Title: Causality constraints and the lightcone

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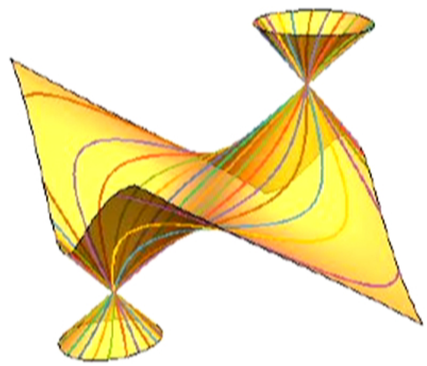
Abstract: It is an attractive idea that effective theories admitting a consistent UV completion require quanta to propagate sub-luminally in non-trivial backgrounds.
However, there is a counter example to this proposition in the form of QED in a curved geometry, a theory that is certainly causal. Nevertheless, Drummond and Hathrell showed that there is always at least one choice of polarization for which low frequency photons propagate super-luminally. Conventional arguments involving dispersion relations would then normally imply that the high frequency phase velocity would also exceed c yielding a contradiction with the UV completion. We show how the contradiction is avoided by a mechanism that relies on the subtle behaviour of the lightcones in the geometry and that, in the end, super-luminal low frequency propagation is perfectly consistent with causality. In particular, time machines cannot be constructed using the effect. The lesson is that causality constraints in low energy effective theories need to be treated with some caution.

Causality constraints and the lightcone

Tim Hollowood (with Graham Shore)

Swansea University — Prifysgol Abertawe



Perimeter Institute
April 2015

Shedding some light ...

- Is superluminal propagation in some effective theory in a non-trivial background pathological?
- Does it rule out a consistent UV completion?

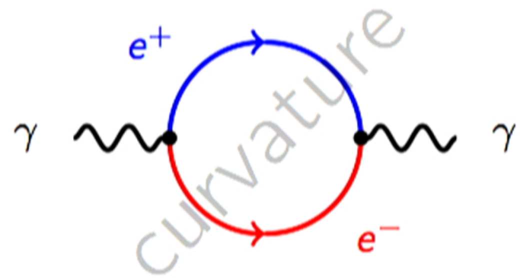
- The answer cannot be YES always because there is a counter example: QED in curved spacetime which is definitely UV complete and causal (at least in P.T.)

Question for today: how does it manage to have superluminal propagation in the effective theory and still be perfectly causal and UV complete?

- This is a causality "violation" story with a nice clean ending [TJH,Shore: to appear]


is a counter example: QED in
UV complete and causal (at least in P.T.)
superluminal propagation in
causal and UV complete?
story with a nice clean ending

Vacuum polarization in curved space



$$\Pi_{\mu\nu}(x,x') = e^2 \,\mathrm{Tr}\left[\gamma_\mu G_{e^+}(x,x')\gamma_\nu G_{e^-}(x,x')\right]$$

Vacuum polarisation generates terms like

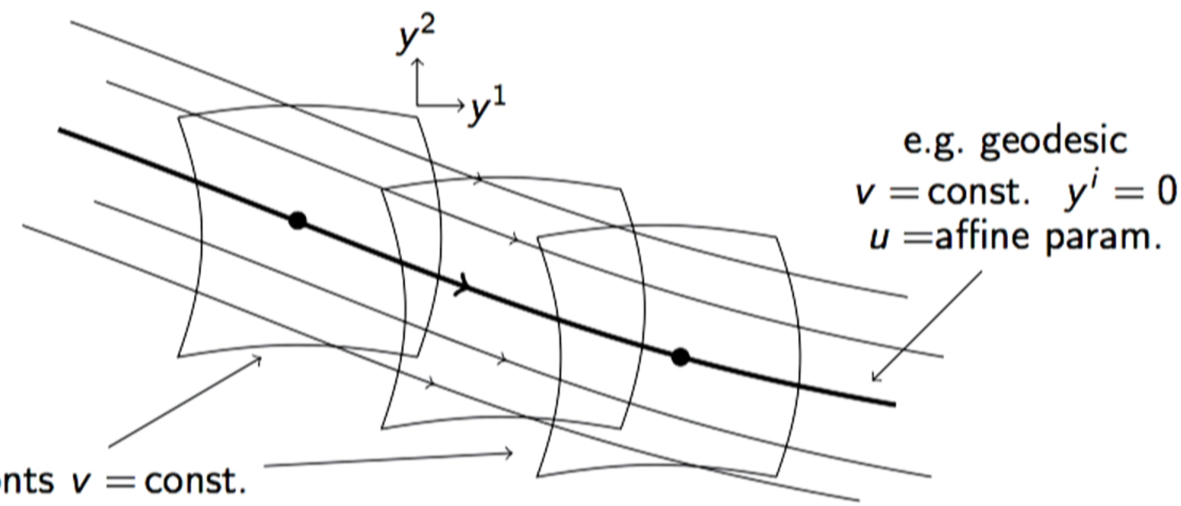
$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{m^{a+2b}}\,\partial^a \cdot R^b \cdot F^2 + \cdots$$

The Drummond Hathrell effect

- In 1980 **Drummond and Hathrell** calculated the first terms

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{13\alpha}{360\pi m^2}R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{\alpha}{360\pi m^2}R_{\mu\nu\lambda\sigma}F^{\mu\nu}F^{\lambda\sigma} + \cdots$$

- Photon propagation before corrections: **geometric optics limit** $\omega \gg \sqrt{R}$; adapted coordinates (u, v, y^i) $\boxed{A_\mu \sim e^{-i\omega v}}$



- Focus on one particular ray $v = \text{const.}$ $y^i = 0$ and u varying
- Now add in the (perturbatively small) correction from $\partial^a \cdot R^b \cdot F^2$ terms

$$A_\mu \sim e^{-i\omega v} e^{i\omega\theta(u)} \qquad \theta(u) = \int^u du' \left(n(u';\omega) - 1\right)$$

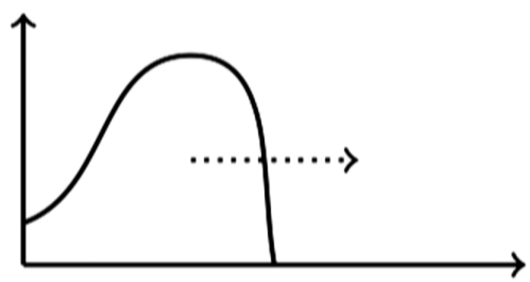
So $n(\omega) = c/v_{\rm ph}(\omega)$ is the instantaneous refractive index along our particular geodesic (depends on affine parameter u)

DH terms give $\qquad n(\omega \to 0) = \delta_{ij} - \dfrac{\alpha}{180\pi m^2}\left[13R_{uu}\delta_{ij} - 4R_{uiuj}\right]$

- At least one polarization is superluminal by null energy condition

So what!

- Causality arguments should be couched in terms of sharp wavefronts which need modes of high frequency (how high?)
- Therefore it is the high frequency behaviour of $n(\omega)$ that is relevant: wavefront velocity $= c/\operatorname{Re} n(\infty)$



Cannot build time machines from low frequency superluminal modes ... need large boosts and/or large curvatures along with sharp wave fronts

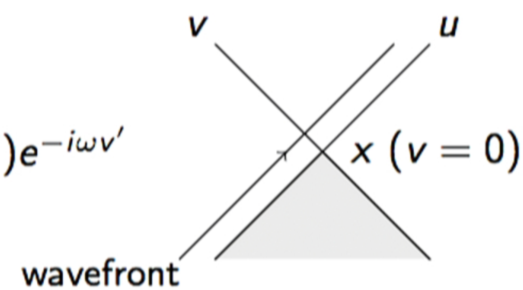
But there is hope ...

We can use low frequency superluminal propagation in a causality argument by using dispersion relations to relate the IR to UV behaviour

Refractive index as a Green function

- Think of $n(x;\omega)$ as the Fourier transform of the response function (**retarded Green function**) $\Pi(x,x')$ of the field at some point x to the signal $A_\mu(x') \sim e^{-i\omega v'}$:

$$n(x;\omega) = 1 + \frac{1}{2\omega}\int d^4x'\,\Pi(x,x')e^{-i\omega v'}$$



- UV completion is QED so can expect $\Pi(x,x')$ to be causal
  - So wavefronts with $v' > 0$ have no causal influence on x at $v = 0$
  - Implies that $n(\omega)$ will be analytic in the UHP

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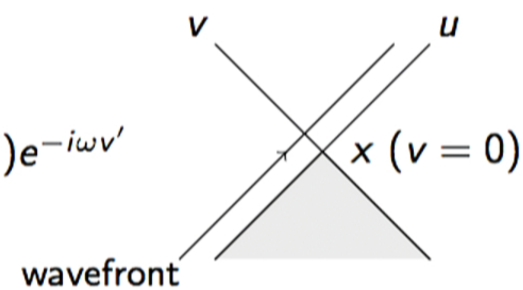

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- If we assume a consistent (perturbative) UV completion then we would expect:

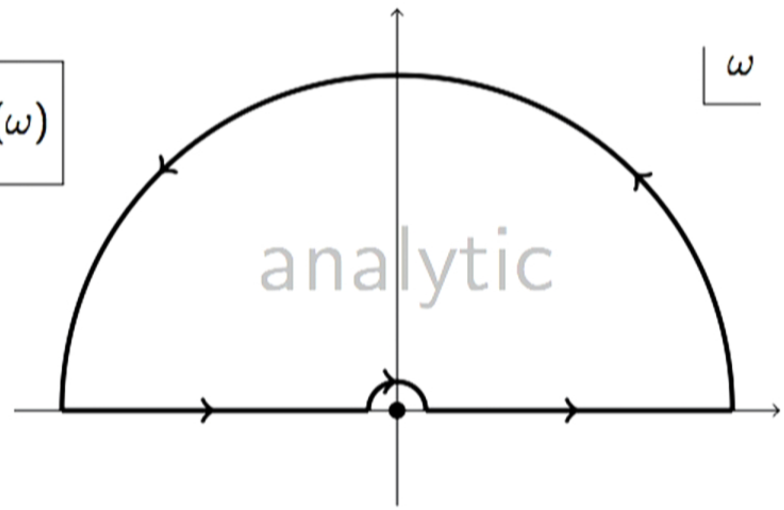
1. $n(\omega)$ is analytic in UHP
2. **Hermiticity:** real space Green function $\Pi(x, x')$ is real so

$$\boxed{n(\omega)^* = n(-\omega) \qquad \omega \in \mathbb{R}}$$

3. High frequency propagation has speed $c = 1$, $\mathrm{Re}\, n(\infty) = 1$
4. **Optical theorem:** propagation is dispersive $\mathrm{Im}\, n(\omega) \geq 0$

The Kramers Kronig dispersion relation

$$\boxed{\oint_{\mathcal{C}} \frac{d\omega}{\omega}\, n(\omega)}$$



$$\int_{-\infty}^{-\epsilon} \frac{d\omega}{\omega}\, n(\omega) - i\pi\, \mathrm{Res}_0 \frac{n(\omega)}{\omega} + \int_{\epsilon}^{\infty} \frac{d\omega}{\omega}\, n(\omega) + i\pi\, \mathrm{Res}_\infty \frac{n(\omega)}{\omega} = 0$$

$$\implies \mathrm{Re}\, n(0) = \mathrm{Re}\, n(\infty) + \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega}\, \mathrm{Im}\, n(\omega)$$

Therefore—subject to the assumptions—we deduce $\mathrm{Re}\, n(0) > 1$

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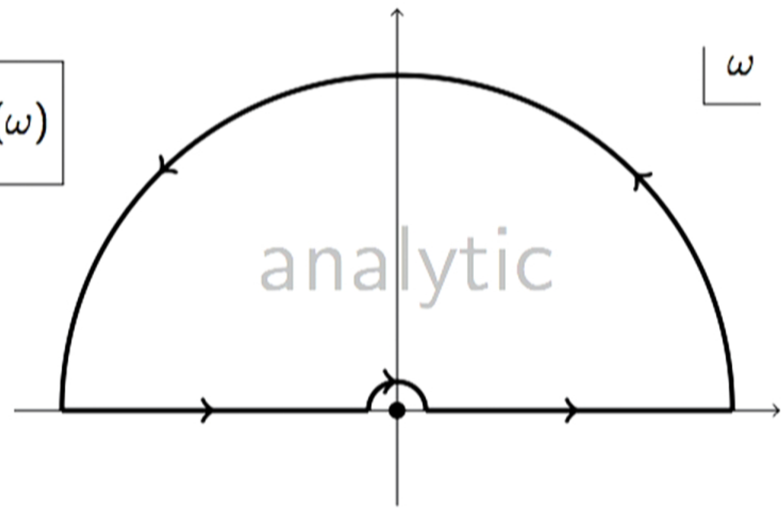

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Therefore—subject to the assumptions—we deduce $\mathrm{Re}\, n(0) > 1$

But DH say Re $n(0) < 1$ so what gives?

- In curved space some of the assumptions are questionable
- Can we do better? Can we calculate $n(\omega)$ exactly (i.e. sum up infinite number of terms in the effective theory)

There is a calculational window where stationary phase (WKB) methods are applicable

1. Eikonal approximation (geometric optics) $\omega \gg \sqrt{R}$
2. Penrose limit $m \gg \sqrt{R}$

- Sums an infinite subset of terms in effective theory

$$\boxed{n(\omega) = 1 + \frac{\alpha R}{m^2} F\left(\frac{\omega}{\sqrt{R}} \cdot \frac{R}{m^2}\right)}$$

(curvature scale R includes derivatives)

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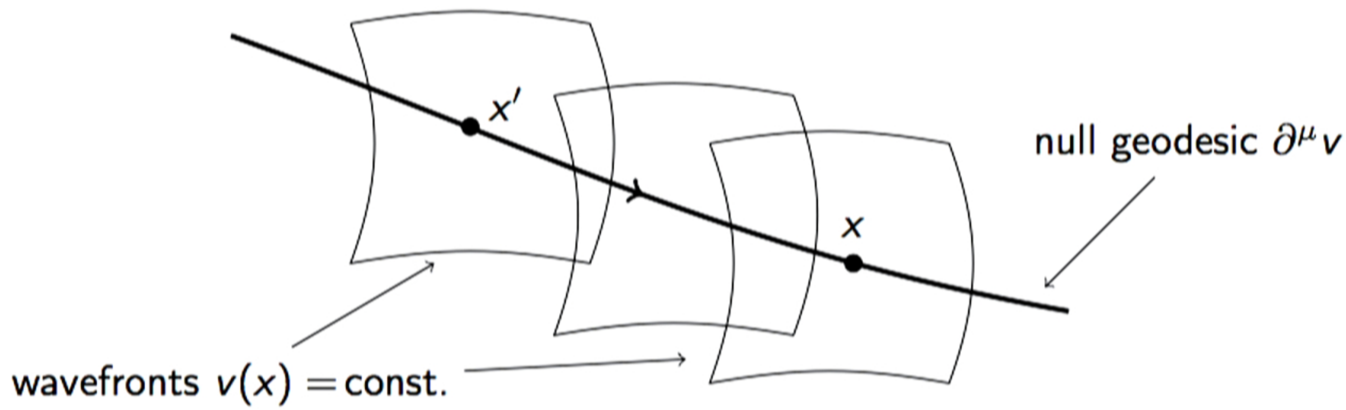
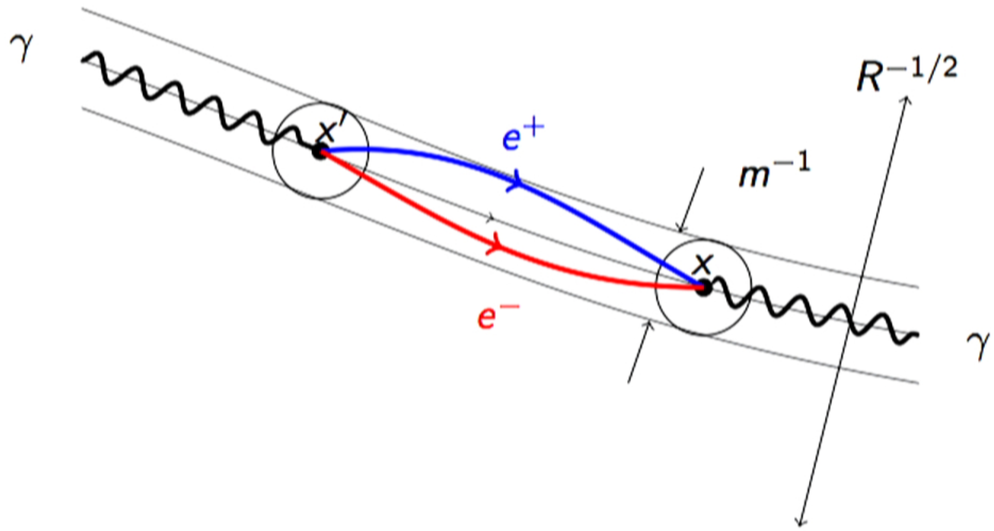
The stationary phase

$$\boxed{\int d^4x' \sqrt{g(x')}\, G_{e^-}(x,x') G_{e^+}(x,x') e^{-i\omega v(x')}}$$

- Vary x' and find it must lie on the null geodesic through x with tangent vector $\partial^\mu v$



- Then since $m \gg \sqrt{R}$ the $e^- e^+$ loop doesn't penetrate far into the geometry and the metric can be approximated by its Penrose limit for the null geodesic



- Penrose limit is a plane wave geometry

$$\boxed{ds^2 = 2du\,dv - h_{ij}(u)\,y^i y^j du^2 - dy^i dy^i}$$

null geodesic is $v = y^i = 0$ with u affine parameter

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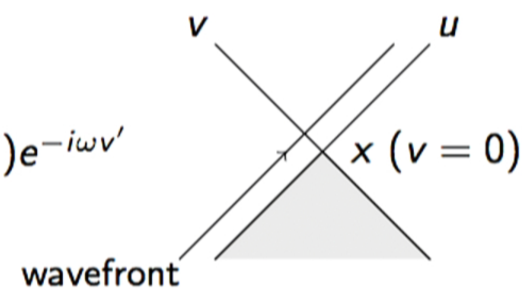

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Plane waves

- Plane waves have WKB exact propagators

$$G(x,x') = \frac{i}{16\pi^2}\int_0^{\infty - i0^+} \frac{dT}{T^2}\sqrt{\Delta(x,x')}\exp\left[-iTm^2 + \frac{i}{T}\sigma(x,x')\right]$$

Van Vleck Morette determinant

geodesic interval

- **Example:** Symmetric plane waves are the closest analogue of a constant background:

$$h_{ij} = \mathrm{diag}(\sigma_1^2, \sigma_2^2) \qquad \sigma_i^2 \lesseqgtr 0 \qquad R_{uiuj} = \sigma_i^2 \delta_{ij}$$

e.g. conformally flat SPW $\sigma_i^2 = \sigma^2 > 0$

- Null energy condition $\sigma_1^2 + \sigma_2^2 \geq 0$

1 Geodesic interval is the world-line action for the geodesic $x^\mu(\tau)$ joining $x = x(0)$ and $x' = x(1)$:

$$\boxed{\sigma(x,x') = \frac{1}{2}\int_0^1 d\tau\, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}$$

- e.g. for conf. flat SPW

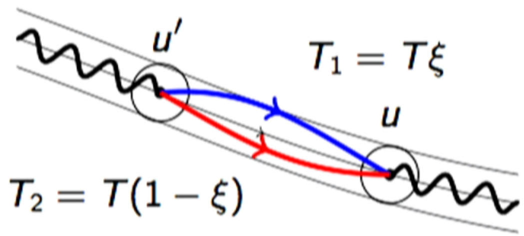
$$\sigma(x,x') = (u-u')\Big[(v-v') - \frac{\sigma}{2}\cot\sigma(u-u')\left(y^i y^i + y'^i y'^i\right) + \sigma\csc\sigma(u-u')\, y^i y'^i\Big]$$

2 Van-Vleck-Morette determinant $\boxed{\Delta(x,x') = \frac{1}{\sqrt{g(x)g(x')}}\det\left|\frac{\partial^2\sigma(x,x')}{\partial x^\mu \partial x'^\nu}\right|}$

- e.g. conf. flat SPW $\qquad \Delta(x,x') = \left[\frac{\sigma(u-u')}{\sin\sigma(u-u')}\right]^2$

- The final answer (scalar QED)

$$n(u;\omega) = 1 - \frac{i\alpha}{2\pi\omega^2}\int_0^{\infty - i0^+}\frac{dT}{T^2}\int_0^1 d\xi\, e^{-iTm^2}\left[\sqrt{\Delta}\frac{\partial^2\sigma}{\partial y^i \partial y'^j}(u,u')\right]_{u'=u-2\omega\xi(1-\xi)T}$$



- e.g. conformally flat symmetric plane wave

$$n(\omega) = 1 - \frac{i\alpha}{2\pi\omega^2}\int_0^{\infty - i0^+}\frac{dT}{T^2}\int_0^1 d\xi\, e^{-iTm^2}\left[\left(\frac{2\sigma\omega\xi(1-\xi)T}{\sin 2\sigma\omega\xi(1-\xi)T}\right)^2 - 1\right]$$

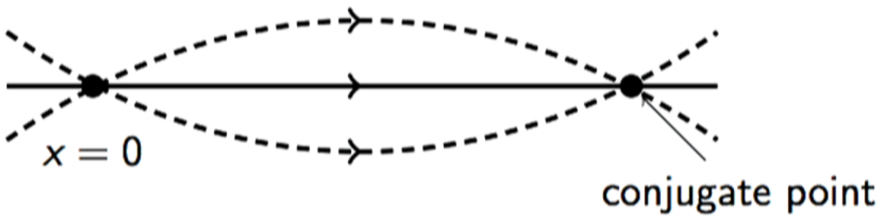
- Expanding in powers of T permits the construction of the effective theory

Conjugate points

- e.g. symmetric plane wave

$$\boxed{\Delta(x,x') = \frac{\sigma_1(u-u')}{\sin\sigma_1(u-u')} \cdot \frac{\sigma_2(u-u')}{\sin\sigma_2(u-u')}}$$

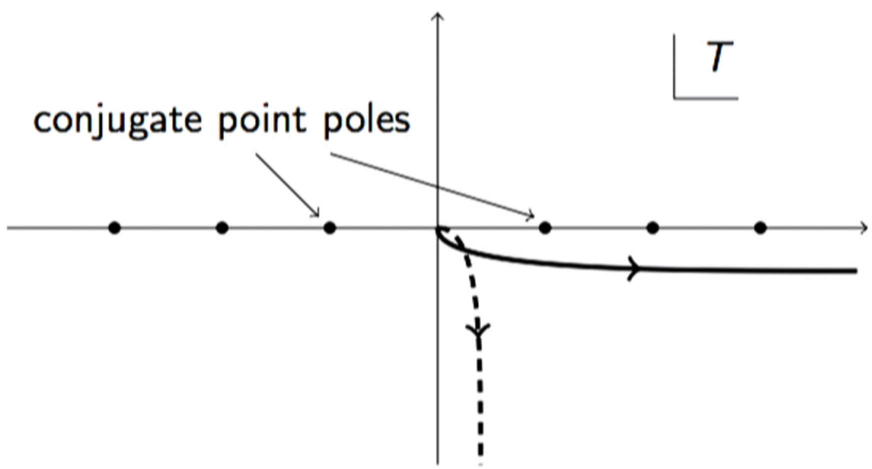
- The poles at $u - u' = \dfrac{n\pi}{\sigma_i}$ have their origin in conjugate points
- If we take the spray of null geodesics through the point $u = v = y^i = 0$ then these focus in the transverse direction y^i onto the points $u = n\pi/\sigma_i$, $v = y^i = 0$:



The null energy condition implies that there is always at least one transverse direction which is focusing (singularity theorems)

Conformally flat symmetric plane wave

$$n(\omega) = 1 - \frac{i\alpha}{2\pi\omega^2} \int_0^{\infty - i0^+} \frac{dT}{T^2} \int_0^1 d\xi \, e^{-iTm^2} \left[\left(\frac{2\sigma\omega\xi(1-\xi)T}{\sin 2\sigma\omega\xi(1-\xi)T} \right)^2 - 1 \right]$$



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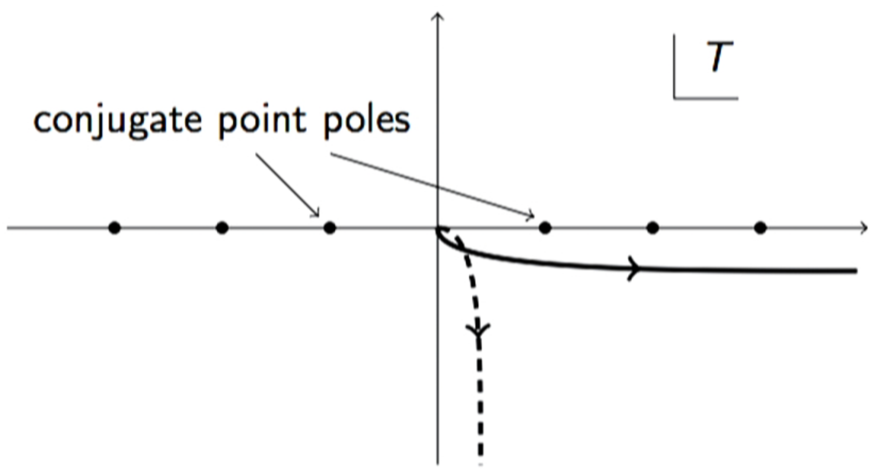
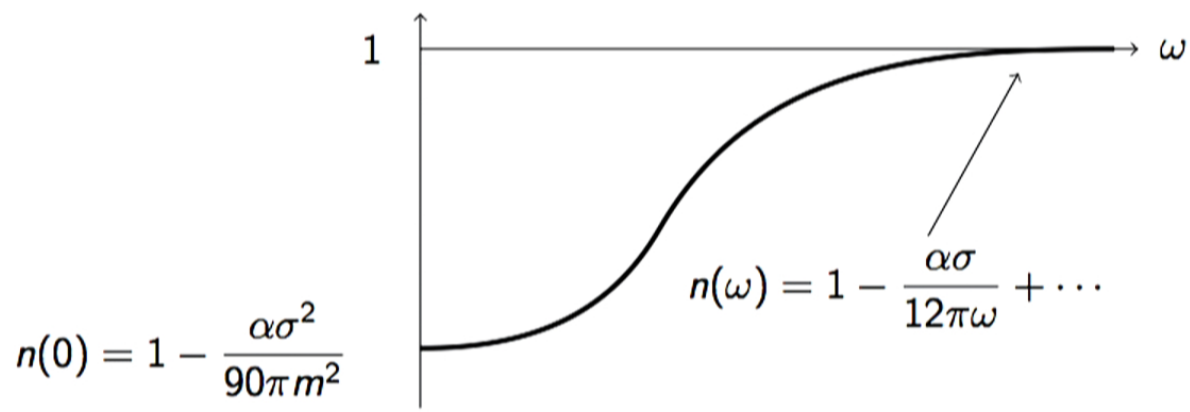


- rotate contour $T \to -iT$

$$\boxed{n(\omega) = 1 + \frac{\alpha}{2\pi\omega^2} \int_0^\infty \frac{dT}{T^2} \int_0^1 d\xi\, e^{-Tm^2} \left[\left(\frac{2\sigma\omega\xi(1-\xi)T}{\sinh 2\sigma\omega\xi(1-\xi)T} \right)^2 - 1 \right]}$$

- So $\mathrm{Im}\, n(\omega) = 0$

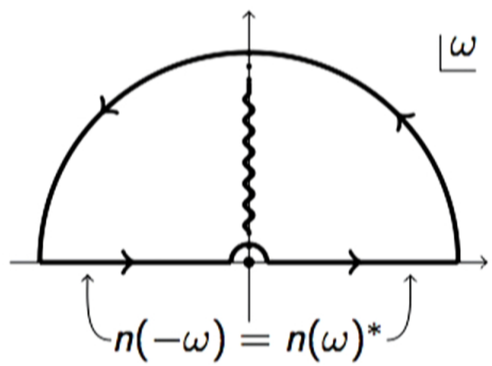


- Now we can see how the KK relation is violated?

$n(\omega)$ is not analytic in UHP (i.e. not apparently causal!)

- Poles in ω of the integrand from $\sinh(2\sigma\omega\xi(1-\xi)T)$ at $\omega = \dfrac{in\pi}{\sigma\xi(1-\xi)T}$,

$n = 1, 2, \ldots$ smear out into a cut



- KK integral picks up the discontinuity across the cut

$$\mathrm{Re}\, n(0) = 1 + \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega} \underbrace{\mathrm{Im}\ n(\omega)}_{0} - \frac{1}{\pi}\int_0^\infty \frac{dy}{y}\ \mathrm{Im}\ \delta_{\mathrm{cut}} n(iy)$$

$$\Longrightarrow \boxed{\mathrm{Re}\, n(0) = 1 - \frac{\alpha\sigma^2}{90\pi m^2}}$$

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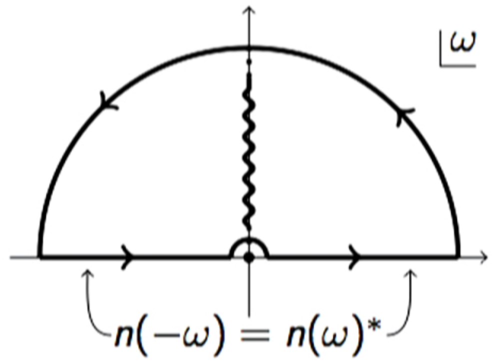

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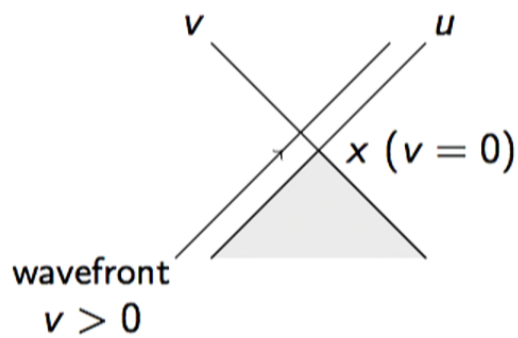
- That explains how KK ends up being “violated”: analyticity in UHP fails
- Fourier transform of $n(\omega)$ does not vanish for $v > 0$:

$$\int d\omega\, e^{i\omega v} n(\omega) \sim e^{-2m\sqrt{2\pi v/\sigma}}$$

- In (u,v) subspace the lightcone is just as in Minkowski space:



The wave front seems to have effects at x before it arrives!

So is there a break down of causality?

How can there be if UV theory is just QED?
There must be a subtlety we have missed!

- The subtlety is caused by the fact that the wavefront is extended in the transverse directions y^i and the conjugate points imply that the light cone in transverse directions is complicated:

$$n(\omega) \sim 1 + \frac{1}{2\omega} \underbrace{\int d^4x'}_{\text{past light cone}} \; \Pi(x = 0, x')\; e^{-i\omega v'}$$

$$\boxed{\sigma(0, x') = 0 \quad \Longrightarrow \quad v' = \frac{\sigma}{2}\cot(\sigma u') y'^i y'^i}$$

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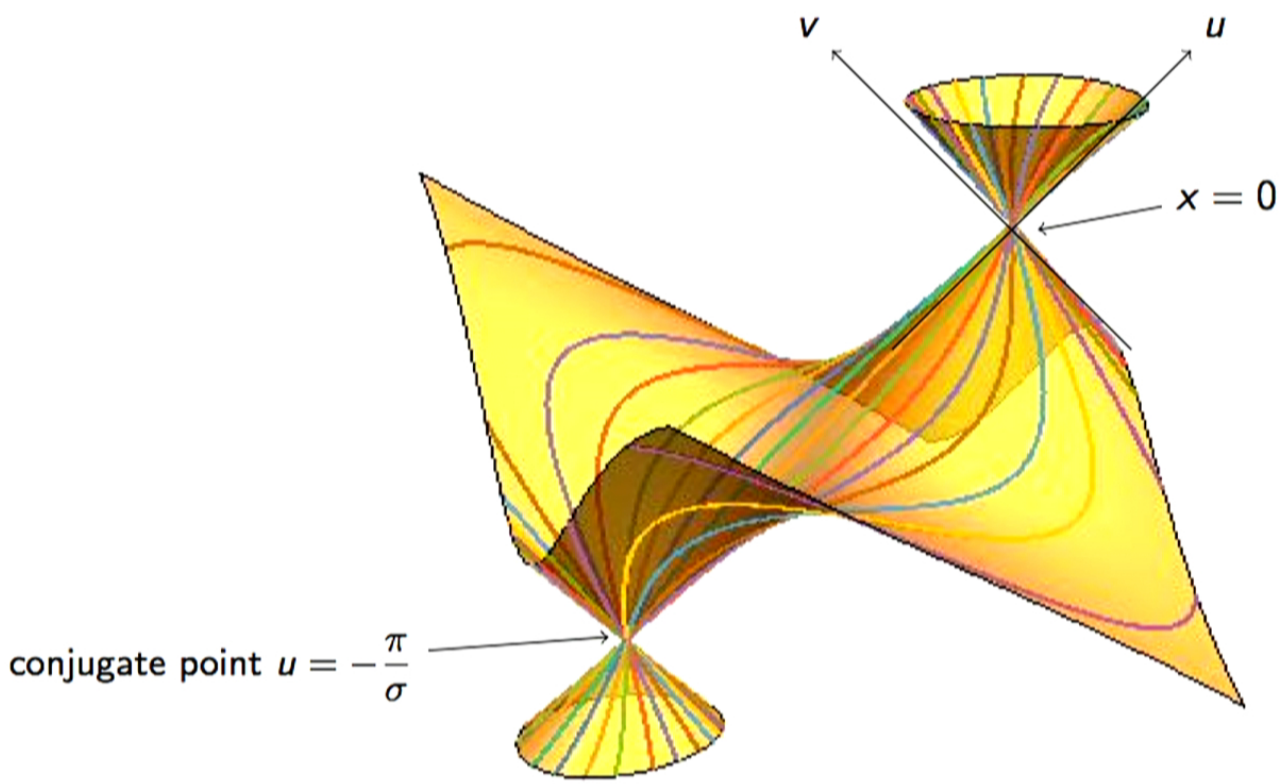
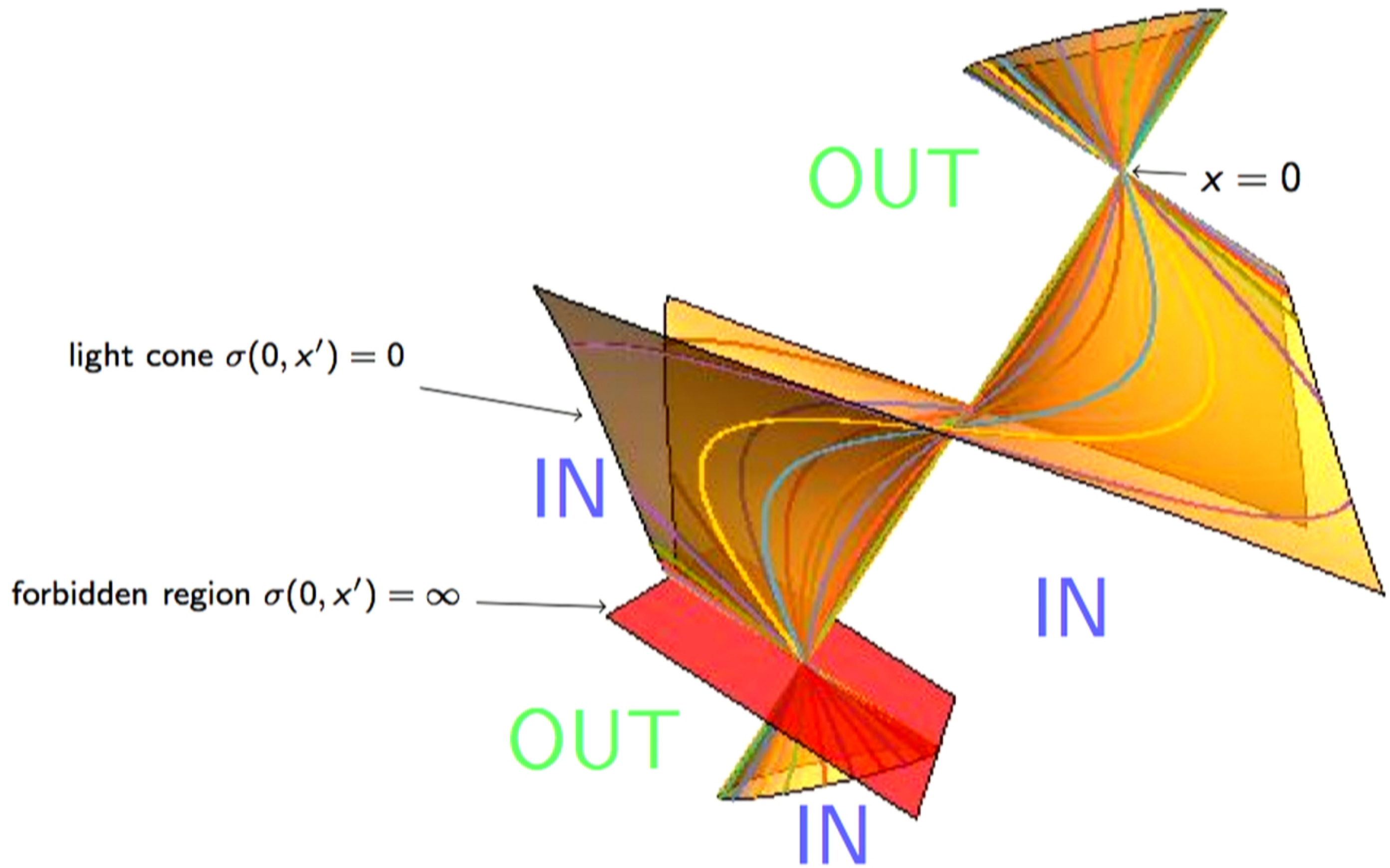
- Even the light cone of the conformally flat plane wave is subtle:

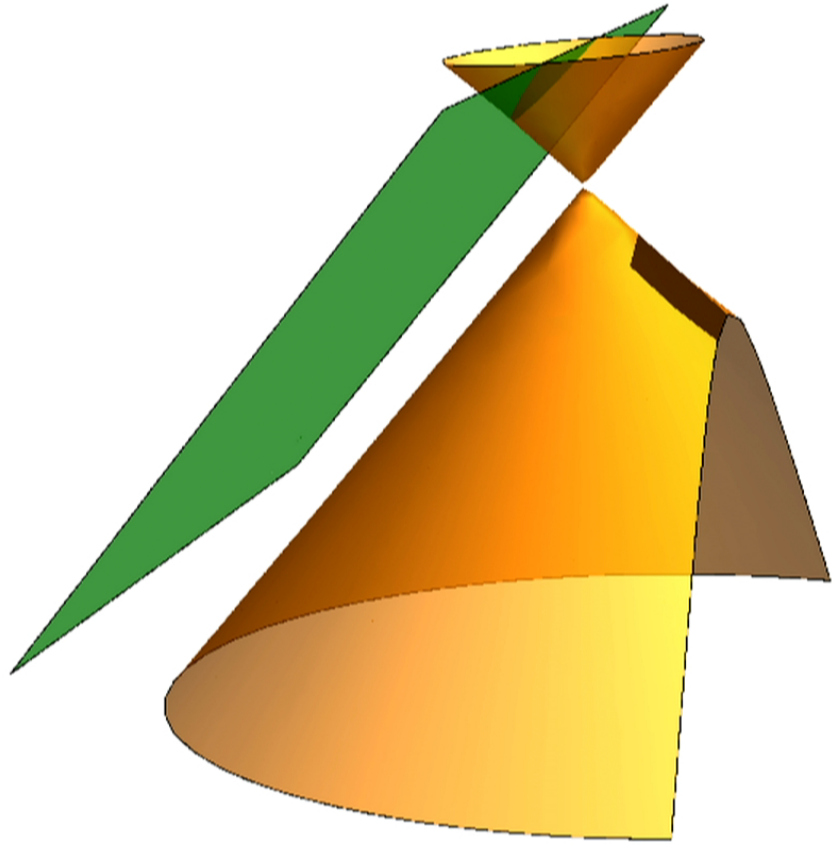
$$v = \frac{\sigma}{2}\cot(\sigma u) y^i y^i$$



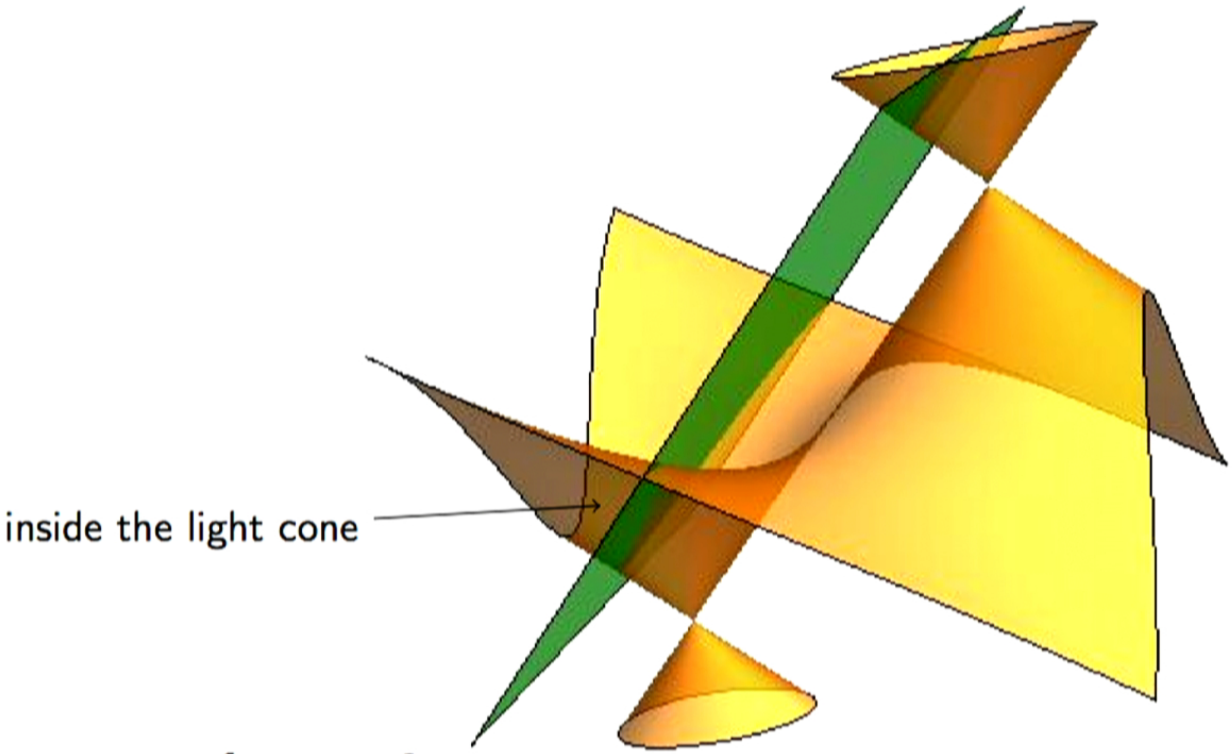
The ins and outs



- In flat space a wave front with $v' > 0$ does not intersect past light cone of $x = 0$



- In the conformally flat SPW



so wavefront $v > 0$

intersects past light cone in the transverse direction

when $u < -\dfrac{\pi}{2\sigma} \implies$ signals can reach $x = 0$

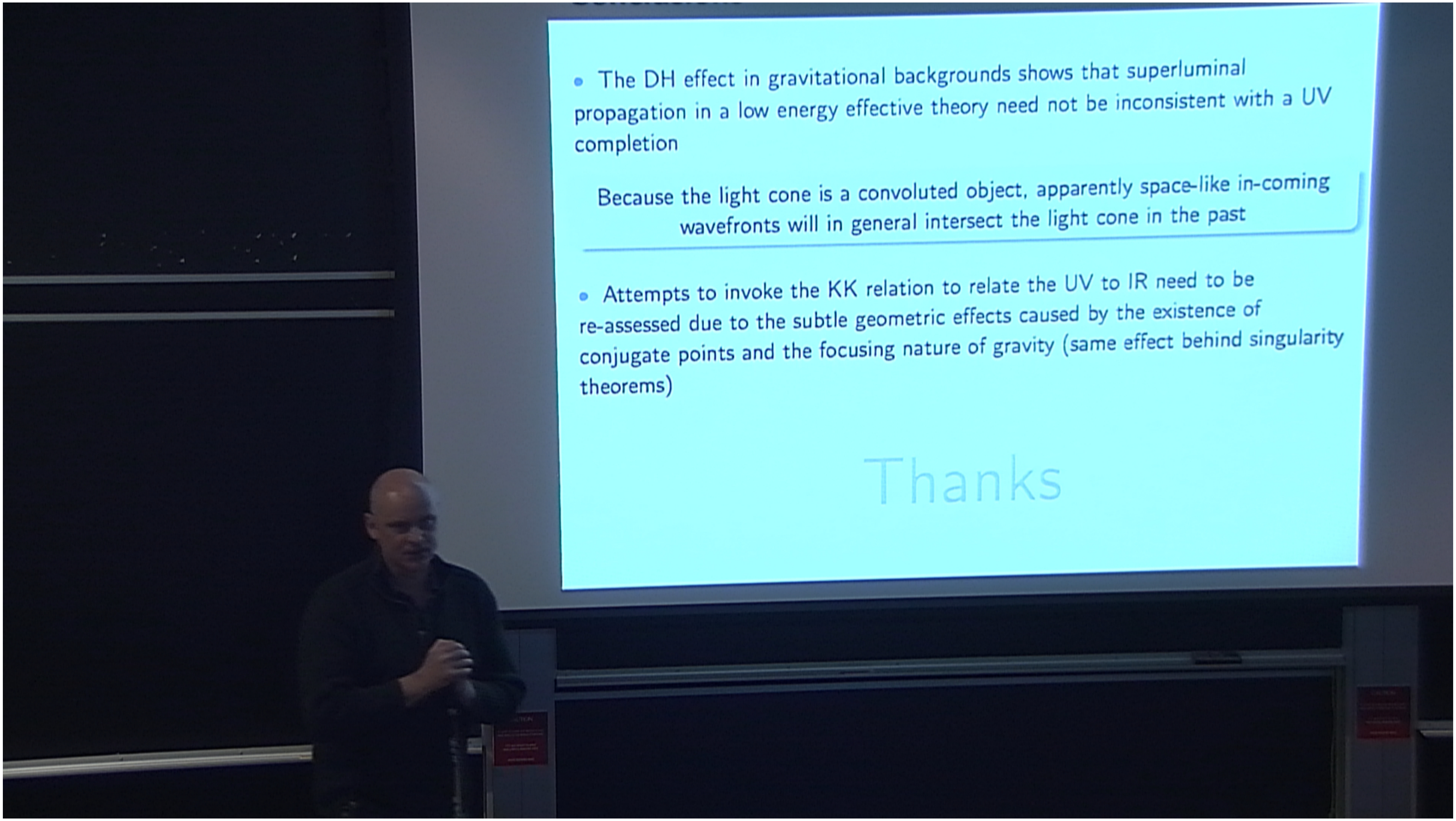
Conclusions

- The DH effect in gravitational backgrounds shows that superluminal propagation in a low energy effective theory need not be inconsistent with a UV completion

Because the light cone is a convoluted object, apparently space-like in-coming wavefronts will in general intersect the light cone in the past

- Attempts to invoke the KK relation to relate the UV to IR need to be re-assessed due to the subtle geometric effects caused by the existence of conjugate points and the focusing nature of gravity (same effect behind singularity theorems)

Thanks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Thanks