Title: Traces of Singular Moduli and Moonshine for the Thompson group

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Abstract: We observe a relationship between the representation theory of the Thompson sporadic group and a weakly holomorphic modular form of weight one-half that appears in Zagier's work on traces of singular moduli and Borcherds products. We conjecture the existence of an infinite dimensional graded module for the Thompson group and use the observed relationship to propose a McKay-Thompson series for each conjugacy class of the Thompson group and then construct weakly holomorphic weight one-half forms at higher level that coincide with the proposed McKay-Thompson series. We also observe a discriminant property in this conjectured moonshine for the Thompson group that is closely related to the discriminant property conjectured to exist in Umbral Moonshine.

Traces of Singular Moduli and Moonshine for the Thompson Group

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(to appear at the next new moon?)

Modularity, Moonshine String Theory

Perimeter Institute

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OUTLINE

1. Traces of Singular Moduli

2. Weakly holomorphic weight 1/2 forms

3. Moonshine for the sporadic Thompson group

   Modules

   McKay-Thompson series and Modularity

   Discriminant property and Singular Moduli

4. Discussion

Traces of Singular Moduli

At a meeting on moonshine I do not have to explain

$$J(\tau) = q^{-1} + 196884q + \cdots , \quad q = e^{2\pi i \tau}$$

But I should explain what a singular modulus is, and how you take its trace.

Let d be a positive integer equal to 0 or 3 mod 4.

Let $Q(X,Y) = [a,b,c] = aX^2 + bXY + cY^2 \qquad a,b,c \in \mathbb{Z}$

denote a binary quadratic form with integer coefficients and let $\mathcal{Q}_d$ denote the space of such forms with discriminant $b^2 - 4ac = -d$. The modular group $SL(2,\mathbb{Z})$ with elements $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ acts on such forms as

$$Q|_\gamma(X,Y) = Q(pX+qY, rX+sY)$$

and preserves the discriminant. This defines an equivalence relation on binary quadratic forms. Since d>0, each $Q \in \mathcal{Q}_d$ gives rise to a quadratic equation $Q(X,1)=0$ with a root α_Q in the upper half plane. A singular modulus is the value $J(\alpha_Q)$ and is an algebraic number depending only on the equivalence class of Q. The number of inequivalent $Q \in \mathcal{Q}_d$ is the class number $h(-d)$. A trace of singular moduli is the basically the sum of singular moduli over inequivalent quadratic forms of given discriminant with one small correction:

order of the stabilizer of Q,=1,2,3.

$$t(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{w_Q} J(\alpha_Q)$$

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We will also require a “twisted” version of such traces:

$$A(D,3) = \frac{1}{\sqrt{D}} \sum_{Q \in Q_{3D}/\Gamma} \chi_{D,-3}(Q) J(\alpha_Q)$$

Genus character taking values plus or minus 1

Examples: $$t(3) = \frac{J((1+i\sqrt{3})/2)}{3} = -248$$

$$D = 5: \;\; Q_1 = [1,1,4], \;\; Q_2 = [2,1,2], \;\; \alpha_1 = (1+i\sqrt{15})/2, \;\; \alpha_2 = (1+i\sqrt{15})/4$$

$$A(5,3) = \frac{J(\alpha_{Q_1}) - J(\alpha_{Q_2})}{\sqrt{5}} = -85995$$

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Weakly holomorphic weight 1/2 forms

Zagier showed these are coefficients of weight 1/2 and weight 3/2 weakly holomorphic modular forms. In particular, there is a weight 1/2 form f_3 that transforms under $\Gamma_0(4)$ like $\theta(\tau)$ and such that coefficients of q^n for n a square can be computed in terms of $t(3)$ and for n not a square can be computed in terms of the twisted" generalization.

$$f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \cdots \in M^!_{1/2}$$

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$$Th \leq E_8(3)$$

The Thompson group is a huge group, with ~ 9×10^{16} elements and 48 conjugacy classes. It occurs naturally as a subgroup of the Monster in that the stabilizer of the 3C class of the Monster is $\mathbb{Z}/3\mathbb{Z} \times Th$

N.B. for experts: The T3C MT series of Monstrous Moonshine exhibits moonshine for the Thompson group since Z3xThompson centralizes the 3C element, and T3C is the Borcherds lift of f3, but the lift only involves the coefficients of square powers of q and so does not explain what is going on.

Things work better for coefficients of square powers if we consider

$$\mathcal{F}_3(\tau) = 2f_3(\tau) + 248\theta(\tau) = \sum_m c(m)q^m$$

The Main Claim: $\mathcal{F}_3$ exhibits moonshine for the Thompson group and shares many features with Umbral Moonshine except that $\mathcal{F}_3$ is modular rather than mock modular and Th is much larger than any of the Umbral groups

The evidence:

1. Decompositions at low levels into irreps
2. Conjectural structure of Modules
3. Modularity of McKay-Thompson series
4. Rademacher series
5. A Discriminant property related to singular moduli.

At low orders there are natural decompositions into either real representations or pairs $V \oplus \overline{V}$

$c(k)$	Decomposition
$c(-3)$	$2 \cdot {}^{1}V_1$
$c(0)$	${}^{248}V_2$
$c(4)$	${}^{27000}V_4 \oplus {}^{27000}V_5$
$-c(5)$	${}^{85995}V_9 \oplus {}^{85995}V_{10}$
$c(8)$	${}^{1707264}V_{17} \oplus {}^{1707264}V_{18}$
$-c(9)$	${}^{4096000}V_{22} \oplus {}^{4096000}V_{23}$
$c(12)$	$2 \cdot {}^{44330496}V_{40}$
$-c(13)$	$2 \cdot {}^{91171899}V_{46} \oplus {}^{779247}V_{14} \oplus {}^{779247}V_{15}$

The alternating signs suggest a (super)-module

$$W = \bigoplus_{\substack{m\geq -3 \\ m\equiv 0,1 \bmod 4}}^{\infty} W_m$$

$$W_m = W_m^{(0)} \oplus W_m^{(1)}$$

even odd

$W_m^{(0)} = \emptyset$ m=1,5,9,13,17,...

$W_m^{(1)} = \emptyset$ m=-3,0,4,8,12,...

As in Umbral Moonshine the coefficient of the most singular term has the "wrong" sign

$$c(m) = str_{W_m} 1 \longrightarrow \mathcal{F}_{3,[g]}(\tau) = \sum_m str_{W_m}(g) q^m$$

Experience with Monstrous and Umbral moonshine suggests that each of these McKay-Thompson series should be a weakly holomorphic weight 1/2 modular form on $\Gamma_0(4h_g o(g))$ for some h_g dividing $2o(g)$.

We use two methods to construct such objects:

A map $Z'_{1,N,3} : M_0^{\natural}(\Gamma_0(N)) \to M^!_{1/2}(\Gamma_0(4N))$ for N such that $\Gamma_0(N)$ is genus 0 (Miller-Pixton).

Coefficients (with multipliers) of weight 1/2 Rademacher series (Rademacher, Knopp, Niebur, Duncan-Frenkel, Cheng-Duncan, Miller-Pixton, Bringmann-Ono, ….)

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In the first method we have a formula for the Fourier coefficients A_m of the image of a hauptmodul $f = q^{-1} + O(q)$ for $\Gamma_0(N)$

$[g]$	$\Gamma_{[g]}$	$T_{[g]}$
$2B$	$\Gamma_0(2)$	$\eta(\tau)^{24}/\eta(2\tau)^{24} + 24$
$3B$	$\Gamma_0(3)$	$\eta(\tau)^{12}/\eta(3\tau)^{12} + 12$
$4C$	$\Gamma_0(4)$	$\eta(\tau)^{8}/\eta(4\tau)^{8} + 8$
$5B$	$\Gamma_0(5)$	$\eta(\tau)^{6}/\eta(5\tau)^{6} + 6$
$6E$	$\Gamma_0(6)$	$\eta(\tau)^{5}\eta(3\tau)/\eta(2\tau)\eta(6\tau)^{5} + 5$
$7B$	$\Gamma_0(7)$	$\eta(\tau)^{4}/\eta(7\tau)^{4} + 4$
$8E$	$\Gamma_0(8)$	$\eta(\tau)^{4}\eta(4\tau)^{2}/\eta(2\tau)^{2}\eta(8\tau)^{4} + 4$
$9B$	$\Gamma_0(9)$	$\eta(\tau)^{3}/\eta(9\tau)^{3} + 3$
$10E$	$\Gamma_0(10)$	$\eta(\tau)^{3}\eta(5\tau)/\eta(2\tau)\eta(10\tau)^{3} + 3$
$12I$	$\Gamma_0(12)$	$\eta(\tau)^{3}\eta(4\tau)\eta(6\tau)^{2}/\eta(2\tau)^{2}\eta(3\tau)\eta(12\tau)^{3} + 3$
$13B$	$\Gamma_0(13)$	$\eta(\tau)^{2}/\eta(13\tau)^{2} + 2$
$16B$	$\Gamma_0(16)$	$\eta(\tau)^{2}\eta(8\tau)/\eta(2\tau)\eta(16\tau)^{2} + 2$
$18D$	$\Gamma_0(18)$	$\eta(\tau)^{2}\eta(6\tau)\eta(9\tau)/\eta(2\tau)\eta(3\tau)\eta(18\tau)^{2} + 2$
$(25Z)$	$\Gamma_0(25)$	$\eta(\tau)/\eta(25\tau) + 1$

$$A_m(Z'_{1,N,3}(f)) = \frac{1}{\sqrt{m}} \sum_{\substack{Q \in \mathcal{Q}_{3m}/\Gamma_0(N) \\ a = 0 mod N}} \frac{\chi_{-3}(Q) f(\alpha_Q)}{\omega_Q}$$

In the second method we compute (numerically) the coefficients of a weight 1/2 Rademacher series with multiplier

$$\psi = \underbrace{\left(\frac{c}{d}\right)\epsilon_d}_{\text{usual multiplier for weight 1/2}} e^{-2\pi i c d v / n_g h_g} \qquad \epsilon_d \equiv \begin{cases} 1 & d \equiv 1 \bmod 4 \\ i & d \equiv 3 \bmod 4 \end{cases}$$

Kronecker symbol

multiplier system familiar from Umbral Moonshine

with n_g the order of and choose (v, h_g) with $h_g | 2n_g$ to reproduce the MT series.

Kloosterman sum

$$c_{\Gamma_0(N),\psi,1/2}(-3,n) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N) / \Gamma_\infty} K_{\gamma,\psi}(-3,n) B_{\gamma,1/2}(-3,n)$$

$$B_{\gamma,1/2}(-3,n) = e^{-2\pi i/8} \frac{2\pi}{c} \left(\frac{3}{n}\right)^{1/4} I_{1/2}\left(\frac{4\pi}{c}\sqrt{3n}\right)$$

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In constructing this class of weight 1/2 forms we are also allowed to add sums of the form

$$\sum_{k,k^2|h_g n_g} c_k\,\theta(k^2\tau)$$

which only affects the coefficients of square powers of q but does not change the modular properties.

We win the moonshine game when we can exhibit modular MT series (i.e. multipliers and constants c_k) such that the there exists decompositions of modules into sums of irreducible representations with positive integer coefficients. We have done this to order q^{32} .

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Discriminant Property

One surprising (to me) aspect of this new kind of moonshine is the fact that it exhibits a discriminant property similar to that of Umbral Moonshine and the discriminant property is linked to the discriminants of quadratic forms used in the computation of the coefficients as traces of singular moduli.

Let me first explain the discriminant property of Umbral Moonshine for the case $X = A_1^{24}, \quad G^X = M_{24}$ at a glance.

Table 8: Character table of $G^{(2)} \simeq M_{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0
χ_3	∘	45	−3	5	0	3	−3	1	1	0	0	−1	b_7	$\overline{b_7}$	−1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	−1	−1
χ_4	∘	45	−3	5	0	3	−3	1	1	0	0	−1	$\overline{b_7}$	b_7	−1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	−1	−1
χ_5	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1
χ_8	+	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	−1	3	0	0	2	−1	−1	1	0	1	0	0	−1	−1	0	0	−1	−1	0	0
χ_{15}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2b_7$	$2\overline{b_7}$	−1	0	1	0	−1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2\overline{b_7}$	$2b_7$	−1	0	1	0	−1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	−15	5	8	−7	1	−3	0	1	0	−2	−2	1	0	0	−1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	−21	11	16	7	3	−5	−1	1	0	−1	0	0	−1	1	0	0	−1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	−1	8	8	0	0	−1	−1	0	1	1	0	−1	0	−1	0	1	1	−1	−1	1	1	0	0
χ_{20}	+	2277	21	−19	0	6	−3	1	−3	−3	0	2	2	2	−1	1	0	0	0	0	0	0	0	−1	−1	0	0
χ_{21}	+	3312	48	16	0	−6	0	0	0	−3	0	−2	1	1	0	1	1	0	0	−1	−1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	−8	0	0	0	0	−2	0	−1	−1	0	0	0	0	0	1	1	0	0	−1	−1	1	1
χ_{23}	+	5313	49	9	−15	0	1	−3	−3	3	1	0	0	0	−1	−1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	−56	24	9	0	−8	0	0	−1	1	0	0	0	0	−1	0	1	0	0	0	−1	−1	0	0	1	1
χ_{25}	+	5796	−28	36	−9	0	−4	4	0	1	−1	0	0	0	0	1	−1	−1	0	0	0	1	1	0	0	0	0
χ_{26}	+	10395	−21	−45	0	0	3	−1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	−1

$q^{n/8}$ $\mathbb{Q}[\sqrt{-7}]$ $\mathbb{Q}[\sqrt{-15}]$ $\mathbb{Q}[\sqrt{-23}]$

Table 48: Decomposition of $K_1^{(2)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
-1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	2	0	0	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10
79	0	0	0	0	2	2	0	2	2	0	0	2	2	2	2	2	4	4	4	6	6	8	12	10	10	24
87	0	0	0	0	0	0	0	0	0	4	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40
95	0	2	0	0	2	2	2	4	4	6	6	8	8	4	8	8	12	12	12	18	26	30	40	38	40	80
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	28	44	44	70	80	84	136
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320
159	0	14	20	20	126	126	128	138	254	390	390	516	516	520	536	536	676	904	1036	1196	1716	1836	2764	2846	2980	5408
167	2	20	40	40	182	182	214	200	396	652	652	814	814	872	860	860	1020	1476	1684	1862	2742	2902	4384	4622	4828	8572
175	2	32	55	55	314	314	328	346	640	988	988	1298	1298	1336	1348	1348	1686	2302	2630	3000	4324	4616	6950	7204	7532	13620
183	2	40	98	98	460	460	512	496	972	1590	1590	2020	2020	2144	2118	2118	2546	3638	4162	4624	6768	7166	10856	11376	11898	21204

$63 = 3^2 \times 7$ $135 = 3^2 \times 15$ $175 = 5^2 \times 7$

$q^{n/8}$ $\mathbb{Q}[\sqrt{-7}]$ $\mathbb{Q}[\sqrt{-15}]$ $\mathbb{Q}[\sqrt{-23}]$

Table 48: Decomposition of $K_1^{(2)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
-1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	2	0	0	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10
79	0	0	0	0	2	2	0	2	2	0	0	2	2	2	2	2	4	4	4	6	6	8	12	10	10	24
87	0	0	0	0	0	0	0	0	0	4	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40
95	0	2	0	0	2	2	2	4	4	6	6	8	8	4	8	8	12	12	12	18	26	30	40	38	40	80
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	28	44	44	70	80	84	136
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320
159	0	14	20	20	126	126	128	138	254	390	390	516	516	520	536	536	676	904	1036	1196	1716	1836	2764	2846	2980	5408
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$63 = 3^2 \times 7$ $135 = 3^2 \times 15$ $175 = 5^2 \times 7$

ℓ	n	(ϱ, ϱ^*)
2	$7, 15, 23$	$(\chi_3, \chi_4), (\chi_5, \chi_6), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13}), (\chi_{15}, \chi_{16})$
3	$5, 8, 11, 20$	$(\chi_4, \chi_5), (\chi_{16}, \chi_{17}), (\chi_{20}, \chi_{21}), (\chi_{22}, \chi_{23}), (\chi_{25}, \chi_{26})$
4	$3, 7$	$(\chi_2, \chi_3), (\chi_{13}, \chi_{14}), (\chi_{15}, \chi_{16})$
5	4	$(\chi_8, \chi_9), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13})$
7	3	$(\chi_2, \chi_3), (\chi_6, \chi_7)$
13	4	(χ_3, χ_4)

Table 7: The irreducible representations of type n.

Armed with the preceding discussion we are now ready to state our main observation for the discriminant property of umbral moonshine. For the purpose of stating this we temporarily write $K^{(\ell)}_{r,d}$ for the ordinary representation of $G^{(\ell)}$ with character $g \mapsto c^{(\ell)}_{g,r}(d)$ where the coefficients $c^{(\ell)}_{g,r}(d)$ are assumed to be those given in §C.

Proposition 5.10. *Let n be one of the integers in Table 7 and let λ_n be the smallest positive integer such that $D = -n\lambda_n^2$ is a discriminant of $H^{(\ell)}$. Then $K^{(\ell)}_{r,-D/4\ell} = \varrho_n \oplus \varrho_n^*$ where ϱ_n and ϱ_n^* are dual irreducible representations of type n. Conversely, if ϱ is an irreducible representation of type n and $-D$ is the smallest positive integer such that $K^{(\ell)}_{r,-D/4\ell}$ has ϱ as an irreducible constituent then there exists an integer λ such that $D = -n\lambda^2$.*

Conjecture 5.11. *If D is a discriminant of $H^{(\ell)}$ which satisfies $D = -n\lambda^2$ for some integer λ then the representation $K^{(\ell)}_{r,-D/4\ell}$ has at least one dual pair of irreducible representations of type n arising as irreducible constituents.*

Conjecture 5.12. *For $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ the representation $K^{(\ell)}_{r,-D/4\ell}$ is a doublet if and only if $D \neq -n\lambda^2$ for any integer λ for any n satisfying the conditions of Proposition 5.7.*

To see some evidence for Conjecture 5.12 one can inspect the proposed decompositions of the representations $K^{(\ell)}_{r,d}$ in the tables in §D for the following discriminants:

- $-D = 7, 15, 23, 63, 135, 175, 207$ for $\ell = 2$,

For $\ell = 2, X = A_1^{24}$ the discriminant property was proved by Creutzig, Hohn and Miezaki

The Discriminant property for Thompson moonshine at a glance.

First, keep in mind that the coefficient of q^n in $\mathcal{F}_3$ can be computed in terms of traces of singular moduli involving quadratic forms of discriminant $3n$

$\sqrt{-3\times 5} = i\sqrt{15}$ $\sqrt{-3\times 20} = 2i\sqrt{15}$ $\sqrt{-3\times 4} = 2i\sqrt{3}$ $\sqrt{-3\times 9} = 3i\sqrt{3}$

Table 14: Decompositions of $W_m^{(0)}, W_m^{(1)}$, part one. The representations appearing in the discriminant conjecture are in bold font.

$\sqrt{-3\times 8} = 2i\sqrt{6}$ $\sqrt{-3\times 32} = 4i\sqrt{6}$

q^n ↓	V_1	V_2	V_3	$\mathbf{V_4}$	$\mathbf{V_5}$	V_6	V_7	V_8	$\mathbf{V_9}$	$\mathbf{V_{10}}$	V_{11}	V_{12}	V_{13}	$\mathbf{V_{14}}$	$\mathbf{V_{15}}$	V_{16}	$\mathbf{V_{17}}$	$\mathbf{V_{18}}$	V_{19}	V_{20}	V_{21}	$\mathbf{V_{22}}$	$\mathbf{V_{23}}$	V_{24}
-3	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	2	0	2	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2
24	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	4	4	4	2	6	4	8	8	8
25	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	4	4	8	8	10	12	12	12
28	0	0	0	2	2	0	2	2	0	0	2	14	14	8	8	10	18	18	38	28	48	54	54	56
29	0	0	2	0	0	0	2	2	2	2	2	18	18	18	18	20	40	40	54	58	74	86	86	86
32	0	0	0	0	0	4	0	4	10	10	10	62	62	78	78	92	173	173	208	256	296	368	368	368

plus one more page to give decompositions into all irreps

The discriminant property for Thompson moonshine in detail

For each m>0 equal to 0 or 1 mod 4 we have a unique decomposition $-3m = D_0(m)\lambda^2$ with $D_0(m)$ a negative fundamental discriminant.

Proposition 4.3. *If $D_0(m)$ is a negative fundamental discriminant satisfying*

1. *there exists an element of Th of order $|D_0(m)|$, and*
2. *there exists a positive integer λ such that $-3m = D_0(m)\lambda^2$ is a discriminant of $\mathcal{F}_3$ and $(\lambda, 3) = 1$,*

then there exists at least one pair of irreducible representations V and $\overline{V}$ of Th and at least one element $g \in Th$ such that $\mathrm{tr}_V(g)$ is not rational but

$$\mathrm{tr}_V(g), \mathrm{tr}_{\overline{V}}(g) \in \mathbb{Q}\left[\sqrt{D_0(m)}\right] \tag{4.4}$$

and $|D_0(m)|$ divides $o(g)$.

Proposition 4.4. *Let $D_0(m)$ be one of the fundamental discriminants satisfying the two conditions of Proposition 4.3 and let λ_m be the smallest positive integer such that $-3m = D_0(m)\lambda_m^2$ is a discriminant of $\mathcal{F}_3$. Then $W_m = V \otimes \overline{V}$ where V and $\overline{V}$ are dual irreducible representations of type $D_0(m)$.*

Remark 4.1. *Since the Schur index of all irreducible representations of Th is one [20], it follows that Th representations of type D_0 can be realized over $\mathbb{Q}[\sqrt{D_0}]$.*

There are analogs of the two discriminant conjectures I quoted for Umbral Moonshine, but they appear to be slightly more subtle here and we are still trying to formulate them carefully.

Proposition 4.4. *Let $D_0(m)$ be one of the fundamental discriminants satisfying the two conditions of Proposition 4.3 and let λ_m be the smallest positive integer such that $-3m = D_0(m)\lambda_m^2$ is a discriminant of $\mathcal{F}_3$. Then $W_m = V \otimes \overline{V}$ where V and $\overline{V}$ are dual irreducible representations of type $D_0(m)$.*

Remark 4.1. *Since the Schur index of all irreducible representations of Th is one [20], it follows that Th representations of type D_0 can be realized over $\mathbb{Q}[\sqrt{D_0}]$.*

There are analogs of the two discriminant conjectures I quoted for Umbral Moonshine, but they appear to be slightly more subtle here and we are still trying to formulate them carefully.

Discussion

The evidence for moonshine for the Thompson group is at the same level as evidence for the various cases of Umbral Moonshine. In particular we have modularity of all MT series, and decompositions with positive integer multiplicities.

This moonshine shares some of the properties of Umbral Moonshine, including the module structure and discriminant property, but the group is much larger and not connected to the Niemeier lattices.

There is however a rank 248 even self-dual lattice with Th (times Z_2) as automorphism group.

Where does this fit into the known moonshine structures?

Monstrous Moonshine and offspring
(Baby Monster, modular moonshine
of Borcherds, Ryba, Griess, Lam …)

Conway moonshine and c=12 relatives
preserving other superconformal algebras

Mathieu moonshine and its extension
to Umbral Moonshine?

I don't know. It seems to be distinct from all of these. It may indicate that the universe of moonshine is larger than we currently think.

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