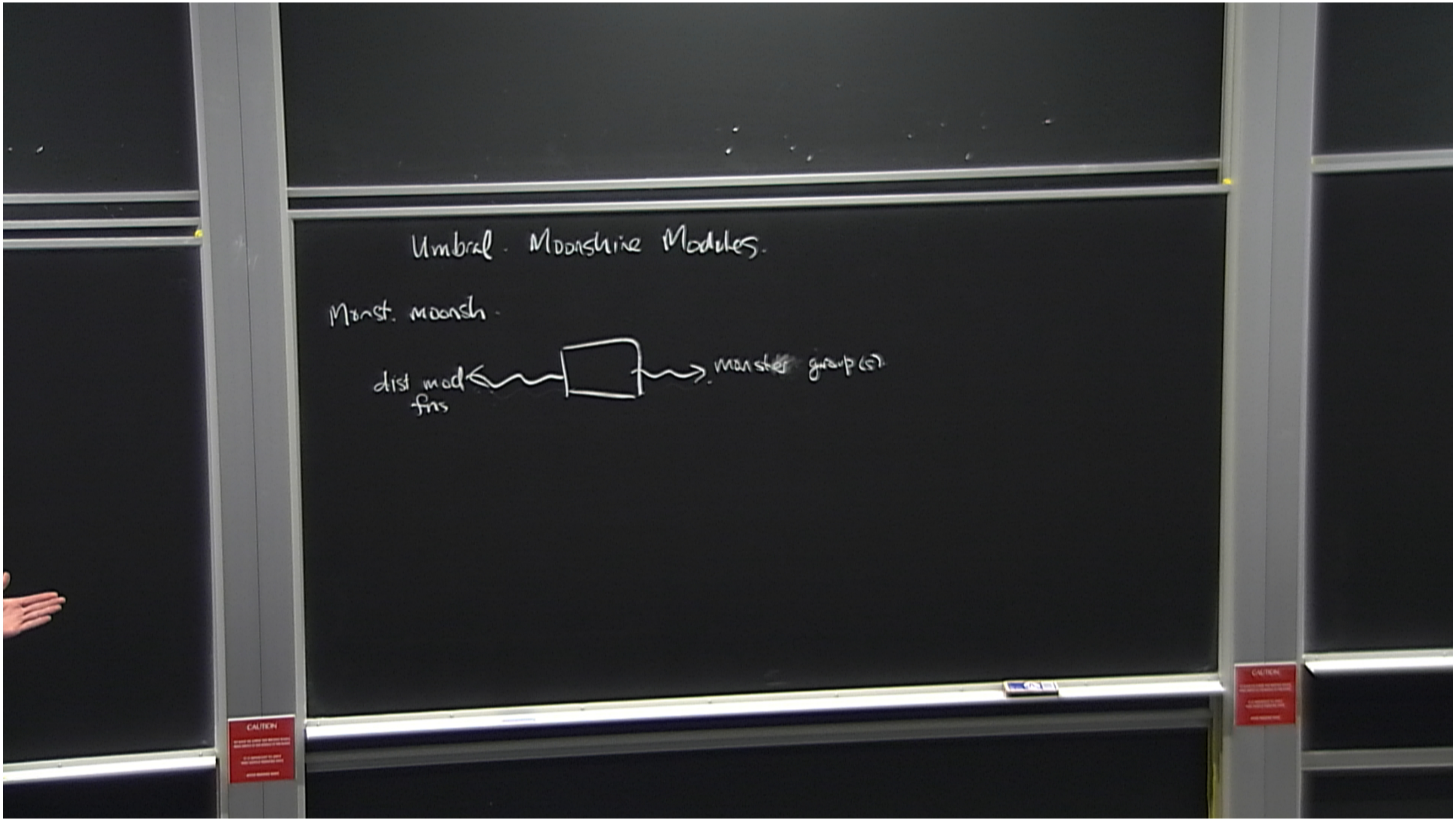
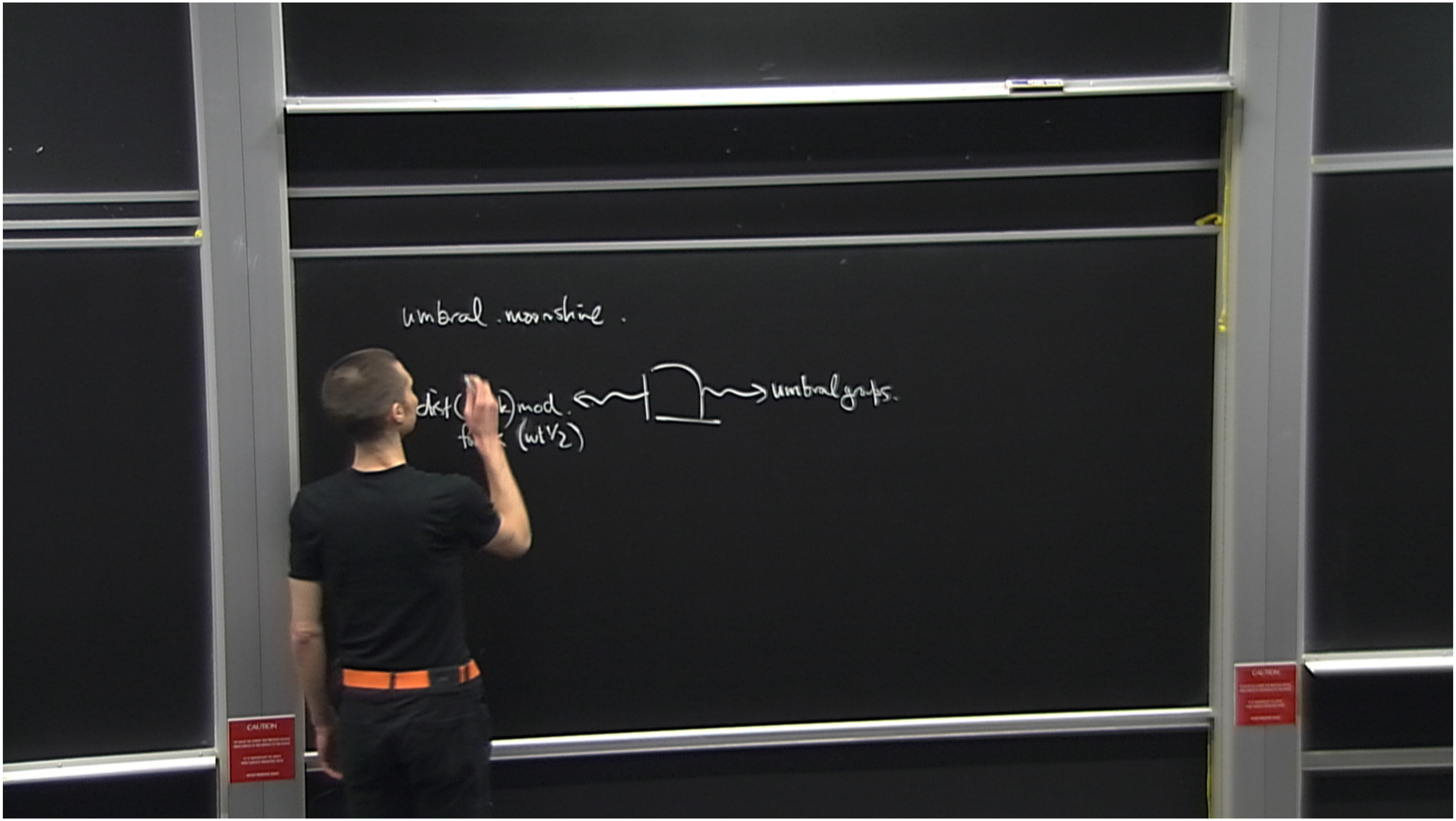
Title: Umbral Moonshine Modules

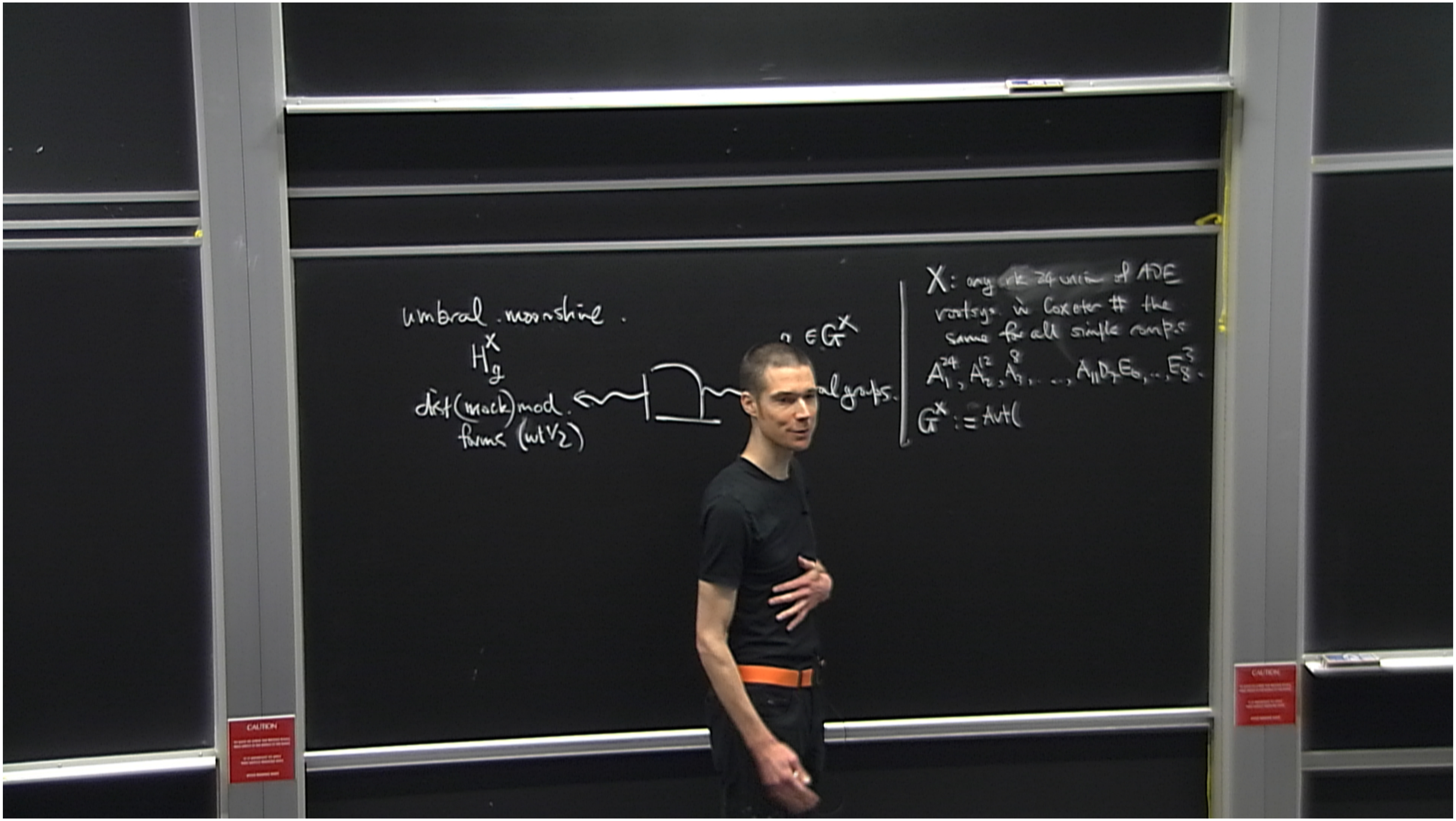
Date: Apr 14, 2015 10:00 AM

URL: http://pirsa.org/15040123

Abstract: Umbral moonshine attaches mock modular forms and meromorphic Jacobi forms to automorphisms of the Niemeier lattices. It is now known that this association can be recovered from specific, graded modules for the Niemeier lattice automorphism groups. We will describe recent progress in a program to realize these modules explicitly.


Umbral Moonshine Modules.
Monst. moonsh.
dist mod fns
monster group(s)


umbral moonshine .
mod.
(wt 1/2)
umbral groups.


umbral moonshine.
H_g^X
dist (mock) mod. forms (wt 1/2)
g ∈ G^X
al grps.
X: any rk 24 union of ADE
rootsys. w/ Coxeter # the
same for all simple comps
A_1^24, A_2^12, A_3^8, ..., A_11 D_7 E_6, ..., E_8^3.
G^X := Aut(

umbral moonshine.

H_g^X $\qquad\qquad\qquad\qquad$ $g \in G^X$

dist (mock) mod. forms (wt $1/2$) $\longleftarrow\!\!\sim\!\sim\!\sim\;\;\sim\!\sim\!\sim\!\!\longrightarrow$ umbral grps.

umbral modules? ✓✓

SCFT/SVOA? ✓ $X = E_8^3, A_6^4, D_{12}^2$.

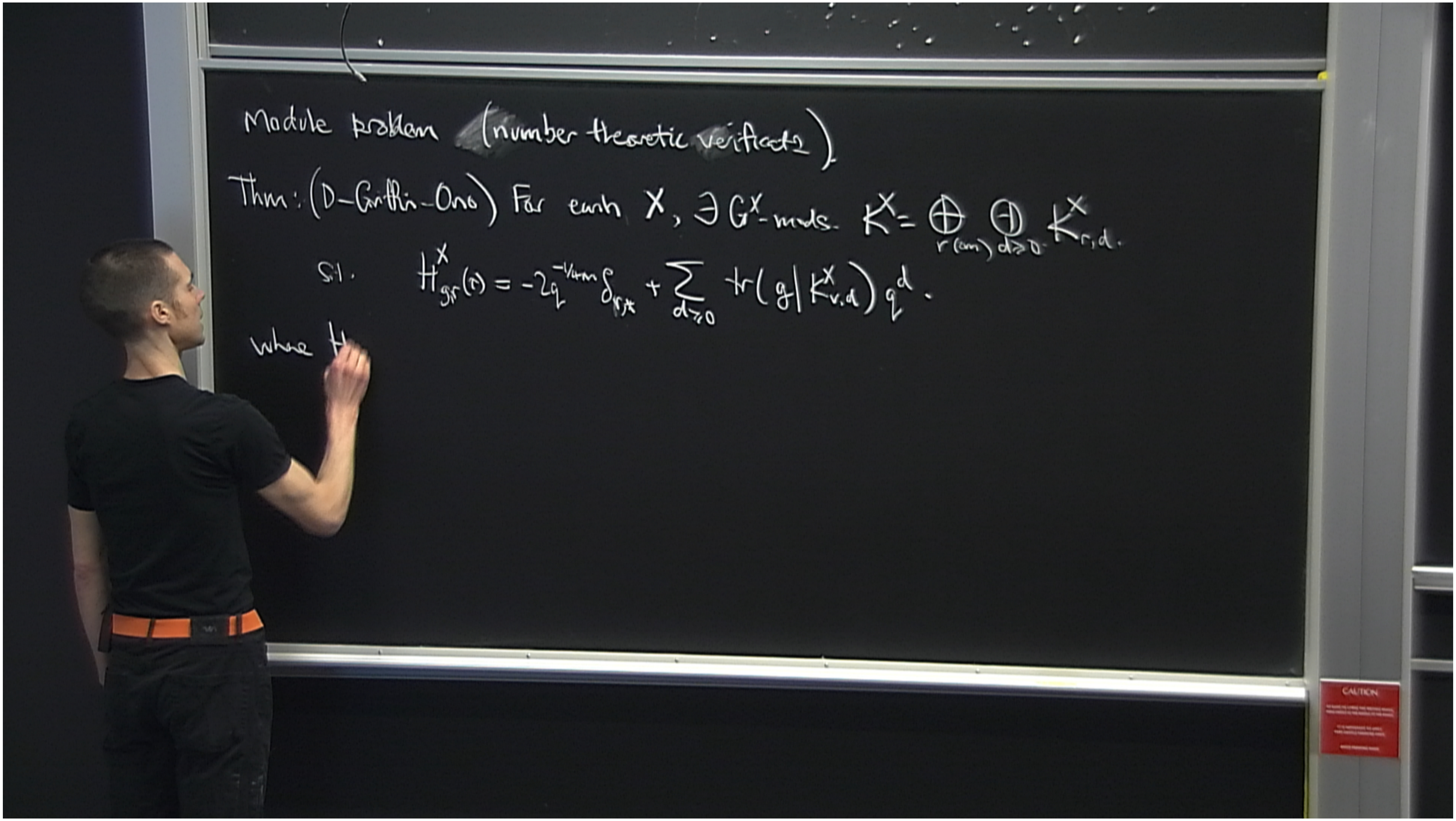
$$G^X = S_3, \underset{\substack{\| \\ 2A_4}}{SL_2(3)}, S_2$$

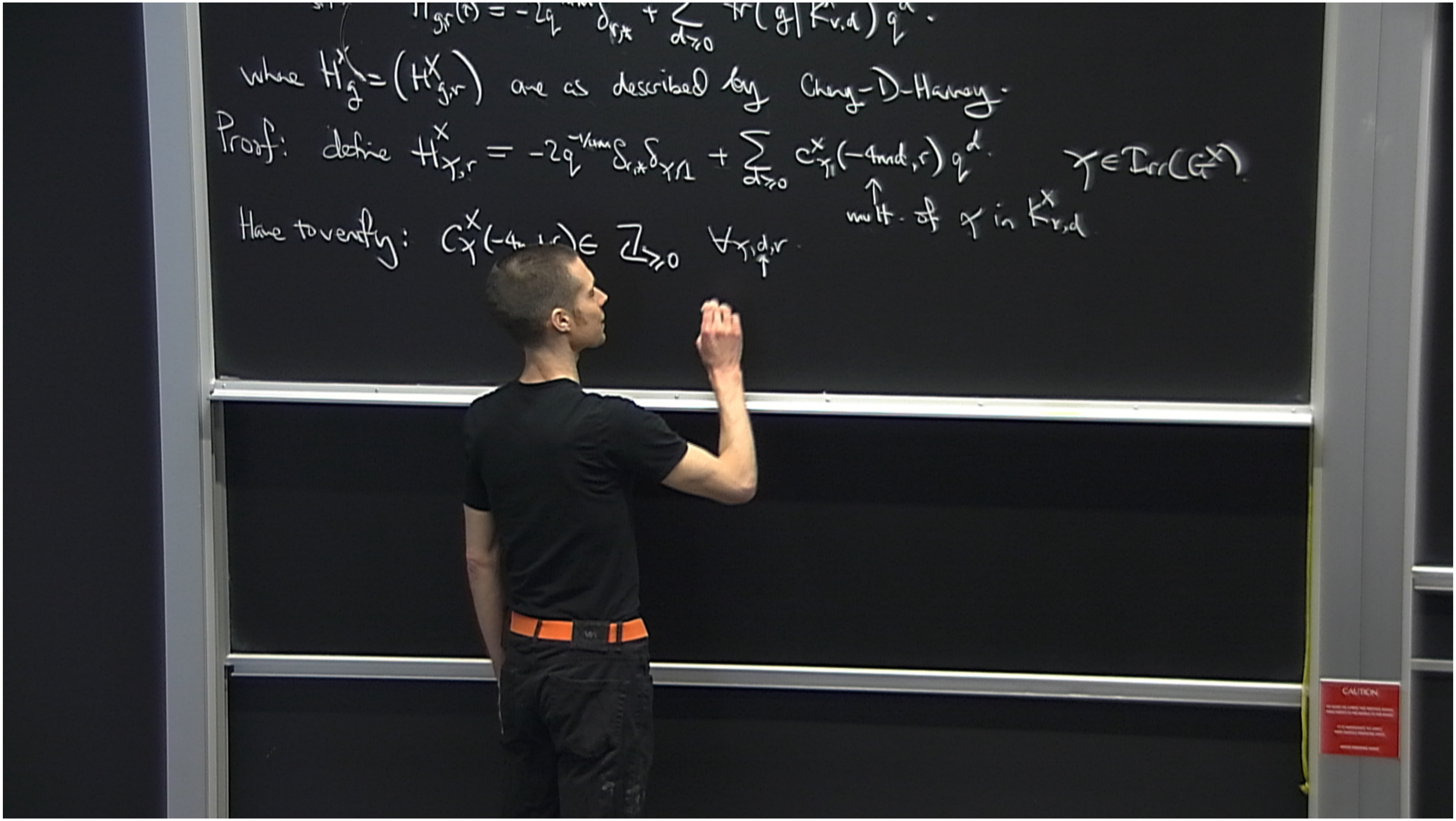
X: any rk 24 union of ADE rootsys. w Coxeter # the same for all simple comps.

$A_1^{24}, A_2^{12}, A_3^{8}, \ldots, A_{11}D_7E_6, \ldots, E_8^3$.

$G^X := \mathrm{Aut}(N^X)/W^X$

$W^X :=$ subgp gen by reflections in root vectors.


Module problem (number theoretic verification)
Thm: (D-Griffin-Ono) For each X, $\exists$ G^X-mods. $K^X = \bigoplus_{r \,(\mathrm{m})} \bigoplus_{d \geq 0} K^X_{r,d}$.
s.t. $H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{d \geq 0} \mathrm{tr}(g | K^X_{r,d})\, q^d$.
where H

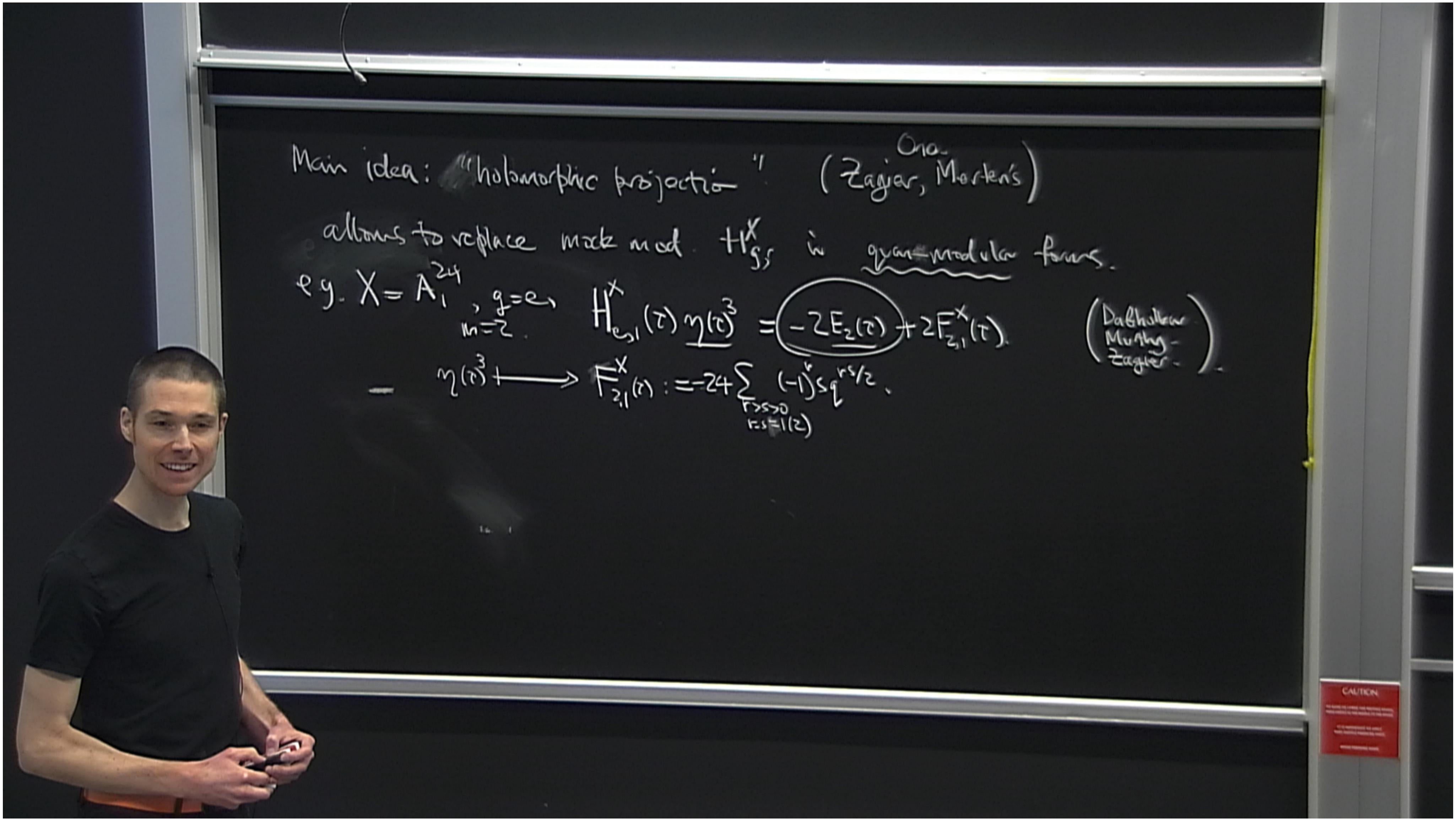

$H_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,*} + \sum_{d\geq 0} \mathrm{tr}(g|K^X_{r,d})\, q^d$.
where $H^X_g = (H^X_{g,r})$ are as described by Cheng-D-Harvey.
Proof: define $H^X_{\chi,r} = -2q^{-1/4m}\delta_{r,*}\delta_{\chi,1} + \sum_{d\geq 0} c^X_{\chi,r}(-4md, r)\, q^d$. $\chi \in \mathrm{Irr}(G^X)$
mult. of χ in $K^X_{r,d}$
Have to verify: $c^X_\chi(-4m\ldots, r) \in \mathbb{Z}_{\geq 0}$ $\forall \chi, d, r$.

Main idea: "holomorphic projection" (Zagier, Ono, Mertens)

allows to replace mock mod. $H^X_{g,s}$ in quasi-modular forms.

e.g. $X = A_1^{24}$, $g = e$, $m = 2$. $\quad H^X_{e,1}(\tau)\,\underline{\eta(\tau)^3} = -2\underline{E_2(\tau)} + 2F^X_{2,1}(\tau)$ (Dabholkar Murthy Zagier).

$$F^X_{2,1}(\tau) := -24 \sum_{\substack{r>s>0 \\ r-s\equiv 1(2)}} (-1)^r s\, q^{rs/2}.$$


Main idea: "holomorphic projection" (Zagier, Mertens, Ono)
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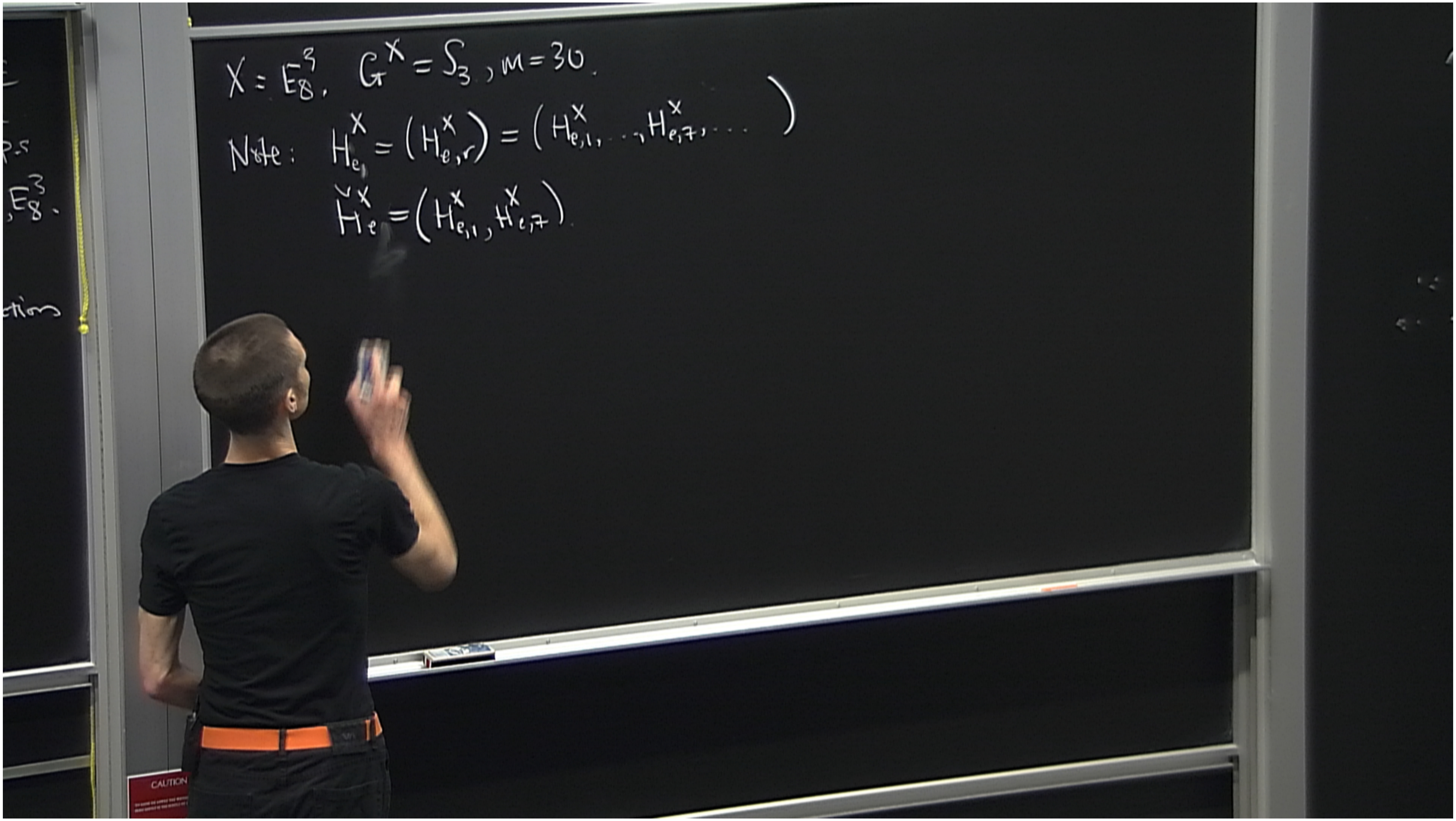
allows to replace mock mod. $H^X_{g,r}$ in quasi-modular forms.

e.g. $X = A_1^{24}$, $g = e$, $m = 2$. $H^X_{e,1}(\tau)\,\eta(\tau)^3 = -2E_2(\tau) + 2F^X_{2,1}(\tau)$. (Dabholkar-Murthy-Zagier).

$$\eta(\tau)^3 \longmapsto F^X_{2,1}(\tau) := -24 \sum_{\substack{r>s>0 \\ r-s \equiv 1 (2)}} (-1)^r s\, q^{rs/2}.$$


$X = E_8^3, \quad G^X = S_3, \quad m = 30.$
Note: $H_e^X = (H_{e,r}^X) = (H_{e,1}^X, \ldots, H_{e,7}^X, \ldots)$
$\check{H}_e^X = (H_{e,1}^X, H_{e,7}^X)$

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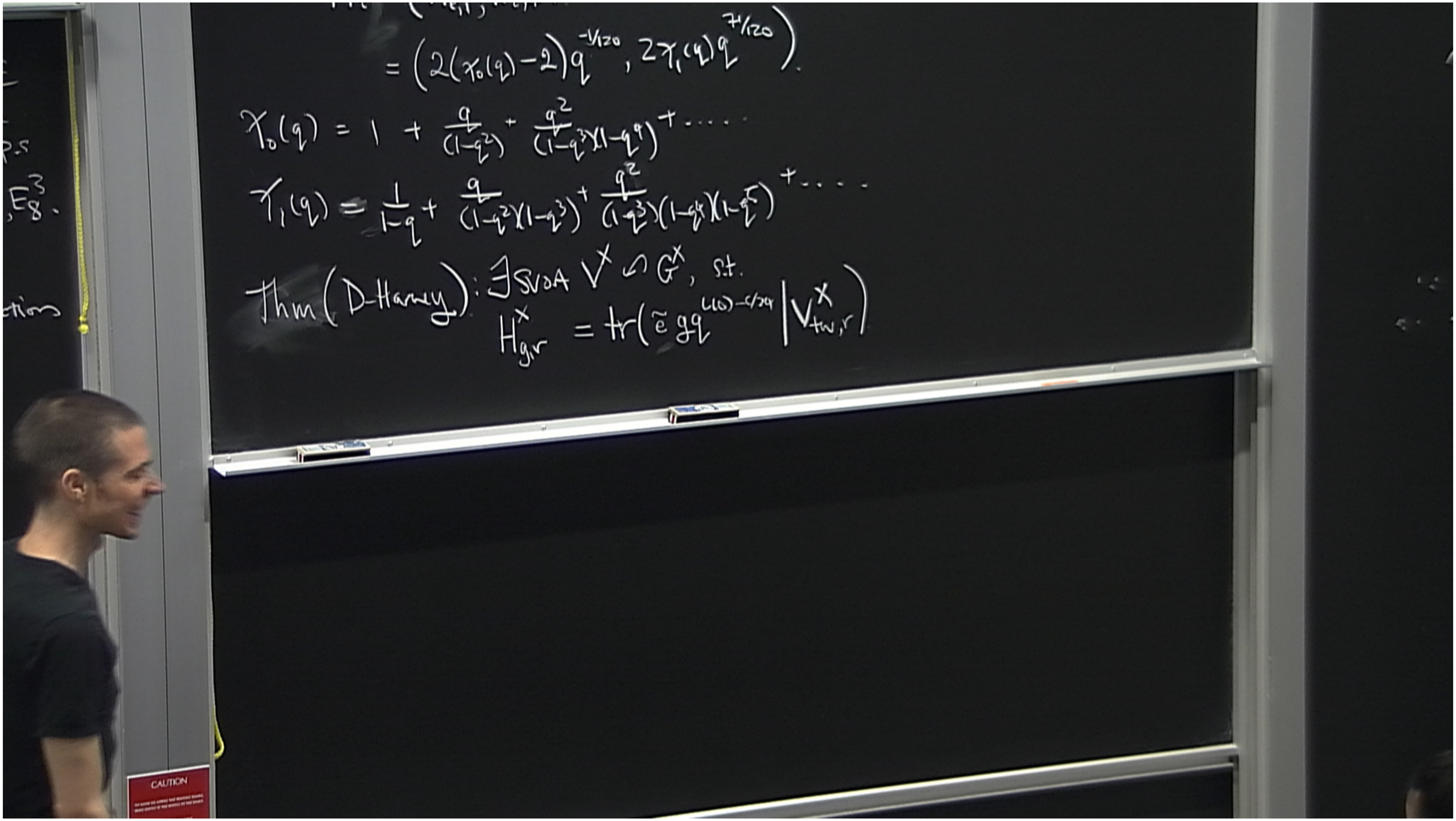
$$= \left(2(\chi_0(q) - 2)q^{-1/120}, 2\chi_1(q) q^{71/120}\right).$$

$$\chi_0(q) = 1 + \frac{q}{(1-q^2)} + \frac{q^2}{(1-q^3)(1-q^4)} + \cdots$$

$$\chi_1(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \cdots$$


$= \left(2(\chi_0(q)-2)q^{-1/120},\ 2\chi_1(q)q^{71/120}\right).$
$\chi_0(q) = 1 + \frac{q}{(1-q^2)} + \frac{q^2}{(1-q^3)(1-q^4)} + \cdots$
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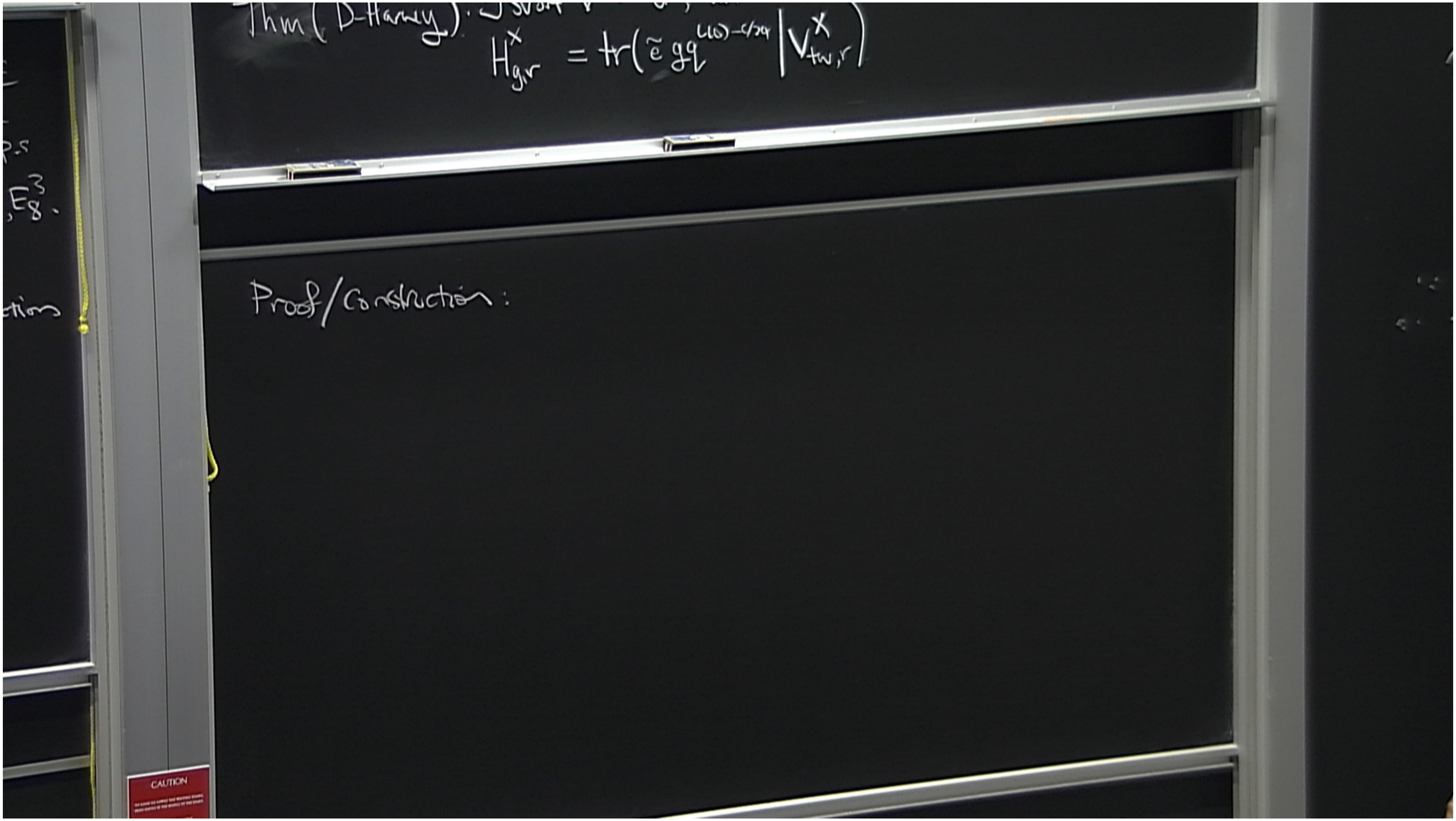
Thm (D-Harvey): $\exists$ SVOA $V^X \circlearrowleft G^X$, s.t.
$H^X_{g,r} = \mathrm{tr}\left(\tilde{e}\, g\, q^{L(0)-c/24} \middle| V^X_{tw,r}\right)$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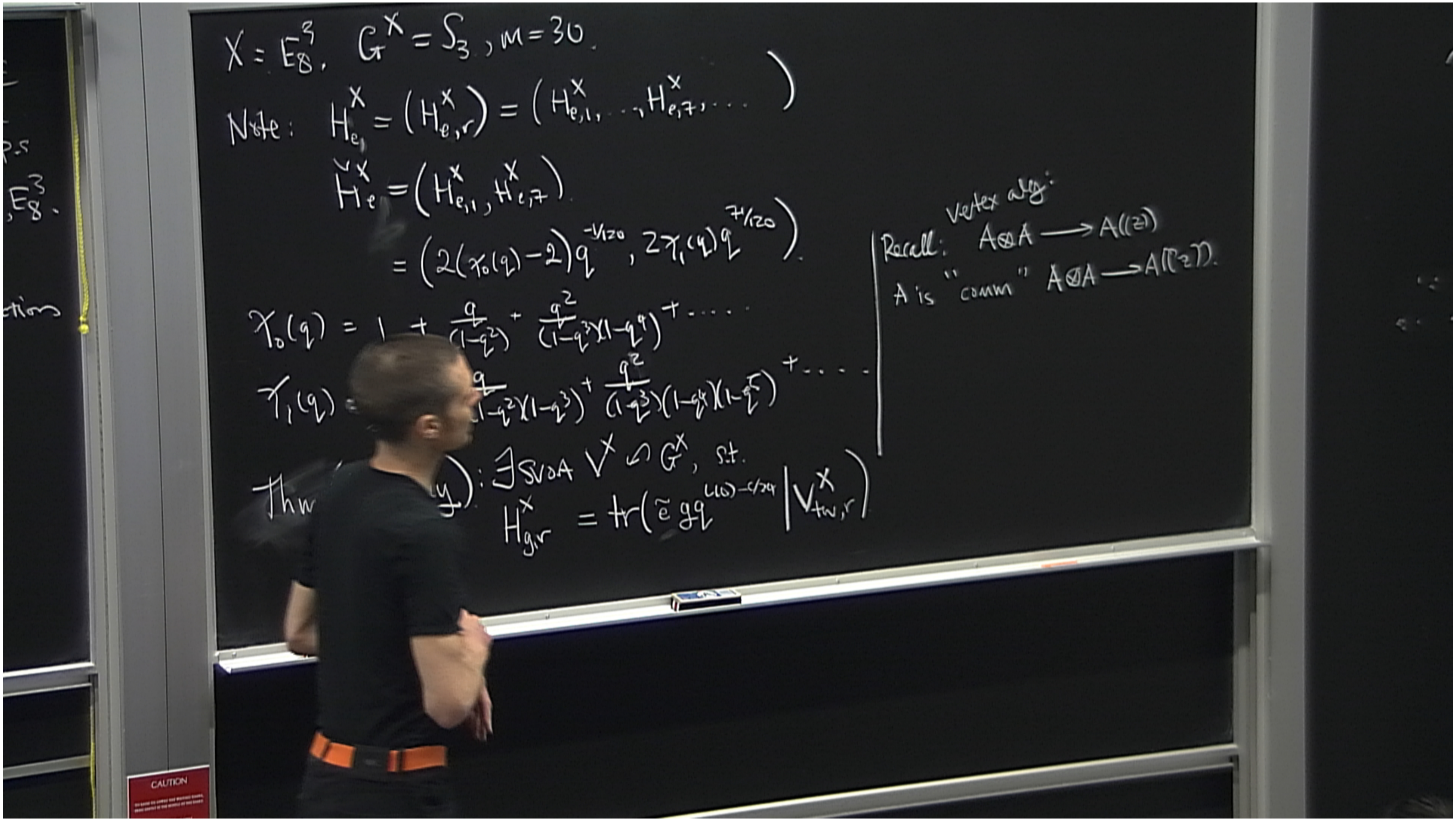
Proof/Construction:


$X = E_8^3,\ G^X = S_3,\ m = 30.$
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$= \left(2(\chi_0(q)-2)q^{-1/120}, 2\chi_1(q)q^{71/120}\right).$
$\chi_0(q) = 1 + \frac{q}{(1-q^2)} + \frac{q^2}{(1-q^3)(1-q^4)} + \cdots$
$\chi_1(q)$
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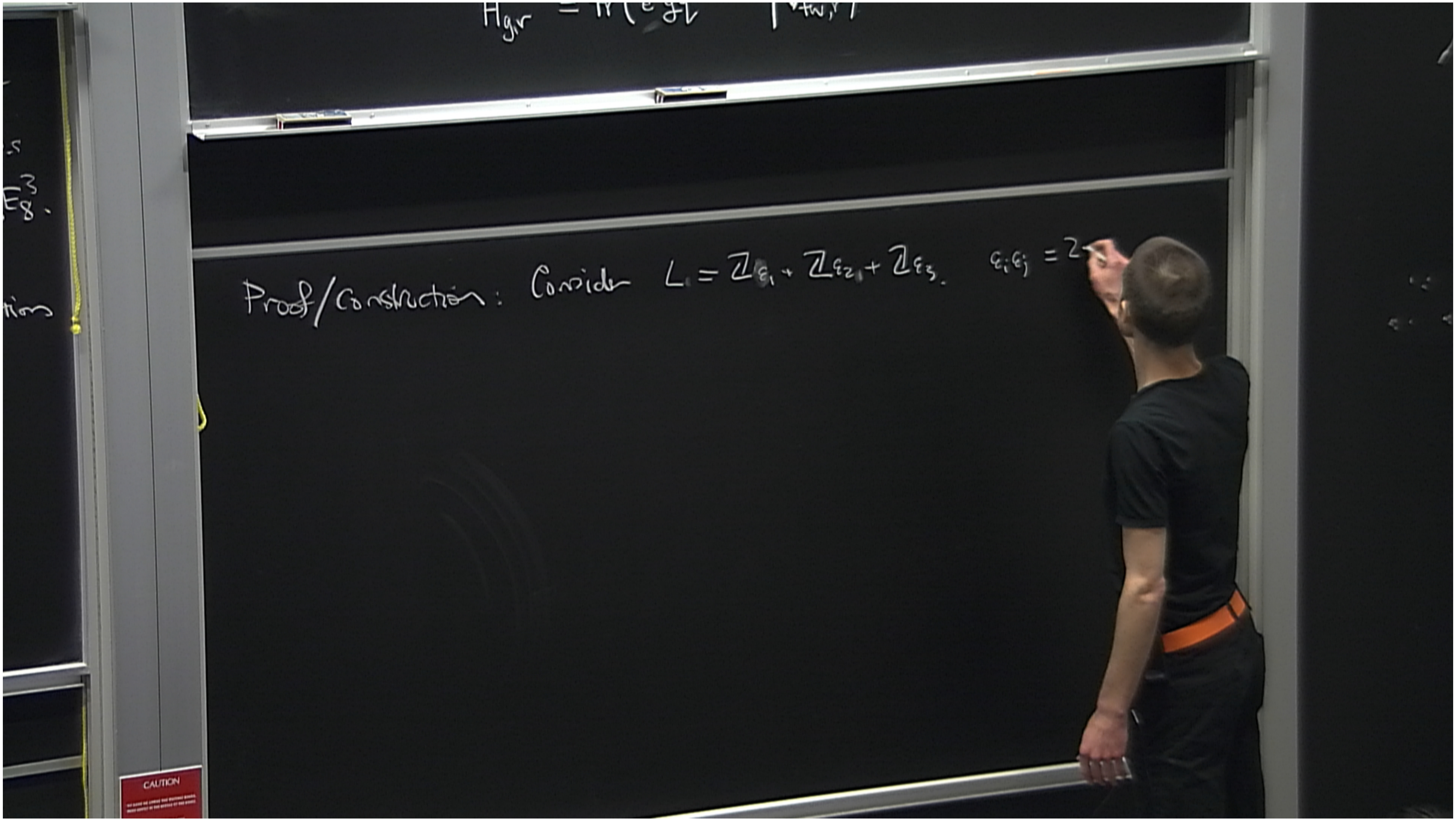
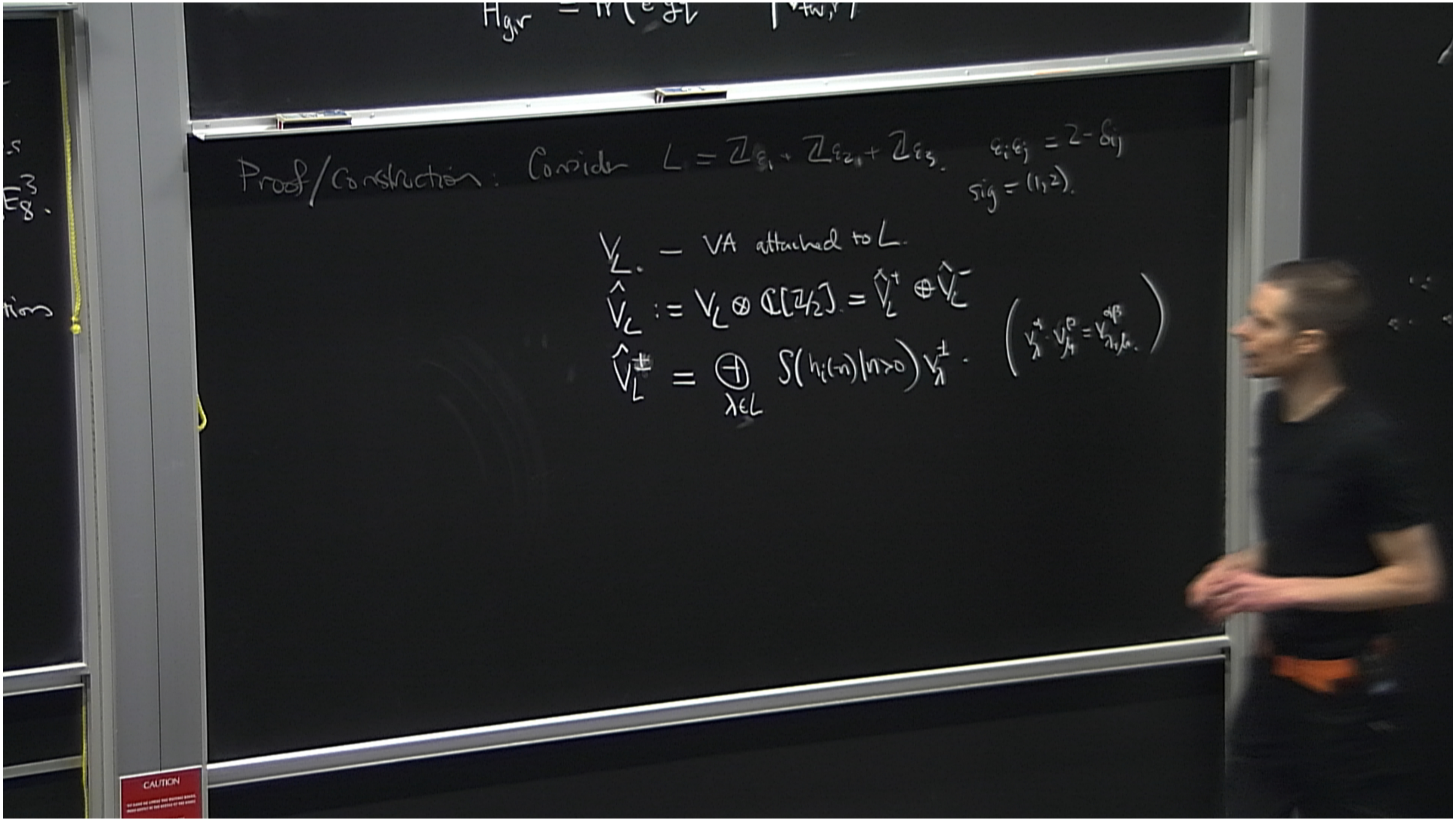
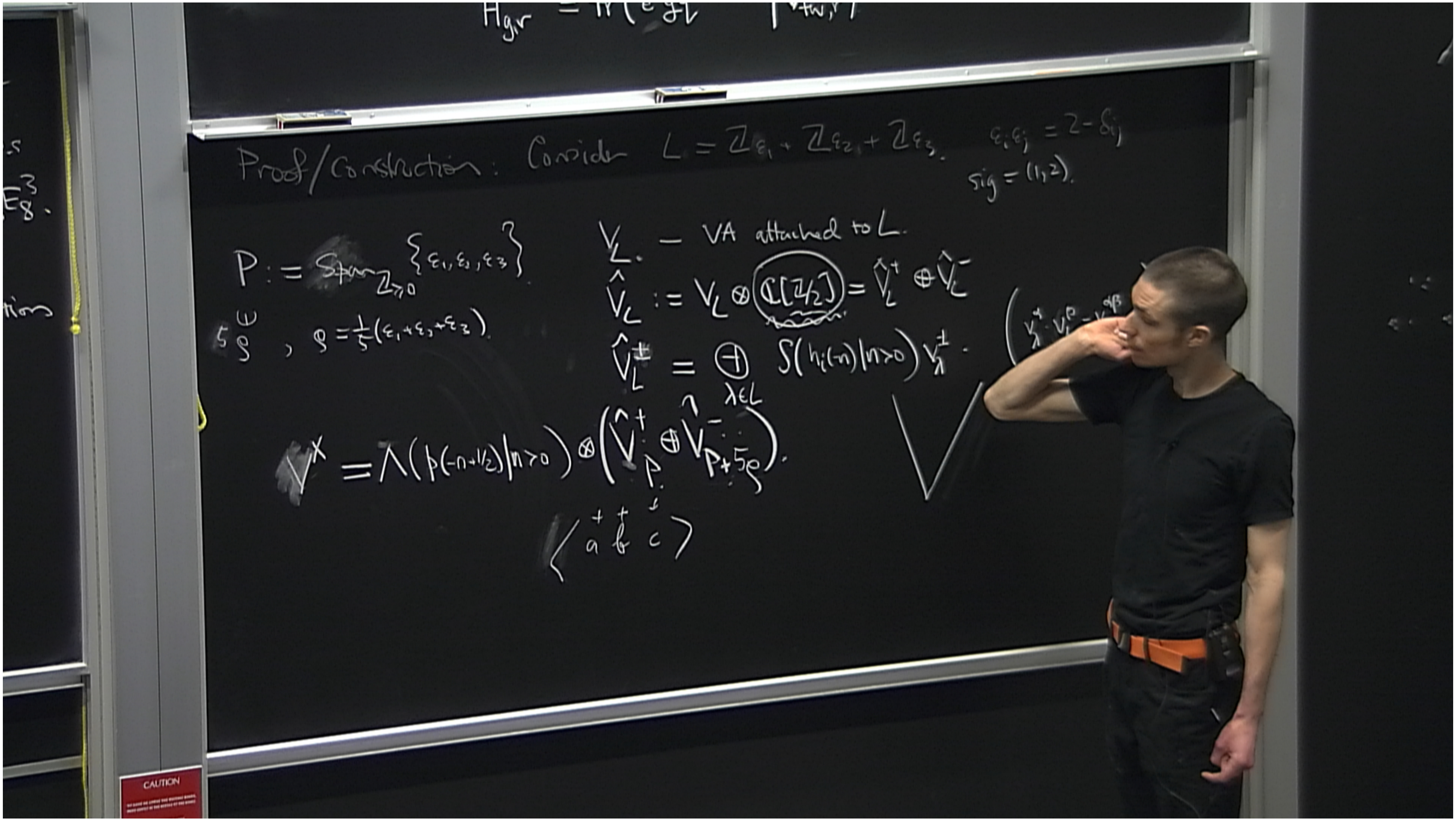
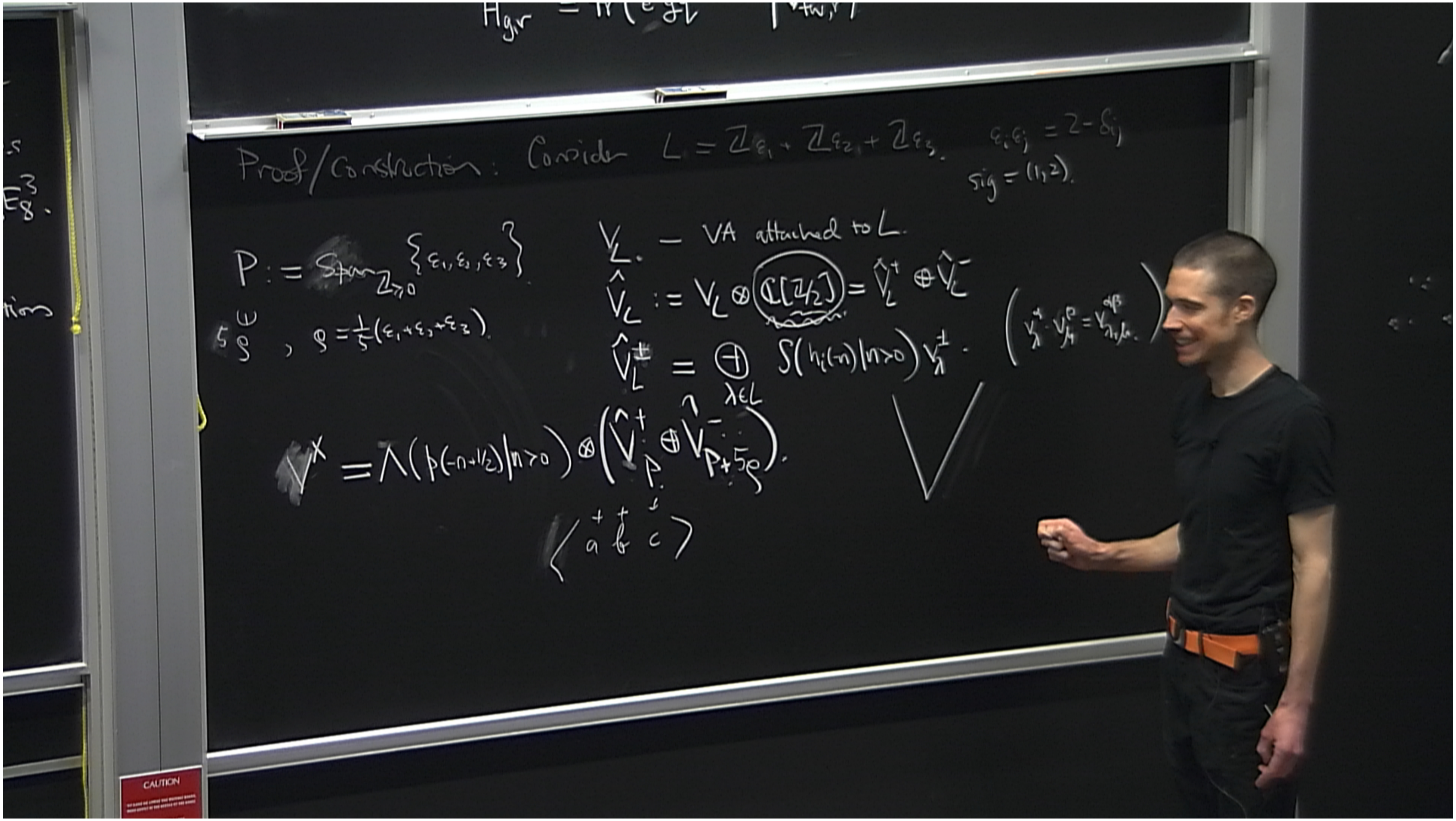
Thm
$: \exists$ SVOA $V^X \curvearrowleft G^X$, s.t.
$H_{g,r}^X = tr\left(\tilde{e}\, g\, q^{L(0)-c/24} \middle| V_{tw,r}^X\right)$
vertex alg:
Recall: $A \otimes A \longrightarrow A((z))$
A is "comm" $A \otimes A \longrightarrow A[[z]]$.


Proof/Construction: Consider L = Zε1 + Zε2 + Zε3. εiεj = 2
CAUTION


Proof/Construction: Consider $L = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \mathbb{Z}\varepsilon_3$.
$\varepsilon_i \varepsilon_j = 2 - \delta_{ij}$
sig $= (1,2)$.
V_L. — VA attached to L.
$\hat{V}_L := V_L \otimes \mathbb{C}[\mathbb{Z}/2] = \hat{V}_L^+ \oplus \hat{V}_L^-$
$\hat{V}_L^\pm = \bigoplus_{\lambda \in L} S(h_i(-n) \mid n > 0)\, v_\lambda^\pm$.
$\left(v_\lambda^\alpha \cdot v_\mu^\beta = v_{\lambda+\mu}^{\alpha\beta} \right)$


Proof/Construction: Consider L = Zε1 + Zε2 + Zε3.
εi εj = 2 − δij
sig = (1,2).
P := Span Z≥0 {ε1, ε2, ε3}
VL − VA attached to L.
ρ = 1/5 (ε1 + ε2 + ε3)
λ∈L
CAUTION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$\mathrm{sig} = (1,2)$.
$P := \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$
5ρ , $\rho = \frac{1}{5}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.
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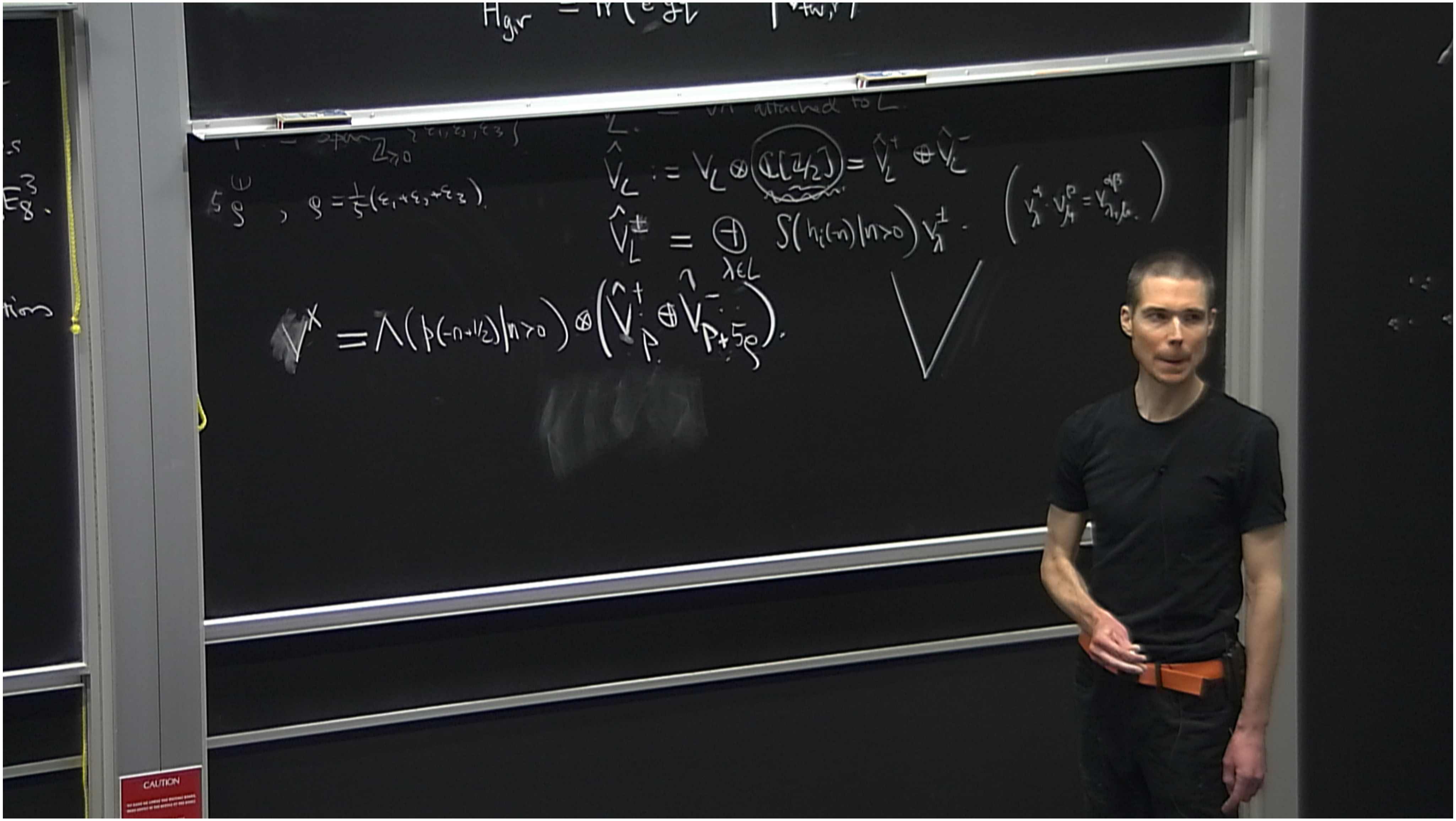
$\left(v_\lambda^{\alpha} \cdot v_\mu^{\beta} = v_{\lambda+\mu}^{\alpha\beta} \right)$
$V^{\natural} = \Lambda(p(-n+1/2) \mid n > 0) \otimes (\hat{V}_P^+ \oplus \hat{V}_{P+5\rho}^-)$.
$\langle \overset{+}{a}\ \overset{+}{b}\ \overset{+}{c} \rangle$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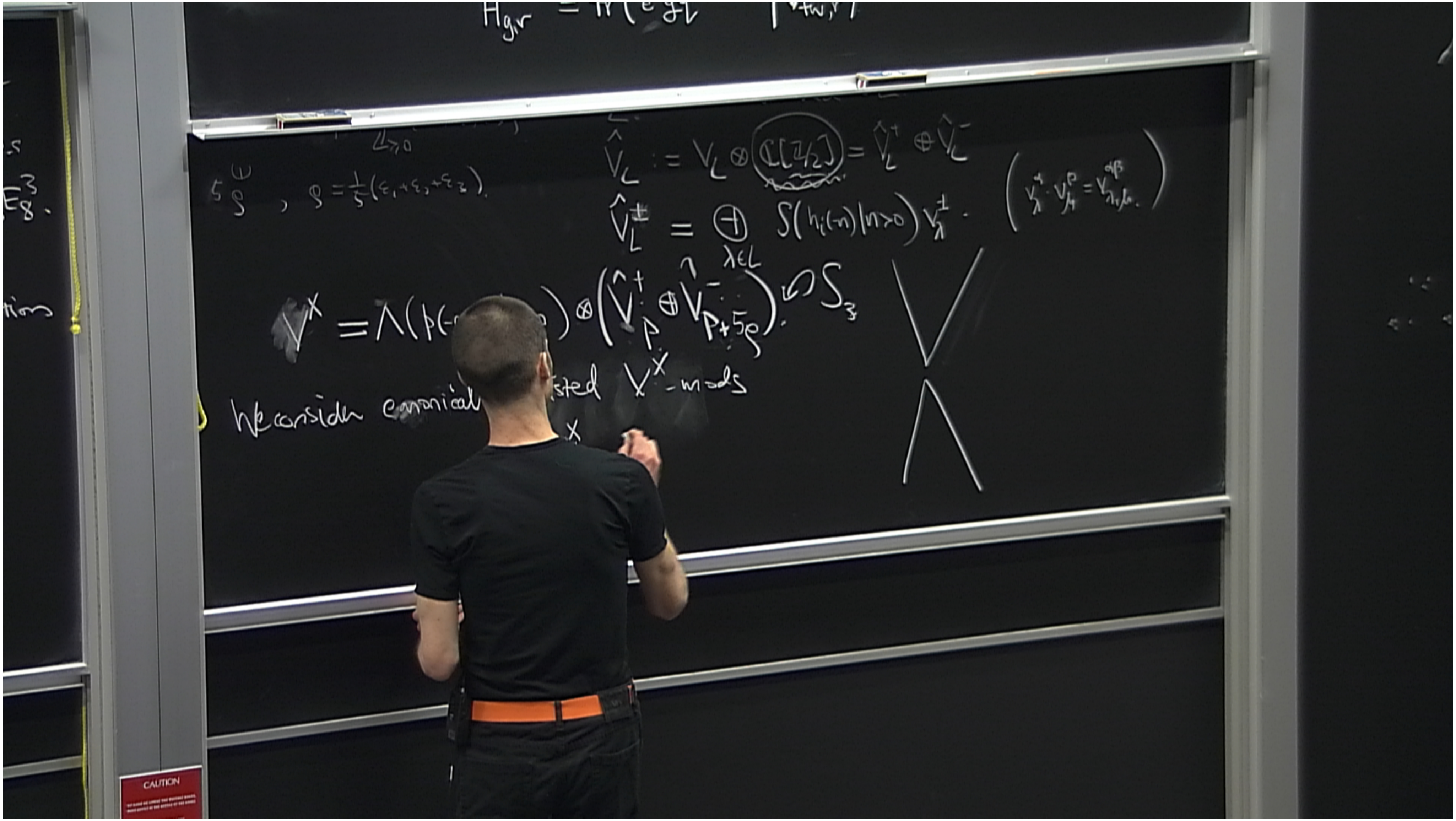
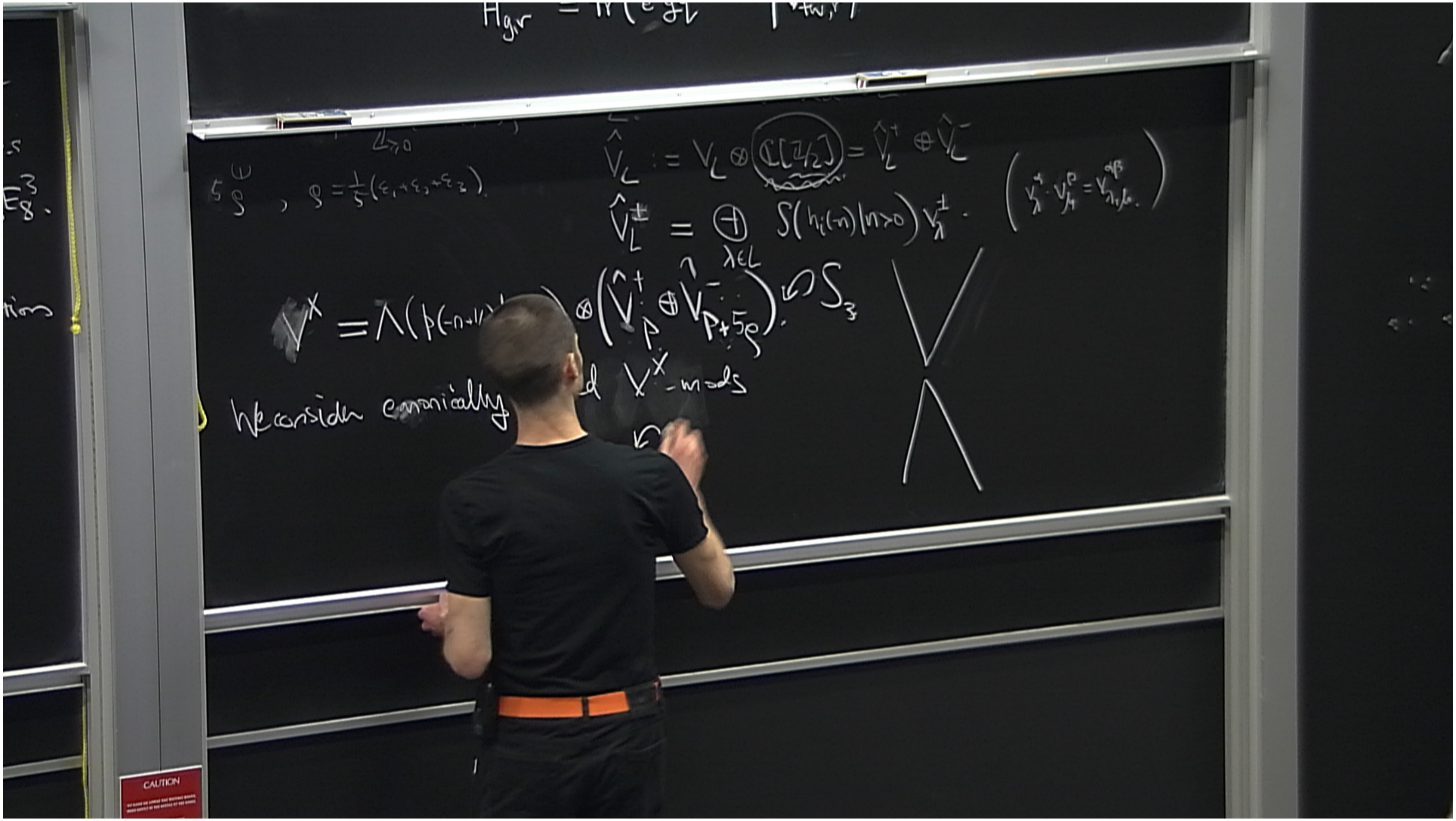
$\rho = \frac{1}{5}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$
$V^\natural = \Lambda(p(-n+1/2)|n>0)\otimes(\hat V_P^+ \oplus \hat V_{P+5\rho}^-)$
CAUTION


We consider canonical twisted V^X-mods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