Title: Moonshine at c=12

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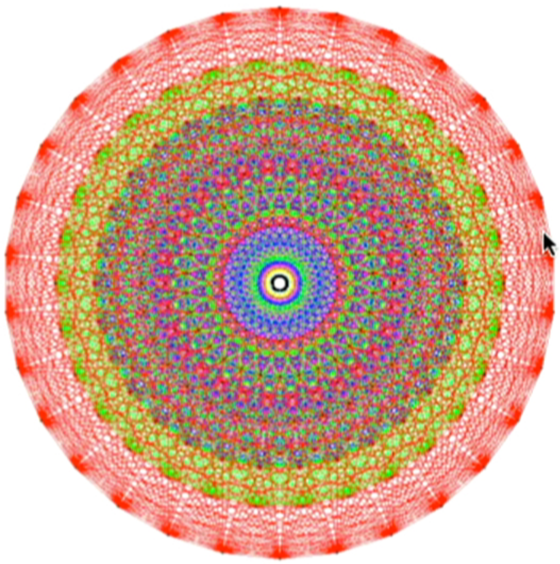
Abstract:

Moonshine at c=12



Shamit Kachru
Stanford & SLAC

Sunday, April 12, 15

Based on three papers

arXiv 1406.5502, 1412.2804, 1503.07219

with a cohort of excellent collaborators at various institutions:

Stanford: N. Benjamin, X. Dong, N. Paquette, D. Whalen, T. Wrase

Harvard: S. Harrison

Amsterdam: M. Cheng

Case Western: J. Duncan

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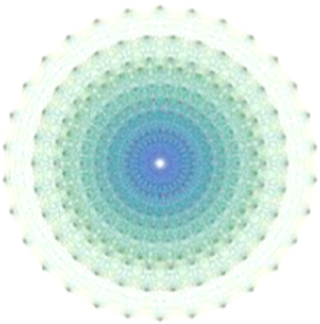
1. Introduction

One of the earliest examples of moonshine, involves a simple c=12 chiral conformal field theory.

Chiral conformal field theory is deeply related to even self-dual lattices. In dimension 8 there is a unique such structure:

E8 root lattice:



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It was discovered long ago that the (supersymmetrized) theory associated to the E8 lattice exhibits moonshine for the Conway group. FLM Duncan '05

More precisely, one should consider the Z2 orbifold of the theory:

$$(X^i, \psi^i) \to (-X^i, -\psi^i) \ .$$

The orbifolding projects out the moduli, as in the analogous story with the Leech lattice.

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The NS sector partition function is:

$$\begin{aligned} Z_{NS,E8}(\tau) &= \mathrm{tr}_{NS}\, q^{L_0 - c/24} = \frac{1}{2}\left(\frac{E_4\theta_3^4}{\eta^{12}} + 16\frac{\theta_4^4}{\theta_2^4} + 16\frac{\theta_2^4}{\theta_4^4}\right)(\tau) \\ &= q^{-1/2} + 0 + 276\, q^{1/2} + 2048\, q + 11202\, q^{3/2} + \cdots , \end{aligned}$$

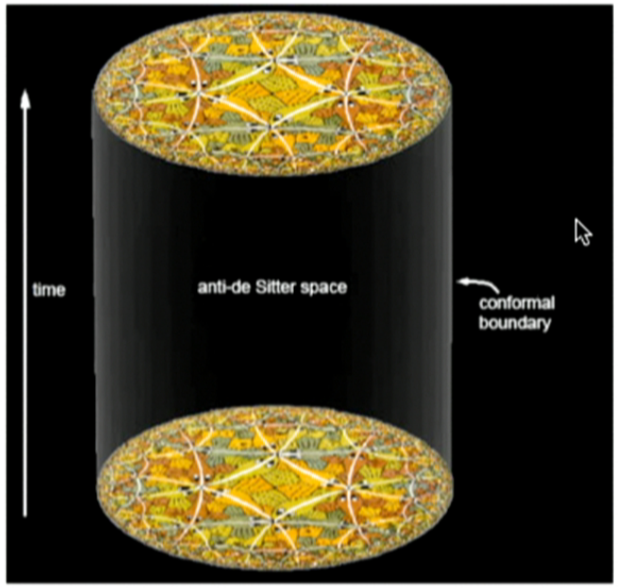
The coefficients in the q-series decompose nicely into representations of the sporadic group Co_0.

For instance, 276=276 while:

$$2048 = 24 + 2024$$

$$11202 = 1 + 276 + 299 + 1771 + 8855$$

This theory has appeared in various interesting contexts.



For instance, it is a candidate dual to (chiral?) supergravity with deep negative cosmological constant. Witten

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A more useful way to view this theory, for our purposes, is as the Z2 orbifold of the theory of 24 chiral fermions.

Then, the partition function is more naturally written as:

$$Z_{NS,\mathrm{fermion}}(\tau) = \frac{1}{2}\sum_{i=2}^{4}\frac{\theta_i^{12}(\tau,0)}{\eta^{12}(\tau)},$$

and the model has a manifest Spin(24) symmetry. If one considers the dimension-3/2 spin fields interpolating to the twisted sector, one can show a choice of N=1 supercurrent breaks the symmetry to Co_0.

There is a natural expression for partition functions

$$Z_g = \mathrm{Tr}\ g q^{L_0 - \frac{c}{24}}$$

twined by group elements g, as well.

Let ϵ_a denote the eigenvalues of g in the 24

$$\epsilon_a = \overline{\epsilon_{a+12}},\ e(z_a) = \epsilon_a,\ z_a \in [0, 1/2)$$

Then the twined partition function is given by: Duncan, Mack-Crane

$$Z_{NS,g}(\tau) = \mathrm{tr}_{NS}\, g q^{L_0 - c/24} = \frac{1}{2}\sum_{i=2}^{4} \varepsilon_i(g) \prod_{a=1}^{12} \frac{\theta_i(\tau, z_a)}{\eta(\tau)} = q^{-\frac{1}{2}} \prod_{n=1}^{\infty}\prod_{a=1}^{24}(1 + \epsilon_a q^{n-\frac{1}{2}}) - \chi_g,$$

$$\chi_g = \mathrm{tr}_{\mathbf{24}} g = \sum_{a=1}^{24} \epsilon_a\ ,\quad \varepsilon_2(g) = \frac{\mathrm{Tr}_{\mathbf{4096}} g}{2^{12}\prod_{a=1}^{12}\cos(\pi z_a)} \in \{-1, 1\}\ ,\quad \varepsilon_3(g) = \varepsilon_4(g) = 1.$$

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These are normalized principal moduli for certain genus zero groups Γ_g .

Our interest in this theory developed in a somewhat roundabout way. Eguchi, Ooguri and Tachikawa made the observation that the elliptic genus of K3 satisfies:

$$\phi(\tau,\gamma) = 8 \sum_{i=2}^{4} \frac{\theta_i(\tau,\gamma)^2}{\theta_i(\tau,0)^2}$$

$$= 20\ ch^{\text{short}}_{1/4,0}(\tau,\gamma) - 2\ ch^{\text{short}}_{1/4,1/2}(\tau,\gamma) + \sum_{n=1}^{\infty} A_n\ ch^{\text{long}}_{1/4+n,1/2}(\tau,\gamma)$$

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}}$$

$$A_3 = 1540 = \mathbf{770} + \overline{\mathbf{770}}$$

← dims of irreps of M24!

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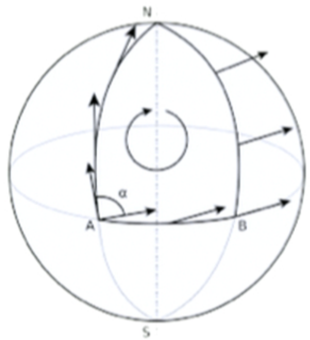
This suggests that there might be more general relations between exceptional geometric objects of interest in string compactification, and examples of moonshine.



There is a famous relationship between special holonomy groups, and space-time supersymmetry in string theory.

E.g. Calabi-Yau threefolds preserve SUSY because

$$\mathbf{16} = (\bar{\mathbf{4}}, \mathbf{2}) \oplus (\mathbf{4}, \mathbf{2}), \quad \mathbf{4} = \mathbf{3} + \mathbf{1}$$

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Berger classified all of the special holonomy groups arising in Riemannian geometry. The list giving supersymmetric string models is not long:

SU(N) Calabi-Yau
Sp(N) hyperKahler
G2
Spin(7) ← Compact examples first constructed by Joyce ~1993

Worldsheet N=1 is necessary for superstrings. But compactifications yielding space-time supersymmetry enjoy an enhanced worldsheet superalgebra.

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The dictionary is roughly

$$\begin{aligned}
&\text{Calabi} - \text{Yau} \leftrightarrow (2,2)\ \text{superconformal algebra}\\
&\text{Hyper} - \text{Kahler} \leftrightarrow (4,4)\ \text{superconformal algebra}\\
&\text{G2} \leftrightarrow (1,1)\ +\ \text{tricritical Ising}\\
&\text{Spin}(7) \leftrightarrow (1,1)\ +\ \text{Ising} \simeq \mathcal{SW}(3/2,2)
\end{aligned}$$

At c=12, the examples which can arise are Calabi-Yau, hyperKahler, and Spin(7) geometries.

We noticed some interesting properties in elliptic genera of these spaces, but had trouble making this precise.

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It turns out the c=12 model I've discussed here is a simplified playground in which we can make these observations precise.

2. Enhancing the worldsheet supersymmetry

What we are going to do is to view the super-E8 theory as furnishing an N=(2,2), (4,4), or SW(3/2,2) superconformal theory. It will be a sort of simplified model realizing (likely) more symmetries than can show up in generic geometries realizing these superalgebras.

We will see that each superalgebra is naturally associated to various sporadic groups.

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The key to enhancing the supersymmetry is to find a suitable R-symmetry in each case. So for instance, it is known that N=4 supersymmetry requires an SU(2) R-symmetry.

In the free fermion description of the theory, we can generate an SU(2) by choosing 3 fermions:

$$J_i = -i\epsilon_{ijk}\lambda_j\lambda_k\,, \qquad i,j,k \in \{1,2,3\}\,.$$

You can check quickly that these have the desired OPE

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij} + \frac{i}{z}\epsilon_{ijk}J_k(0)\,.$$

By bosonizing the fermions, writing the currents in bosonic language, and writing the N=1 supercurrent in the same way, one can show that one obtains from OPEs

$$J_i(z)W(0) \sim -\frac{i}{2z}W_i(0)$$

The algebra of the supercurrents is then calculated to be:

$$W_i(z)W_j(0) \sim \delta_{ij}\left[\frac{8}{z^3} + \frac{8}{z}T(0)\right] + 2i\epsilon_{ijk}\left[\frac{2}{z^2}J_k(0) + \frac{1}{z}\partial J_k(0)\right]$$

$$W(z)W_i(0) \sim -2i\left(\frac{2}{z^2} + \frac{\partial}{z}\right)J_i(0)\,,$$

$$J_i(z)W_j(0) \sim \frac{i}{2z}(\delta_{ij}W + \epsilon_{ijk}W_k)\,.$$

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Together with the stress-energy tensor

$$T = -\frac{1}{2}\lambda_\alpha \partial \lambda_\alpha = -\frac{1}{2}\partial H_a \partial H_a$$

these yield a copy of the N=4 superconformal algebra.

It should be clear that by choosing two fermions and bosonizing, one could similarly obtain a U(1) current to extend N=1 to the N=2 superconformal algebra; or by choosing a single fermion, an Ising sector to extend N=1 to the SW(3/2,2) algebra.

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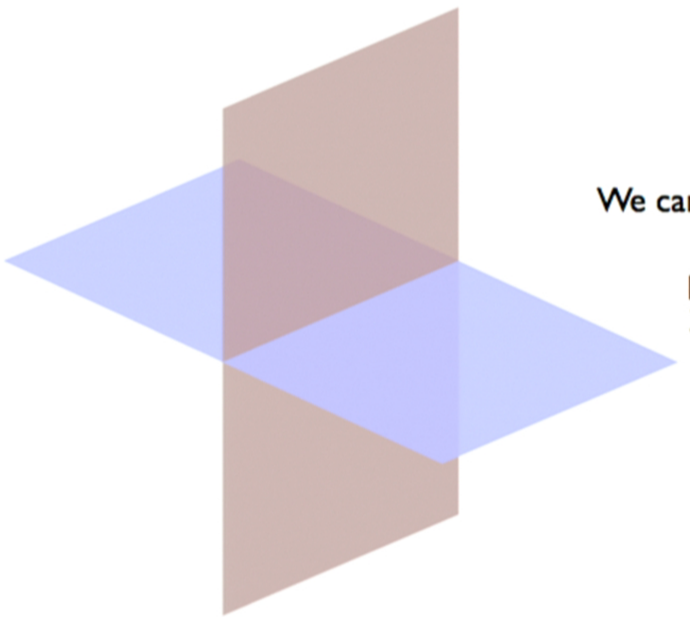
3. Global symmetries

The Conway symmetry of this theory plays nicely with an N=1 superconformal algebra. The other superconformal algebras play nicely with different symmetries.



We can think of this in the following way. Choosing a triplet (doublet, singlet) of fermions out of the 24 present in the theory, is the same as choosing a 3-plane (2-plane, 1-plane) in the 24 dimensional representation of Co_0.

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IV. N=4 superconformal moonshine for M22

In the cases that the superconformal algebra has a U(1) symmetry, it is natural to write a U(1) graded partition function:

$$
\begin{aligned}
Z(\tau, z) &= \mathrm{Tr}_V (-1)^F q^{L_0 - c/24} y^{J_0} \\
&= \frac{1}{2\eta^{12}(\tau)} \sum_{i=1}^{4} (-1)^{i+1} \theta_i(\tau, 2z) \theta_i^{11}(\tau, 0) \\
&= \frac{1}{2} \frac{E_4(\tau)\theta_1^4(\tau, z)}{\eta^{12}(\tau)} + 8 \sum_{i=2}^{4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^4
\end{aligned}
$$

Here,

$$y = e(z) \ .$$

We can expand this thing in terms of characters of the N=4 superconformal algebra.

* The unitary highest weight irreducible representations are labelled by h, j (eigenvalues of $L_0, \frac{1}{2}J_0^3$).

* There are short (or BPS) representations, with

$$h = \frac{c}{24} = \frac{m-1}{4}$$

$$j \in \{0, \frac{1}{2}, \cdots, \frac{m-1}{2}\},$$

* as well as long (non-BPS) representations, with

$$h > \frac{m-1}{4}$$

$$j \in \{\frac{1}{2}, 1, \cdots, \frac{m-1}{2}\}.$$

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The graded characters are given by:

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = \mathrm{tr}_{\mathcal{V}^{\mathcal{N}=4}_{m;h,j}}\left((-1)^{J_0^3} y^{J_0^3} q^{L_0-c/24}\right)$$

We will discuss their form in a moment.

Now, we can decompose the graded partition function:

$$Z(\tau,z) = 21\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},1} + \left(560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},\frac{1}{2}} + 8470\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},\frac{1}{2}} + 70576\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},\frac{1}{2}} + \dots\right)$$
$$+ \left(210\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},1} + 4444\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},1} + 42560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},1} + \dots\right)$$

We recognize M22 representations:

c.f. Jeff Harvey 2010, unpublished

$$21 = 21$$
$$560 = 280 + \overline{280}$$
$$210 = 210$$

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To understand the modular properties, we need to define some objects. The characters are naturally written as:

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = \left(\Psi_{1,1}(\tau,z)\right)^{-1}\mu_{m;j}(\tau,z)$$

Eguchi, Taormina

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = \left(\Psi_{1,1}(\tau,z)\right)^{-1} q^{h-\frac{c}{24}-\frac{j^2}{m}} \left(\theta_{m,2j}(\tau,z) - \theta_{m,-2j}(\tau,z)\right)$$

in the short and long cases, respectively.

μ is given by

$$\mu_{m;j}(\tau,z) = (-1)^{1+2j}\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{(yq^k)^{-2j} + (yq^k)^{-2j+1} + \cdots + (yq^k)^{1+2j}}{1-yq^k}$$

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while $\Psi_{1,1}$ is a meromorphic Jacobi form of weight 1 and index 1, given by

$$\Psi_{1,1}(\tau,z) = -i\,\frac{\theta_1(\tau,2z)\,\eta(\tau)^3}{(\theta_1(\tau,z))^2}$$

Finally we have used the standard theta functions

$$\theta_{m,r}(\tau,z) = \sum_{k=r \pmod{2m}} \mathrm{e}(\tfrac{k}{2})\, q^{k^2/4m} y^k,$$

which satisfy

$$\theta_{m,r}(\tau,z) = \theta_{m,r+2m}(\tau,z) = \mathrm{e}(m)\,\theta_{m,-r}(\tau,-z).$$

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We can usefully think of the $\theta_m = (\theta_{m,r}),\ r - m \in \mathbb{Z}/2m\mathbb{Z},$ as a vector-valued Jacobi form of weight 1/2 and index m.

It satisfies:

$$\begin{aligned}\theta_m(\tau, z) &= \sqrt{\frac{1}{2m}}\sqrt{\frac{i}{\tau}}\, \mathrm{e}(-\tfrac{m}{\tau}z^2)\, \mathcal{S}_\theta.\theta_m(-\tfrac{1}{\tau}, \tfrac{z}{\tau}) \\ &= \mathcal{T}_\theta.\theta_m(\tau+1, z) \\ &= \theta_m(\tau, z+1) = \mathrm{e}(m(\tau + 2z + 1))\theta_m(\tau, z+\tau),\end{aligned}$$

where the 2m x 2m matrices S, T are:

$$(\mathcal{S}_\theta)_{r,r'} = \mathrm{e}(\tfrac{rr'}{2m})\,\mathrm{e}(\tfrac{-r+r'}{2}) \quad , \quad (\mathcal{T}_\theta)_{r,r'} = \mathrm{e}(-\tfrac{r^2}{4m})\,\delta_{r,r'}.$$

The object μ also has interesting modular properties.

We can define its non-holomorphic completion by:

$$\hat{\mu}_{m;0}(\tau,\bar{\tau},z) = \mu_{m;0}(\tau,z) - \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\sum_{r\in\mathbb{Z}/2m\mathbb{Z}}\theta_{m,r}(\tau,z)\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau'\ .$$

This transforms as a Jacobi form of weight 1 and index m under the Jacobi group. Here, $S_m = (S_{m,r})$ is a vector-valued cusp form (with nontrivial multiplier) under SL(2,Z):

$$S_{m,r}(\tau) = \sum_{k=r \pmod{2m}} \mathrm{e}(\tfrac{k}{2})\,k\,q^{k^2/4m} = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)|_{z=0}.$$

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We can then re-write our graded partition function as:

$$Z(\tau,z) = 21\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},1} + \left(560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},\frac{1}{2}} + 8470\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},\frac{1}{2}} + 70576\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},\frac{1}{2}} + \dots\right)$$
$$+ \left(210\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},1} + 4444\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},1} + 42560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},1} + \dots\right)$$
$$= \left(\Psi_{1,1}(\tau,z)\right)^{-1}\left(24\,\mu_{3;0}(\tau,z) + \sum_{r\in\mathbb{Z}/6\mathbb{Z}} h_r(\tau)\theta_{3,r}(\tau,z)\right)$$

Based on the properties we discussed, we see that h is naturally thought of as a weight-1/2 vector-valued mock modular form with 6 components, 2 of which are independent:

$$h_0 = h_3 = 0,\ h_{-1} = -h_1,\ h_{-2} = -h_2$$

Its shadow is given by 24 S_3.

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We know from our earlier discussion that we can view the CFT as furnishing an N=4 supersymmetric M22 module

$$V^G = \bigoplus_{r=1,2} \bigoplus_{n=1}^{\infty} V^G_{r,n}$$

So we expect that we can study twined partition functions,

$$Z_g(\tau, z) = (\Psi_{1,1}(\tau, z))^{-1} \left((\mathrm{Tr}_{\mathbf{24}} g)\, \mu_{3;0}(\tau, z) + \sum_{r \in \mathbb{Z}/6\mathbb{Z}} h_{g,r}(\tau) \theta_{3,r}(\tau, z) \right),$$

and the coefficients of the twined h

$$h_{g,r}(\tau) = a_r q^{-r^2/12} + \sum_{n=1}^{\infty} (\mathrm{Tr}_{V^G_{r,n}} g)\, q^{n - r^2/12}$$

will naturally be given by characters of the M22 module.

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We checked this explicitly for the first 30 or so coefficients for all conjugacy classes of M22.

The functions $h_g = (h_{g,r})$ have interesting mock modular properties as well.

We can define the Hecke congruence subgroups as

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), n|c \right\}.$$

Then we find that Z_g is a weak Jacobi form of weight zero and index two for the group $\Gamma_0(o_g)$, and h_g are vector-valued mock modular forms of weight 1/2 with shadow $\mathrm{Tr}_{\mathbf{24}}(g)S_3$.

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V. Sketch of the results for N=2 and a special property

Precisely analogous considerations hold in the other two cases of interest here.

In the N=2 case, there is a decomposition:

$$
\begin{aligned}
Z(\tau,z) &= 23\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},2} + \Big(770\,(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},-1}) + 13915\,(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},-1}) + \dots\Big) \\
&\quad + \Big(231\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},2} + 5796\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},2} + \dots\Big) \qquad (7.11) \\
&= \mathrm{e}(\tfrac{3}{4})\Psi^{-1}_{1,-\frac{1}{2}}\Big(24\,\tilde{\mu}_{\frac{3}{2};0} + (-q^{-\frac{1}{24}} + 770\,q^{\frac{23}{24}} + 13915\,q^{\frac{47}{24}} + \dots)\,(\theta_{\frac{3}{2},\frac{1}{2}} + \theta_{\frac{3}{2},-\frac{1}{2}}) \\
&\quad + (q^{-\frac{3}{8}} + 231\,q^{\frac{5}{8}} + 57962 q^{\frac{13}{8}} + \dots)\,\theta_{\frac{3}{2},\frac{3}{2}}\Big) \qquad (7.12)
\end{aligned}
$$

One recognizes irreps of M23 immediately.

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Again, one finds a pair of vector-valued mock modular forms, and finds twinings governed by mock modular forms under suitable congruence subgroups of SL(2,Z).

In both the M22 and M23 cases, there is an important additional property that the special functions which arise here satisfy.

First, let us recall the celebrated genus zero condition of monstrous moonshine:

"The MT series are Hauptmoduln with only a polar term at the cusp representative of i-infinity, and no poles at any other cusps."

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In Umbral moonshine, there is also a special condition governing the modular forms which arise:

"(i) $q^{1/4m} H^X_{g,r}(\tau) = O(1)$ as $\tau \to i\infty$ for all r,

(ii) $H^X_{g,r}(\tau) = O(1)$ for all r as $\tau \to \alpha \in \mathbb{Q}$, whenever $\infty \notin \Gamma^X_g \alpha$."

Following the idea of Duncan and Frenkel, it is then perhaps the case that the unifying property of the twining functions in moonshine, is the constructibility of the functions as Rademacher sums starting with a single polar piece.

Are there any analogous special properties for the functions arising in the M22 and M23 mock modular moonshine we've constructed here?

It turns out that among all of the possible symmetry groups stabilizing 2 or 3-planes in the 24, the M22 and M23 functions are unique in satisfying:

(i) $q^{1/6} h_{g,r}(\tau) = O(1)$ as $\tau \to i\infty$ for all r,

(ii) $h_{g,r}(\tau) = O(1)$ for all r as $\tau \to \alpha \in \mathbb{Q}$, whenever $\infty \notin \Gamma(\mathrm{o}_g)\alpha$,

(i) $q^{3/8} \tilde{h}_{g,j}(\tau) = O(1)$ as $\tau \to i\infty$ for all j,

(ii) $\tilde{h}_{g,j}(\tau) = O(1)$ for all j as $\tau \to \alpha \in \mathbb{Q}$, whenever $\infty \notin \Gamma(\mathrm{o}_g)\alpha$,

for all g in M22, M23 respectively. (Other groups fail on ii).

VI. M24 and other groups

Finally, we come to the case of the superalgebra relevant for compactification on manifolds of Spin(7) holonomy, which I'll call (by abuse) the "Spin(7) algebra." It was connected to manifolds of Spin(7) holonomy by Shatashvili and Vafa in 1994.

It is a sort of odd algebra, adjoining to the spin-2 stress tensor and the normal supercharge G, a second spin-2 field and a spin-5/2 fermionic generator:

$$L_n, G_n, X_n, M_n$$

The full set of commutation relations are:

$$[L_n, L_m] = (n-m)L_{n+m} + (n^3 - n)\delta_{m+n}$$

$$[X_n, X_m] = \tfrac{8}{3}(n^3 - n)\delta_{n+m} + 8(n-m)X_{n+m}$$

$$[L_n, X_m] = \tfrac{1}{3}(n^3 - n)\delta_{n+m} + (n-m)X_{n+m}$$

$$[G_n, X_m] = \tfrac{1}{2}\left(n + \tfrac{1}{2}\right)G_{n+m} + M_{n+m}$$

$$[X_n, M_m] = \left(\tfrac{15}{4}(n+1)\left(m + \tfrac{3}{2}\right) - \tfrac{5}{4}\left(n + m + \tfrac{3}{2}\right)\left(n + m + \tfrac{5}{2}\right)\right)G_{n+m} - \left(-8(n+1) + \tfrac{11}{2}\left(n + m + \tfrac{5}{2}\right)\right)M_{n+m} - 6{:}GX{:}_{n+m}$$

$$[L_n, G_m] = \left(\tfrac{1}{2}n - m\right)G_{n+m}$$

$$[L_n, M_m] = \tfrac{1}{4}n(n+1)G_{n+m} + \left(\tfrac{3}{2}n - m\right)M_{n+m}$$

$$\{G_n, M_m\} = \tfrac{2}{3}\left(n^2 - \tfrac{1}{4}\right)\left(n - \tfrac{3}{2}\right)\delta_{n+m} - \left(n + \tfrac{1}{2}\right)L_{n+m} + (3n - m)X_{n+m}$$

$$\{M_n, M_m\} = -\tfrac{8}{3}\left(n^2 - \tfrac{9}{4}\right)\left(n^2 - \tfrac{1}{4}\right)\delta_{n+m} + \Big(\tfrac{15}{2}\left(m + \tfrac{3}{2}\right)\left(n + \tfrac{3}{2}\right) - \tfrac{5}{2}(n+m+2)(n+m+3)\Big)L_{n+m} + \Big(16\left(m + \tfrac{3}{2}\right)\left(n + \tfrac{3}{2}\right) - \tfrac{5}{2}(n+m+2)(n+m+3)\Big)X_{n+m} - 12{:}LX{:}_{n+m} + 6{:}GM{:}_{n+m}$$

$$\{G_n, G_m\} = (4n^2 - 1)\delta_{n+m} + 2L_{n+m}.$$

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The NS sector and R sector characters were derived in our recent paper. There are three massless representations and two continuous families of massive representations.

The NS partition function can be expanded as:

$$Z_{NS} = a_0 \tilde{\chi}_0^{NS} + a_{\frac{1}{16}} \tilde{\chi}_{\frac{1}{16}}^{NS} + a_{\frac{1}{2}} \tilde{\chi}_{\frac{1}{2}}^{NS} + \sum_{n=1}^{\infty} b_n \chi_{0,n}^{NS} + \sum_{n=1}^{\infty} c_n \chi_{\frac{1}{16},\frac{1}{2}+n}^{NS}$$

where:

$$b_1 = 253, b_2 = 7359, b_3 = 95128, \ldots$$

$$c_1 = 1771, c_2 = 35650, c_3 = 374141, \ldots .$$

M24 decompositions:

$$253 = \mathbf{253}, \quad 7359 = \mathbf{23} \oplus \mathbf{252} \oplus \mathbf{770} \oplus \overline{\mathbf{770}} \oplus \mathbf{5544}$$

$$95128 = \mathbf{253} \oplus \mathbf{990} \oplus \overline{\mathbf{990}} \oplus 2 \times \mathbf{1265} \oplus \mathbf{1771} \oplus \mathbf{2024} \oplus \mathbf{2277} \oplus \mathbf{3312}$$

$$\oplus 2 \times \mathbf{3520} \oplus 2 \times \mathbf{5313} \oplus \mathbf{5544} \oplus \mathbf{5796} \oplus 5 \times \mathbf{10395}, \ldots$$

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By analogy with our previous constructions, these can be collected to give the q-series for a 2 component vector-valued weight 1/2 mock modular form:

$$f_1(\tau) = q^{-\frac{1}{120}}(-1 + c_1 q + c_2 q^2 + c_3 q^3 + \ldots)$$
$$= q^{-\frac{1}{120}}(-1 + 1771q + 35650q^2 + 374141q^3 + \ldots)$$

$$f_7(\tau) = q^{-\frac{49}{120}}(1 + b_1 q + b_2 q^2 + b_3 q^3 + \ldots),$$
$$= q^{-\frac{49}{120}}(1 + 253q + 7359q^2 + 95128q^3 + \ldots).$$

with shadow 24 S:

$S_\alpha(\tau) = \sum_{k\in\mathbb{Z}} k\epsilon^R_\alpha(k) q^{k^2/120}$ for $\alpha = 1, 7$ and

$$\epsilon^R_1(k) = \begin{cases} 1 & k = 1, 29 \pmod{60} \\ -1 & k = -11, -19 \pmod{60} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon^R_7(k) = \begin{cases} 1 & k = -7, -23 \pmod{60} \\ -1 & k = 17, 13 \pmod{60} \\ 0 & \text{otherwise} \end{cases} .$$

In this case (as well as the N=2, 4 cases) there are also groups besides the Mathieu group which can appear. Unlike the previous cases, here we do not know of a special characteristic of the twining functions that distinguishes M24 from the other groups.

CONCLUSIONS

* We can give explicitly realized modules underlying moonshine relating mock modular forms to sporadic simple groups.

* The geometry of k-planes in the 24 determines both the extended superalgebra, and the subgroups of the Conway group that appear.

* Via the relation between 2d superalgebras and manifolds of special holonomy, it is tempting to think that these symmetries (broken to smaller subgroups) underlie the geometry of some hyperKahler, Calabi-Yau, or Spin(7) manifolds. But this is difficult to make precise.

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