Title: Symplectic automorphisms of some hyperk\"ahler manifolds

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Abstract:

Symplectic automorphisms of some hyperkähler manifolds

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Mukai's Theorem

$\Omega = 24$ letters permuted by $M_{24} \supseteq M_{23}$.

$\mathfrak{X} = \{G \subseteq M_{23} \mid G \text{ has} \geq 5 \text{ orbits on } \Omega\}$.

Theorem. (Mukai)
• A finite group of symplectic automorphisms of a $K3$ surface is isomorphic to a group in $\mathfrak{X}$.

• If $G \in \mathfrak{X}$, there is a $K3$ surface X such that G is isomorphic to a group of symplectic automorphisms of X.

$K3^{[n]}$

In this talk, we discuss an extension of Mukai's Theorem to certain *higher dimensional* CY manifolds.

Let X be a $K3$ surface. We can construct (Grothendieck) a complex manifold, informally defined as follows:

$$K3^{[n]} := \left\{ \underbrace{X \times \ldots \times X}_{n}, \text{blow-up diagonal} \right\} / S_n$$

This is the Hilbert scheme of n points of X. It is a CY, hyperkähler (holonomy in Sp_n) complex manifold of complex dimension 2n.

$K3^{[2]}$

There is a good chance that an analog of Mukai's Theorem can be proved for *all* $K3^{[n]}$.

Here we deal with the case $n = 2$.

The geometry of $K3^{[2]}$ is tied to that of $K3$, but is more complicated......

$K3$ Hodge diamond

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

$K3^{[2]}$ Hodge diamond

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 232 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Mukai-type Theorem for $K3^{[2]}$

$\mathfrak{X}' = \{G \subseteq M_{23} \mid G \text{ has exactly 4 orbits on } \Omega\}$.

$\Lambda =$ Leech lattice. $Aut(\Lambda) = Co_0$.

Main Theorem. (Höhn-GM)

- A finite group of symplectic automorphisms of a $K3^{[2]}$ is isomorphic to a subgroup of one of 15 iso classes:

  (a) a group in $\mathfrak{X}'$

  (b) one of two $\mathcal{S}$-lattice groups in Co_0 isomorphic to $3^{1+4}.2.2^2$ and $3^4.A_6$.

- If G is as in (a) or (b), there is a $K3^{[2]}$ X' such that G is isomorphic to a group of symplectic automorphisms of X'.

In addition to Mukai's Theorem, another related result due to Gaberdiel-Hohenegger-Volpato:

Theorem (G-H-V) The group G of symmetries of a non-linear σ-model on $K3$ preserving the $N = (4, 4)$ superconformal algebra and the spectral flow operators satisfies
(i) $G = G'.G''$; $G' \subseteq 2^{11}, M_{24} \supseteq G'' \geq 4$ orbits on Ω
(ii) $G = 5^{1+2}.4$
(iii) $G = 3^4.A_6$
(iv) $G = 3^{1+4}.Z_2.G''$, $G'' = 1, 2, 2^2, 4$

The idea in both the Main Theorem and G-H-V is to establish that $G \subseteq Co_0$ with $rk(\Lambda^G) \geq 4$, then classify such G.

Coordinate frames in Λ

A *coordinate frame* in Λ is a set of 24 pairs of *mutually orthogonal* vectors $\{\pm v_i\}$ of squared-length 8:

$$(v_i, v_j) = \pm 8\delta_{ij}.$$

Co_0 acts *transitively* on such sets, with stabilizer

$$2^{12}.M_{24}.$$

Proposition. (Conway) *Exactly one* of the following holds for each coset $v + 2\Lambda \in \Lambda/2\Lambda$.

$v + 2\Lambda = 2\Lambda$,

$v + 2\Lambda = w + 2\Lambda$, $\quad (w, w) = 4, \pm w$ unique,

$v + 2\Lambda = w + 2\Lambda$, $\quad (w, w) = 6, \pm w$ unique,

$v + 2\Lambda$ contains a *unique* coordinate frame.

$\mathcal{S}$-lattices

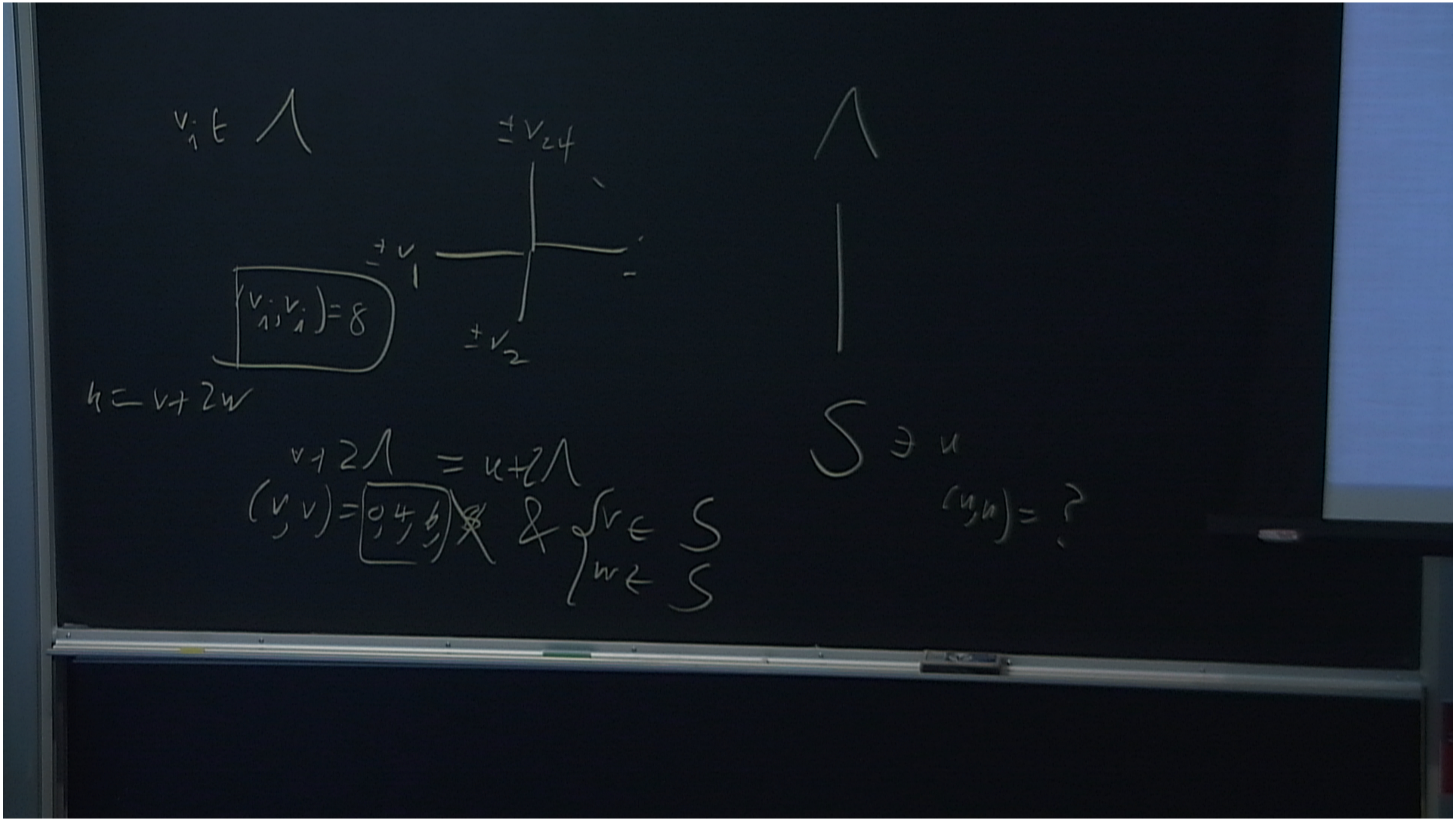
An $\mathcal{S}$-lattice is a sublattice $S \subseteq \Lambda$ satisfying

$$\begin{aligned} &\text{if} \quad v \in S, v = 2u + w,\ u, w \in \Lambda,\ (w, w) \leq 8 \\ &\text{then } (w, w) \leq 6,\ u, w \in S. \end{aligned}$$

Example. Suppose $G \subseteq Co_0$ has no subgroups of index 2. One of the following holds:

- $G \subseteq 2^{12}.M_{24}$
- Λ^G is an $\mathcal{S}$-lattice

This is an easy consequence of Conway's Proposition.



v_i ∈ Λ
±v_24
±v_1
±v_2
(v_i, v_i) = 8
u = v + 2w
v + 2Λ = u + 2Λ
(v, v) = 0, 4, 6, 8 & v ∈ S
w ∈ S
Λ
S ∋ u
(u, u) = ?

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• If G is as in (a) or (b), there is a $K3^{[2]}$ X' such that G is isomorphic to a group of symplectic automorphisms of X'.

Curtis's Theorem

Curtis has *classified* the maximal $\mathcal{S}$-lattices. They are as follows.

rank	stabilizer
4	$3^4.A_6$
4	$5^{1+2}.4$
6	$3^{1+4}.2$

$K3^{[2]}$ and Λ

X is a $K3^{[2]}$ with *symplectic* automorphism group $Aut(X)$.

$$L := H^2(X, \mathbb{Z}) \cong 2E_8(-1) \oplus 3U \oplus \langle -2 \rangle$$

with respect to the *Bogomolov-Beauville* integral bilinear form.

Theorem. (Mongardi-Beauville-Hassett-Tschinkel) There is an embedding

$$Aut(X) \to O(L).$$

$K3^{[2]}$ and Λ

Theorem (Mongardi, G-H-V, Huybrechts-Höhn-GM)
There is a commuting diagram

$$\begin{array}{ccc} G & \hookrightarrow & L_G(-1) \\ \downarrow & & \downarrow \\ Co_0 & \hookrightarrow & \Lambda \end{array}$$

In other words, there is an *identification* $G \subseteq Co_0$ such that

$$(L_G(-1), G) \xrightarrow{\cong} (\Lambda_G, G)$$

In particular $rk(\Lambda^G) \geq 4$.

Summary so far

For finite group of symplectic automorphisms $G \subseteq Aut(X)$ of a $K3^{[2]}$ X, we have

- $G \subseteq Co_0$
- $rk(\Lambda^G) \geq 4$
- $G \subseteq 2^{12}.M_{24}$ or $\Lambda^G \subseteq \mathcal{S}-$ lattice

Conjugacy classes and some equivariant topology

The next step is to consider the possible conjugacy classes of Co_0 that could possibly meet G.

Co_0 has 167 conjugacy classes.

Only 42 of them satisfy $rk(\Lambda^g) \geq 4$.

Assume that $g \in Aut(X)$ 'corresponds' to $g \in Co_0$.

The equivariant Hirzebruch χ_y-genus of X is:

$$\chi_y(g|X) := \sum_{p,q=0}^{4} (-1)^q Tr(g|H^{p,q}(X))y^p$$

(This is Dolbeault cohomology, the analog of de Rham cohomology for cmpx mnflds.)

$\chi_y(g|X)$ is determined by the eigenvalues of g on $H^{1,1}(X)$:

$$\chi_y(g|X) := 3 - 2ty + \frac{6 + t^2 + s}{2} y^2 - 2ty^3 + 3y^4$$

$$t := Tr(g|H^{1,1}(X))$$

$$s := Tr(g^2|H^{1,1}(X))$$

These we know explicitly for each g (they are in $\mathbb{Z}$).

Equivariant topology

On the other hand, the Atiyah-Singer-Segal fixed-point Theorem determines $\chi_y(g|X)$ *locally* in terms of data associated with the connected components of

$$X^g := \{x \in X | g.x = x\}.$$

At an *isolated fixed-point*, g acts on the tangent space $T_x = \mathbb{C}^4$ with eigenvalues $\lambda_x, \lambda_x^{-1}, \mu_x, \mu_x^{-1}$.

Otherwise, we consider the eigenvalues occurring in the *normal bundle*.

Usually X^g is *finite*. The only possible connected components are $K3$ or T^2 (complex 2-torus).

Equivariant topology

ASS gives a formula of the type

$$\chi_y(g|X) = \sum_{\{conn \ cmpnts\}} (\text{local data determined by evs})$$

There are also *consistency checks* e.g., the identities have to holds for all powers g^n:

$$\chi_y(g^n|X) = \sum_{\{conn \ cmpnts\}} (\text{local data determined by evs})$$

The bottom line is that for each of the 42 classes we get some equations, sometimes a large number (several hundred) that have to be satisfied.

Theorem (Höhn-GM) Of the 42 classes with $rk(\Lambda^g) \geq 4$, only 15 'admissible' classes can possibly act symplectically on $K3^{[2]}$.

Camere and Mongardi had done some of these cases before.

This Theorem is *sharp*, because all of the remaining 15 classes *really do* occur as symplectic automorphisms of $K3^{[2]}$.

11 of the classes meet $2^{12}{:}M_{24}$; 4 of them do not, but fix an $\mathcal{S}$-lattice pointwise.

Some group theory

Now use group theory to classify the possible subgroups $G \subseteq Co_0$ *maximal* subject to $rk(\Lambda^G) \geq 4$ and G meets only the 15 'admissible' classes.

For the possibilities $|G| = 2^k, 3.2^k$ the computer is essential. (There are literally millions of conjugacy classes of such subgroups in Co_0.) We obtain

Theorem(Höhn-GM) There are exactly 22 classes of such G.
- thirteen subgroups $G \subseteq M_{23}$ with 4 orbits,
- two groups $3^4.A_6$ and $3^{1+4} : 2.2^2$ related to S-lattices,
- two groups of order 48 and five 2-groups.

Existence

There are necessary and sufficient conditions, in terms of the structure of the lattice L_G, for G to act symplectically on a $K3^{[2]}$. Recall that

$$L = H^2(X, \mathbb{Z}) \cong 2E_8(-1) \oplus 3U \oplus \langle -2 \rangle$$

Theorem (Mongardi-Höhn-GM) Let $G \subseteq Co_0$ be admissible. Then G is a group of symplectic automorphisms of a $K3^{[2]}$ with $(\Lambda_G(-1), G)) \cong (L_G, G)$ if, and only if,

$$\alpha(L_G) := 24 - rk(L_G) - rk(L_G^*/L_G) \geq 1$$

This criterion can be applied to the 22 choices of G. It reduces us to the final list.

We need existence of X for the remaining 15 classes of G.

Several were already known to act on $K3^{[2]}$. Some of our examples are new. E.g., with $\zeta = e^{2\pi i/24}$,

$$
\begin{aligned}
&f := (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3) \\
&+\frac{1}{5}(-3\zeta^7 - 3\zeta^5 + 3\zeta^4 - 3\zeta^3 + 6\zeta - 3) \times \\
&\{x_1x_2x_3 + x_1x_2x_4 + (\zeta^4 - 1)x_1x_2x_5 + x_1x_2x_6 + (\zeta^4 - 1)x_1x_3x_4 \\
&+x_1x_3x_5 + x_1x_3x_6 + (\zeta^4 - 1)x_1x_4x_5 - \zeta^4 x_1x_4x_6 - \zeta^4 x_1x_5x_6 \\
&+(\zeta^4 - 1)x_2x_3x_4 + (\zeta^4 - 1)x_2x_3x_5 - \zeta^4 x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 \\
&-\zeta^4 x_2x_5x_6 + x_3x_4x_5 - \zeta^4 x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6\}
\end{aligned}
$$

defines a smooth cubic in $\mathbb{CP}^5$ that is a $K3^{[2]}$ admitting M_{10}.

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& \{x_1x_2x_3 + x_1x_2x_4 + (\zeta^4 - 1)x_1x_2x_5 + x_1x_2x_6 + (\zeta^4 - 1)x_1x_3x_4 \\
& + x_1x_3x_5 + x_1x_3x_6 + (\zeta^4 - 1)x_1x_4x_5 - \zeta^4 x_1x_4x_6 - \zeta^4 x_1x_5x_6 \\
& + (\zeta^4 - 1)x_2x_3x_4 + (\zeta^4 - 1)x_2x_3x_5 - \zeta^4 x_2x_3x_6 + x_2x_4x_5 + x_2x_4x_6 \\
& - \zeta^4 x_2x_5x_6 + x_3x_4x_5 - \zeta^4 x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6\}
\end{aligned}
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Summary and prospects

From the perspective of symplectic automorphisms, there is a clear causal connection between $K3^{[2]}$ (and even $K3^{[n]}$), and Co_0. The connection with M_{24} is incidental.

It seems likely that all finite symplectic automorphism groups of $K3^{[n]}$ can be determined for all n.

The list of groups that will appear will almost certainly be very similar to the case $n = 2$ together with a couple more $\mathcal{S}$-lattice groups corresponding to the cases when $\alpha(L_G) \geq 1$.

The elliptic genus (in the sense of chiral de Rham complex) of a hyperkähler manifold of cmpx dim d is a vertex algebra with a pair of commuting $N = 4, c = 3d$ structures that depend on the hyperkähler metric.

$\boxed{\text{Thank you}}$