Title: Null Canonical Gravity, Integrability and Quantization

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Abstract:

NULL CANONICAL GRAVITY, INTEGRABILITY AND QUANTIZATION

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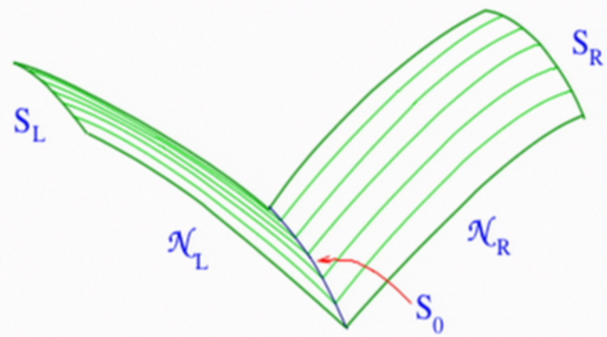
PLAN OF TALK

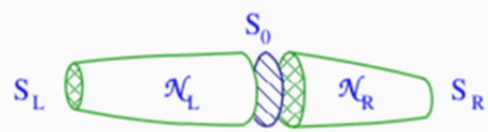
- Review of canonical GR in terms of unconstrained initial data on null hypersurfaces
- The Poisson brackets of the main data
- Poisson brackets in cylindrically symmetric gravity
- Transformation to new data
- Poisson brackets of new data
- Quantization of new data
- Things to do

Double null sheets as initial data hypersurfaces

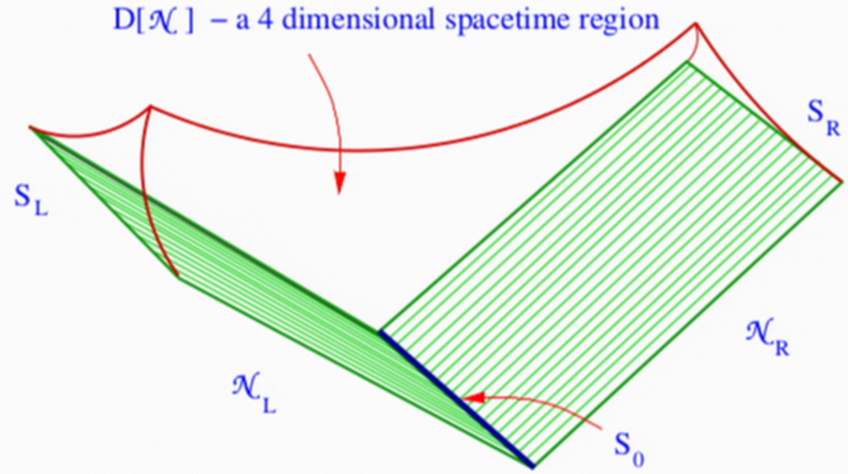
- A double null sheet is a pair of intersecting null hypersurfaces (or "lightfronts") - like an open book in spacetime.



- $\mathcal{N}_R, \mathcal{N}_L$ are 3-surfaces swept out by null geodesics emerging normally from the two sides of 2-disk S_0.

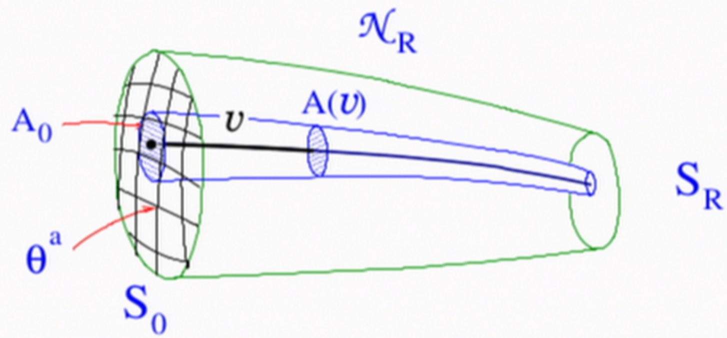


- initial data on $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_R$ specifies solution in domain of dependence $D[\mathcal{N}]$



THE FREE INITIAL DATA

Coordinates adapted to $\mathcal{N}$



- θ^1, θ^2 coordinates on S_0. Held constant on generators.
- v is a parameter along each generator defined so that the cross sectional area of an infinitesimal bundle of neighboring generators is

$$A(v) = A_0 v^2$$

where A_0 is the cross sectional area at S_0.

Data

- "Bulk" data on the 3-manifolds $\mathcal{N}_L$ and $\mathcal{N}_R$. "Surface" data on S_0.
- Bulk data = conformal 2-metric $e_{ab}(\theta^1, \theta^2, v)$
  - Induced metric on $\mathcal{N}$ degenerate because $\mathcal{N}$ is null, so

$$ds^2 = h_{ab} d\theta^a d\theta^b \qquad \text{- no } dv \text{ terms}$$

  - Definition:

$$e_{ab} = h_{ab}/\sqrt{\det h} \qquad \text{- makes } \det e = 1$$

  - Parametrize e_{ab} by a complex scalar μ

$$ds^2 = h_{ab} d\theta^a d\theta^b = \frac{\rho}{1 - \mu\bar{\mu}}(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz)$$

  with $z = \theta^1 + i\theta^2$ and $\rho = \sqrt{det\, h_{ab}}$

- Surface data on S_0: ρ_0, λ, τ_a.

THE POISSON BRACKETS FOR FREE DATA ON $\mathcal{N}$ FOR CLASSICAL VACUUM GR

Brackets not shown vanish.

$$\{\mu(\mathbf{1}),\bar{\mu}(\mathbf{2})\} = 4\pi G\,\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\,H(\mathbf{1},\mathbf{2})\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{1}} \times\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{2}} e^{\int_1^2(\bar{\mu}d\mu-\mu d\bar{\mu})/(1-\mu\bar{\mu})}$$

for $\mathbf{1}$, $\mathbf{2}$ in the same branch, $\mathcal{N}_A$.

$$\{\rho_0(\theta_1),\lambda(\theta_2)\} = 8\pi G\delta^2(\theta_2-\theta_1)$$

$$\{\rho_0(\theta),\tau[f]\} = -8\pi G\pounds_f\rho_0(\theta)$$

$$\{\lambda(\theta),\tau[f]\} = -8\pi G\Big[\pounds_f\lambda+\frac{\pounds_f\mu}{(1-\mu\bar{\mu})^2}(\partial_{v_R}\bar{\mu}-\partial_{v_L}\bar{\mu})\Big]_\theta$$

$$\{\tau[f_1],\tau[f_2]\} = -16\pi G\Big[\tau[[f_1,f_2]]-\frac{1}{2}\int_{S_0}\pounds_{[f_1,f_2]}\epsilon + \int_{S_0}\big[\frac{\pounds_{f_1}\mu}{(1-\mu\bar{\mu})^2}\{\epsilon\pounds_{f_2}\bar{\mu}-\frac{1}{2}\pounds_{f_2}\epsilon(\partial_{v_R}\bar{\mu}+\partial_{v_L}\bar{\mu})\}-(1\leftrightarrow 2)\big]\Big].$$

For $\mathbf{1}$ in $\mathcal{N}_R - S_0$

$$\{\mu(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_R\partial_{v_R}\mu]_{\mathbf{1}}$$

$$\{\mu(\mathbf{1}),\tau[f]\} = -16\pi G\Big[\pounds_f\mu-\frac{1}{4}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\mu\Big]_{\mathbf{1}}.$$

For $\mathbf{1}$ in S_0

$$\{\mu(\mathbf{1}),\lambda(\mathbf{2})\} = 0$$

$$\{\mu(\mathbf{1}),\tau[f]\} = -8\pi G[\pounds_f\mu]_{\mathbf{1}}.$$

For $\mathbf{1}$ in $\mathcal{N}_L - S_0$

$$\{\mu(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_L\partial_{v_L}\mu]_{\mathbf{1}}$$

$$\{\mu(\mathbf{1}),\tau[f]\} = -4\pi G\Big[\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\mu\Big]_{\mathbf{1}}.$$

For $\mathbf{1}\in\mathcal{N}_R$ (including $\mathbf{1}\in S_0$)

$$\{\bar{\mu}(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_R\partial_{v_R}\bar{\mu})_{\mathbf{1}} +\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}(\partial_{v_L}\bar{\mu})_{\mathbf{1}_0}\Big]$$

$$\{\bar{\mu}(\mathbf{1}),\tau[f]\} = -8\pi G\Big[\Big(2\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\bar{\mu}\Big)_{\mathbf{1}} -\Big(\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_L}\bar{\mu}\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}\Big]$$

where $\mathbf{1}_0\in S_0$ is the origin of the generator through $\mathbf{1}$.
For $\mathbf{1}\in\mathcal{N}_L$

$$\{\bar{\mu}(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_L\partial_{v_L}\bar{\mu})_{\mathbf{1}} +\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}(\partial_{v_R}\bar{\mu})_{\mathbf{1}_0}\Big]$$

$$\{\bar{\mu}(\mathbf{1}),\tau[f]\} = -8\pi G\Big[\Big(\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\bar{\mu}\Big)_{\mathbf{1}} +\Big(\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_R}\bar{\mu}\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}\Big].$$

THE POISSON BRACKETS FOR FREE DATA ON $\mathcal{N}$ FOR CLASSICAL VACUUM GR

Brackets not shown vanish.

$$\begin{aligned}\{\mu(\mathbf{1}),\bar{\mu}(\mathbf{2})\} &= 4\pi G\,\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\,H(\mathbf{1},\mathbf{2})\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{1}} \\ &\quad\times\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{2}}e^{\int_{\mathbf{1}}^{\mathbf{2}}(\bar{\mu}d\mu-\mu d\bar{\mu})/(1-\mu\bar{\mu})}\end{aligned}$$

for $\mathbf{1}$, $\mathbf{2}$ in the same branch, $\mathcal{N}_A$.

$$\begin{aligned}\{\rho_0(\theta_1),\lambda(\theta_2)\} &= 8\pi G\delta^2(\theta_2-\theta_1) \\ \{\rho_0(\theta),\tau[f]\} &= -8\pi G\pounds_f\rho_0(\theta) \\ \{\lambda(\theta),\tau[f]\} &= -8\pi G\Big[\pounds_f\lambda+\frac{\pounds_f\mu}{(1-\mu\bar{\mu})^2}(\partial_{v_R}\bar{\mu}-\partial_{v_L}\bar{\mu})\Big]_\theta\end{aligned}$$

$$\begin{aligned}\{\tau[f_1],\tau[f_2]\} = -16\pi G\Big[&\tau[[f_1,f_2]]-\frac{1}{2}\int_{S_0}\pounds_{[f_1,f_2]}\epsilon \\ &+\int_{S_0}\big[\frac{\pounds_{f_1}\mu}{(1-\mu\bar{\mu})^2}\{\epsilon\pounds_{f_2}\bar{\mu}-\frac{1}{2}\pounds_{f_2}\epsilon(\partial_{v_R}\bar{\mu}+\partial_{v_L}\bar{\mu})\}-(1\leftrightarrow 2)\big]\Big].\end{aligned}$$

For $\mathbf{1}$ in $\mathcal{N}_R - S_0$

$$\{\mu(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_R\partial_{v_R}\mu]_{\mathbf{1}}$$

$$\{\mu(\mathbf{1}),\tau[f]\} = -16\pi G\Big[\pounds_f\mu-\frac{1}{4}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\mu\Big]_{\mathbf{1}}.$$

For $\mathbf{1}$ in S_0

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\mathbf{2})\} &= 0 \\ \{\mu(\mathbf{1}),\tau[f]\} &= -8\pi G[\pounds_f\mu]_{\mathbf{1}}.\end{aligned}$$

For $\mathbf{1}$ in $\mathcal{N}_L - S_0$

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_L\partial_{v_L}\mu]_{\mathbf{1}} \\ \{\mu(\mathbf{1}),\tau[f]\} &= -4\pi G\Big[\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\mu\Big]_{\mathbf{1}}.\end{aligned}$$

For $\mathbf{1}\in\mathcal{N}_R$ (including $\mathbf{1}\in S_0$)

$$\begin{aligned}\{\bar{\mu}(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_R\partial_{v_R}\bar{\mu})_{\mathbf{1}} \\ &\quad+\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}(\partial_{v_L}\bar{\mu})_{\mathbf{1}_0}\Big] \\ \{\bar{\mu}(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(2\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\bar{\mu}\Big)_{\mathbf{1}} \\ &\quad-\Big(\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_L}\bar{\mu}\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}\Big]\end{aligned}$$

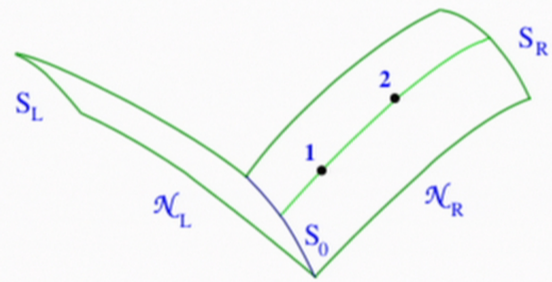
where $\mathbf{1}_0\in S_0$ is the origin of the generator through $\mathbf{1}$.
For $\mathbf{1}\in\mathcal{N}_L$

$$\begin{aligned}\{\bar{\mu}(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_L\partial_{v_L}\bar{\mu})_{\mathbf{1}} \\ &\quad+\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}(\partial_{v_R}\bar{\mu})_{\mathbf{1}_0}\Big] \\ \{\bar{\mu}(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\bar{\mu}\Big)_{\mathbf{1}} \\ &\quad+\Big(\pounds_f\bar{\mu}-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_R}\bar{\mu}\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar{\mu})/(1-\mu\bar{\mu})}\Big].\end{aligned}$$

POISSON BRACKETS OF THE BULK DATA

$$\{\mu(\mathbf{1}),\mu(\mathbf{2})\} = \{\bar\mu(\mathbf{1}),\bar\mu(\mathbf{2})\} = 0$$
$$\{\mu(\mathbf{1}),\bar\mu(\mathbf{2})\} = 4\pi G\frac{1}{\sqrt{\rho_1\rho_2}}\delta^2(\theta_2-\theta_1)\,H(\mathbf{1},\mathbf{2})$$
$$\times[1-\mu\bar\mu]_{\mathbf{1}}[1-\mu\bar\mu]_{\mathbf{2}}\,e^{\int_{\mathbf{1}}^{\mathbf{2}}(\bar\mu d\mu-\mu d\bar\mu)/(1-\mu\bar\mu)}.$$

$H(\mathbf{1},\mathbf{2})$ step function = 1 if **2** follows **1** along the generator, 0 otherwise.



- Only data on same generator have non-zero bracket. Consistent with causality since points on distinct generators are spacelike separated.
- Bracket does not quite preserve reality of induced metric on $\mathcal{N}$, but imaginary mode is shock wave that does not enter interior of domain of dependence. Bracket preserves reality of spacetime metric there.

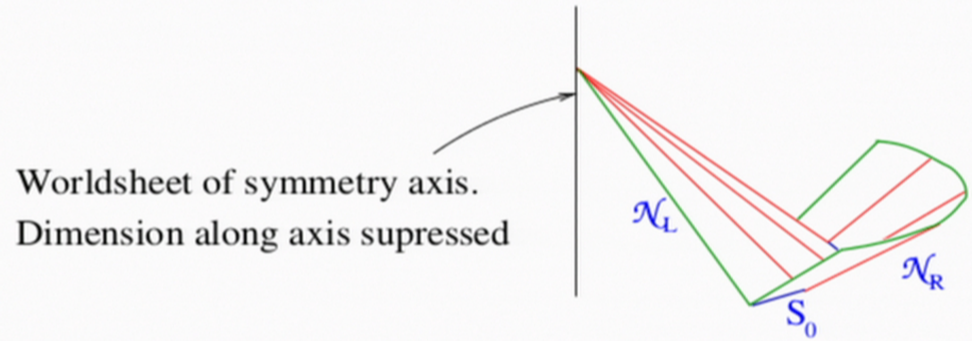
A SIMPLER PROBLEM

- Step toward quantization: quantize the "one generator algebra"

$$\{\mu(\mathbf{1}), \mu(\mathbf{2})\} = \{\bar{\mu}(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 0$$

$$\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 4\pi G \frac{1}{\sqrt{\rho_1 \rho_2}} H(\mathbf{1}, \mathbf{2}) \times [1 - \mu\bar{\mu}]_{\mathbf{1}} [1 - \mu\bar{\mu}]_{\mathbf{2}} \, e^{\int_{\mathbf{1}}^{\mathbf{2}} (\bar{\mu} d\mu - \mu d\bar{\mu})/(1 - \mu\bar{\mu})}.$$

  - Brackets with $\delta^2(\theta_2 - \theta_1)$ removed. μ, $\bar{\mu}$ functions on a single line.
- This is the bracket in cylindrically symmetric GR on $\mathcal{N}$ swept out by radial light rays from symmetry axis.



- Bracket obtained as bracket of averages $\langle\mu\rangle$ and $\langle\bar{\mu}\rangle$ over symmetry orbits at symmetric solutions, or from symmetry reduced action.

Transformation to new variables

- Cylindrically symmetric GR is an integrable system. Quantization exists [Korotkin and Samtleben 1998].
- Transform in steps from μ, $\bar{\mu}$ to variables with known quantization

$$\mu, \bar{\mu} \mapsto \mathcal{V} \mapsto \hat{\mathcal{V}} \mapsto \mathcal{M}$$

- $\mu, \bar{\mu} \mapsto \mathcal{V}$: $\mathcal{V}$ is zweibein for conformal 2-metric e_{ab} on symmetry orbits - $e = \mathcal{V}\mathcal{V}^T$.

$$\mathcal{V} = \frac{1}{\sqrt{1-\mu\bar{\mu}}} \frac{1}{\sqrt{(1-\mu)(1-\bar{\mu})}} \begin{bmatrix} 1-\mu\bar{\mu} & -i(\mu-\bar{\mu}) \\ 0 & (1-\mu)(1-\bar{\mu}) \end{bmatrix}$$

- A natural boundary condition: 4-metric regular on axis. But we will use $\mathcal{V}$ regular at axis which implies 4-metric not regular.
- We treat only the branch $\mathcal{N}_L$ which touches symmetry axis.

- $\mathcal{V} \mapsto \hat{\mathcal{V}}$: Define

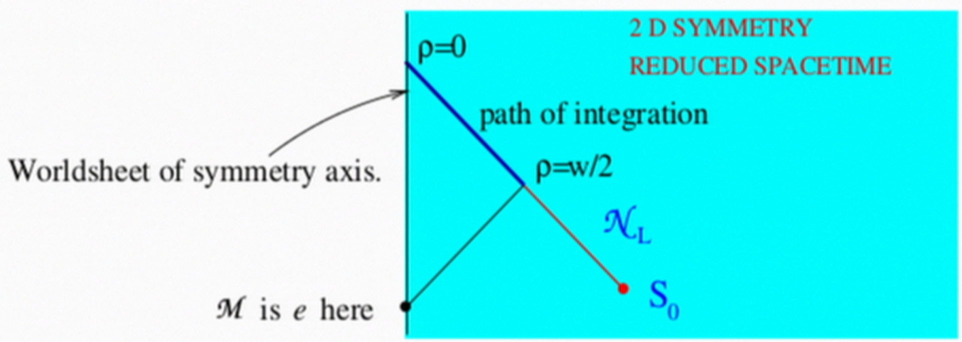
$$J = \mathcal{V}^{-1} d\mathcal{V}$$

$$P = \tfrac{1}{2}(J + J^T) \quad Q = \tfrac{1}{2}(J - J^T)$$

Introduce *spectral parameter* w. We use only real w in classical theory

$$\hat{J}(x; w) = Q(x) + \sqrt{\frac{w}{w - 2\rho}} P(x) \quad \hat{\mathcal{V}}(w) = \mathcal{V}(0)\, \mathcal{P} e^{\int_{\rho=0}^{\rho=w/2} \hat{J}(w)}$$

- $\hat{\mathcal{V}} \mapsto \mathcal{M}$: “Monodromy matrix” $\mathcal{M}(w) = \hat{\mathcal{V}}(w)\hat{\mathcal{V}}^T(w)$
- Interpretation of $\mathcal{M}$: It is e_{ab} on symmetry axis at instant t, connected by future directed light ray to point on $\mathcal{N}_L$ where $\rho = w/2$.



- Transformation $\mu, \bar{\mu} \mapsto \mathcal{M}$ is invertible, modulo imaginary shockwave mode.

Poisson algebra of new variables

- A lengthy calculation yields

$$\{\overset{1}{\mathcal{M}}(v), \overset{2}{\mathcal{M}}(w)\} = \frac{8\pi G}{v-w}\Big[\Omega\overset{1}{\mathcal{M}}(v)\overset{2}{\mathcal{M}}(w) + \overset{1}{\mathcal{M}}(v)\overset{2}{\mathcal{M}}(w)\Omega - \overset{1}{\mathcal{M}}(v)\Omega^{\eta}\overset{2}{\mathcal{M}}(w) - \overset{2}{\mathcal{M}}(w)\Omega^{\eta}\overset{1}{\mathcal{M}}(v)\Big]$$

- Matrices $\overset{1}{A}$ and $\overset{2}{B}$ act on different spaces, so their product is a tensor product: $(\overset{1}{A}\overset{2}{B})_{ab,cd} = A_{ab}B_{cd}$.

- Ω and Ω^{η} acts in the product of the two spaces.
  - $\Omega_{ab,cd} = \delta_{ad}\delta_{cb} - 1/2\delta_{ab}\delta_{cd}$. Contracting Ω on both indices in space 2 with a matrix there gives the trace free part of the matrix acting in space 1.
  - $\Omega^{\eta}_{ab,cd} = -\delta_{bd}\delta_{ac} + 1/2\delta_{ab}\delta_{cd}$ projects on trace free matrices and also takes the transpose and multiplies by -1.

QUANTIZATION

- Korotkin and Samtleben 1998 have presented a consistent operator algebra that quantizes the Poisson algebra of the $\mathcal{M}$s:

$$R(v-w)\overset{1}{\mathcal{M}}(v)R^{\eta}(w-v+2ia)\overset{2}{\mathcal{M}}(w)$$
$$= \overset{2}{\mathcal{M}}(w)R^{\eta}(v-w+2ia)\overset{1}{\mathcal{M}}(v)R(w-v)\frac{v-w-2ia}{v-w+2ia}$$

  - $a = 8\pi G\hbar$ - $8\pi\times$ Planck area.
  - $R(u)_{ab,cd} = u\delta_{ab}\delta_{cd} - ia\delta_{ad}\delta_{bc}$
  - $R(u)^{\eta}_{ab,cd} = (u-ia)\delta_{ab}\delta_{cd} + ia\delta_{ac}\delta_{bd}$

  $\mathcal{M}_{ab} = \mathcal{M}_{ba}$, *-algebra with $\mathcal{M}^* = \mathcal{M}$,
  $\mathcal{M}^{11}(u-ia)\mathcal{M}^{22}(u) - \mathcal{M}^{12}(u-ia)\mathcal{M}^{21}(u) = 1$.

- The quantization is known at the algebraic level, but the representations of the algebra are not well explored. Algebra closely related to $\mathfrak{sl}(2)$ Yangian double.

To do

- Study the representations of the $\mathcal{M}$ algebra, in general, and in single polarization model [Kuchar 1971].
- S_0 data λ and ρ_0 are present in cylindrically symmetric GR. Here we have ignored them. They should be incorporated into Poisson algebra and quantization.
- Use the results from cylindrically symmetric GR to quantize data in full GR. The chief obstacle seems to be that our formalism requiers that $\mathcal{V}$ be regular at $\rho = 0$, and thus that the caustic at the end of the generator be of a special type. This requierment is not generally met in the absence of symmetry.