

Title: Symmetry, Defects, and Gauging of Topological Phases

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Abstract: <p>We examine the interplay of symmetry and topological order in 2+1D topological phases of matter. We define the topological symmetry group, characterizing symmetry of the emergent topological quantum numbers, and describe its relation with the microscopic symmetry of the physical system.</p>

<p>We then derive a general classification of symmetry fractionalization in topological phases, including phases that are non-Abelian and symmetries that permute the quasiparticle types and/or are anti-unitary. We develop a general algebraic theory of extrinsic defects (fluxes) associated with elements of the symmetry group, which provides a general classification of symmetry-enriched topological phases derived from a topological phase of matter with symmetry. We also examine the promotion of the global symmetry to a local gauge invariance, wherein the extrinsic defects are turned into deconfined quasiparticle excitations, which results in a different topological phase.</p>

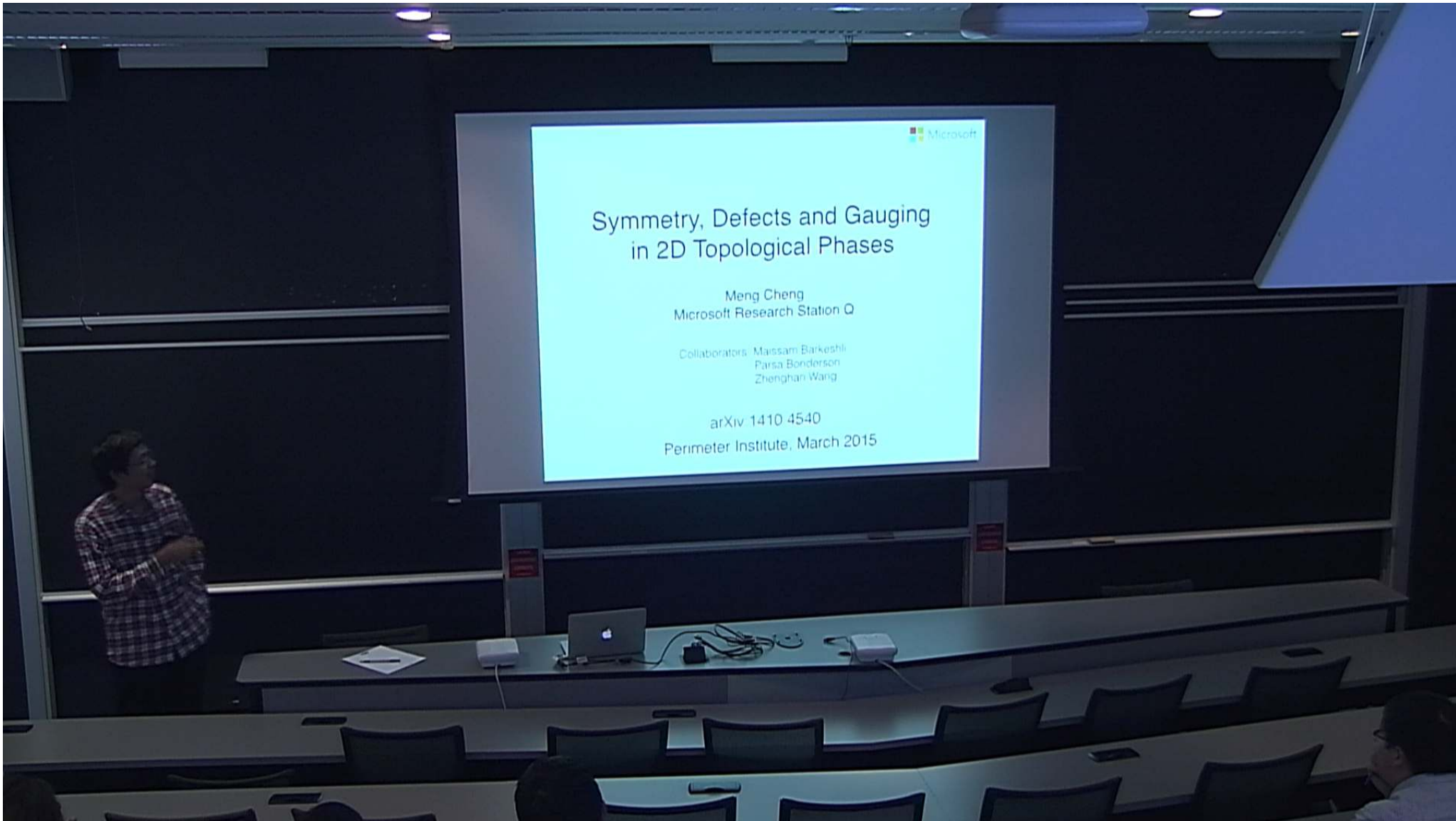
# Symmetry, Defects and Gauging in 2D Topological Phases

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Parsa Bonderson  
Zhenghan Wang

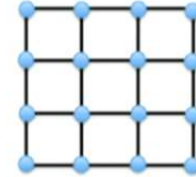
arXiv:1410.4540

Perimeter Institute, March 2015



# Topological Phases of Matter

Local Hilbert space  $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$



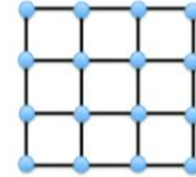
Hamiltonian with short-range interactions

Topological phases =  $\frac{\text{Gapped quantum phases}}{\sim \text{by adiabatic continuity}}$

- Two-dimensions:
- Chiral central charge on the edge
  - Anyonic quasiparticles in the bulk

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# Topological Phases + Symmetry

Symmetric gapped quantum phases  
~ by symmetric adiabatic continuity



No symmetry



With symmetry

Trivial phases

- Topological insulators  $U(1) \times \mathbb{Z}_2^T$
- Haldane phases  $SO(3)$

Symmetry **P**rotected **T**opological Phases

Topological phases

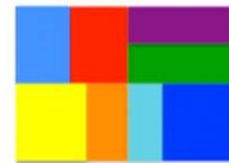
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# Symmetry Fractionalization

## Fractional Quantum Hall States

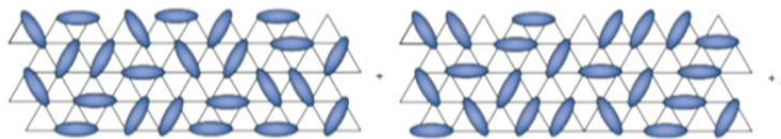
sistent with all the experimental facts and explain the effect. The ground state is a new state of matter, a quantum fluid the elementary excitations of which, the quasielectrons and quasiholes, are **fractionally charged**. I have verified

R. Laughlin, PRL 1983

A quasihole in 1/3 Laughlin state carries 1/3 charge

$$\prod_i (z_i - \xi) \prod_{i < j} (z_i - z_j)^3$$

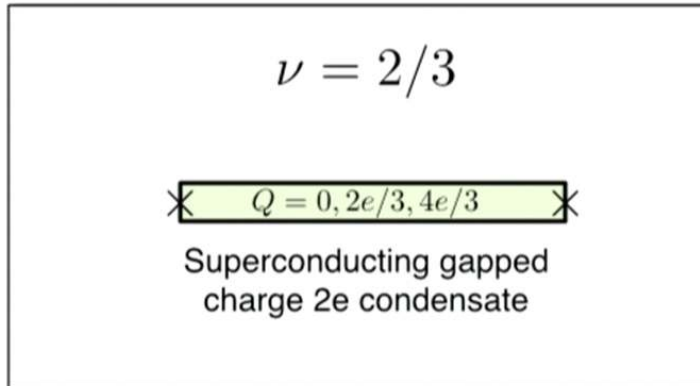
## Gapped $Z_2$ Spin Liquid



Short-range RVB state (L. Balents 2013)

SO(3) symmetry  
Deconfined spinons

# Twist Defects: Parafermion Zero Modes



Setup in Clarke et. al. arXiv:1312.6123

$Z_3$  anyons in the FQH bulk

Charge  $0, 2/3, 4/3$

$n$  parafermion zero modes

$$\dim \mathcal{H}_n = 3^{n/2-1}$$

Quantum dimension  $d_\sigma = \sqrt{3}$

Conjectured fusion rules:

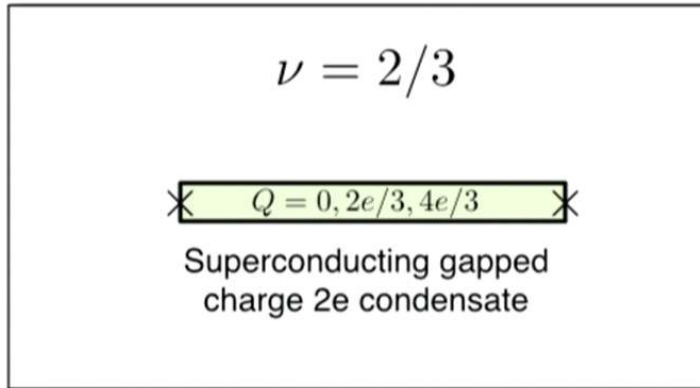
$$\sigma \times \sigma = [0] + [2/3] + [4/3]$$

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However, this theory does not admit braiding!

Clarke et. al. 2012, 2013; Lindner et. al. 2012; MC 2012

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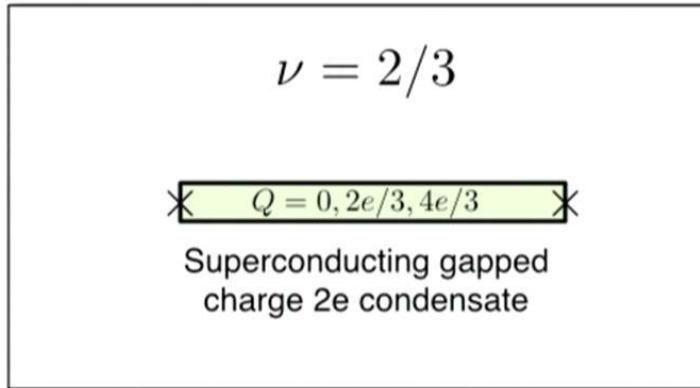
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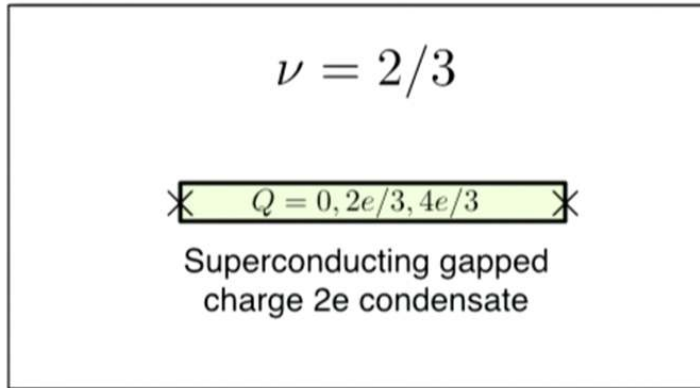
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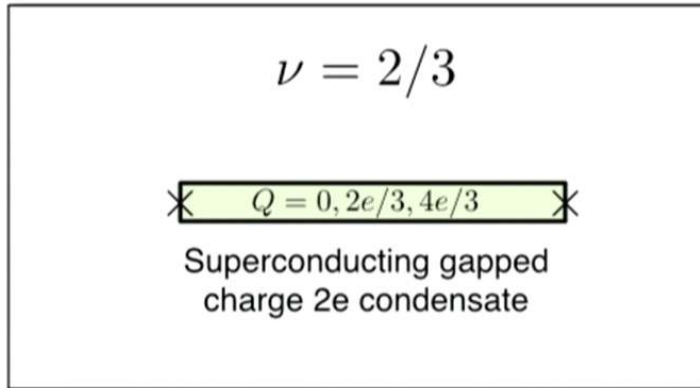
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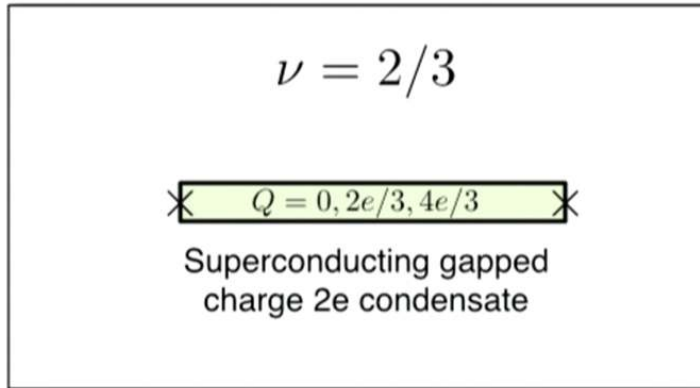
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# Outline

- Review of anyon models
- Topological symmetry and fractionalization
- Algebraic theory of defects
- Modular transformations

Disclaimer: Only bosonic systems; on-site unitary symmetries

# Anyon Model

Anyon types =  $\frac{\text{Finite-energy quasiparticle excitations}}{\sim \text{by local operators}}$

labeled by  $a, b, c, \dots$

Fusion rules:  $a \times b = \sum_c N_{ab}^c c, N_{ab}^c \geq 0$

Splitting/fusion spaces:  $\begin{array}{c} a \\ \swarrow \\ \mu \\ \uparrow \\ c \end{array} \begin{array}{c} b \\ \searrow \\ \mu \\ \uparrow \\ c \end{array} = |a, b; c, \mu\rangle \in V_c^{ab}$   $\begin{array}{c} c \\ \uparrow \\ \mu \\ \swarrow \\ a \end{array} \begin{array}{c} \mu \\ \searrow \\ b \end{array} = \langle a, b; c, \mu| \in V_{ab}^c$

F symbols:  
(basis transformations)

$$\begin{array}{c} a & b & c \\ \swarrow & \searrow & \uparrow \\ \alpha & \beta & \mu \\ \swarrow & \searrow & \uparrow \\ e & \beta & \mu \\ \swarrow & \searrow & \uparrow \\ d \end{array} = \sum_{f, \mu, \nu} [F_d^{abc}]_{(e, \alpha, \beta)(f, \mu, \nu)} \begin{array}{c} a & b & c \\ \swarrow & \searrow & \uparrow \\ \nu & \mu & \mu \\ \swarrow & \searrow & \uparrow \\ d \end{array}$$

R symbols:  
(braiding)

$$\begin{array}{c} a & b \\ \swarrow & \searrow \\ \mu \\ \uparrow \\ c \end{array} = \sum_{\nu} [R_c^{ab}]_{\mu\nu} \begin{array}{c} a & b \\ \swarrow & \searrow \\ \nu \\ \uparrow \\ c \end{array}$$

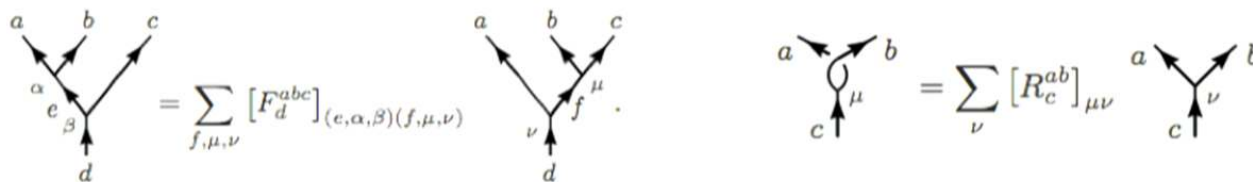
# Gauge Transformations

Redefine basis states:

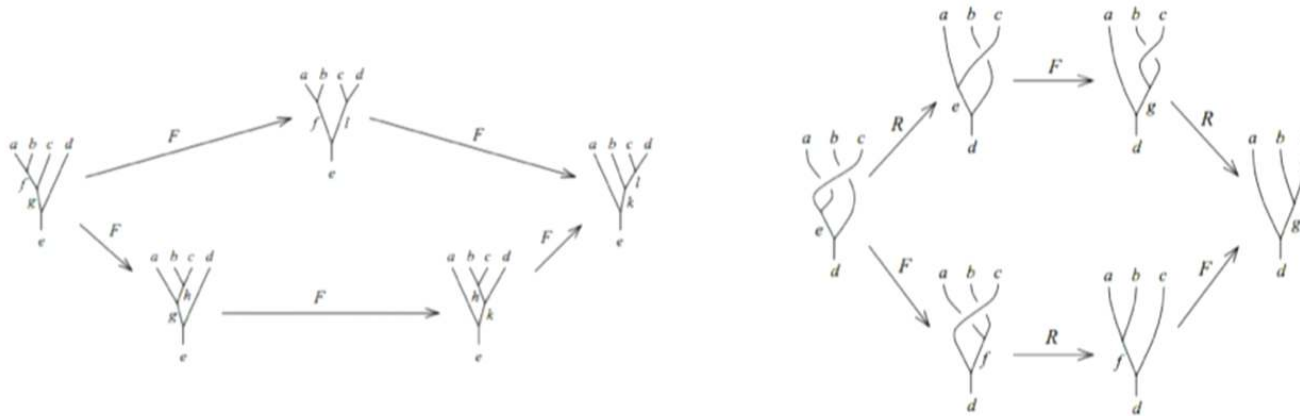
$$|\widetilde{a, b; c, \mu}\rangle = \sum_{\mu\nu} [\Gamma_c^{ab}]_{\mu\nu} |a, b, c; \nu\rangle$$

$$[\widetilde{F}_d^{abc}]_{ef} = \Gamma_e^{ab} \Gamma_d^{ec} [F_d^{abc}]_{ef} (\Gamma_f^{bc})^{-1} (\Gamma_d^{af})^{-1}$$

$$\widetilde{R}_c^{ab} = \Gamma_c^{ba} R_c^{ab} (\Gamma_c^{ab})^{-1}$$



# Consistency equations



No further conditions are needed

**MacLane's coherence theorem**

Solutions/ $\sim$  by gauge transformations are discrete

**Ocneanu rigidity**

# Topological Invariants

Quantum dimensions  $d_a = a$  

Topological twists  
(exchange statistics)  $\theta_a = \frac{1}{d_a} \text{ (figure-eight diagram) } = \sum_{c,\mu} \frac{d_c}{d_a} [R_c^{aa}]_{\mu\mu}$

Topological S matrix  
(full braiding)  $S_{ab} = \mathcal{D}^{-1} \sum_c N_{\bar{a}b}^c \frac{\theta_c}{\theta_a \theta_b} d_c = \frac{1}{\mathcal{D}} \text{ (braiding diagram) }$

Braiding non-degeneracy  
(Modularity)

S is non-singular

Unitary Modular Tensor Category

# Topological Symmetry

$\text{Aut}(\mathcal{C}) =$  Permutations of anyon labels that leave physical quantities invariant

$$\varphi(a) = a' \qquad \varphi(|a, b; c\rangle) = u_{c'}^{a'b'} |a', b'; c'\rangle$$

$$\varphi(F_d^{abc}) = \tilde{F}_{d'}^{a'b'c'} = F_d^{abc}$$

$$\theta_{a'} = \theta_a, S_{a'b'} = S_{ab}$$

$$\varphi(R_c^{ab}) = \tilde{R}_{c'}^{a'b'} = R_c^{ab}$$

$\text{Aut}(\mathcal{C}) = \frac{\text{Auto-equivalences } \varphi}{\sim \text{ by natural isomorphism}}$

$$\Upsilon(a) = a$$

Natural isomorphisms as the identity:

$$\Upsilon(|a, b; c\rangle) = \frac{\gamma_a \gamma_b}{\gamma_c} |a, b; c\rangle$$

Emergent symmetries of the topological order

# Global Symmetry

G: on-site unitary symmetry group

$$R_g = \prod_i R_g^{(i)} \quad R_g R_h = R_{gh}$$

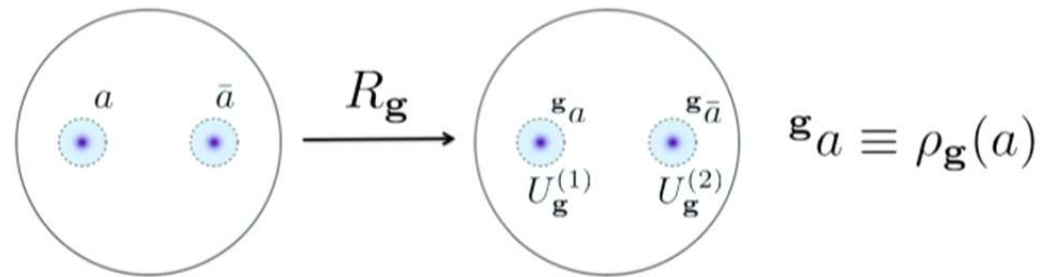
How symmetries permute anyons?

Group homomorphism  $\rho : G \rightarrow \text{Aut}(\mathcal{C})$

$$\rho_g \text{ has } \begin{cases} \text{actions on anyon types } \mathfrak{g}a \equiv \rho_g(a) \\ \rho_g(|a, b; c\rangle) = U_g(\mathfrak{g}a, \mathfrak{g}b; \mathfrak{g}c) | \mathfrak{g}a, \mathfrak{g}b; \mathfrak{g}c \rangle \end{cases}$$

$$\rho_{gh} = \kappa_{g,h} \rho_g \rho_h \quad \kappa_{g,h} |a, b; c\rangle = \frac{\beta_a \beta_b}{\beta_c} |a, b; c\rangle$$

# Symmetry Localization



$$\begin{aligned} R_{\mathfrak{g}}|\Psi_{a,\bar{a};0}\rangle &= U_{\mathfrak{g}}^{(1)}U_{\mathfrak{g}}^{(2)}\rho_{\mathfrak{g}}|\Psi_{a,\bar{a};0}\rangle \\ &= U_{\mathfrak{g}}^{(1)}U_{\mathfrak{g}}^{(2)}U_{\mathfrak{g}}(\mathfrak{g}a,\mathfrak{g}\bar{a};0)|\Psi_{\mathfrak{g}a,\mathfrak{g}\bar{a};0}\rangle \end{aligned}$$

$$U_{\mathfrak{g}}^{(j)}\rho_{\mathfrak{g}}U_{\mathfrak{h}}^{(j)}\rho_{\mathfrak{g}}^{-1} = \eta_{a_j}(\mathfrak{g},\mathfrak{h})U_{\mathfrak{gh}}^{(j)}$$

Local actions can be projective

# Symmetry Localization

Given a  $[\rho]$ , we identify an obstruction class

$$[\mathcal{O}] \in H_\rho^3(G, \mathcal{A})$$

If  $[\mathcal{O}]$  does not vanish, the symmetry can not be localized. **End of story.**

When  $[\mathcal{O}]$  vanishes

Symmetry fractionalization are classified by  $H_\rho^2(G, \mathcal{A})$

$$1/m \text{ Laughlin state} \quad G = \text{U}(1) \quad H^2(\text{U}(1), \mathbb{Z}_m) = \mathbb{Z}_m$$

E.g fractional 1/m charges

$$\mathbb{Z}_2 \text{ spin liquids} \quad G = \text{SO}(3) \quad H^2(\text{SO}(3), \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^2$$

Spin 1/2 projective rep. of SO(3)

Barkeshli, Bonderson, MC, Wang (2014)

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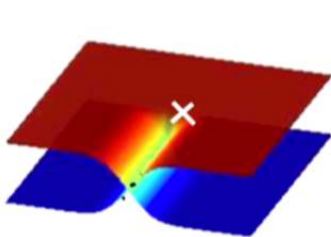
Barkeshli, Bonderson, MC, Wang (2014)

# Why Defects?

- They can be used to characterize SPT/SET phases.

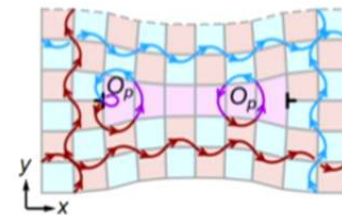
Levin and Gu 2012, 2013; Cheng and Gu, 2013; Hung and Wan 2012;  
Wen 2012; Zaletel 2013; Chen et. al. 2014

- Non-Abelian defects can be exploited for topological quantum information processing.



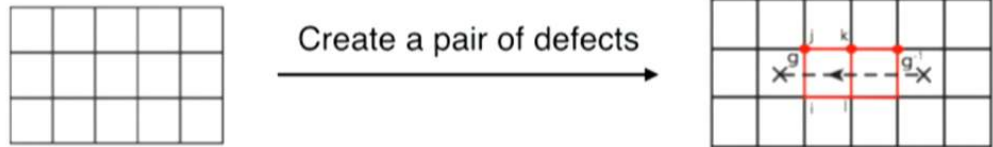
$$\nu = 2/3$$
$$Q = 0, 2e/3, 4e/3$$

Superconducting gapped



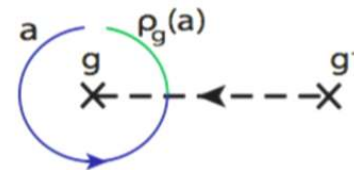
Barkeshli-Qi, 2011; Barkeshli-Qi, 2013; Bombin 2010; Kitaev-Kong 2011; You-Wen 2012

# Extrinsic Symmetry Defects



$$H_0 = \sum_i h_i + \sum_{\langle ij \rangle} h_{ij}$$

$$H_{g, g^{-1}} = H_0 + \sum_{\substack{\langle ij \rangle: \\ i \in C_l; j \in C_r}} [R_g^{(j)} h_{ij} R_g^{(j)-1} - h_{ij}]$$



Generalized Aharonov-Bohm effect

$$\nu = 2/3$$

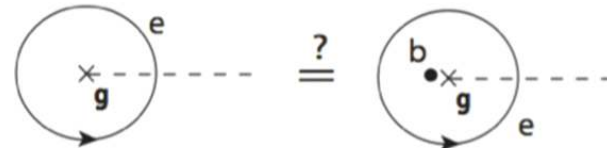
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Superconducting gapped

$$[2e/3] \rightarrow [-2e/3] \equiv [4e/3]$$

# G-graded Fusion of Defects

There may be distinct  $\mathfrak{g}$ -defects  
measured by  $\mathfrak{g}$ -invariant Wilson loops



$\mathcal{C}_{\mathfrak{g}}$  Collection of  $\mathfrak{g}$ -defects  $\mathcal{C}_G = \bigoplus_{\mathfrak{g} \in G} \mathcal{C}_{\mathfrak{g}}$

G-graded fusion rules  $a_{\mathfrak{g}} \times b_{\mathfrak{h}} = \sum_{c_{\mathfrak{gh}}} N_{a_{\mathfrak{g}} b_{\mathfrak{h}}}^{c_{\mathfrak{gh}}} c_{\mathfrak{gh}}$

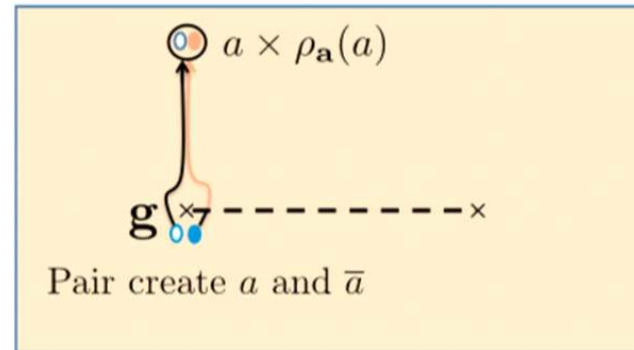
- Equal total quantum dimension in each sector  $\sum_{a_{\mathfrak{g}}} d_a^2 = \mathcal{D}_0$
- # of  $\mathfrak{g}$ -defects = # of  $\mathfrak{g}$ -invariant anyons (require modularity)

Barkeshli, Bonderson, MC, Wang (2014)

# Example: $\mathbb{Z}_2$ Toric Code with EM Duality

Toric code  $1, e, m, \psi$

$\mathbb{Z}_2$  symmetry  $e \leftrightarrow m$

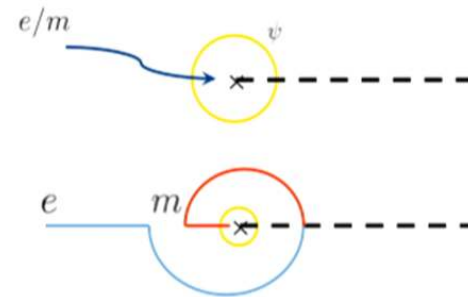


Fusion of defects

$$\sigma_{\pm} \times e = \sigma_{\pm} \times m = \sigma_{\mp}$$

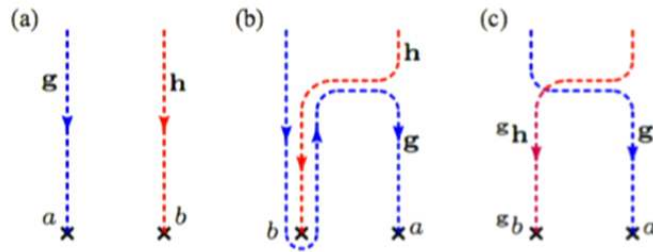
$$\sigma_{+} \times \sigma_{+} = \sigma_{-} \times \sigma_{-} = 1 + \psi$$

$\sigma_{\pm}$  carries a Majorana zero mode



Bombin 2010; Kitaev-Kong 2011; You-Wen 2012

# G-crossed Braiding



Exchanging **g**- and **h**-defects

$${}^g h \equiv g h g^{-1}$$

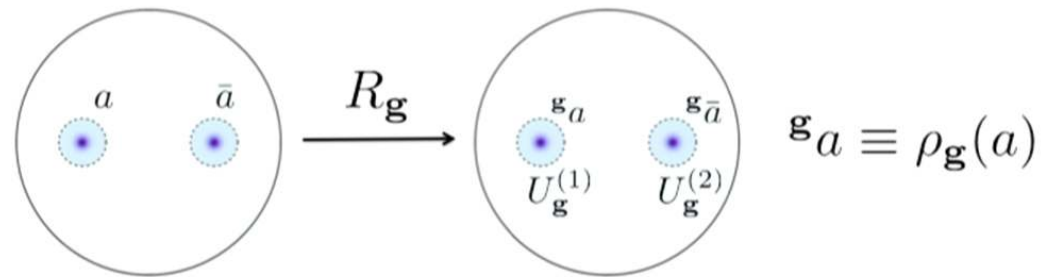
$$R^{a_g b_h} = \begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ b_h \quad \bar{h} a_g \end{array}$$

Sliding rules **NEW!**

Symmetry action on splitting space:

Projective action on anyons/defects:

# Symmetry Localization

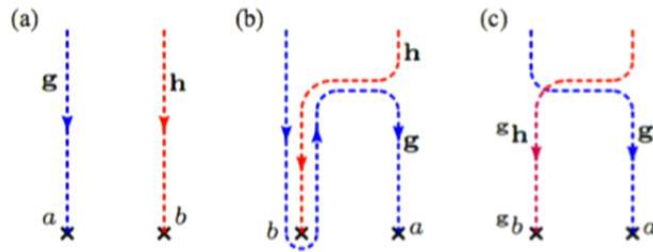


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Local actions can be projective

# G-crossed Braiding



$$R^{a_g b_h} = \begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ b_h \quad \bar{h} a_g \end{array}$$

Exchanging **g**- and **h**-defects

$${}^g h \equiv g h g^{-1}$$

Sliding rules **NEW!**

Symmetry action on splitting space:

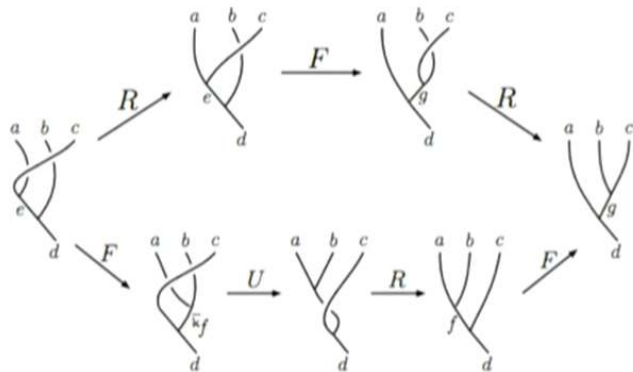
$$\begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ x_k \quad \mu \\ \bar{k} c_{gh} \end{array} = \sum_{\nu} [U_{\mathbf{k}}(a, b; c)]_{\mu\nu} \begin{array}{c} a_g \quad b_h \\ \diagdown \quad \diagup \\ x_k \quad \nu \\ \bar{k} c_{gh} \end{array}$$

Projective action on anyons/defects:

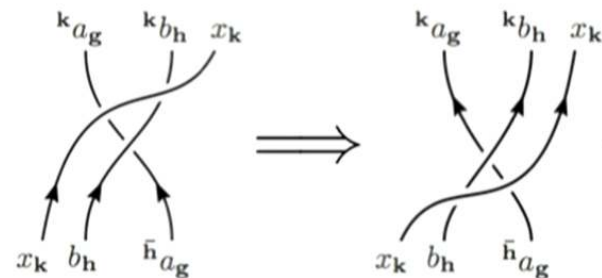
$$\begin{array}{c} a_g \quad b_h \quad h_g x_k \\ \diagdown \quad \diagup \\ x_k \quad \mu \\ c_{gh} \end{array} = \eta_x(\mathbf{g}, \mathbf{h}) \begin{array}{c} a_g \quad b_h \quad h_g x_k \\ \diagdown \quad \diagup \\ x_k \quad \mu \\ c_{gh} \end{array}$$

# New Consistency Equations

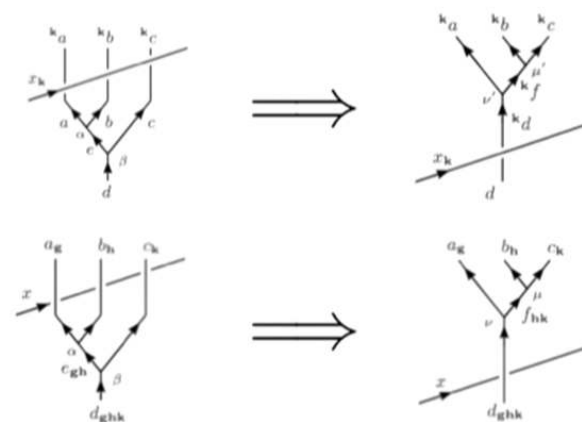
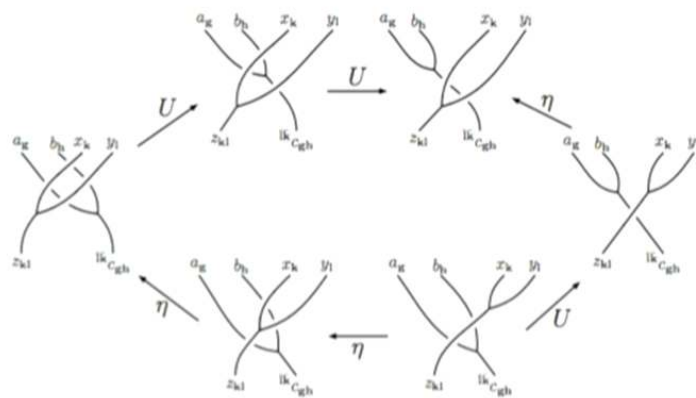
G-crossed heptagon equation



Yang-Baxter equation



Sliding consistency



# Gauge Transformations

Vertex-basis gauge transformations

$$|\widetilde{a, b; c, \mu}\rangle = \sum_{\mu'} [\Gamma_c^{ab}]_{\mu\mu'} |a, b; c, \mu'\rangle$$

$$\tilde{U}_{\mathbf{k}}(a, b; c) = \Gamma_{\mathbf{k}_c}^{k_a k_b} U_{\mathbf{k}}(a, b; c) (\Gamma_c^{ab})^{-1}$$

$$[\tilde{F}_d^{abc}]_{ef} = \Gamma_e^{ab} \Gamma_d^{ec} [F_d^{abc}]_{ef} (\Gamma_f^{bc})^{-1} (\Gamma_d^{af})^{-1}$$

$$\tilde{R}_{c_{\mathbf{g}\mathbf{h}}}^{a_{\mathbf{g}} b_{\mathbf{h}}} = \Gamma_c^{b_{\mathbf{h}} a_{\mathbf{g}}} R_{c_{\mathbf{g}\mathbf{h}}}^{a_{\mathbf{g}} b_{\mathbf{h}}} (\Gamma_c^{ab})^{-1}$$

Symmetry-action gauge transformations

$$\tilde{\rho}_{\mathbf{g}} = \Upsilon_{\mathbf{g}} \rho_{\mathbf{g}}$$

$$\tilde{U}_{\mathbf{k}}(a, b; c) = \frac{\gamma_a(\mathbf{k}) \gamma_b(\mathbf{k})}{\gamma_c(\mathbf{k})} U_{\mathbf{k}}(a, b; c)$$

$$\tilde{\eta}_x(\mathbf{g}, \mathbf{h}) = \frac{\gamma_x(\mathbf{g}\mathbf{h})}{\gamma_{\mathbf{g}\mathbf{x}}(\mathbf{h}) \gamma_x(\mathbf{g})} \eta_x(\mathbf{g}, \mathbf{h})$$

$$\tilde{R}_{c_{\mathbf{g}\mathbf{h}}}^{a_{\mathbf{g}} b_{\mathbf{h}}} = \gamma_a(\mathbf{h}) R_{c_{\mathbf{g}\mathbf{h}}}^{a_{\mathbf{g}} b_{\mathbf{h}}}$$

What are physical quantities in the G-crossed theory?

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What are physical quantities in the G-crossed theory?

## Gauge-Invariant Quantities

$$\theta_a = \frac{1}{d_a} \text{ (diagram of a figure-eight loop with arrow 'a') }$$

$$\check{\theta}_{a_g} = \gamma_{a_g}(\mathbf{g}) \theta_{a_g}$$

$$S_{a_g b_h} = \frac{1}{\mathcal{D}_0} \text{ (diagram of two overlapping loops with arrows 'a' and 'b') }$$

$$\check{S}_{a_g b_h} = \gamma_{\bar{a}}(\mathbf{h}) \gamma_b(\bar{\mathbf{g}}) S_{a_g b_h}$$

Systematically construct gauge-invariant quantities:

- G-crossed modular transformations
- Gauging (“equivariantization”)

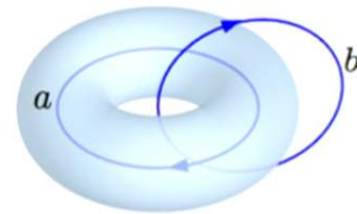
# Modular Transformation on a Torus

“Large” diffeomorphisms:  $SL(2, \mathbb{Z})$



Ground states on a torus labeled by anyon fluxes (MES)

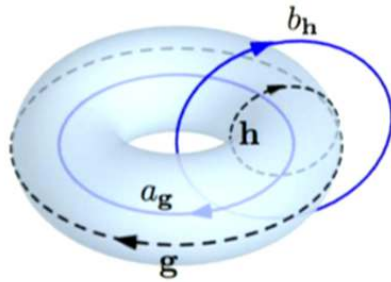
$$|a\rangle_l = \sum_b S_{ab} |b\rangle_m \quad |a\rangle_l = \sum_b T_{ab} |b\rangle_{l+m}$$



Same as the diagrammatic  $S, T$  matrices

$$(ST)^3 = e^{\frac{i\pi c_-}{4}} C, S^2 = C, C^2 = 1$$

# Torus with Defect Lines



$$\mathcal{V}^{\text{ext}} = \bigoplus_{(\mathbf{g}, \mathbf{h}), \mathbf{gh}=\mathbf{hg}} \mathcal{V}_{(\mathbf{g}, \mathbf{h})}$$

$$\dim \mathcal{V}_{(\mathbf{g}, \mathbf{h})} = |\mathcal{C}_{\mathbf{g}}^{\mathbf{h}}| \quad \mathcal{C}_{\mathbf{g}}^{\mathbf{h}} = \{ a \in \mathcal{C}_{\mathbf{g}} \mid \mathbf{h}a = a \}$$

Extended modular transformations

$$\mathcal{S}^{(\mathbf{g}, \mathbf{h})} : \mathcal{V}_{(\mathbf{g}, \mathbf{h})} \rightarrow \mathcal{V}_{(\mathbf{h}, \bar{\mathbf{g}})}$$

$$\mathcal{T}^{(\mathbf{g}, \mathbf{h})} : \mathcal{V}_{(\mathbf{g}, \mathbf{h})} \rightarrow \mathcal{V}_{(\mathbf{g}, \mathbf{gh})}$$

$$\mathcal{S}_{a_{\mathbf{g}} b_{\mathbf{h}}}^{(\mathbf{g}, \mathbf{h})} = \frac{S_{a_{\mathbf{g}} b_{\mathbf{h}}}}{U_{\mathbf{h}}(a, \bar{a}; 0)} \quad \mathcal{T}_{a_{\mathbf{g}} b_{\mathbf{g}}}^{(\mathbf{g}, \mathbf{h})} = \eta_a(\mathbf{g}, \mathbf{h}) \theta_{a_{\mathbf{g}}} \delta_{a_{\mathbf{g}} b_{\mathbf{g}}}$$

$$\mathcal{C}_{a_{\mathbf{g}} b_{\bar{\mathbf{g}}}}^{(\mathbf{g}, \mathbf{h})} = \frac{1}{U_{\mathbf{h}}(\bar{b}, b; 0) \eta_b(\mathbf{h}, \bar{\mathbf{h}})} \delta_{a_{\mathbf{g}} b_{\bar{\mathbf{g}}}}$$

# Topological Invariants from Modular Transformations

Invariance under symmetry-action gauge transformations

$$Q^{(\mathbf{g}, \mathbf{h})} : \mathcal{V}_{(\mathbf{g}, \mathbf{h})} \rightarrow \mathcal{V}_{(\mathbf{g}', \mathbf{h}')} \quad \check{Q}_{a_{\mathbf{g}} b_{\mathbf{g}'}}^{(\mathbf{g}, \mathbf{h})} = \frac{\gamma_b(\mathbf{h}')}{\gamma_a(\mathbf{h})} Q_{a_{\mathbf{g}} b_{\mathbf{g}'}}^{(\mathbf{g}, \mathbf{h})}$$

If  $\mathbf{h} = \mathbf{h}'$  and  $a = b$ ,  $Q_{a_{\mathbf{g}} a_{\mathbf{g}}}^{(\mathbf{g}, \mathbf{h})}$  is an invariant

If  $\mathbf{h} = \mathbf{h}' = \mathbf{0}$ ,  $Q_{a_{\mathbf{g}} b_{\mathbf{g}}}^{(\mathbf{g}, \mathbf{0})}$  is an invariant

$$\mathbf{g}^n = \mathbf{0} \quad [\mathcal{T}^n]_{a_{\mathbf{g}} a_{\mathbf{g}}}^{(\mathbf{g}, \mathbf{h})} = \theta_{a_{\mathbf{g}}}^n \prod_{j=0}^{n-1} \eta_{a_{\mathbf{g}}}(\mathbf{g}, \mathbf{g}^j \mathbf{h})$$

# Application I: Classification of Bosonic SPT

Defect theory:  $\mathcal{C} = \{1\}$   $\mathcal{C}_{\mathbf{g}} = \{\mathbf{g}\}$

$$\mathbf{g} \times \mathbf{h} = \mathbf{gh}$$

$$[F_{\mathbf{ghk}}^{\mathbf{g},\mathbf{h},\mathbf{k}}]_{\mathbf{gh},\mathbf{hk}} \equiv \omega(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in Z^3(G, U(1))$$

Gauge transformations:  $B^3(G, U(1))$

Bosonic SPT phases are classified by  $H^3(G, U(1))$

$$R_{\mathbf{gh}}^{\mathbf{g},\mathbf{h}} = 1$$

$$U_{\mathbf{k}}(\mathbf{g}, \mathbf{h}; \mathbf{gh}) = \frac{\alpha(\mathbf{g}, \mathbf{k}, \bar{\mathbf{k}}\mathbf{h})}{\alpha(\mathbf{g}, \mathbf{h}, \mathbf{k})\alpha(\mathbf{k}, \bar{\mathbf{k}}\mathbf{g}, \bar{\mathbf{k}}\mathbf{h})}$$

$$\eta_{\mathbf{k}}(\mathbf{g}, \mathbf{h}) = \frac{\alpha(\mathbf{g}, \bar{\mathbf{g}}\mathbf{k}, \mathbf{h})}{\alpha(\mathbf{g}, \mathbf{h}, \bar{\mathbf{h}}\bar{\mathbf{g}}\mathbf{k})\alpha(\mathbf{k}, \mathbf{g}, \mathbf{h})}$$

Chen, Gu, Liu and Wen 2011

# Application II: $Z_N$ Parafermion Zero Modes

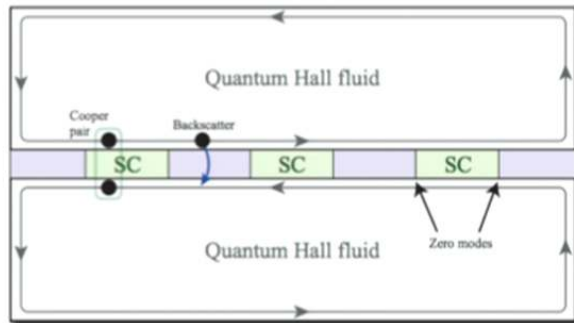


Figure from arXiv:1410.0359

$$\mathcal{C}_{\mathbf{g}} = \{\sigma\}$$

$$\sigma \times a = \sigma, \sigma \times \sigma = \sum_{a=0}^{N-1} a$$

Clarke et. al. 2012; Lindner et. al. 2012; MC 2012

Particle-hole symmetry in a  $1/N$  Laughlin

$$a = 0, 1, \dots, N - 1$$

$$F_{[a+b+c]}^{abc} = 1, R_{[a+b]}^{ab} = e^{\frac{2\pi i p a b}{N}}$$

$$\rho_{\mathbf{g}} : [a] \rightarrow [-a]$$

$$S = \left[ \begin{array}{c|ccc} S_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma^2 e^{\frac{\pi i}{4} c} \end{array} \right] \begin{array}{l} |0^{(0,\mathbf{g})}\rangle \\ |\sigma^{(\mathbf{g},0)}\rangle \\ |\sigma^{(\mathbf{g},\mathbf{g})}\rangle \end{array}$$

$$\mathcal{T} = \left[ \begin{array}{c|ccc} T_0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \gamma^* \\ 0 & 0 & \gamma^* & 0 \end{array} \right]$$

$$\gamma^2 = \pm \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (-1)^{pn} e^{-\frac{\pi i p n^2}{N}}$$

# Conclusions

- General classification of symmetry fractionalization
- Develop an algebraic theory of defect
  - New data and consistency conditions
  - Topological invariants from modular transformations
  - [Obstruction theory](#)
  - [Construction of the gauged theory](#)
- Related works
  - Etingof, Nikshych and Ostrik, arXiv:0909.3140 (MATH)
  - Fidkowski, Lindner and Kitaev, unpublished
  - Teo, Taylor and Fradkin, arXiv:1503.06812