Title: Symmetry, Defects, and Gauging of Topological Phases

Date: Apr 07, 2015 03:30 PM

URL: http://pirsa.org/15040088

Abstract: <p>We examine the interplay of symmetry and topological order in 2+1D topological phases of matter. We define the topological symmetry group, characterizing symmetry of the emergent topological quantum numbers, and describe its relation with the microscopic symmetry of the physical system.</p>

<p>We then derive a general classification of symmetry fractionalization in topological phases, including phases that are non-Abelian and symmetries that permute the quasiparticle types and/or are anti-unitary. We develop a general algebraic theory of extrinsic defects (fluxes) associated with elements of the symmetry group, which provides a general classification of symmetry-enriched topological phases derived from a topological phase of matter with symmetry. We also examine the promotion of the global symmetry to a local gauge invariance, wherein the extrinsic defects are turned into deconfined quasiparticle excitations, which results in a different topological phase.</p>

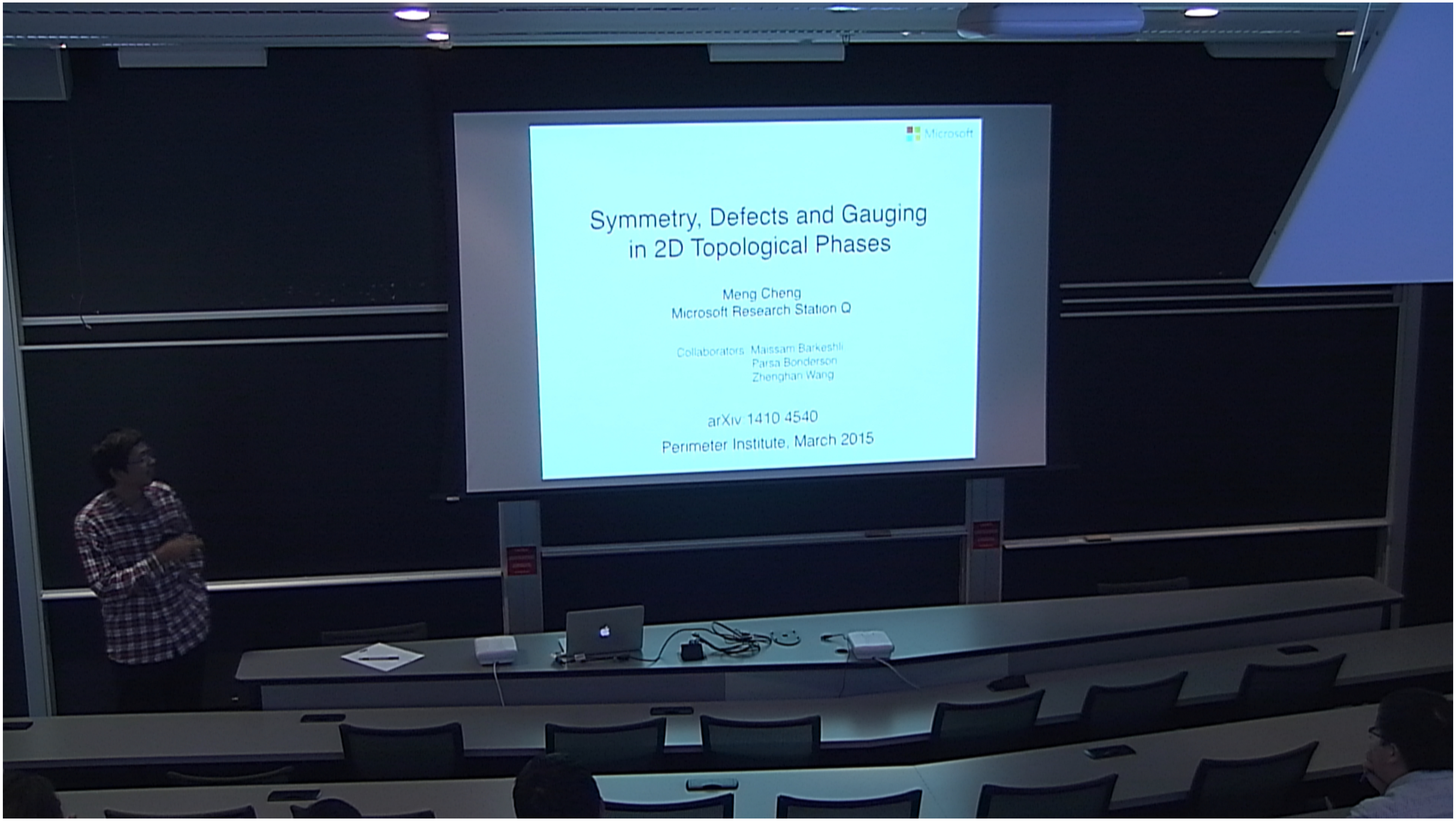
Symmetry, Defects and Gauging in 2D Topological Phases

Meng Cheng
Microsoft Research Station Q

Collaborators: Maissam Barkeshli,
Parsa Bonderson
Zhenghan Wang

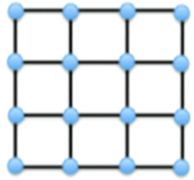
arXiv:1410.4540

Perimeter Institute, March 2015


Microsoft
Symmetry, Defects and Gauging
in 2D Topological Phases
Meng Cheng
Microsoft Research Station Q
Collaborators: Maissam Barkeshli
Parsa Bonderson
Zhenghan Wang
arXiv 1410 4540
Perimeter Institute, March 2015

Topological Phases of Matter

Local Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$



Hamiltonian with short-range interactions

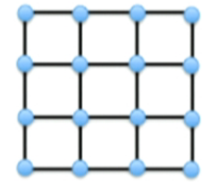
$$\text{Topological phases} = \frac{\text{Gapped quantum phases}}{\sim \ \text{by adiabatic continuity}}$$

Two-dimensions:

- Chiral central charge on the edge
- Anyonic quasiparticles in the bulk

Topological Phases of Matter

Local Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$



Hamiltonian with short-range interactions

$$\text{Topological phases} = \frac{\text{Gapped quantum phases}}{\sim \ \text{by adiabatic continuity}}$$

Two-dimensions:

- Chiral central charge on the edge
- Anyonic quasiparticles in the bulk

Topological Phases + Symmetry

$$\frac{\text{Symmetric gapped quantum phases}}{\sim \ \text{by symmetric adiabatic continuity}}$$

Trivial phases

- Topological insulators $\mathrm{U}(1) \rtimes \mathbb{Z}_2^T$
- Haldane phases $\mathrm{SO}(3)$

Symmetry **P**rotected **T**opological Phases

Topological phases

Symmetry **E**nriched **T**opological Phases

Topological Phases + Symmetry

$$\frac{\text{Symmetric gapped quantum phases}}{\sim \text{ by symmetric adiabatic continuity}}$$

No symmetry

With symmetry

Trivial phases

- Topological insulators $\mathrm{U}(1) \rtimes \mathbb{Z}_2^T$
- Haldane phases $\mathrm{SO}(3)$

Symmetry **P**rotected **T**opological Phases

Topological phases

Symmetry **E**nriched **T**opological Phases

Symmetry Fractionalization

Fractional Quantum Hall States

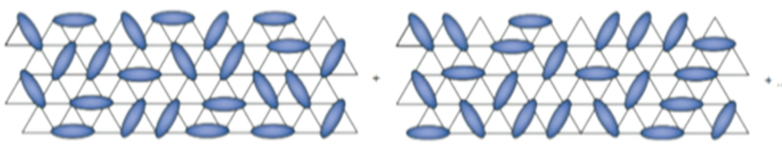
sistent with all the experimental facts and explain the effect. The ground state is a new state of matter, a quantum fluid the elementary excitations of which, the quasielectrons and quasiholes, are fractionally charged. I have verified

R. Laughlin, PRL 1983

A quasihole in 1/3 Laughlin state carries 1/3 charge

$$\prod_i (z_i - \xi) \prod_{i<j} (z_i - z_j)^3$$

Gapped Z_2 Spin Liquid

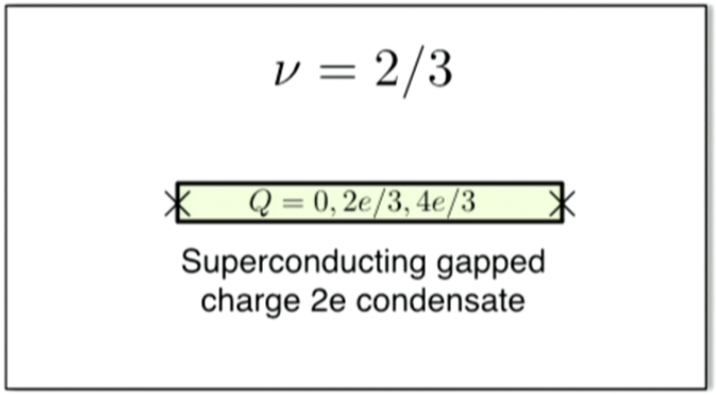


Short-range RVB state (L. Balents 2013)

$\mathrm{SO}(3)$ symmetry

Deconfined spinons

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

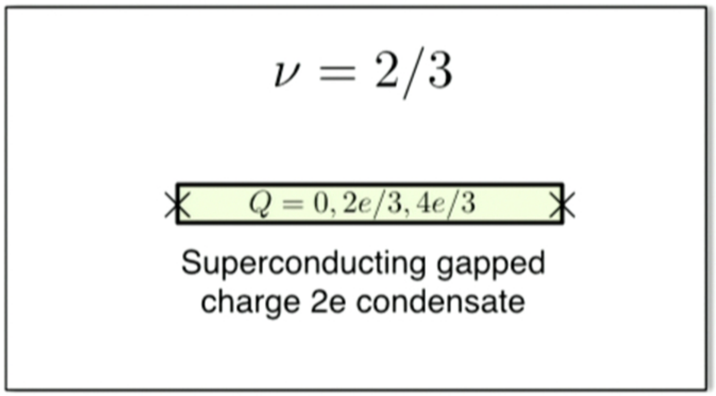
$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$

$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

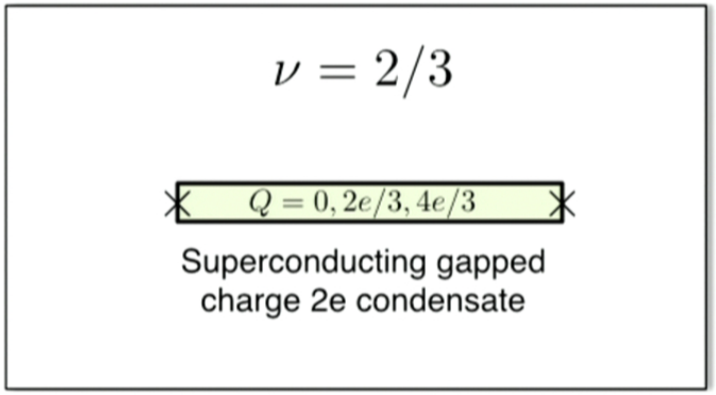
$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$

$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

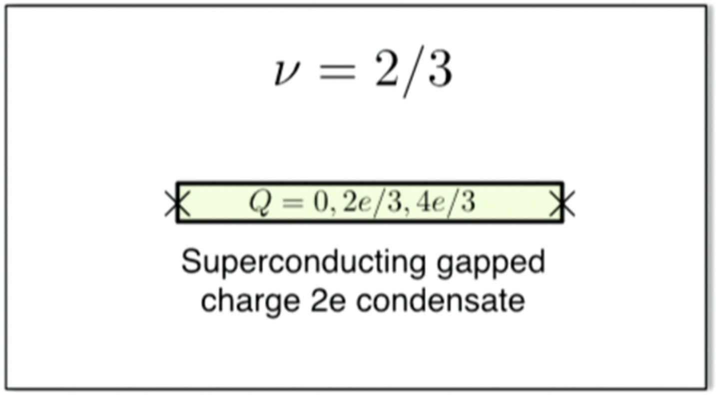
$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$

$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

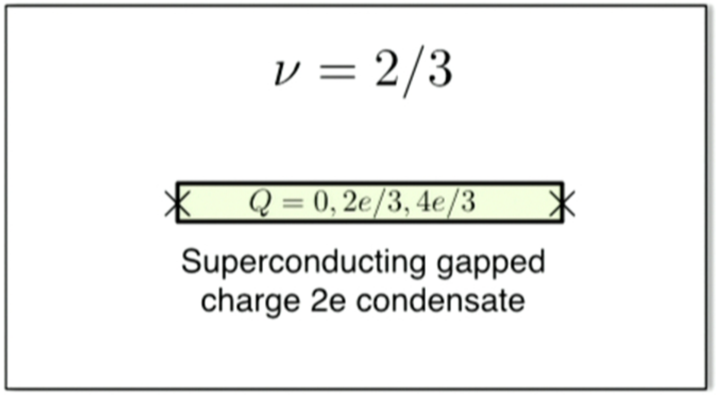
Conjectured fusion rules:

$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$
$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

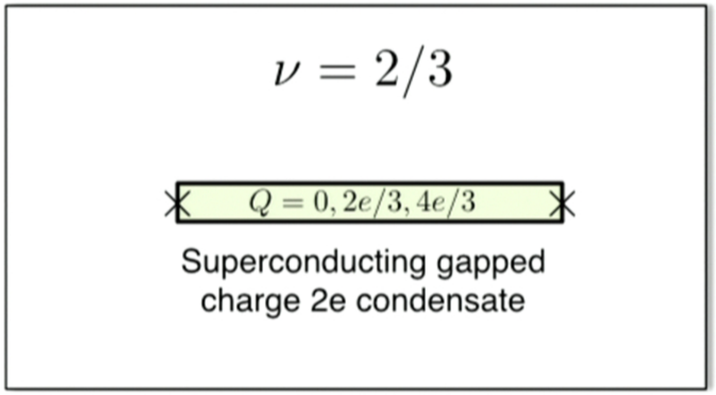
$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$

$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

Z_3 anyons in the FQH bulk

Charge $0, 2/3, 4/3$

n parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$
$$[\sigma] \times [2/3] = [\sigma] \times [4/3] = [\sigma]$$

However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

Outline

- Review of anyon models
- Topological symmetry and fractionalization
- Algebraic theory of defects
- Modular transformations

Disclaimer: Only bosonic systems; on-site unitary symmetries

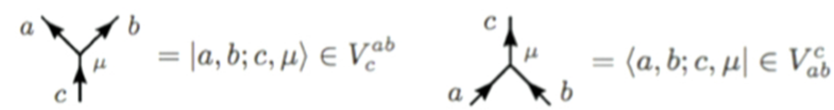
Anyon Model

$$\text{Anyon types} = \frac{\text{Finite-energy quasiparticle excitations}}{\sim \text{ by local operators}}$$

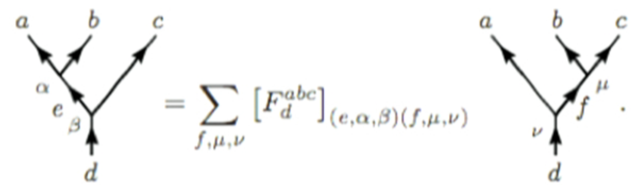
labeled by $a, b, c, \ldots$

Fusion rules: $a \times b = \sum_c N_{ab}^c c,\ N_{ab}^c \geq 0$

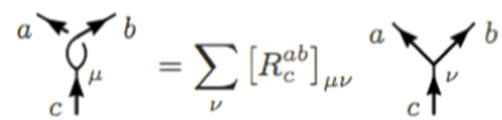
Splitting/fusion spaces:



F symbols:
(basis transformations)



R symbols:
(braiding)



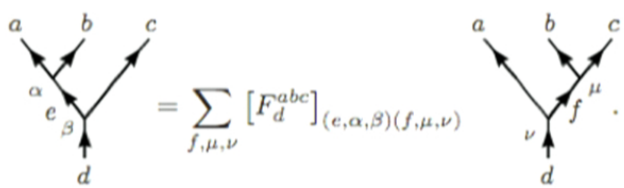
Gauge Transformations

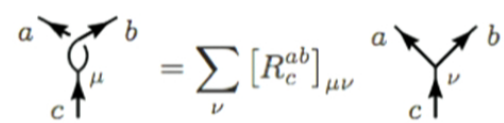
Redefine basis states:

$$|\widetilde{a, b; c, \mu}\rangle = \sum_{\mu\nu} [\Gamma^{ab}_{c}]_{\mu\nu} |a, b, c; \nu\rangle$$

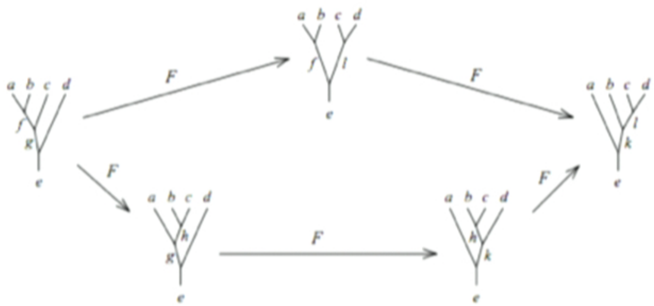
$$[\tilde{F}^{abc}_{d}]_{ef} = \Gamma^{ab}_{e} \Gamma^{ec}_{d} [F^{abc}_{d}]_{ef} (\Gamma^{bc}_{f})^{-1} (\Gamma^{af}_{d})^{-1}$$

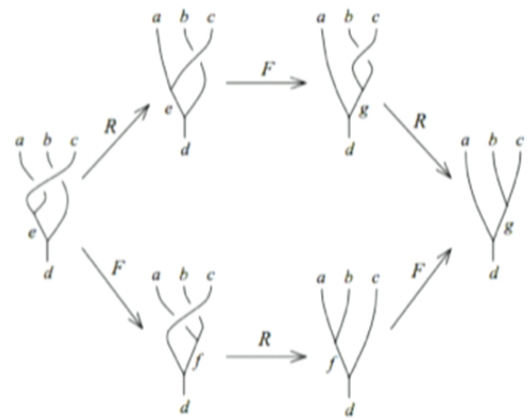
$$\tilde{R}^{ab}_{c} = \Gamma^{ba}_{c} \, R^{ab}_{c} (\Gamma^{ab}_{c})^{-1}$$





Consistency equations





No further conditions are needed

MacLane's coherence theorem

Solutions/~ by gauge transformations are discrete

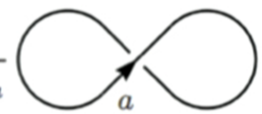
Ocneanu rigidity

Topological Invariants

Quantum dimensions $d_a =$

Topological twists
(exchange statistics)

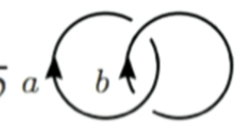
$\theta_a = \frac{1}{d_a}$



$= \sum_{c,\mu} \frac{d_c}{d_a} [R_c^{aa}]_{\mu\mu}$

Topological S matrix
(full braiding)

$S_{ab} = \mathcal{D}^{-1} \sum_c N_{\bar{a}b}^c \frac{\theta_c}{\theta_a \theta_b} d_c = \frac{1}{\mathcal{D}}$



Braiding non-degeneracy
(Modularity)

S is non-singular

Unitary Modular Tensor Category

Topological Symmetry

$$\mathrm{Aut}(\mathcal{C}) =$$ Permutations of anyon labels that leave physical quantities invariant

$$\varphi(a) = a'$$

$$\theta_{a'} = \theta_a, S_{a'b'} = S_{ab}$$

$$\varphi(|a,b;c\rangle) = u_{c'}^{a'b'} |a',b';c'\rangle$$

$$\varphi(F_d^{abc}) = \tilde{F}_{d'}^{a'b'c'} = F_d^{abc}$$

$$\varphi(R_c^{ab}) = \tilde{R}_{c'}^{a'b'} = R_c^{ab}$$

$$\mathrm{Aut}(\mathcal{C}) = \frac{\text{Auto-equivalences } \varphi}{\sim \text{ by natural isomorphism}}$$

Natural isomorphisms as the identity:

$$\Upsilon(a) = a$$

$$\Upsilon(|a,b;c\rangle) = \frac{\gamma_a \gamma_b}{\gamma_c} |a,b;c\rangle$$

Emergent symmetries of the topological order

Global Symmetry

G: on-site unitary symmetry group

$$R_{\mathbf{g}} = \prod_i R_{\mathbf{g}}^{(i)} \qquad R_{\mathbf{g}} R_{\mathbf{h}} = R_{\mathbf{gh}}$$

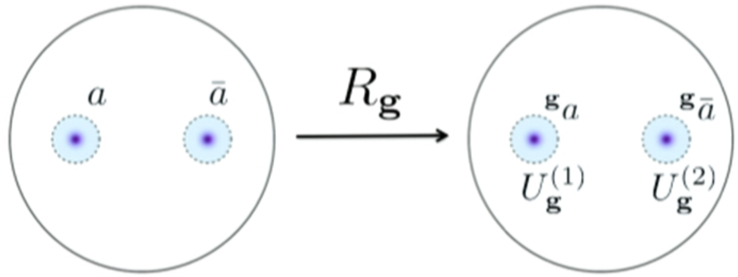
How symmetries permute anyons?

Group homomorphism $\rho : G \to \mathrm{Aut}(\mathcal{C})$

$\rho_{\mathbf{g}}$ has
- actions on anyon types ${}^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$
- $\rho_{\mathbf{g}}(|a, b; c\rangle) = U_{\mathbf{g}}({}^{\mathbf{g}}a, {}^{\mathbf{g}}b; {}^{\mathbf{g}}c)|{}^{\mathbf{g}}a, {}^{\mathbf{g}}b; {}^{\mathbf{g}}c\rangle$

$$\rho_{\mathbf{gh}} = \kappa_{\mathbf{g},\mathbf{h}} \rho_{\mathbf{g}} \rho_{\mathbf{h}} \qquad \kappa_{\mathbf{g},\mathbf{h}} |a, b; c\rangle = \frac{\beta_a \beta_b}{\beta_c} |a, b; c\rangle$$

Symmetry Localization



$${}^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$$

$$R_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle = U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}\ \rho_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle$$
$$= U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}U_{\mathbf{g}}({}^{\mathbf{g}}a,\ {}^{\mathbf{g}}\bar{a};0)|\Psi_{{}^{\mathbf{g}}a,\,{}^{\mathbf{g}}\bar{a};0}\rangle$$

$$U_{\mathbf{g}}^{(j)}\rho_{\mathbf{g}}U_{\mathbf{h}}^{(j)}\rho_{\mathbf{g}}^{-1} = \eta_{a_j}(\mathbf{g},\mathbf{h})U_{\mathbf{gh}}^{(j)}$$

Local actions can be projective

Symmetry Localization

Given a $[\rho]$, we identify an obstruction class

$$[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$$

If $[\mathfrak{O}]$ does not vanish, the symmetry can not be localized. End of story.

When $[\mathfrak{O}]$ vanishes

Symmetry fractionalization are classified by $H^2_\rho(G, \mathcal{A})$

$1/m$ Laughlin state $\quad G = \mathrm{U}(1) \quad H^2(\mathrm{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$

E.g fractional 1/m charges

Z_2 spin liquids $\quad G = \mathrm{SO}(3) \quad H^2(\mathrm{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$

Spin 1/2 projective rep. of SO(3)

Barkeshli, Bonderson, MC, Wang (2014)

Symmetry Localization

Given a $[\rho]$, we identify an obstruction class

$$[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$$

If $[\mathfrak{O}]$ does not vanish, the symmetry can not be localized. End of story.

When $[\mathfrak{O}]$ vanishes

Symmetry fractionalization are classified by $H^2_\rho(G, \mathcal{A})$

$1/m$ Laughlin state $G = \mathrm{U}(1)$ $H^2(\mathrm{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$

E.g fractional 1/m charges

Z_2 spin liquids $G = \mathrm{SO}(3)$ $H^2(\mathrm{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$

Spin 1/2 projective rep. of SO(3)

Barkeshli, Bonderson, MC, Wang (2014)

Symmetry Localization

Given a $[\rho]$, we identify an obstruction class

$$[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$$

If $[\mathfrak{O}]$ does not vanish, the symmetry can not be localized. End of story.

When $[\mathfrak{O}]$ vanishes

Symmetry fractionalization are classified by $H^2_\rho(G, \mathcal{A})$

$1/m$ Laughlin state $G = \mathrm{U}(1)$ $H^2(\mathrm{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$

E.g fractional 1/m charges

Z_2 spin liquids $G = \mathrm{SO}(3)$ $H^2(\mathrm{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$

Spin 1/2 projective rep. of SO(3)

Barkeshli, Bonderson, MC, Wang (2014)

Symmetry Localization

Given a $[\rho]$, we identify an obstruction class

$$[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$$

If $[\mathfrak{O}]$ does not vanish, the symmetry can not be localized. End of story.

When $[\mathfrak{O}]$ vanishes

Symmetry fractionalization are classified by $H^2_\rho(G, \mathcal{A})$

$1/m$ Laughlin state $G = \mathrm{U}(1)$ $H^2(\mathrm{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$

E.g fractional 1/m charges

Z_2 spin liquids $G = \mathrm{SO}(3)$ $H^2(\mathrm{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$

Spin 1/2 projective rep. of SO(3)

Barkeshli, Bonderson, MC, Wang (2014)

Symmetry Localization

Given a $[\rho]$, we identify an obstruction class

$$[\mathfrak{O}] \in H^3_\rho(G, \mathcal{A})$$

If $[\mathfrak{O}]$ does not vanish, the symmetry can not be localized. End of story.

When $[\mathfrak{O}]$ vanishes

Symmetry fractionalization are classified by $H^2_\rho(G, \mathcal{A})$

$1/m$ Laughlin state $G = \mathrm{U}(1)$ $H^2(\mathrm{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$

E.g fractional 1/m charges

Z_2 spin liquids $G = \mathrm{SO}(3)$ $H^2(\mathrm{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$

Spin 1/2 projective rep. of SO(3)

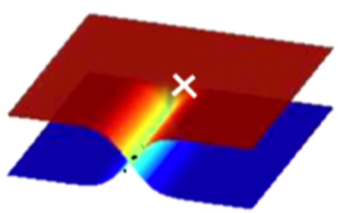
Barkeshli, Bonderson, MC, Wang (2014)

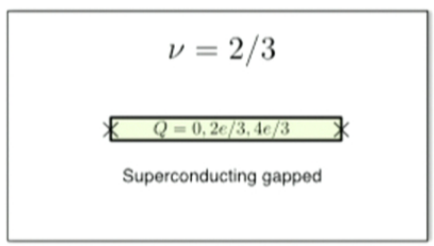
Why Defects?

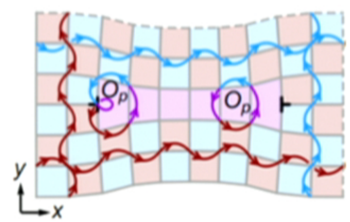
- They can be used to characterize SPT/SET phases.

  Levin and Gu 2012, 2013; Cheng and Gu, 2013; Hung and Wan 2012; Wen 2012; Zaletel 2013; Chen et. al. 2014

- Non-Abelian defects can be exploited for topological quantum information processing.

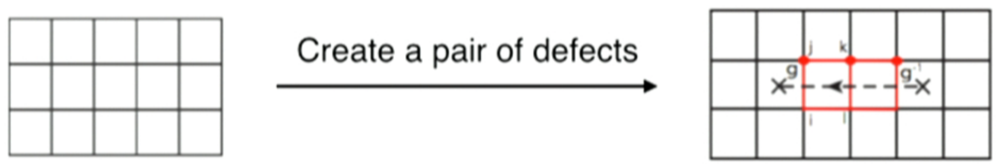






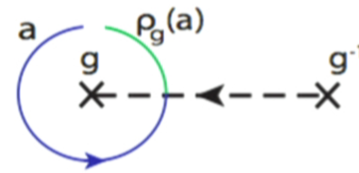
Barkeshli-Qi, 2011; Barkeshli-Qi, 2013; Bombin 2010; Kitaev-Kong 2011; You-Wen 2012

Extrinsic Symmetry Defects

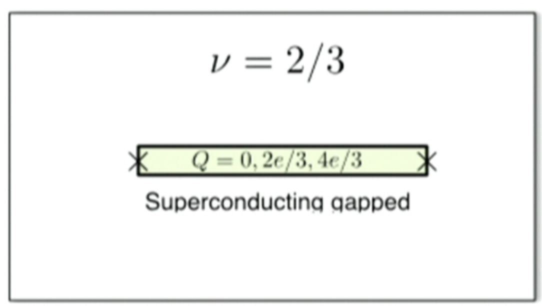


$$H_0 = \sum_i h_i + \sum_{\langle ij \rangle} h_{ij}$$

$$H_{\mathbf{g},\mathbf{g}^{-1}} = H_0 + \sum_{\substack{\langle ij \rangle: \\ i \in C_l; j \in C_r}} [R_{\mathbf{g}}^{(j)} h_{ij} R_{\mathbf{g}}^{(j)-1} - h_{ij}]$$



Generalized Aharonov-Bohm effect

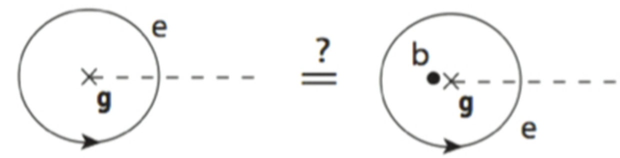


$$[2e/3] \to [-2e/3] \equiv [4e/3]$$

G-graded Fusion of Defects

There may be distinct **g**-defects

measured by **g**-invariant Wilson loops



$\mathcal{C}_{\mathbf{g}}$ Collection of g-defects $\quad \mathcal{C}_G = \bigoplus_{\mathbf{g}\in G} \mathcal{C}_{\mathbf{g}}$

G-graded fusion rules $\quad a_{\mathbf{g}} \times b_{\mathbf{h}} = \sum_{c_{\mathbf{gh}}} N_{a_{\mathbf{g}} b_{\mathbf{h}}}^{c_{\mathbf{gh}}} c_{\mathbf{gh}}$

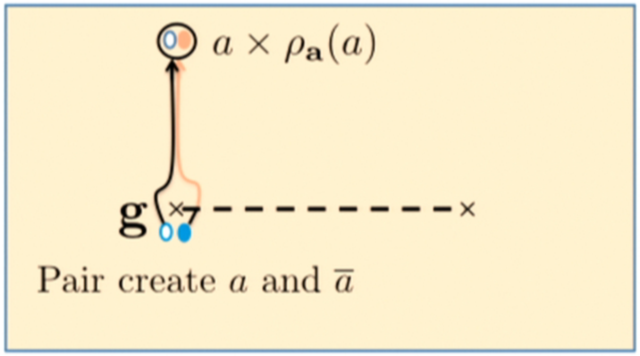
- Equal total quantum dimension in each sector $\sum_{a_{\mathbf{g}}} d_a^2 = \mathcal{D}_0$
- # of **g**-defects = # of **g**-invariant anyons (require modularity)

Barkeshli, Bonderson, MC, Wang (2014)

Example: Z_2 Toric Code with EM Duality

Toric code $1, e, m, \psi$

Z_2 symmetry $e \leftrightarrow m$

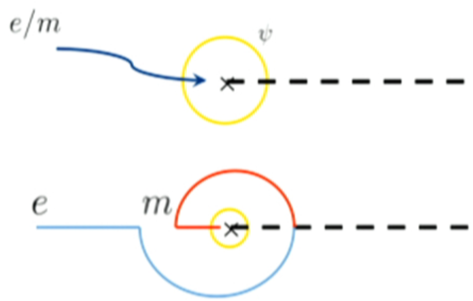


Fusion of defects

$$\sigma_\pm \times e = \sigma_\pm \times m = \sigma_\mp$$

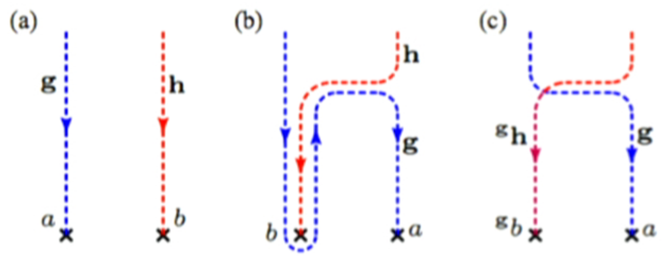
$$\sigma_+ \times \sigma_+ = \sigma_- \times \sigma_- = 1 + \psi$$

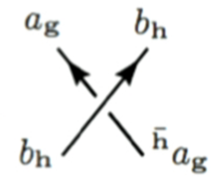
$\sigma_\pm$ carries a Majorana zero mode



Bombin 2010; Kitaev-Kong 2011; You-Wen 2012

G-crossed Braiding



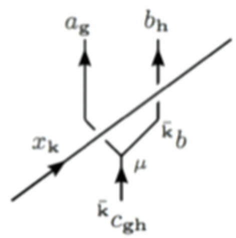
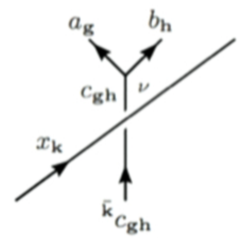
$R^{a_{\mathbf{g}} b_{\mathbf{h}}} =$ 

Exchanging **g**- and **h**-defects

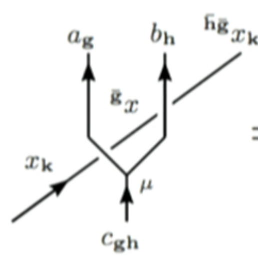
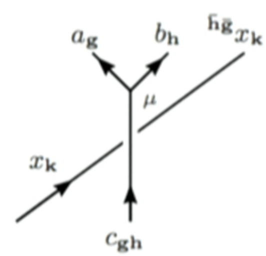
$^{\mathbf{g}}\mathbf{h} \equiv \mathbf{g}\mathbf{h}\mathbf{g}^{-1}$

Sliding rules 

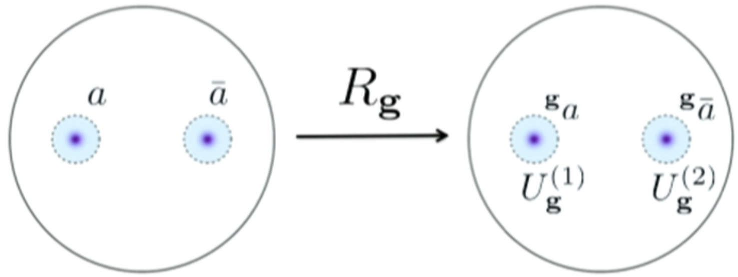
Symmetry action on splitting space:

 $= \sum_{\nu} \left[U_{\mathbf{k}}(a,b;c)\right]_{\mu\nu}$ 

Projective action on anyons/defects:

 $= \eta_x(\mathbf{g},\mathbf{h})$ 

Symmetry Localization



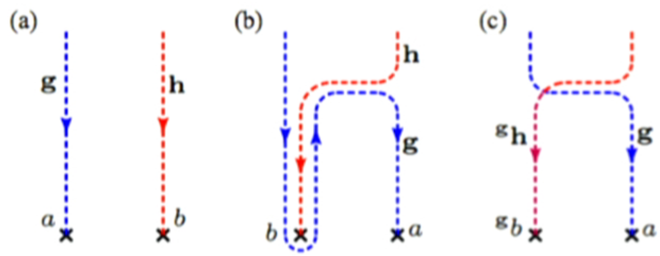
$${}^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$$

$$R_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle = U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}\,\rho_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle$$
$$= U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}U_{\mathbf{g}}({}^{\mathbf{g}}a,\,{}^{\mathbf{g}}\bar{a};0)|\Psi_{{}^{\mathbf{g}}a,\,{}^{\mathbf{g}}\bar{a};0}\rangle$$

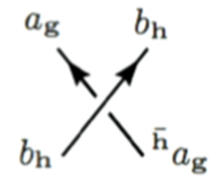
$$U_{\mathbf{g}}^{(j)}\rho_{\mathbf{g}}U_{\mathbf{h}}^{(j)}\rho_{\mathbf{g}}^{-1} = \boxed{\eta_{a_j}(\mathbf{g},\mathbf{h})}U_{\mathbf{gh}}^{(j)}$$

Local actions can be projective

G-crossed Braiding



$R^{a_{\mathbf{g}} b_{\mathbf{h}}} =$

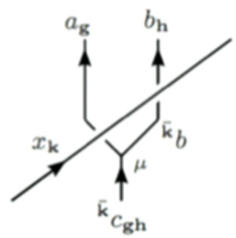


Exchanging **g**- and **h**-defects

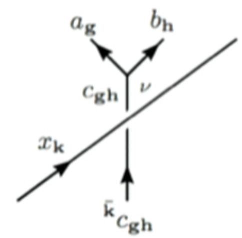
$^{\mathbf{g}}\mathbf{h} \equiv \mathbf{g}\mathbf{h}\mathbf{g}^{-1}$

Sliding rules

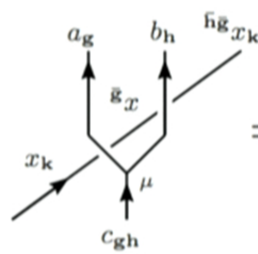
Symmetry action on splitting space:



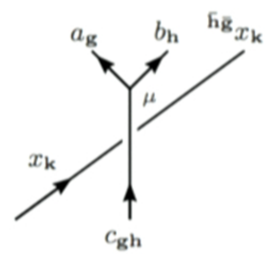
$$= \sum_{\nu} \left[U_{\mathbf{k}}\left(a, b; c\right)\right]_{\mu\nu}$$



Projective action on anyons/defects:

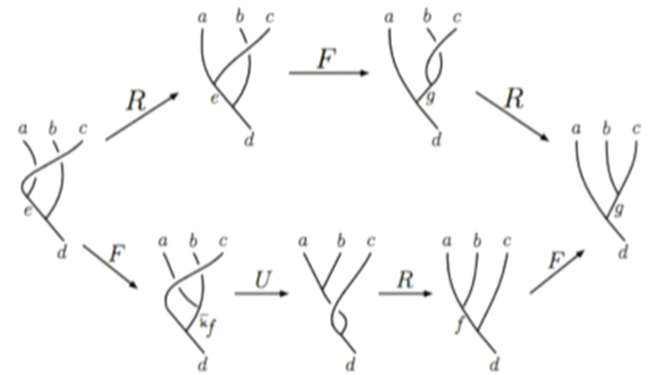


$$= \eta_x\left(\mathbf{g}, \mathbf{h}\right)$$

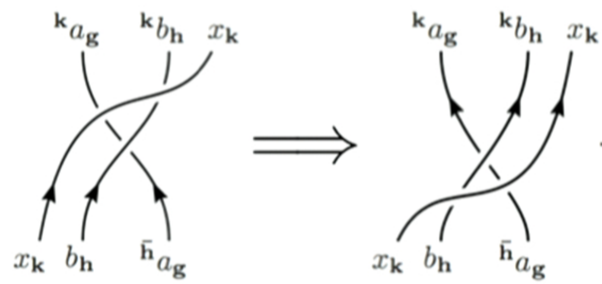


New Consistency Equations

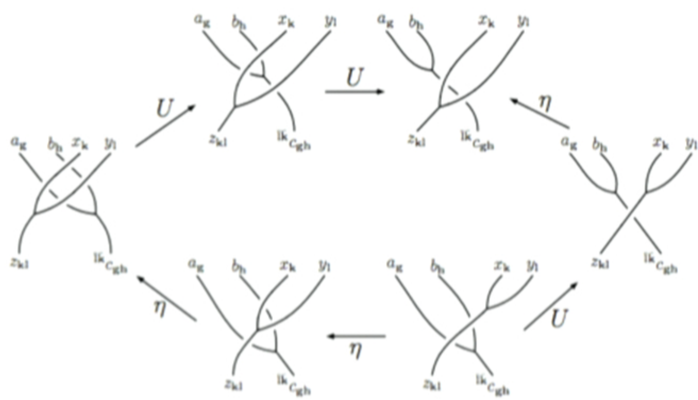
G-crossed heptagon equation

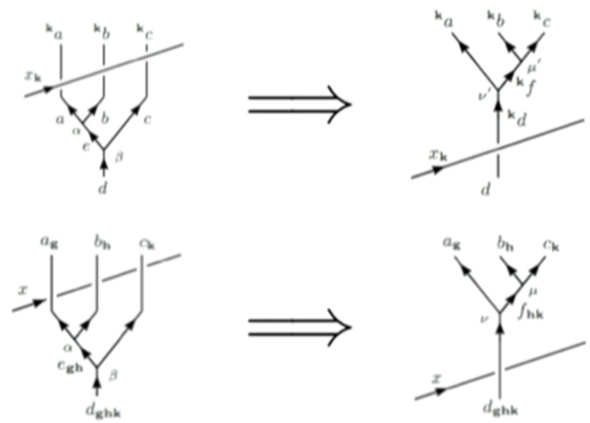


Yang-Baxter equation



Sliding consistency





Gauge Transformations

Vertex-basis gauge transformations

$$\widetilde{|a, b; c, \mu\rangle} = \sum_{\mu'} \left[\Gamma^{ab}_{c}\right]_{\mu\mu'} |a, b; c, \mu'\rangle$$

$$\tilde{U}_{\mathbf{k}}(a, b; c) = \Gamma^{{}^{\bar{\mathbf{k}}}a\,{}^{\bar{\mathbf{k}}}b}_{{}^{\bar{\mathbf{k}}}c} U_{\mathbf{k}}(a, b; c)(\Gamma^{ab}_{c})^{-1}$$

$$[\tilde{F}^{abc}_{d}]_{ef} = \Gamma^{ab}_{e}\Gamma^{ec}_{d}[F^{abc}_{d}]_{ef}(\Gamma^{bc}_{f})^{-1}(\Gamma^{af}_{d})^{-1}$$

$$\tilde{R}^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}} = \Gamma^{{}^{\bar{\mathbf{g}}}b\,a}_{c} R^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}}(\Gamma^{ab}_{c})^{-1}$$

Symmetry-action gauge transformations

$$\tilde{\rho}_{\mathbf{g}} = \Upsilon_{\mathbf{g}}\rho_{\mathbf{g}}$$

$$\tilde{U}_{\mathbf{k}}(a, b; c) = \frac{\gamma_a(\mathbf{k})\gamma_b(\mathbf{k})}{\gamma_c(\mathbf{k})} U_{\mathbf{k}}(a, b; c)$$

$$\tilde{\eta}_x(\mathbf{g}, \mathbf{h}) = \frac{\gamma_x(\mathbf{gh})}{\gamma_{{}^{\bar{\mathbf{g}}}x}(\mathbf{h})\gamma_x(\mathbf{g})} \eta_x(\mathbf{g}, \mathbf{h})$$

$$\tilde{R}^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}} = \gamma_a(\mathbf{h}) R^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}}$$

What are physical quantities in the G-crossed theory?

Gauge Transformations

Vertex-basis gauge transformations

$$\widetilde{|a,b;c,\mu\rangle} = \sum_{\mu'} \left[\Gamma^{ab}_{c}\right]_{\mu\mu'} |a,b;c,\mu'\rangle$$

$$\tilde{U}_{\mathbf{k}}(a,b;c) = \Gamma^{{}^{\bar{\mathbf{k}}}a\,{}^{\bar{\mathbf{k}}}b}_{{}^{\bar{\mathbf{k}}}c} U_{\mathbf{k}}(a,b;c)(\Gamma^{ab}_{c})^{-1}$$

$$[\tilde{F}^{abc}_{d}]_{ef} = \Gamma^{ab}_{e}\Gamma^{ec}_{d}[F^{abc}_{d}]_{ef}(\Gamma^{bc}_{f})^{-1}(\Gamma^{af}_{d})^{-1}$$

$$\tilde{R}^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}} = \Gamma^{{}^{\bar{\mathbf{h}}}b\,a}_{c} R^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}}(\Gamma^{ab}_{c})^{-1}$$

Symmetry-action gauge transformations

$$\tilde{\rho}_{\mathbf{g}} = \Upsilon_{\mathbf{g}}\rho_{\mathbf{g}}$$

$$\tilde{U}_{\mathbf{k}}(a,b;c) = \frac{\gamma_a(\mathbf{k})\gamma_b(\mathbf{k})}{\gamma_c(\mathbf{k})} U_{\mathbf{k}}(a,b;c)$$

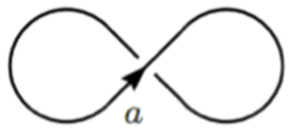
$$\tilde{\eta}_x(\mathbf{g},\mathbf{h}) = \frac{\gamma_x(\mathbf{gh})}{\gamma_{{}^{\bar{\mathbf{g}}}x}(\mathbf{h})\gamma_x(\mathbf{g})}\eta_x(\mathbf{g},\mathbf{h})$$

$$\tilde{R}^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}} = \gamma_a(\mathbf{h}) R^{a_{\mathbf{g}}b_{\mathbf{h}}}_{c_{\mathbf{gh}}}$$

What are physical quantities in the G-crossed theory?

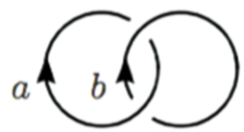
Gauge-Invariant Quantities

$$\theta_a = \frac{1}{d_a}$$



$$\check{\theta}_{a_{\mathbf{g}}} = \gamma_{a_{\mathbf{g}}}(\mathbf{g})\theta_{a_{\mathbf{g}}}$$

$$S_{a_{\mathbf{g}}b_{\mathbf{h}}} = \frac{1}{\mathcal{D}_0}$$



$$\check{S}_{a_{\mathbf{g}}b_{\mathbf{h}}} = \gamma_{\bar{a}}(\mathbf{h})\gamma_{b}(\bar{\mathbf{g}})S_{a_{\mathbf{g}}b_{\mathbf{h}}}$$

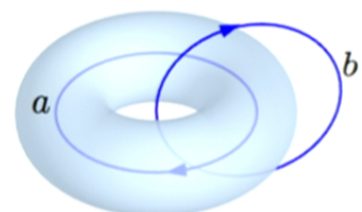
Systematically construct gauge-invariant quantities:

- G-crossed modular transformations
- Gauging (“equivariantization”)

Modular Transformation on a Torus

"Large" diffeomorphisms: $\mathrm{SL}(2,\mathbb{Z})$

Ground states on a torus labeled by anyon fluxes (MES)

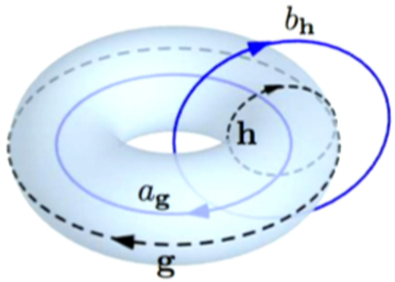


$$|a\rangle_l = \sum_b S_{ab}|b\rangle_m \quad |a\rangle_l = \sum_b T_{ab}|b\rangle_{l+m}$$

Same as the diagrammatic *S*, *T* matrices

$$(ST)^3 = e^{\frac{i\pi c_-}{4}} C, S^2 = C, C^2 = 1$$

Torus with Defect Lines



$$\mathcal{V}^{\text{ext}} = \bigoplus_{(\mathbf{g},\mathbf{h}),\mathbf{gh}=\mathbf{hg}} \mathcal{V}_{(\mathbf{g},\mathbf{h})}$$

$$\dim \mathcal{V}_{(\mathbf{g},\mathbf{h})} = |\mathcal{C}_{\mathbf{g}}^{\mathbf{h}}| \qquad \mathcal{C}_{\mathbf{g}}^{\mathbf{h}} = \{\ a \in \mathcal{C}_{\mathbf{g}} \mid {}^{\mathbf{h}}a = a\ \}$$

Extended modular transformations

$$\mathcal{S}^{(\mathbf{g},\mathbf{h})} \ : \ \mathcal{V}_{(\mathbf{g},\mathbf{h})} \to \mathcal{V}_{(\mathbf{h},\bar{\mathbf{g}})}$$

$$\mathcal{T}^{(\mathbf{g},\mathbf{h})} \ : \ \mathcal{V}_{(\mathbf{g},\mathbf{h})} \to \mathcal{V}_{(\mathbf{g},\mathbf{gh})}$$

$$\mathcal{S}_{a_{\mathbf{g}}b_{\mathbf{h}}}^{(\mathbf{g},\mathbf{h})} = \frac{S_{a_{\mathbf{g}}b_{\mathbf{h}}}}{U_{\mathbf{h}}(a,\bar{a};0)} \qquad \mathcal{T}_{a_{\mathbf{g}}b_{\mathbf{g}}}^{(\mathbf{g},\mathbf{h})} = \eta_a(\mathbf{g},\mathbf{h})\theta_{a_{\mathbf{g}}}\delta_{a_{\mathbf{g}}b_{\mathbf{g}}}$$

$$C_{a_{\mathbf{g}}b_{\bar{\mathbf{g}}}}^{(\mathbf{g},\mathbf{h})} = \frac{1}{U_{\mathbf{h}}(\bar{b},b;0)\eta_b(\mathbf{h},\bar{\mathbf{h}})}\delta_{a_{\mathbf{g}}\overline{b_{\bar{\mathbf{g}}}}}$$

Topological Invariants from Modular Transformations

Invariance under symmetry-action gauge transformations

$$\mathcal{Q}^{(\mathbf{g},\mathbf{h})} : \mathcal{V}_{(\mathbf{g},\mathbf{h})} \to \mathcal{V}_{(\mathbf{g}',\mathbf{h}')} \qquad \check{\mathcal{Q}}^{(\mathbf{g},\mathbf{h})}_{a_\mathbf{g} b_{\mathbf{g}'}} = \frac{\gamma_b(\mathbf{h}')}{\gamma_a(\mathbf{h})} \mathcal{Q}^{(\mathbf{g},\mathbf{h})}_{a_\mathbf{g} b_{\mathbf{g}'}}$$

If $\mathbf{h} = \mathbf{h}'$ and $a = b$, $\mathcal{Q}^{(\mathbf{g},\mathbf{h})}_{a_\mathbf{g} a_\mathbf{g}}$ is an invariant

If $\mathbf{h} = \mathbf{h}' = \mathbf{0}$, $\mathcal{Q}^{(\mathbf{g},\mathbf{0})}_{a_\mathbf{g} b_\mathbf{g}}$ is an invariant

$$\mathbf{g}^n = \mathbf{0} \qquad [\mathcal{T}^n]^{(\mathbf{g},\mathbf{h})}_{a_\mathbf{g} a_\mathbf{g}} = \theta^n_{a_\mathbf{g}} \prod_{j=0}^{n-1} \eta_{a_\mathbf{g}}(\mathbf{g}, \mathbf{g}^j \mathbf{h})$$

Application I: Classification of Bosonic SPT

Defect theory: $\mathcal{C} = \{1\} \quad \mathcal{C}_{\mathbf{g}} = \{\mathbf{g}\}$

$$\mathbf{g} \times \mathbf{h} = \mathbf{gh}$$

$$[F^{\mathbf{g},\mathbf{h},\mathbf{k}}_{\mathbf{ghk}}]_{\mathbf{gh},\mathbf{hk}} \equiv \omega(\mathbf{g},\mathbf{h},\mathbf{k}) \in Z^3(G, \mathrm{U}(1))$$

Gauge transformations: $B^3(G, U(1))$

Bosonic SPT phases are classified by $H^3(G, \mathrm{U}(1))$

$$R^{\mathbf{g},\mathbf{h}}_{\mathbf{gh}} = 1$$

$$U_{\mathbf{k}}(\mathbf{g},\mathbf{h};\mathbf{gh}) = \frac{\alpha(\mathbf{g},\mathbf{k},{}^{\bar{\mathbf{k}}}\mathbf{h})}{\alpha(\mathbf{g},\mathbf{h},\mathbf{k})\alpha(\mathbf{k},{}^{\bar{\mathbf{k}}}\mathbf{g},{}^{\bar{\mathbf{k}}}\mathbf{h})}$$

$$\eta_{\mathbf{k}}(\mathbf{g},\mathbf{h}) = \frac{\alpha(\mathbf{g},{}^{\bar{\mathbf{g}}}\mathbf{k},\mathbf{h})}{\alpha(\mathbf{g},\mathbf{h},{}^{\bar{\mathbf{h}}\bar{\mathbf{g}}}\mathbf{k})\alpha(\mathbf{k},\mathbf{g},\mathbf{h})}.$$

Chen, Gu, Liu and Wen 2011

Application II: Z_N Parafermion Zero Modes

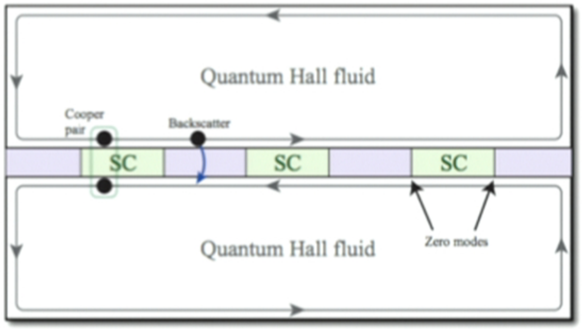


Figure from arXiv:1410.0359

Particle-hole symmetry in a 1/N Laughlin

$$a = 0, 1, \ldots, N-1$$

$$F^{abc}_{[a+b+c]} = 1, R^{ab}_{[a+b]} = e^{\frac{2\pi i p a b}{N}}$$

$$\rho_{\mathbf{g}} : [a] \to [-a]$$

$$\mathcal{C}_{\mathbf{g}} = \{\sigma\}$$

$$\sigma \times a = \sigma, \sigma \times \sigma = \sum_{a=0}^{N-1} a$$

$$\mathcal{S} = \left[\begin{array}{c|ccc} S_{\mathbf{0}} & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma^2 e^{\frac{\pi i}{4} c_-} \end{array}\right] \begin{array}{c} |0^{(0,\mathbf{g})}\rangle \\ |\sigma^{(\mathbf{g},0)}\rangle \\ |\sigma^{(\mathbf{g},\mathbf{g})}\rangle \end{array}$$

$$\mathcal{T} = \left[\begin{array}{c|ccc} T_{\mathbf{0}} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma^* \\ 0 & 0 & \gamma^* & 0 \end{array}\right]$$

$$\gamma^2 = \pm \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (-1)^{pn} e^{-\frac{\pi i p n^2}{N}}$$

Clarke et. al. 2012; Lindner et. al. 2012; MC 2012

Conclusions

- General classification of symmetry fractionalization
- Develop an algebraic theory of defect
  - New data and consistency conditions
  - Topological invariants from modular transformations
  - Obstruction theory
  - Construction of the gauged theory
- Related works
  - Etingof, Nikshych and Ostrik, arXiv:0909.3140 (MATH)
  - Fidkowski, Lindner and Kitaev, unpublished
  - Teo, Taylor and Fradkin, arXiv:1503.06812