series and U-duality
2. Automorphic forms on Kac-Moody groups
3. Conclusions and future directions

Higher-derivative action in type II string theory on tori

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

Higher-derivative action in type II string theory on tori

$$\int d^{10-n}x\sqrt{G}\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

- → $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- → must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- → supersymmetry requires that they are **Laplacian eigenfunctions**
- → well-defined **weak-coupling expansions** as $g_s \to 0$

defining properties of an **automorphic form!**

Example: type IIB in D=10

$$\int d^{10}x\sqrt{G}f_0(\tau)\mathcal{R}^4$$

- $f_0(\tau)$ function of $\tau = \chi + ie^{-\phi} \in \mathbb{H} \cong SL(2,\mathbb{R})/SO(2)$ **"axio-dilaton"**
- $f_0\left(\frac{a\tau+b}{c\tau+d}\right) = f_0(\tau)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$
- $\Delta_{\mathbb{H}} f_0(\tau) = -\frac{3}{4} f_0(\tau)$ $\quad \Delta_{\mathbb{H}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ $\quad \tau = x + iy$
- $f_0(\tau) \sim 2\zeta(3)y^{3/2}$ as $g_s = e^{\phi} = y^{-1} \to 0$ **weak string coupling limit**

unique solution!

$$f_0(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^{3/2}}{|m+n\tau|^3}$$

[Green, Gutperle]
[Green, Sethi]
[Pioline]

Non-holomorphic Eisenstein series

Consider the sum:

$$E_s(\tau) = \sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}}$$

non-holomorphic Eisenstein series

$s \in \mathbb{C}$

→ a function on $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \,|\, y > 0\}$

→ invariant under $\tau \mapsto \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Invariance under $\tau \mapsto \tau + 1$ yields the **Fourier expansion**

$$E_s(\tau) = \underbrace{C(y;s)}_{\substack{\text{constant term}\\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} F_n(y;s) e^{2\pi i n x}}_{\text{non-zero mode}}$$

→ $$C(y;s) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

completed zeta-function:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Physical interpretation: $$f_0(\tau) = 2\zeta(3)E_{3/2}(\tau)$$

Perturbative quantum effects (weak-coupling limit $y \to \infty$)

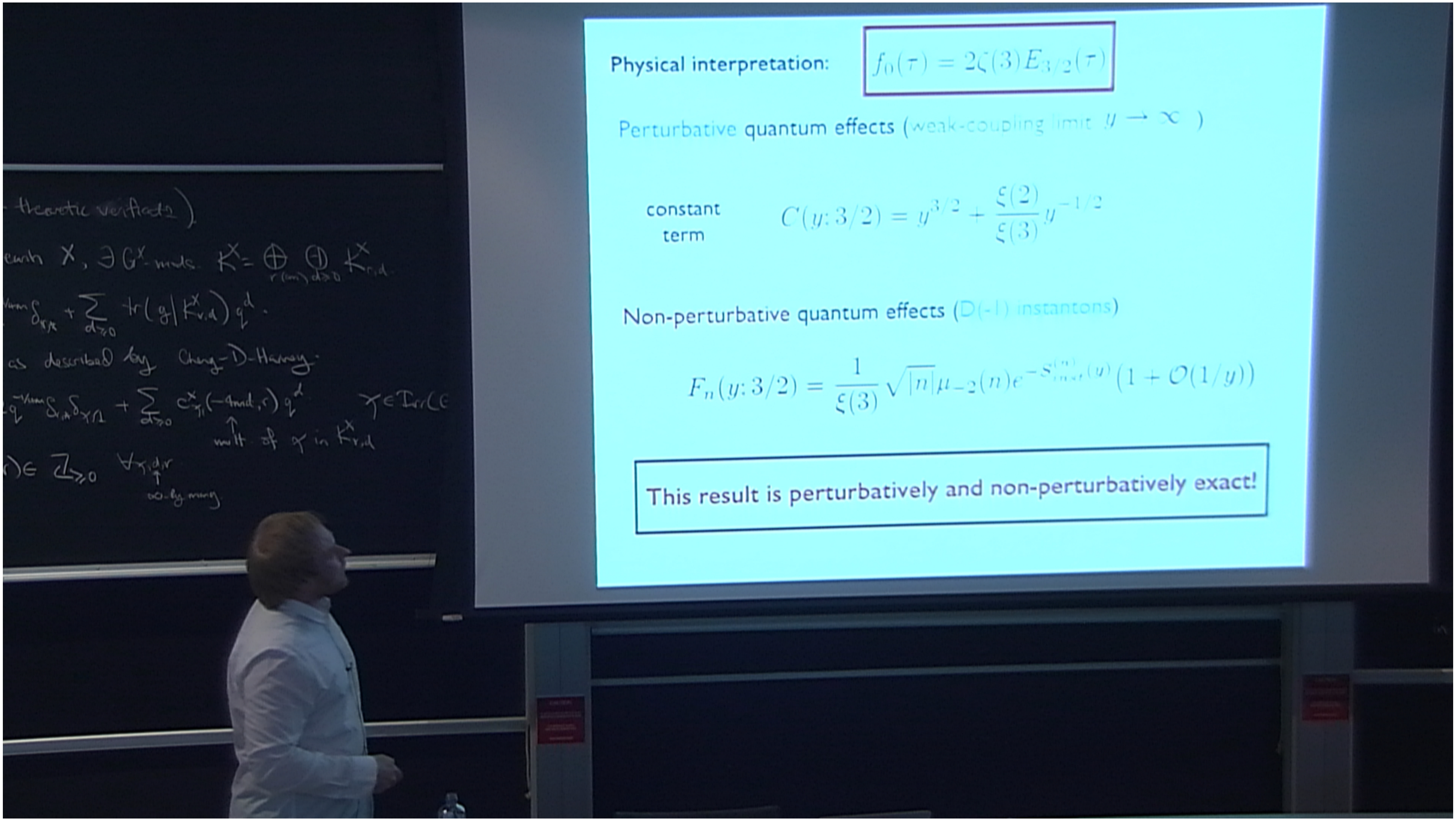
constant term

$$C(y;3/2) = y^{3/2} + \frac{\xi(2)}{\xi(3)}y^{-1/2}$$

Non-perturbative quantum effects (D(-1) instantons)

$$F_n(y;3/2) = \frac{1}{\xi(3)}\sqrt{|n|}\mu_{-2}(n)e^{-S^{(n)}_{inst}(y)}\left(1+\mathcal{O}(1/y)\right)$$

instanton measure $$\mu_{-2}(n) = \sum_{d|n} d^{-2}$$


Physical interpretation: $f_0(\tau) = 2\zeta(3)E_{3/2}(\tau)$
Perturbative quantum effects (weak-coupling limit $y \to \infty$)
constant term $C(y; 3/2) = y^{3/2} + \frac{\xi(2)}{\xi(3)} y^{-1/2}$
Non-perturbative quantum effects (D(-1) instantons)
$F_n(y; 3/2) = \frac{1}{\xi(3)} \sqrt{|n|}\mu_{-2}(n)e^{-S^{(n)}_{inst}(y)}(1 + \mathcal{O}(1/y))$
This result is perturbatively and non-perturbatively exact!

Eisenstein series on semi-simple Lie groups

The **Langlands Eisenstein series** on a semi-simple Lie group is defined by:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

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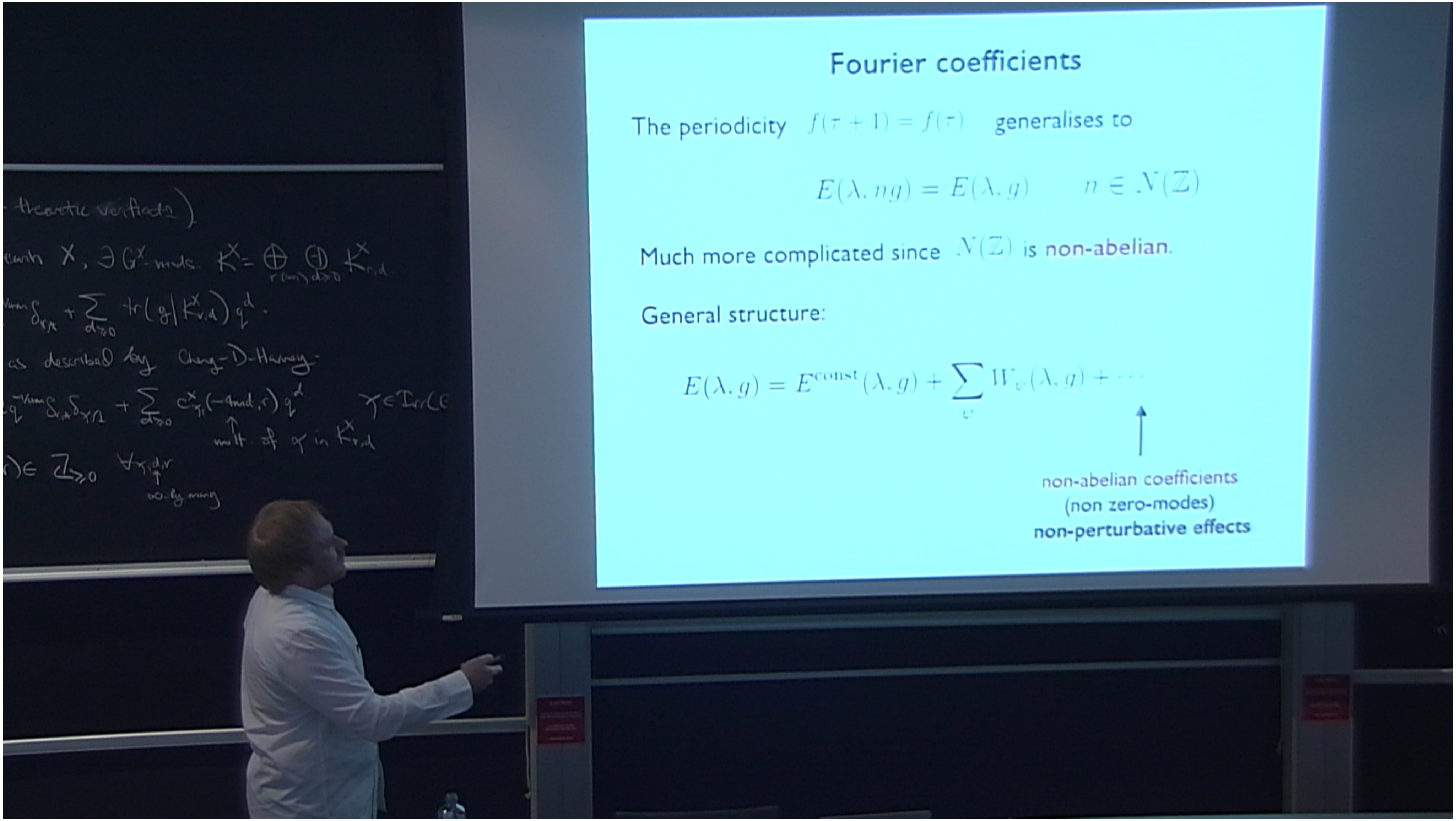
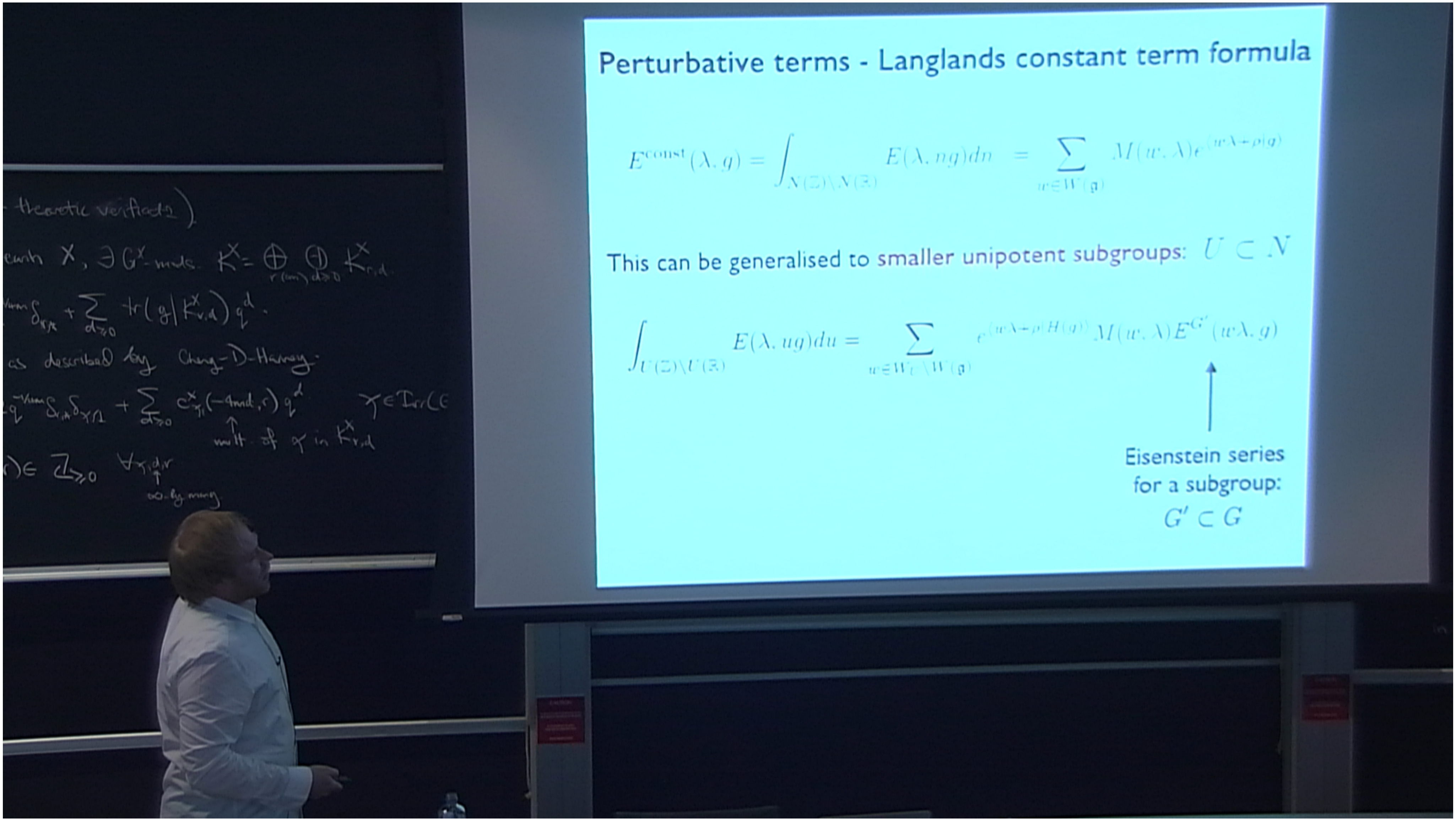
Iwasawa decomposition: $G = BK = NAK$

Logarithm map: $H \, : \, G \to \mathfrak{h} = \mathrm{Lie}\, A$ $\qquad H(nak) = \log a$

Weight: $\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$

Weyl vector: $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$


Fourier coefficients
The periodicity $f(\tau+1)=f(\tau)$ generalises to
$E(\lambda, ng) = E(\lambda, g) \qquad n \in N(\mathbb{Z})$
Much more complicated since $N(\mathbb{Z})$ is non-abelian.
General structure:
$E(\lambda, g) = E^{\text{const}}(\lambda, g) + \sum_{\psi} W_{\psi}(\lambda, g) + \cdots$
non-abelian coefficients
(non zero-modes)
non-perturbative effects


Perturbative terms - Langlands constant term formula
$E^{\text{const}}(\lambda, g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)dn = \sum_{w \in W(\mathfrak{g})} M(w,\lambda) e^{\langle w\lambda+\rho | H(g) \rangle}$
This can be generalised to smaller unipotent subgroups: $U \subset N$
$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\lambda, ug)du = \sum_{w \in W_{\ell} \backslash W(\mathfrak{g})} e^{\langle w\lambda+\rho | H(g) \rangle} M(w,\lambda) E^{G'}(w\lambda, g)$
Eisenstein series
for a subgroup:
$G' \subset G$

Perturbative limit - choices of unipotent subgroups

→ Decompactification limit

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

→ String perturbation limit

- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

→ M-theory limit

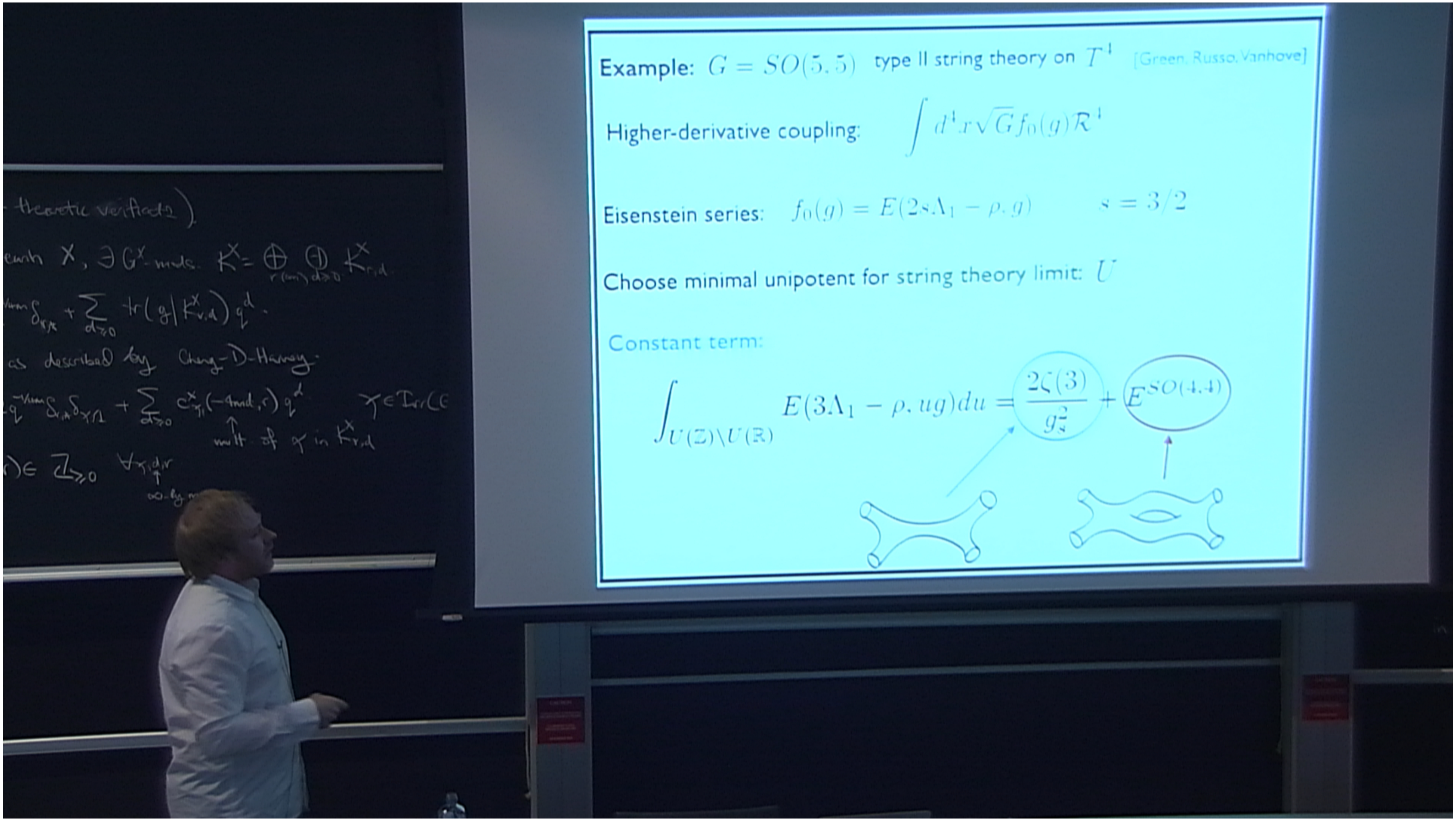
- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

Example: $G = SO(5,5)$ type II string theory on T^4 [Green, Russo, Vanhove]

Higher-derivative coupling: $\int d^4x\sqrt{G}f_0(g)\mathcal{R}^4$

Eisenstein series: $f_0(g) = E(2s\Lambda_1 - \rho, g)$ $s = 3/2$

Choose minimal unipotent for **string theory limit**: U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Choose minimal unipotent for string theory limit: U
Constant term:
$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(3\Lambda_1 - \rho, ug)du = \frac{2\zeta(3)}{g_s^2} + E^{SO(4,4)}$

Much more is known!

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4 \qquad f_0(g) = E(2s\Lambda_1 - \rho, g) \qquad s = 3/2$$

$$\int d^{11-n}x\sqrt{G}f_4(g)\mathcal{R}^4 \qquad f_4(g) = E(2s\Lambda_1 - \rho, g) \qquad s = 5/2$$

Successfully checked against perturbative string calculations for all

$$G = E_n(\mathbb{R}) \qquad n \leq 8$$

[Green, Russo, Vanhove][Green, Miller, Vanhove][Pioline]

$\partial^6\mathcal{R}^4$ also works but more complicated story…

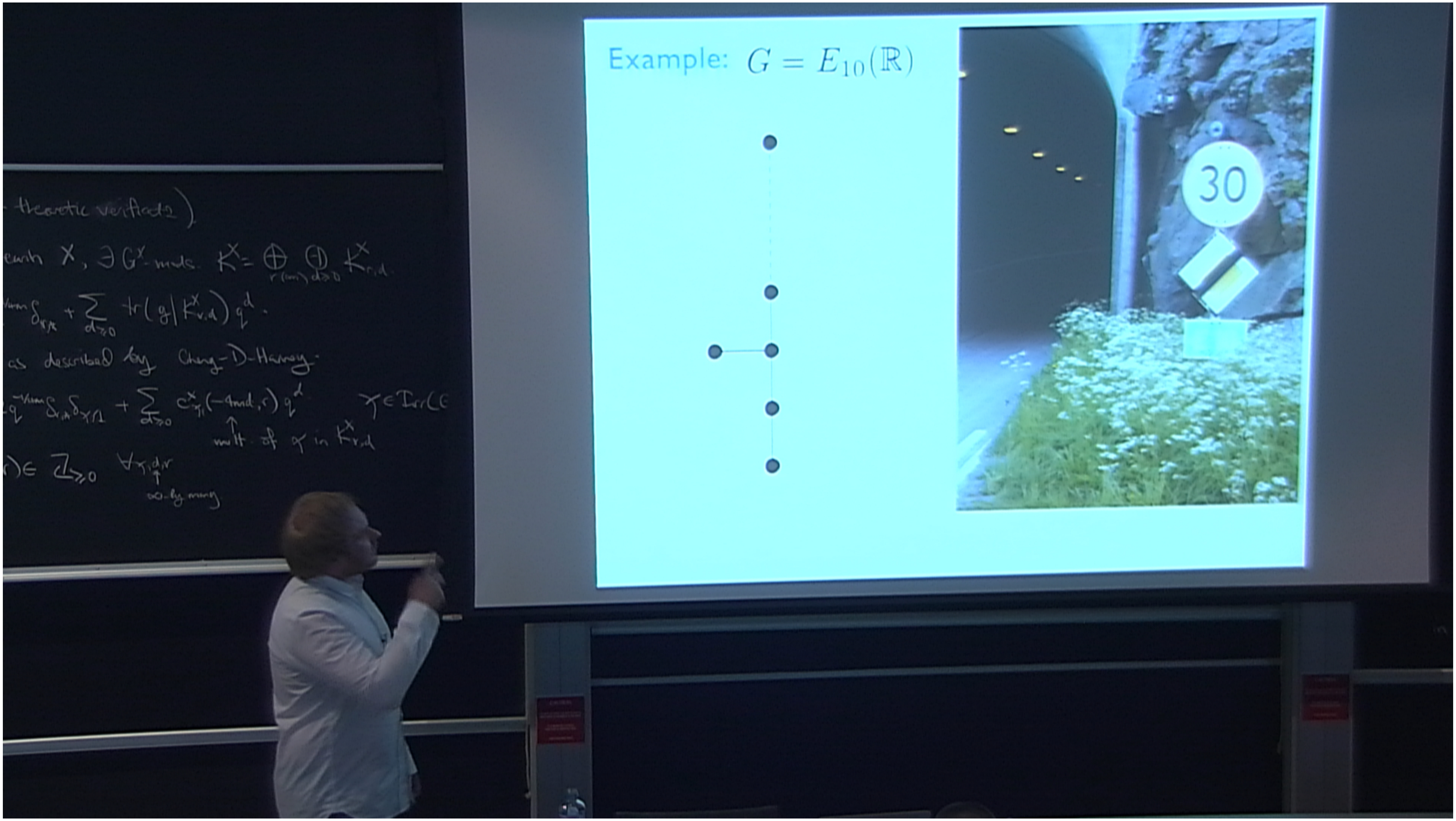
Eisenstein series can formally be defined for any **Kac-Moody group**

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

→ $G(\mathbb{Z}) \subset G(\mathbb{R})$ defined as a **Chevalley group**

→ convergence established by Garland in the affine case and by Carbone, Lee, Liu for rank 2 hyperbolic

→ generalization of Langlands constant term formula established by Garland in the affine case



Example: $G = E_{10}(\mathbb{R})$
30

Example: $G = E_{10}(\mathbb{R}) \quad s = 3/2 \longleftrightarrow \lambda = 3\Lambda_1 - \rho$

$$B_{s,m}(a) \sim \Big(\sum_{d|m} d^s\Big) K_s(ma)$$

ψ	$W_\psi(\chi_{3/2}, a)$
$(m,0,0,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0,0,0)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3) v_7^2}$
$(0,0,0,0,0,0,m,0,0,0)$	$v_7^4 v_8^{-3} B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)$
$(0,0,0,0,0,0,0,m,0,0)$	$\frac{\xi(4) v_8^5 v_9^{-4} B_{-3/2,m}\left(v_8^2 v_7^{-1} v_9^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,0,0,m,0)$	$\frac{\xi(5) v_9^6 v_{10}^{-5} B_{-2,m}\left(v_9^2 v_8^{-1} v_{10}^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,0,0,0,m)$	$\frac{\xi(6) v_{10}^7 B_{-5/2,m}\left(v_{10}^2 v_9^{-1}\right)}{\xi(3)}$

String theory interpretation

Expansion of Bessel functions reveal **D-instanton effects**

$$W_\psi(a) \sim e^{-1/g_s}$$

but we can also obtain **arbitrary powers of the coupling!**

$$\boxed{e^{-1/g_s^n}}$$

These might correspond to
exotic instantons

[Obers, Pioline][Shigemori, de Boer]

Conclusions and speculations

What does our results mean mathematically?

Conjecture: *For* $G = E_9, E_{10}, E_{11}$ *the Eisenstein series* $E(3\Lambda_1 - \rho, g)$ *is attached to the* ***minimal representation***

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Minimal representations can be defined for all simply laced finite simple Lie groups. [Joseph]

Automorphic realisations of minimal representations have **very few Fourier coefficients** [Ginzburg, Rallis, Soudry]

Conjecture holds for E_n with $n \leq 8$ [Green, Miller, Vanhove][Pioline]

Conclusions and future directions

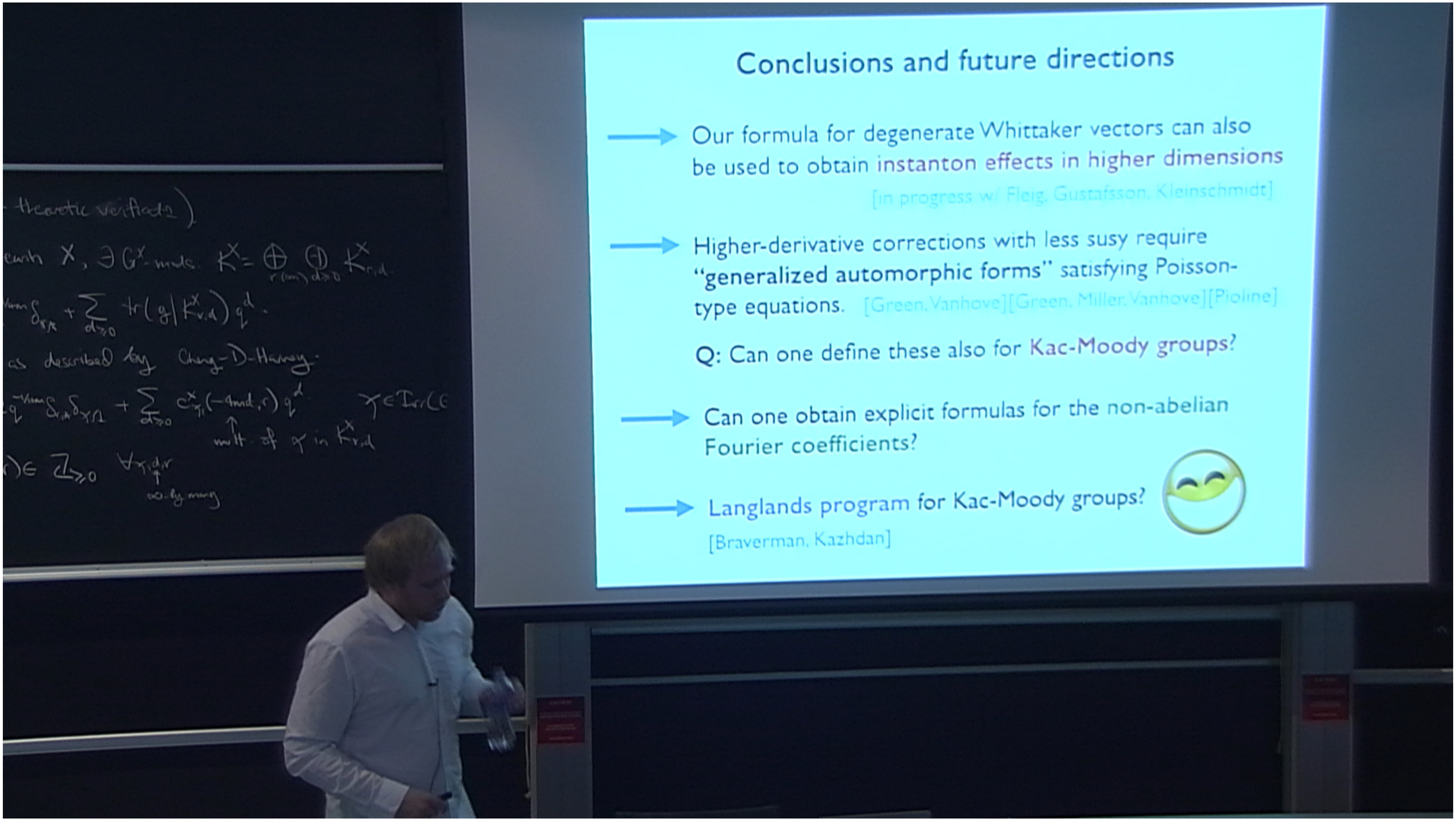
- Our formula for degenerate Whittaker vectors can also be used to obtain **instanton effects in higher dimensions**
  [in progress w/ Fleig, Gustafsson, Kleinschmidt]

- Higher-derivative corrections with less susy require **"generalized automorphic forms"** satisfying Poisson-type equations. [Green, Vanhove][Green, Miller, Vanhove][Pioline]

  Q: Can one define these also for **Kac-Moody groups**?

- Can one obtain explicit formulas for the **non-abelian Fourier coefficients**?

- **Langlands program** for Kac-Moody groups?
  [Braverman, Kazhdan]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