Title: Expansion of Lie algebras and accidental symmetries in Lovelock theories

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Abstract: <p>Main properties of generalized contraction methods of Lie algebras, known also as expansion methods, are briefly introduced. Between some of their physical applications, one might study the nature of solutions in theories constructed with those expanded algebras. In particular, as we are interested in solutions that could be relevant in the context of AdS/CFT and Holographic Superconductors, we would like to study the holographic QFT dual to Chern-Simons gravity for an expansion of AdS algebra. As a first step, we studied charged static spherically symmetric BH solutions of a CS theory for the most simple extension of AdS symmetry: AdSÃ—U(1). It is shown that in this kind of higher dimensional gravity, degeneracy in some sectors of the space of solutions can appear. In fact, arbitrary functions remain undetermined after the field equations are imposed. This is related to an increase in local symmetries and it is shown that the knowledge of these "accidental symmetries" can help to formulate a simple criterion that avoids unwanted degenerate ansatze. Finally, main properties of Pure Lovelock gravity are presented and some issues about black hole solutions this theory are also discussed.</p>

Expansion of Lie algebras and accidental symmetries in Lovelock theories

Nelson Merino

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April 2015

Contents

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For example, in $D = 4$ Einstein-Cartán theory

$$\mathcal{L}_{EC}(e,\omega) = \kappa \left(\epsilon_{abcd} R^{ab} e^c e^d - \frac{\Lambda}{6} e^a e^b e^c e^d \right)\ ;\ \ \kappa = \frac{1}{32\pi G}\ ,$$

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Generalization to D-dimensions: Lanczos-Lovelock (L-L) gravity

$$\mathcal{L} = \sum_{p=0}^{\left[\frac{D}{2}\right]} \alpha_p \mathcal{L}^{(p)} \, ,$$

$$\mathcal{L}^{(p)} = \epsilon_{a_1 \cdots a_D} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D} \; .$$

Requiring the theory to have the maximum possible number of degrees of freedom, leads to a special choosing[1] of α_p which in even D gives to Born-Infeld (BI) gravity; while in odd D it gives AdS-Chern-Simons (CS) gravity, described by

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Ex. in $D = 5$: $\mathcal{L}_{\text{CS}}(\mathbf{A}) = k\epsilon_{abcde}\left(\frac{1}{5l^5} e^a e^b e^c e^d e^e + \frac{2}{3l^3} R^{ab} e^c e^d e^e + \frac{1}{l} R^{ab} R^{cd} e^e\right)$.

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Another interesting family is Pure Lovelock theory[2], which have only α_0 and α_N $(N = [(D-1)/2])$ non-vanishing.

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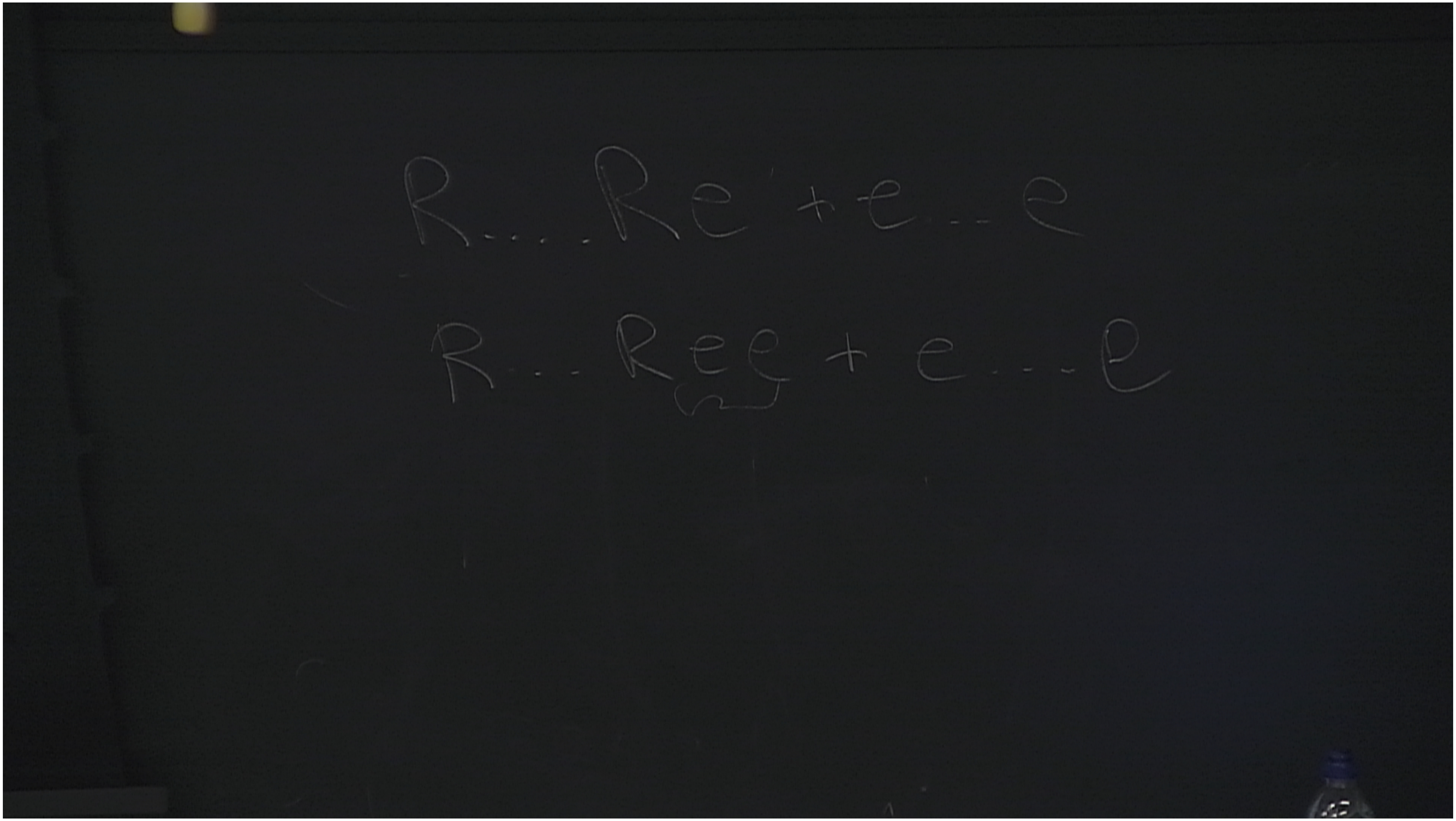
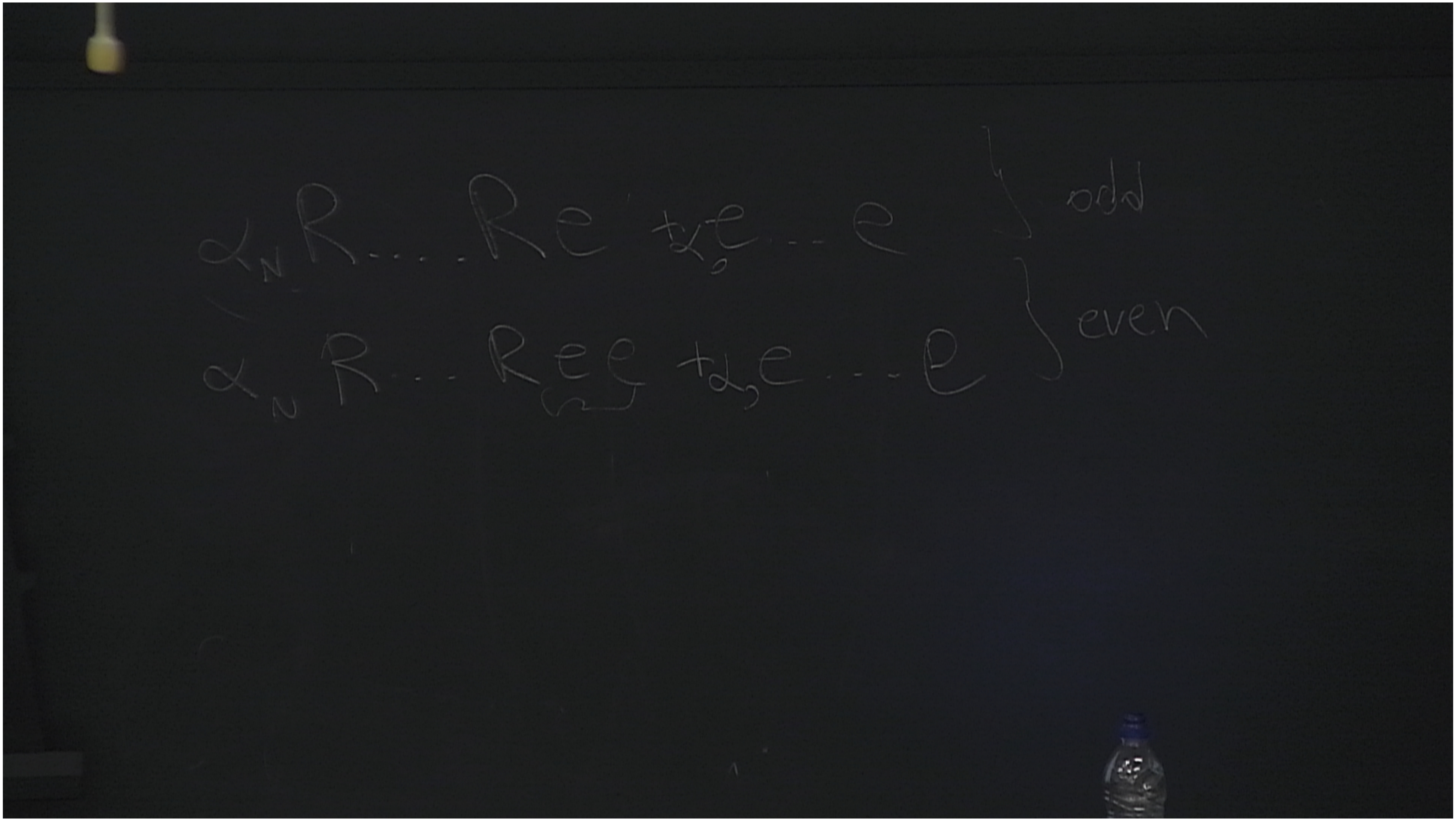

R....Re'+e...e
R...Ree+e...e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$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2\, d\Omega^2_{D-2} \quad ; \quad \text{with}\, f^2 = 1 - \psi\,(r)\, r^2\,.$$

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L-L field equations reduces to,

$$\mathcal{F}(\psi(r)) = \sum_p \tilde{\alpha}_p \psi^p = \frac{\mu}{r^{D-1}}\,,$$

where μ is the mass parameter and $\tilde{\alpha}_p = \alpha_p 2\kappa\,(D-2p)\,(D-2)!\,.$

When coupling constants are such that $F(\psi)$ has a degenerate zero, the field equations leaves g_{tt} arbitrary (and were called "geometrically free" solutions by J. T. Wheeler).

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$$f^2 = 1 - \beta r^2 - r^2 \left(\frac{\mu}{r^{D-1}}\right)^{\frac{1}{N}} \underset{\text{large } r}{\nrightarrow} \underbrace{f^2 = 1 - ar^2 - \frac{b}{r^{D-3}}}_{\text{Schwarzschild dS/AdS BH}}\,, \text{where} \begin{cases} a \sim \Lambda \\ b \sim \mu \end{cases}$$

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- The other way is to restrict to non vanishing coeficients α_0 and α_N (with $N = [(D-1)/2]$), known as Pure Lovelock case[4]:

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This work also deals with expansion methods of Lie algebras[5] and gravity theories constructed with them (for which BH and a cosmological solutions have already been studied[6]).

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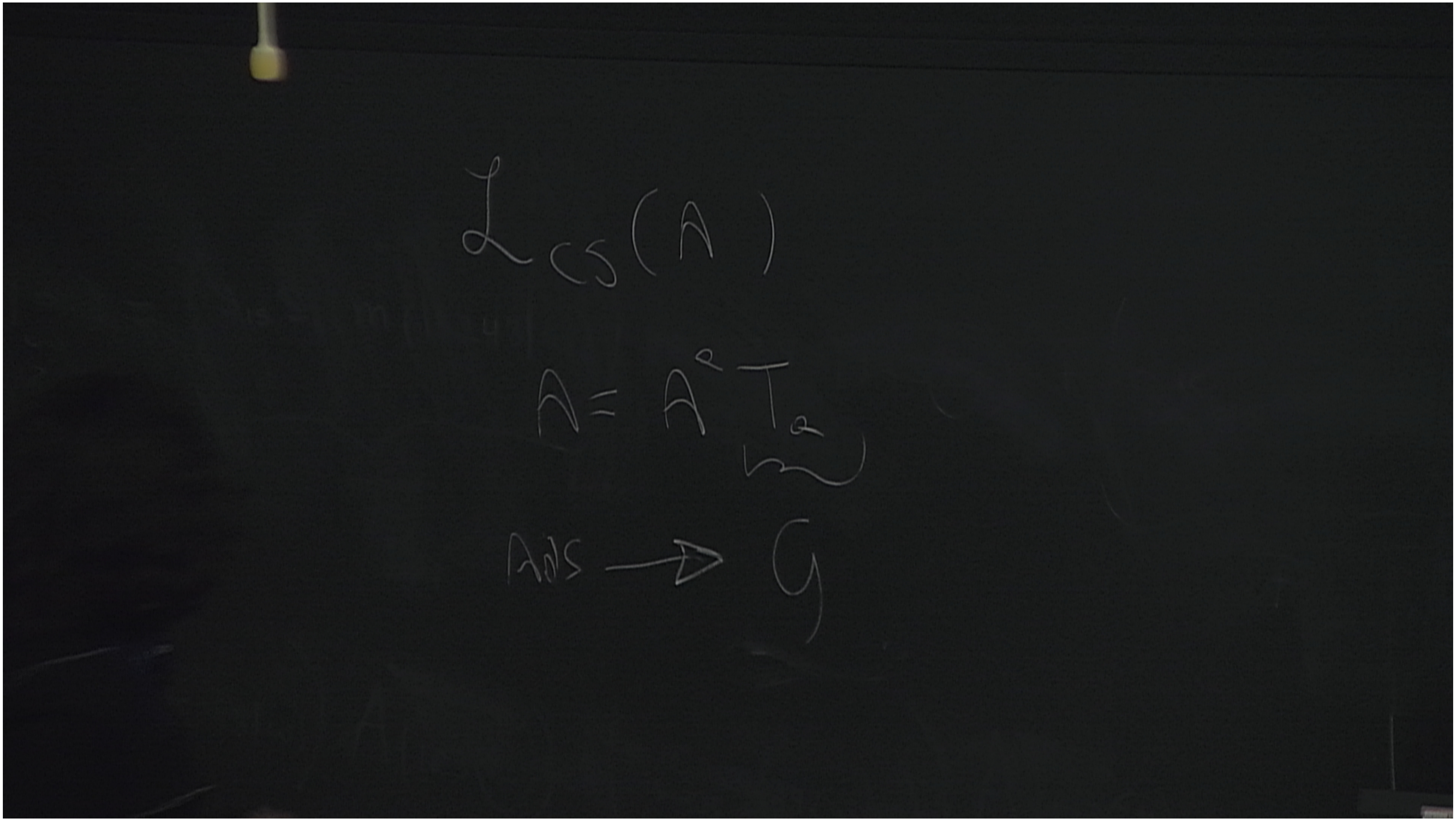
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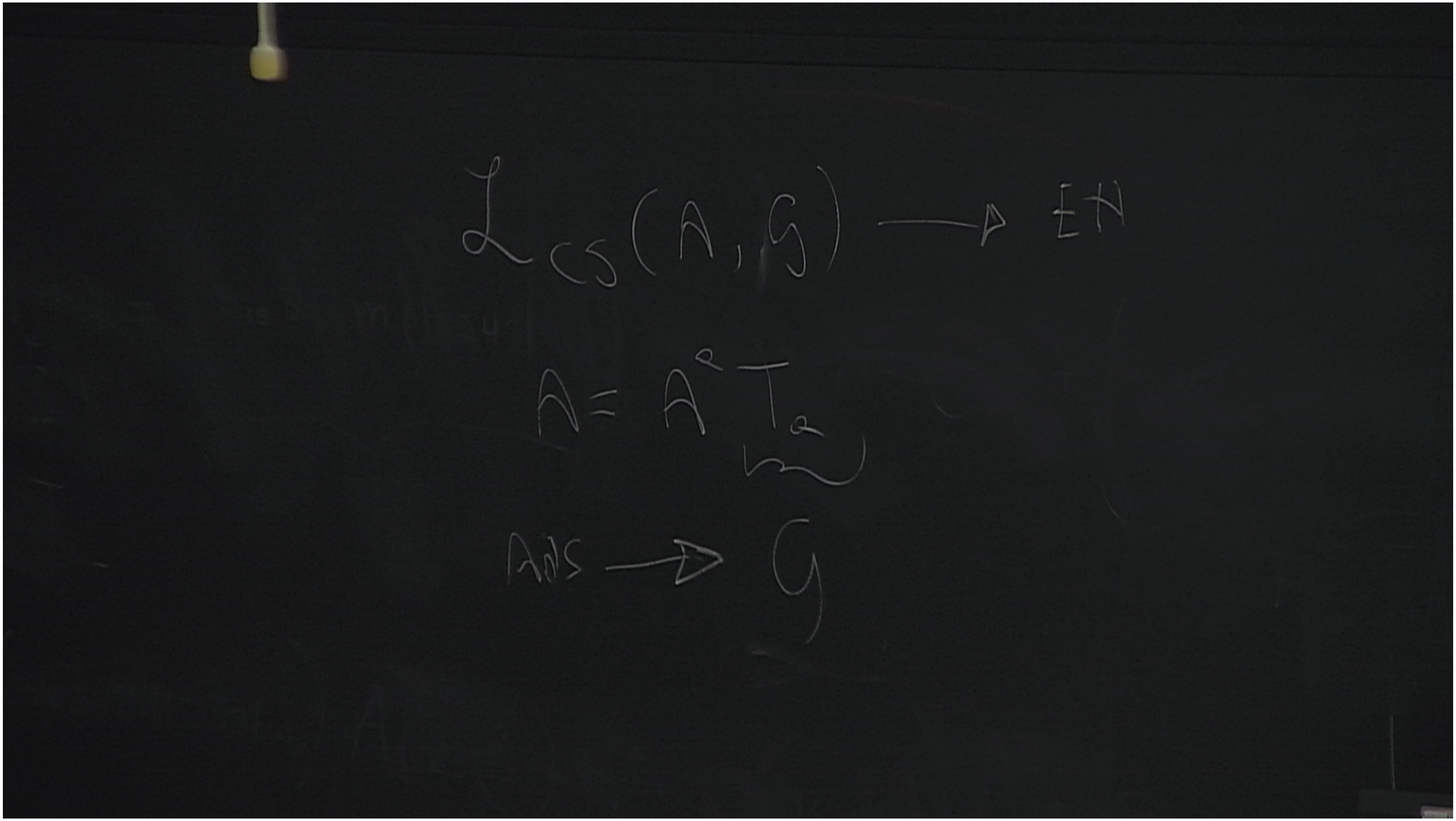
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- There are some unconventional superconductors (discovered '78 and 86') whose theoretical description so far is based on holography [e.g., arXiv:1308.2976 for a recent review].

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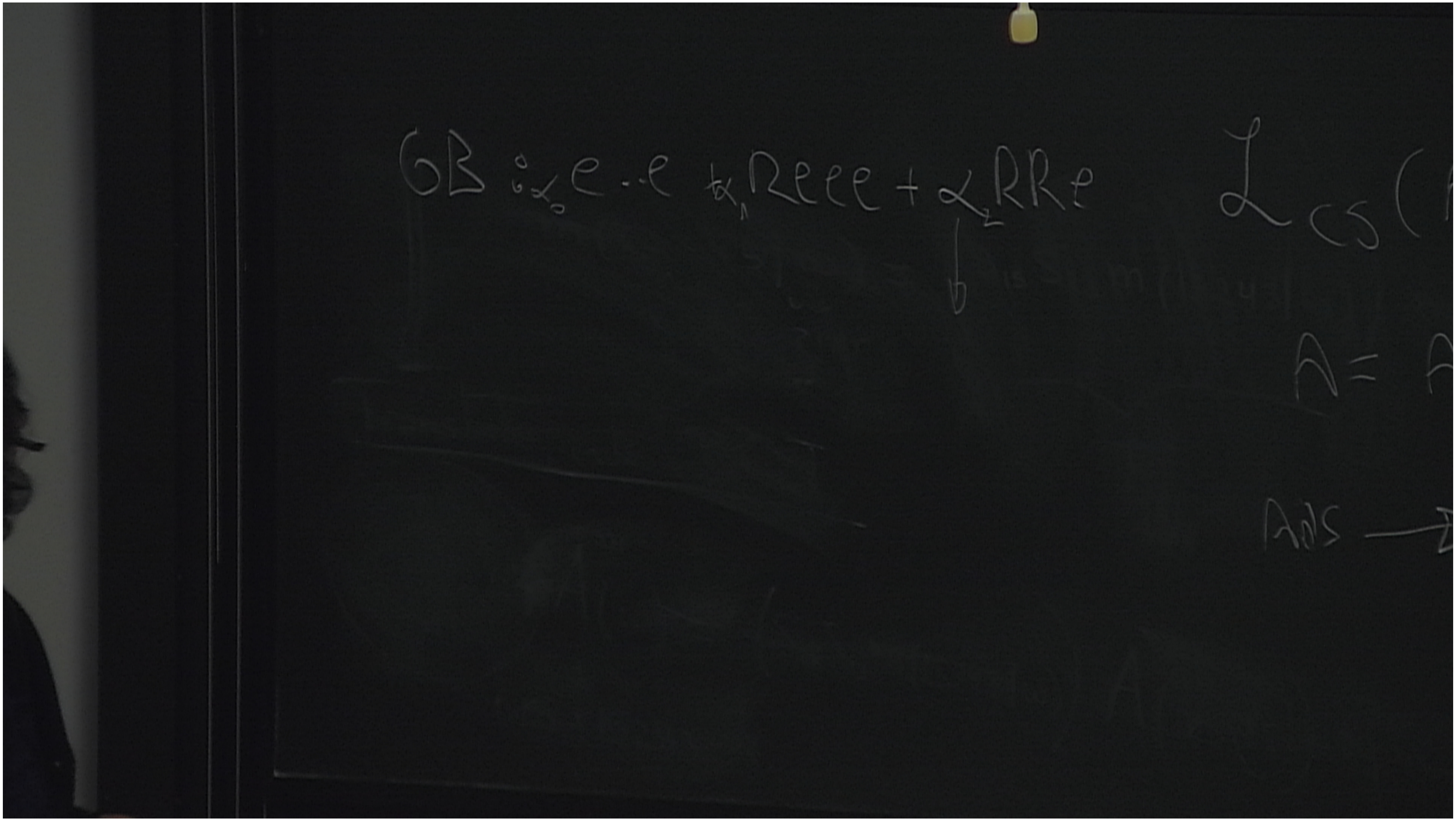
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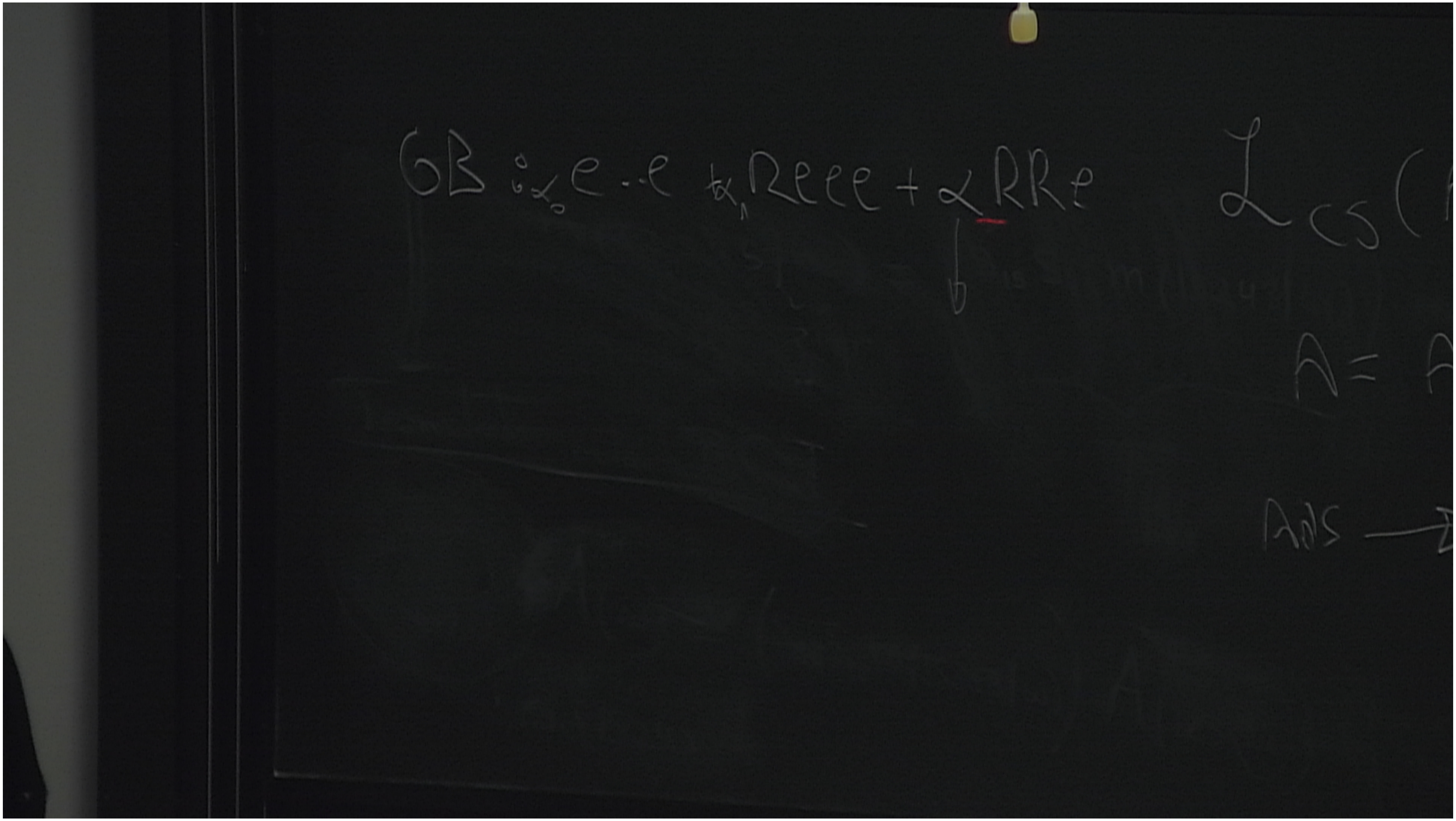
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As a first step in arXiv:1406.3096 - also in collaboration with G. Giribet (UBA, Argentina) and J. Zanelli (CECs, Chile) - we studied static spherically symmetric BH solutions of a CS theory for the most simple extension of AdS symmetry: $\mathfrak{so}(4,2) \times U(1)$.

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- To explain how that knowledge was useful to identify a new physical solution ; and then why it should be useful in the next step, when consdering CS theory based on expanded algebras.

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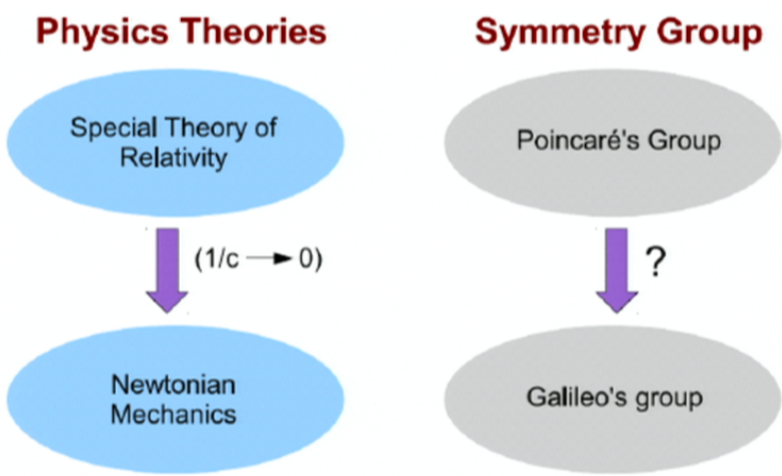

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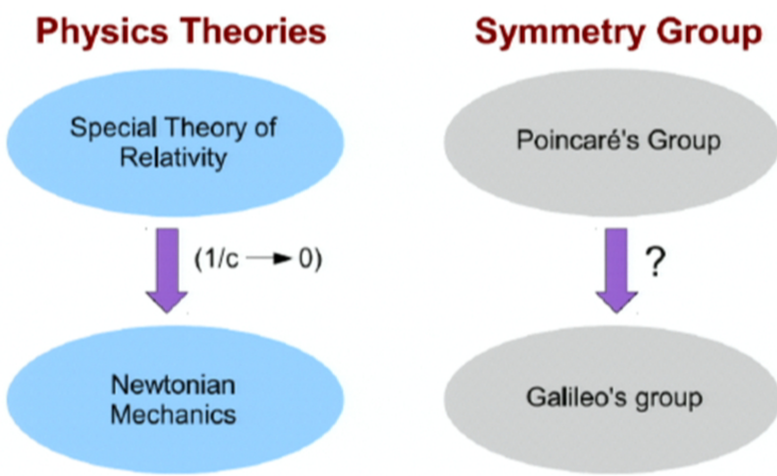

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$$[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{bc}M_{ad} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac}\,, \quad a,b,\ldots = 1,\ldots,5$$

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$$[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{bc}M_{ad} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac}\,, \quad a,b,\ldots = 1,\ldots,5$$

Reescaling $M_{5\mu} = RP_\mu$, with $\mu,\nu,\ldots = 1,\ldots,4$ leads to

$$[P_\mu, P_\nu] = \frac{1}{R^2}\eta_{55}M_{\mu\nu}$$

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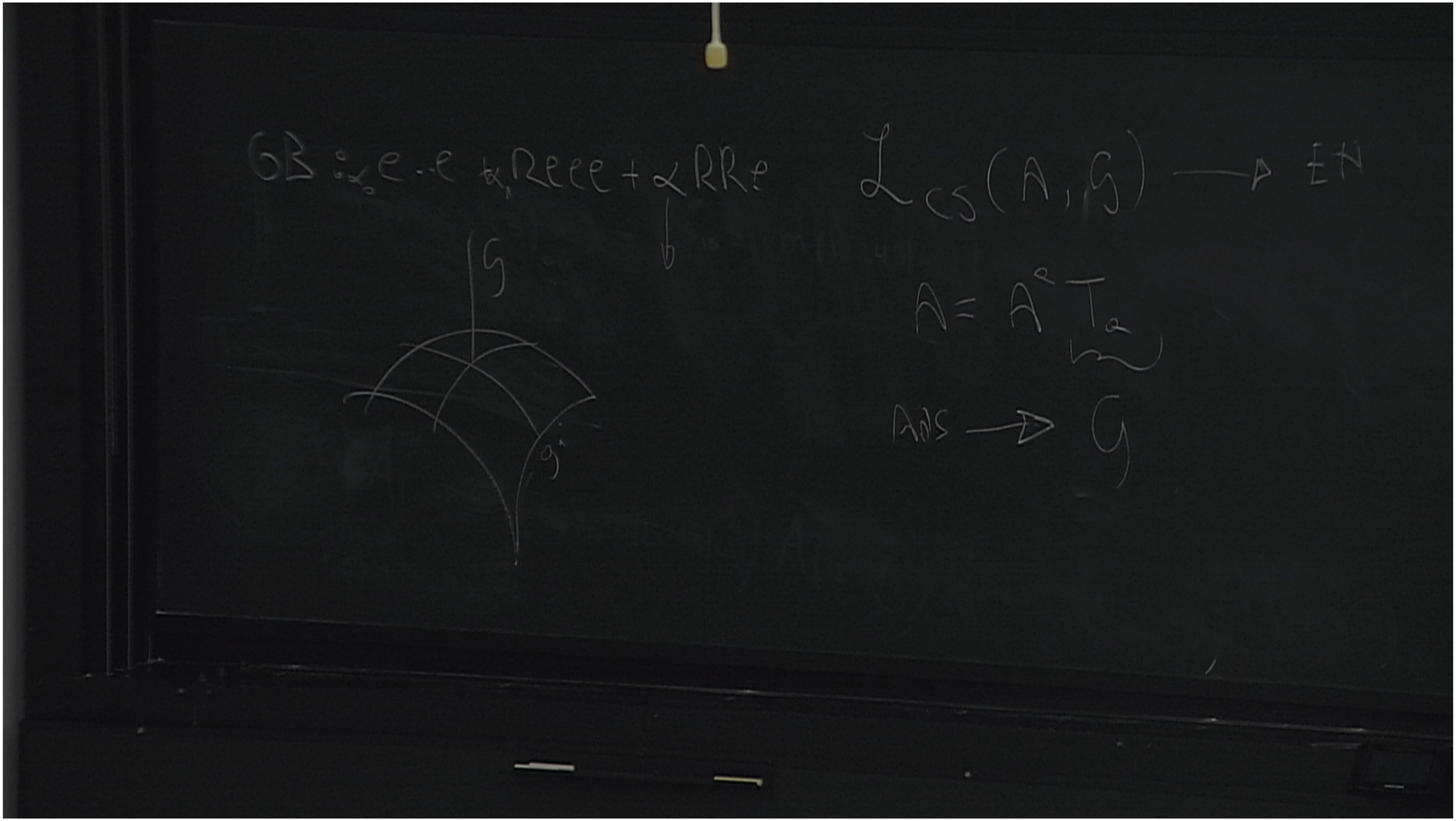
GB : e..e + Reee + α RRe
L CS(A, G) → EH
A = A^a T_a
AdS → G

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- 2006: S-expansion method, Izaurieta, Rodriguez, Salgado [arXiv:heo-th/0606215]

Advantages and some applications of the S-expansion method:

- Direct supersymmetric extension
- It provides non trivial invariant tensors for the expanded algebra (different from the (super)symmetrized trace)
- That was useful to construct[8] a CS action for the $\mathcal{M}$-algebra (which was obtained as an expansion of $\mathfrak{osp}(32|1)$)

[8]F. Izaurieta, *et al*, *arXiv: 0606225*; see also de Azcárraga *et al*, *arXiv:0212347*

[9]F. Izaurieta *et al*, *arXiv:0903.4712*

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- By constructing lagrangians invariant under expansions of $\mathfrak{so}(D-1,2)$, it has been found standard General Relativity (SGR) in even and odd dimensions as a special limit of a BI and CS Lagrangian respectively[10].

---

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All generalized contractions, mentioned before, can be reproduced in the frame of the S-expansion method by using one of the semigroups $S_E^{(N)} = \{\lambda_0, \ldots, \lambda_{N+1}\}$ defined by:

$$\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}, \text{ if } \alpha + \beta < N+1$$
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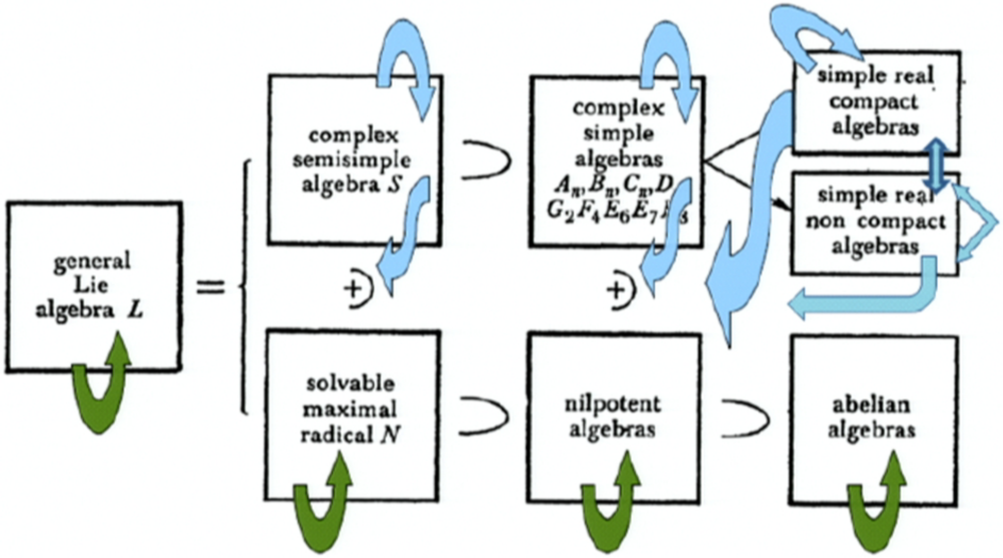
As many physical applications have been appeared by using this methods, we wanted to answer the following question:

Given two Lie algebras, can they be related by an S-expansion?

To answer that, one should consider the complete family of abelian semigrops, i.e., to take into account the history of their classification:

order	$Q = \#$ semigroups	
1	1	
2	4	
3	18	
4	126	[Forsythe '54]
5	1,160	[Motzkin, Selfridge '55]
6	15,973	[Plemmons '66]
7	836,021	[Jurgensen, Wick '76]
8	1,843,120,128	[Satoh, Yama, Tokizawa '94]
9	**52,989,400,714,478**	[Distler, Kelsey, Mitchell '09]

In particular, we have found[11] some criteria, related to the preservation of some properties of the Lie algebra, that allows to answer ithat question:



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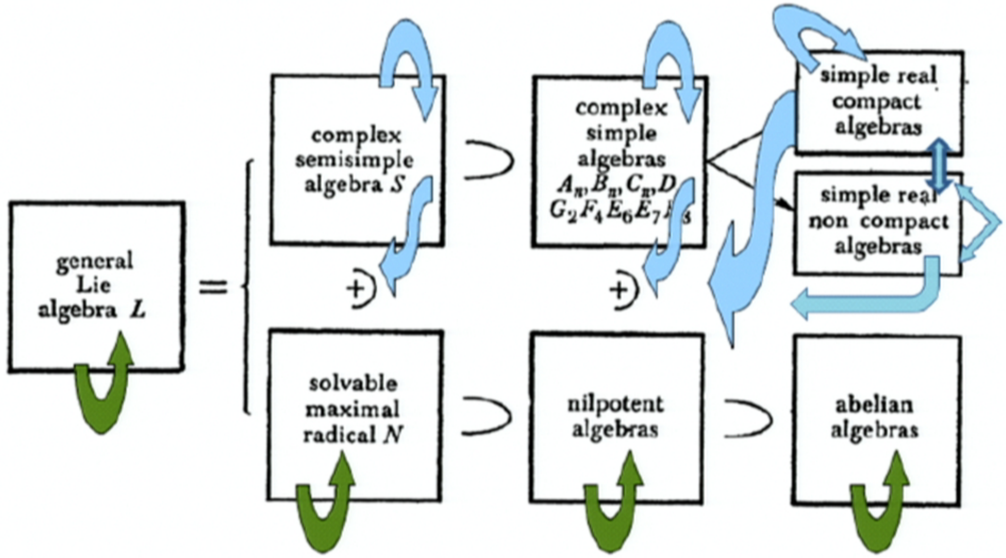

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We have also implemented computer programs to perform expansion with any semigroup (up to order 6) that allow us:

- to study all resonant decompositions of a semigroup,
- to establish the isomorphism of an arbitrary semigroup with one of those classified in the literature and

As an example, we have applied the procedure[12] in the context of Bianchi algebras:

Principal Idea: can be related 2 and 3-dimensional isometries?

Considering the two set of algebras: $[X_1, X_2] = 0$ and $[X_1, X_2] = X_1$

Group	Algebra
→ type I	$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$
→ type II	$[X_1, X_2] = [X_1, X_3] = 0, \quad [X_2, X_3] = X_1$
→ type III	$[X_1, X_2] = [X_2, X_3] = 0, \quad [X_1, X_3] = X_1$
type IV	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2$
→ type V	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2$
type VI	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = hX_2,$ where $h \neq 0, 1$
type VII_1	$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1$
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type VIII	$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3$
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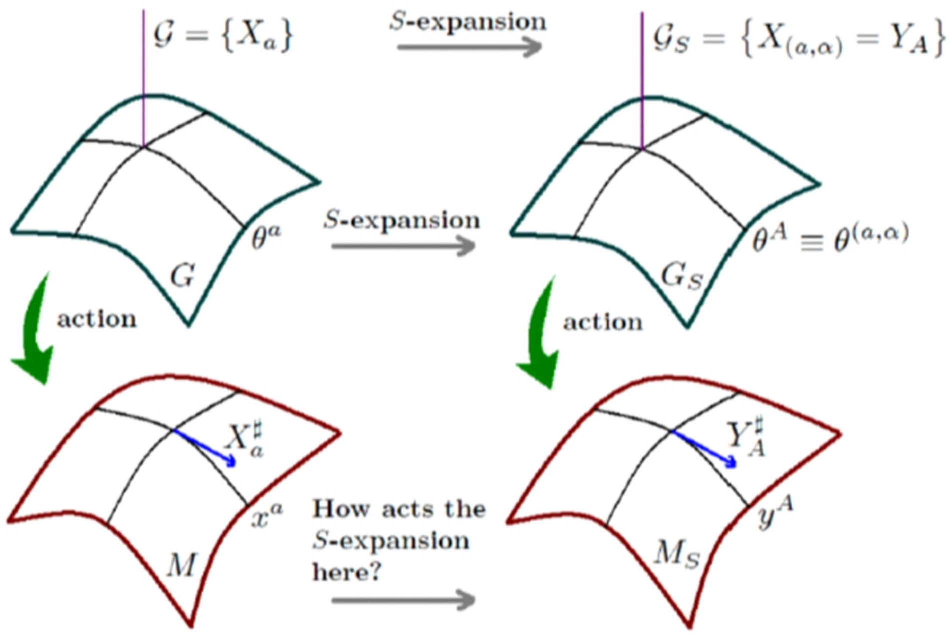
In particular, we currently working[13] in finding how the mechanism works in the case of Lie group transformations.



[13] Discussions with M. Calderón, I. Kondrashuk and M. Trigiante.

As the metric could represent a solution of either GR or Cosmology, we expect this extension to be useful to relate solutions in different kind of theories.

In fact, a BH[14] and a cosmological[15] solution have already been studied for a CS theory based on expanded algebras.

However, an important issue related to degeneracy in some sectors of the space of solutions in higher curvature theories must be taken into account. This is the subject of the next part of the talk.

[14] Quinzacara *et al*, arXiv:1401.1797
[15] Crisóstomo *et al*, arXiv:1401.2128

Action, field equations and some solutions

The Lagrangian is a CS density constructed from $\langle \mathbf{F} \wedge \cdots \wedge \mathbf{F} \rangle_g = dL_{\text{CS}}(\mathbf{A})$,

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$$\begin{aligned}I_{\mathrm{CS}}[\mathbf{A}] = \int_M \mathcal{L}_{\mathrm{CS}}(\mathbf{A}) &= \frac{i}{3}\int\limits_M \left\langle \mathbf{A}\mathbf{F}^2 - \frac{1}{2}\,\mathbf{F}\mathbf{A}^3 + \frac{1}{10}\,\mathbf{A}^5\right\rangle\\ &= \int\limits_M \left[\mathcal{L}_{\mathrm{AdS}}(e,\omega) + \mathcal{L}_{U(1)}(A) + \mathcal{L}_{\mathrm{int}}(e,\omega,A)\right]\,,\end{aligned}$$

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$$\begin{aligned} \mathcal{L}_{\mathrm{AdS}}(e,\omega) &= \frac{k}{4\ell}\,\epsilon_{abcde} \left(R^{ab}R^{cd} + \frac{2}{3\ell^2}\,R^{ab}e^c e^d + \frac{1}{5\ell^4}\,e^a e^b e^c e^d \right) e^e \,, \\ \mathcal{L}_{U(1)}(A) &= \beta A F^2 \,, \\ \mathcal{L}_{\mathrm{int}} &= \frac{\alpha}{2} \left[R^{ab}R_{ab} + \frac{2}{\ell^2} \left(R^{ab} e_a e_b - T^a T_a \right) \right] A \,. \end{aligned}$$

Field equations are

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$$= \int\limits_M \left[\mathcal{L}_{\mathrm{AdS}}(e,\omega) + \mathcal{L}_{U(1)}(A) + \mathcal{L}_{\mathrm{int}}(e,\omega,A)\right] \,,$$

In the local coordinates $x^\mu = (t, r, x^m)$ (with $x^m = x, y, z$), the ansatz is:

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In order to use Riemann-Cartan formalism, we split group indices as $a = (0, 1, i)$ (with $i = 2, 3, 4$), so

$$e^0 = f(r)\,dt\,, \qquad e^1 = \frac{dr}{f(r)}\,, \qquad e^i = r\,\delta^i_m\,dx^m := r\,dx^i\,.$$

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$f(r), A_t(r), A_r(r), \phi(r), \chi_r(r), \chi_t(r), \psi_r(r), \psi_t(r)$.

Field equations are

$$\delta e^a : \qquad 0 = \frac{k}{4}\,\epsilon_{abcde}\,F^{bc}F^{de} - \frac{2\alpha}{\ell}\,T_a F\,,$$
$$\delta\omega^{ab} : \qquad 0 = \frac{k}{\ell}\,\epsilon_{abcde}\,F^{cd}T^e + 2\alpha\,F_{ab}F\,,$$
$$\delta A : \qquad 0 = FF + \frac{\alpha}{2}\,R^{ab}R_{ab} - \frac{\alpha}{\ell^2}\,d(T^a e_a)\,.$$

with $F^{ab} = R^{ab} + \frac{1}{\ell^2}e^a e^b$.

We look for exact spherically symmetric charged BH solutions to these field equations.

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$$
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0 &= \left(-\frac{\psi_t^2}{f^2} + f^2\left(\psi_r - r\right)^2 + \frac{\phi^2}{4r^2} - \frac{r^4}{\ell^2}\right)\psi_r + \frac{\phi}{r}\left(\frac{r}{2}\phi' - \phi\right) , \\
0 &= \left(-\frac{\psi_t^2}{f^2} + f^2\left(\psi_r - r\right)^2 + \frac{\phi^2}{4r^2} - \frac{r^4}{\ell^2}\right)\psi_t , \\
0 &= \frac{C\ell\alpha}{k} r^2 f F_{tr} - rf\chi_r\psi_t + f^2\eta - rf^2\eta' \\
&\quad - f\frac{r^3}{\ell^2} + r^2 f\chi_t + r\chi_t\eta - r^2f^2f' - rff'\eta , \\
0 &= \eta\chi_t - f\chi_r\psi_t - r^2 f\chi_t'^2 ff'^2 + r^2f^2f'' - \frac{r^2}{\ell^2}f .
\end{aligned}
$$

A solution with axial torsion $(\phi(r) \neq 0)$ was considered by Canfora et al [arXiv:0707.1056], however, is uncharged.

Case with non-vanishing ψ_r and ϕ

$$
\begin{aligned}
f^2(r) &= \frac{r^2}{\ell^2} + br - \mu\,, \\
\phi(r) &= 2Cr^2\,, \\
\psi_r &= r\frac{\sqrt{r^2+\ell^2 br-\ell^2\mu} - \varepsilon_\psi\sqrt{r^2-\ell^2C^2}}{\sqrt{r^2+\ell^2 br-\ell^2\mu}}\,, \\
A_t &= \Phi - \frac{k}{C\ell\alpha}\left[\frac{r^2}{\ell^2}+\frac{br}{2}-\sqrt{\left(\frac{r^2}{\ell^2}+br-\mu\right)\left(\frac{r^2}{\ell^2}-C^2\right)}\right]\,, \\
A_r &= 0\,.
\end{aligned}
$$

This $f(r)$ represents the five-dimensional analogue of the hairy BH solution considered in conformal gravity and massive gravity in three dimensions by Oliva et al [arXiv:0905.1510, 0905.1545].

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Study of the corresponding horizons, asymptotic behavior, calculation of the mass (on the curve $\ell b = \pm 2\sqrt{C^2 - \mu}$, where electric field vanishes), Hawking temperature and entropy of this black branes was also made in arXiv:1406.3096.

Torsion and degeneracy

By adding non-vanishing ψ_t one gets

$$
\begin{aligned}
f^2 &= \frac{r^2}{\ell^2} + br - \mu + \theta\,, & \phi &= 2Cr^2\,,\\
A_t &= \Phi - \frac{k}{C\ell\alpha}\left(rff' + \frac{f\eta}{r}\right)\,, & \psi_r &= r + \frac{\eta}{f}\,,\\
\chi_r &= \frac{r^2\theta''}{2\psi_t}\,, & \psi_t &= \varepsilon_\psi f\,\sqrt{\eta^2 + C^2r^2 - \frac{r^4}{\ell^2}}\,,
\end{aligned}
$$

Torsion and degeneracy

By adding non-vanishing ψ_t one gets

$$\begin{aligned} f^2 &= \tfrac{r^2}{\ell^2} + br - \mu + \theta\,, & \phi &= 2Cr^2\,, \\ A_t &= \Phi - \tfrac{k}{C\ell\alpha}\left(rff' + \tfrac{f\eta}{r}\right), & \psi_r &= r + \tfrac{\eta}{f}\,, \\ \chi_r &= \tfrac{r^2\theta''}{2\psi_t}\,, & \psi_t &= \varepsilon_\psi f\sqrt{\eta^2 + C^2r^2 - \tfrac{r^4}{\ell^2}}\,, \end{aligned}$$

where $\theta(r)$ is an arbitrary function.

General solution

$$f^2 = \frac{r^2}{\ell^2} + br - \mu + \theta\,,$$

$$A_t = \Phi - \frac{k}{C\ell\alpha}\left[\frac{r^2}{\ell^2} + \frac{br}{2} + \frac{r\theta_t'}{\theta' - \theta_r' + \theta_t}\left(\frac{r^2}{\ell^2} + br - \mu + \theta\right) + \frac{r(\theta_r' - \theta_t)}{2}\right],$$

$$A_r = 0\,,$$

$$\phi = 2Cr^2\,,$$

$$\psi_t = \varepsilon_\psi \varepsilon_f\, r\sqrt{\frac{r^2}{\ell^2} + br - \mu + \theta}\sqrt{\left(\frac{r^2}{\ell^2} + br - \mu + \theta\right)\left(\frac{r\,\theta_t'}{\theta' - \theta_r' + \theta_t}\right)^2 + C^2 - \frac{r^2}{\ell^2}}\,,$$

$$\psi_r = r\left(1 + \frac{r\,\theta_t'}{\theta' - \theta_r' + \theta_t}\right),$$

$$\chi_t = \frac{\theta' - \theta_r' + \theta_t}{2}\,,$$

$$\chi_r = \frac{\varepsilon_\psi \varepsilon_f\, r\,\theta_r''}{2\sqrt{\frac{r^2}{\ell^2} + br - \mu + \theta}\sqrt{\left(\frac{r^2}{\ell^2} + br - \mu + \theta\right)\left(\frac{r\,\theta_t'}{\theta' - \theta_r' + \theta_t}\right)^2 + C^2 - \frac{r^2}{\ell^2}}}\,,$$

where $\theta_t(r), \theta_r(r)$, and $\theta(r)$ are arbitrary functions.

In CS (super)gravity theories, the appearance of arbitrary functions arise from degeneracies in the symplectic structure on certain sectors of phase space[16].

In those sectors the system acquires extra gauge symmetry and looses dynamical degrees of freedom.

[16] Bañados, Garay, Henneaux / arXiv:hep-th/9506187, 9605159.
[17] Saavedra-Troncoso-Zanelli (2001).

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However it is not an exclusive feature of CS theory since, as mention at the begining, arbitrary functions (called "geometrically free solutions" by J. T. Wheeler) are known to exist in some sectors of general Lovelock gravity.

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Furthermore, this behavior also exist in many mechanical systems [17].

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[16] Bañados, Garay, Henneaux / arXiv:hep-th/9506187, 9605159.
[17] Saavedra-Troncoso-Zanelli (2001).

General Lovelock theory has a pathological structure of its phase space because of the non-invertible relation between the metric and its conjugate momentum [18].

This introduces an indeterminacy in the dynamical evolution and leads to degenerate dynamics.

Metrics with undetermined components were reported in higher-dimensional theories in the torsionless case as well, e.g., in Einstein-Gauss-Bonnet (EGB) AdS gravity when the transverse section of the metric is maximally symmetric[19].

[18]Teitelboim-Zanelli (1987).
[19]Zegers, gr-qc/0505016.
[20]Oliva, arXiv:1210.4123.

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In case $f(t, r)$, there are still branches with undetermined components in CS theories[20].

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[18] Teitelboim-Zanelli (1987).
[19] Zegers, gr-qc/0505016.
[20] Oliva, arXiv:1210.4123.

It has been argued that the arbitrariness in the metric that appear in five-dimensional CS AdS gravity can be removed by:

- changing the cosmological constant, so that CS gravity becomes effectively EGB gravity[21].
- gauge-fixing [22], however a solution obtained in this way is still degenerate, i.e., the gauge-fixing hides the original arbitrariness in the metric.

[21] Bañados, hep-th/0310160.
[22] Aros-Contreras, gr-qc/0601135.

Accidental symmetries

The presence of three arbitrary functions in the general solution is consequence of a local symmetry.

This symmetry cannot be a restriction of the gauge transformation $A' = g^{-1}(A + d)g$ that preserves the form of the spherically symmetric ansatz A.

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We have shown that the infinitesimal gauge transformations that preserve this ansatz are necessarily rigid ($g = Const$).

Thus, residual gauge symmetries of this kind cannot explain this.

Accidental symmetries

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→ Our background is not generic and it possesses additional local symmetries (different from Λ and ξ) called "accidental" because they happen to exist only in certain sectors.

In fact, we proved that field equations (in the branch where all spherically symmetric torsion components are swiched on) are insensitive to the infinitesimal changes

$$\delta\theta = 2\sigma(r)\,,$$

$$\delta\theta_t = 2\int dr\,\tau(r)\,,$$

$$\delta\theta_r = -2\int dr\,\rho(r) + 2\int dr \int_0^r ds\,\tau(s) + 2\sigma(r)\,,$$

Accidental symmetries

which induces the following local transformations on the fields

$$
\begin{aligned}
\delta f &= \frac{\sigma}{f}\,, \\
\delta A_t &= -\frac{k}{C\ell\alpha}\left[r\sigma' + \frac{2\eta}{rf}\,\sigma + \frac{rf^2}{\chi_t}\,\tau - \left(r + \frac{f\eta}{r\chi_t}\right)\rho\right], \\
\delta\psi_r &= \frac{r^2}{\chi_t}\,\tau - \frac{\eta}{f\chi_t}\,\rho\,, \\
\delta\psi_t &= \left(\frac{\psi_t}{f^2} + \frac{\eta^2}{\psi_t}\right)\sigma + \frac{f^2\eta}{\psi_t\chi_t}\left(r^2 f\,\tau - \eta\rho\right), \\
\delta\chi_r &= \frac{r^2}{\psi_t}\,\sigma'' - \chi_r\left(\frac{1}{f^2} + \frac{\eta^2}{\psi_t^2}\right)\sigma + \frac{r^2}{\psi_t}\left(1 - \frac{f^3\eta\chi_r}{\psi_t\chi_t}\right)\tau + \frac{f^2\eta^2\chi_r}{\psi_t^2\chi_t}\,\rho - \frac{r^2}{\psi_t}\,\rho'\,, \\
\delta\chi_t &= \rho\,,
\end{aligned}
$$

with local parameters $\sigma(r)$, $\tau(r)$ and $\rho(r)$.

Accidental symmetries

The transformations are Abelian because $[\delta_1, \delta_2] = 0$ upon acting on any field.

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The transformations are Abelian because $[\delta_1, \delta_2] = 0$ upon acting on any field.

This new unexpected on-shell symmetry $U(1) \times U(1) \times U(1)$ cannot be a Cartan subgroup of $SO(2,4) \times U(1)$ because we already showed that there are no residual gauge symmetries.

In fact, we found that on this sector there is only one degree of freedom and that Hamiltonian is (off-shell) invariant under 4-parameter local symmetry that on-shell reduces to the 3-parameter transformations $(\delta f, \delta A_t, \delta\psi_r, \delta\psi_t, \delta\chi_r, \delta\chi_t)$.

Conclusion and future directions

Degeneracy in the space of solutions may appear in different families of L-L gravity in cases with and without torsion.

[23] Bañados-Garay-Henneaux, *arXiv:hep-th/9506187, 9605159*.

Conclusion and future directions

Degeneracy in the space of solutions may appear in different families of L-L gravity in cases with and without torsion.

We studied charged BH in CS AdS$\times U(1)$ gravity and saw there exist degenerated branches in the static spherically symmetric sector.

In contrast with a generic CS AdS gravity with a $U(1)$ field[23], that possesses maximal number of degrees of freedom (14 in this case), we found that there is only one dynamically propagating mode in the static symmetric sector of phase space.

[23] Bañados-Garay-Henneaux, *arXiv:hep-th/9506187*, *9605159*.

Conclusion and future directions

Indeed, missing degrees of freedom were related to an increase in local symmetries and thus our example provide an explicit realization of a non-generic CS gravity.

In particular this shows that the knowledge of these "accidental symmetries" can help to formulate a simple criterion that avoids unwanted degenerate ansatze.

Conclusion and future directions

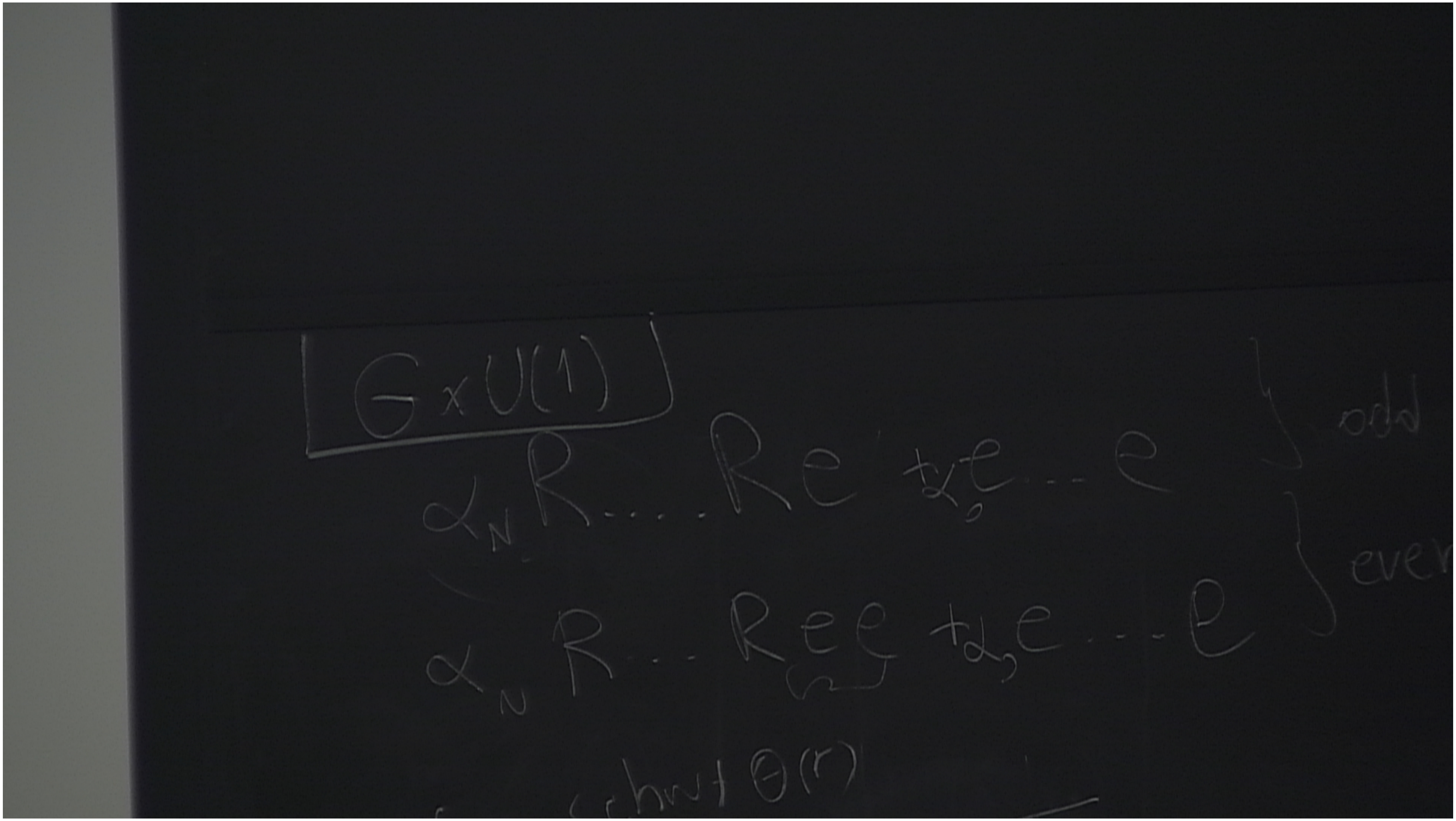
Indeed, missing degrees of freedom were related to an increase in local symmetries and thus our example provide an explicit realization of a non-generic CS gravity.

In particular this shows that the knowledge of these "accidental symmetries" can help to formulate a simple criterion that avoids unwanted degenerate ansatze.

This way, we identified two interesting solutions: the axial torsion one - already known in the literature -, and a new 2-components torsion solution.

Conclusion and future directions

This issue of accidental symmetries should then be taken into account when studying charged solutions in CS theory based on expanded algebras, which is the next step to find an application in the Holographic context.


G x U(1)
odd
even