Title: Quantum Measurements from a Logical Point of View

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Abstract: <p>We present a formal logic modeling some aspects of the behavior of the quantum measurement process, and study some properties of the models of this logic, from which we deduce some characteristics that any such model should verify. In the case of a Hilbert space of dimension at least 3, we then show that no model can lead to the prediction with certainty of more than one atomic outcome. Moreover, if the Hilbert space is finite dimensional, we can precisely describe the structure of the predictions of any model of our logic. As a consequence, we also prove that all the models of our logic make exactly the same predictions regarding whether a given sequence of outcomes is possible.</p>

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QUANTUM MEASUREMENTS FROM A LOGICAL POINT OF VIEW

Olivier Brunet

April 21, 2015

Quantum Foundations Seminar
Perimeter Institute
Waterloo, Ontario, Canada

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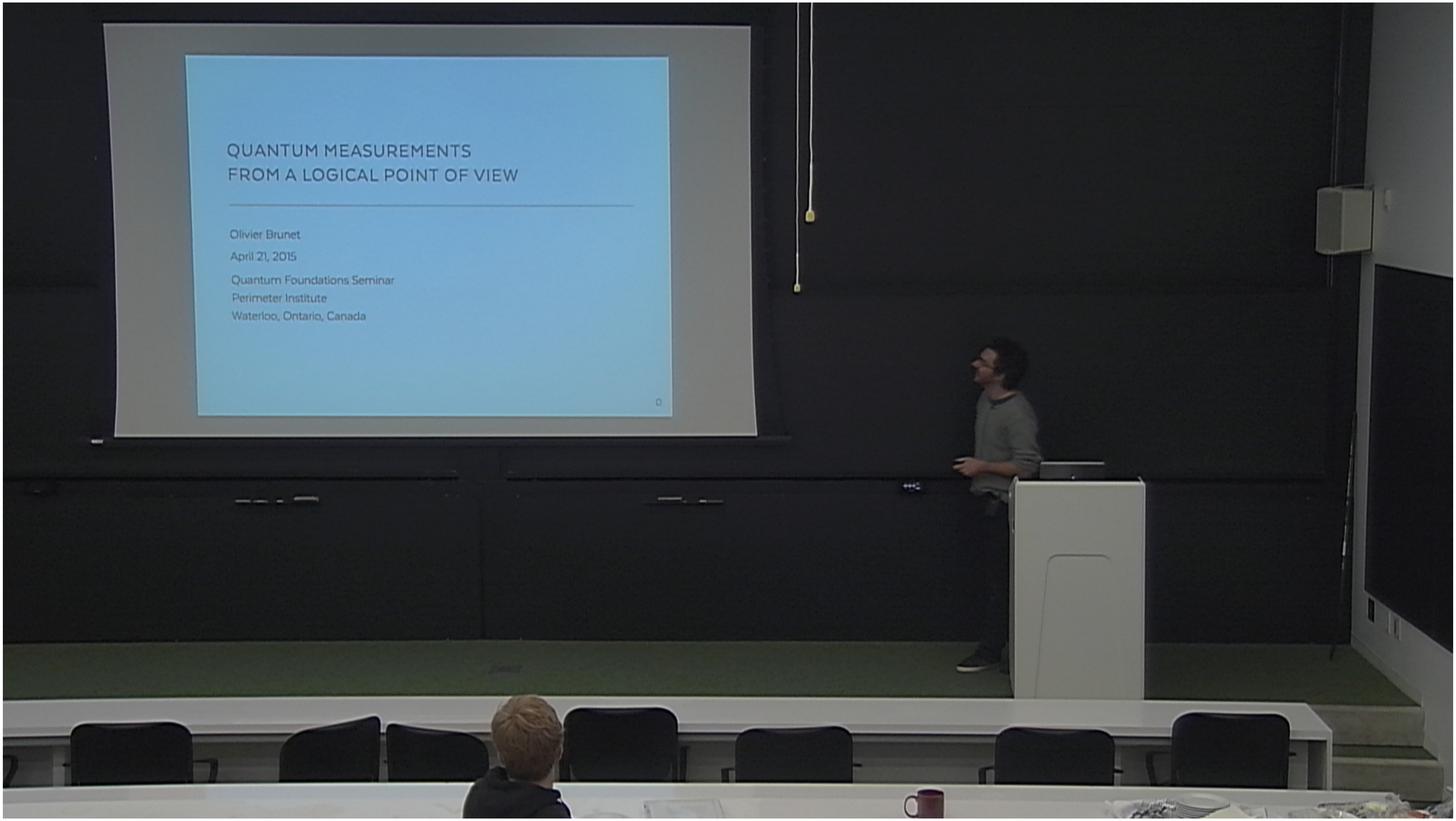
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BASIC INGREDIENTS

Logical Constructor

$$\mathrm{Mes}(s, \mathcal{O}, p, t)$$

System s has been measured with observable $\mathcal{O}$, yielding outcome $p \in \mathcal{O}$. Label t denotes the resulting system.

Examples

$$\forall\, s, \mathcal{O}, \exists\, p \in \mathcal{O}, t:\ \mathrm{Mes}(s, \mathcal{O}, p, t)$$

$$\forall\, p, \neg\big(\exists\, s, t, u, \mathcal{O}, \mathcal{O}':\ \mathrm{Mes}(s, \mathcal{O}, p, t)\ \text{ and }\ \mathrm{Mes}(t, \mathcal{O}', p^{\perp}, u)\big)$$

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VERIFICATION STATEMENT

Definition

$$s \blacktriangleright p \overset{\Delta}{\iff} \neg\left(\exists\, t, \mathcal{O} :\ \mathrm{Mes}\left(s, \mathcal{O}, p^{\perp}, t\right)\right)$$

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VERIFICATION STATEMENT

Definition

$$s \blacktriangleright p \overset{\Delta}{\Longleftrightarrow} \neg\left(\exists\, t, \mathcal{O} :\ \mathrm{Mes}(s, \mathcal{O}, p^{\perp}, t)\right)$$

Meaning in a Hilbert Lattice

If a system s is in a state $|\varphi\rangle$,
then $s \blacktriangleright p$ corresponds to $|\varphi\rangle \in p$, or $\Pi_p|\varphi\rangle = |\varphi\rangle$.

STRUCTURAL AXIOMS

$$s \blacktriangleright \top$$

$$\neg(s \blacktriangleright \bot)$$

$$\forall p \neq \bot, \exists s : s \blacktriangleright p$$

WEAK NONCONTEXTUALITY

Claim

The **certainty/impossibility** of an outcome is **independent** of the measured observable.

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Motivation: the Born Rule

The probability of obtaining outcome P in state $|\varphi\rangle$ is $\langle\varphi|\Pi_P|\varphi\rangle$.

WEAK NONCONTEXTUALITY

Axiom

If $s \blacktriangleright p$ and $p \leq q$, then $s \blacktriangleright q$.

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Justification

$$\{p, p^{\perp} \wedge q, q^{\perp}\}$$

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$$\{q, q^\perp\}$$

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COMPATIBLE PRESERVATION

Axiom

If p and q are compatible, $s \blacktriangleright p$ and $\text{Mes}(s, \mathcal{O}, q, t)$, then $t \blacktriangleright p$.

Justification

p and q are compatible iff Π_p and Π_q commute

If $\Pi_p|\varphi\rangle = |\varphi\rangle$, then $\Pi_p(\Pi_q|\varphi\rangle) = \Pi_q|\varphi\rangle$.

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SASAKI PROJECTION

Definition

$$\forall a, b \in L, \qquad a \,\&\, b \triangleq b \wedge (a \vee b^{\perp})$$

Intuition

It's the lattice theoretic equivalent of the orthogonal projection.

"Simplifications"

Weak Noncontextuality 2 $\quad s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q$

Predictivity $\quad s \blacktriangleright p \text{ and } \mathrm{Mes}(s, \mathcal{O}, q, t) \implies t \blacktriangleright p \,\&\, q$

SUMMARY

Axioms of $\mathcal{T}_L$

$$s \blacktriangleright \top$$

$$\neg(s \blacktriangleright \bot)$$

$$p \neq \top \implies \exists s : s \blacktriangleright p$$

$$s \blacktriangleright p \text{ and } p \leq q \implies s \blacktriangleright q$$

$$s \blacktriangleright p \text{ and } s \blacktriangleright q \implies s \blacktriangleright p \,\&\, q$$

$$s \blacktriangleright p \text{ and } \mathrm{Mes}(s, \mathcal{O}, q, t) \implies t \blacktriangleright p \,\&\, q$$

EXAMPLE: THE HILBERT MODEL

Definition

Given a Hilbert space $\mathcal{H}$, we define the model $\mathfrak{H}_{\mathcal{H}} = (A_{\mathfrak{H}}, M_{\mathfrak{H}})$ by

$$A_{\mathfrak{H}} \stackrel{\Delta}{=} \{|\varphi\rangle \mid \langle\varphi|\varphi\rangle = 1\}$$

$$M_{\mathfrak{H}}(|\varphi\rangle, p, |\psi\rangle) \stackrel{\Delta}{\iff} \Pi_p|\psi\rangle \neq |0\rangle \text{ and } |\varphi\rangle = \frac{\Pi_p|\psi\rangle}{\|\Pi_p|\psi\rangle\|}$$

EXAMPLE: THE LATTICE MODEL

Definition

Given an *orthomodular lattice L*, we define the model $\mathfrak{L}_L = (A_{\mathfrak{L}}, M_{\mathfrak{L}})$ by

$$A_{\mathfrak{L}} \triangleq L^{\star} \qquad \text{where} \qquad L^{\star} \triangleq L \setminus \{\perp\}$$

$$M_{\mathfrak{L}}(a, p, b) \overset{\Delta}{\iff} b \leq a \,\&\, p$$

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Verification Relation

$$a \blacktriangleright_\mathfrak{L} p \iff p \leq a$$

SASAKI FILTERS

Proposition

Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, for all $a \in A$,

$$[\![a]\!]_{\mathfrak{G}} \triangleq \{p \in L \mid a \blacktriangleright_{\mathfrak{G}} p\}$$

is a consistent Sasaki filter of L.

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Proof

Nonempty	$a \blacktriangleright_{\mathfrak{G}} \top$
Upper set	$a \blacktriangleright_{\mathfrak{G}} p$ and $p \leq q \implies a \blacktriangleright_{\mathfrak{G}} q$
&-stable	$a \blacktriangleright_{\mathfrak{G}} p$ and $a \blacktriangleright_{\mathfrak{G}} q \implies a \blacktriangleright_{\mathfrak{G}} p \,\&\, q$
Consistent	$\neg(a \blacktriangleright_{\mathfrak{G}} \bot)$

SASAKI FILTERS OF A HILBERT LATTICE

Theorem[1]

If $\mathcal{H}$ is a Hilbert space such that $\dim \mathcal{H} \geq 3$, then any consistent Sasaki filter of $L(\mathcal{H})$ contains at most one vector ray.

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any consistent Sasaki filter of $L(\mathcal{H})$ contains at most one vector ray.

Corollary

If $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$ with $\dim \mathcal{H} \geq 3$, then
for all $a \in A$, $[\![a]\!]_{\mathfrak{G}}$ contains at most one vector ray.

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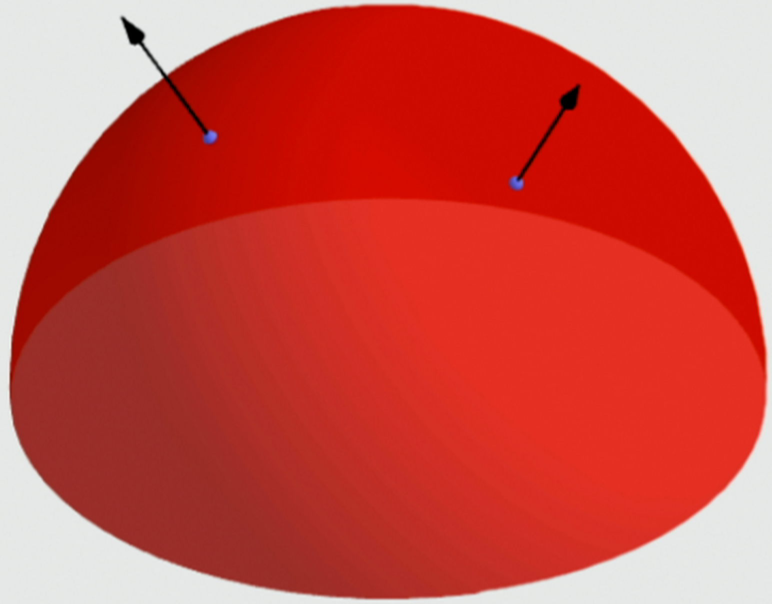
PROOF OF THE THEOREM



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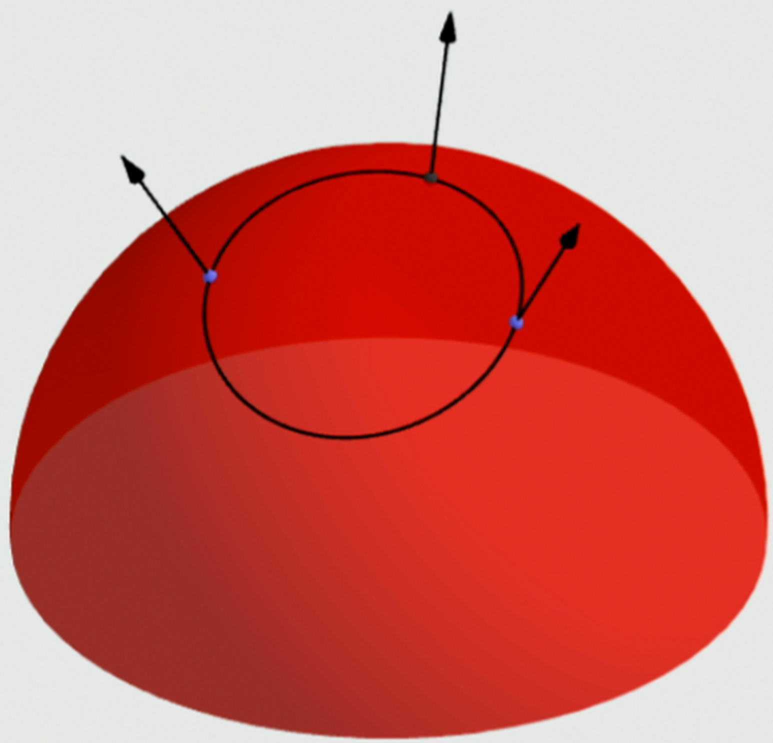

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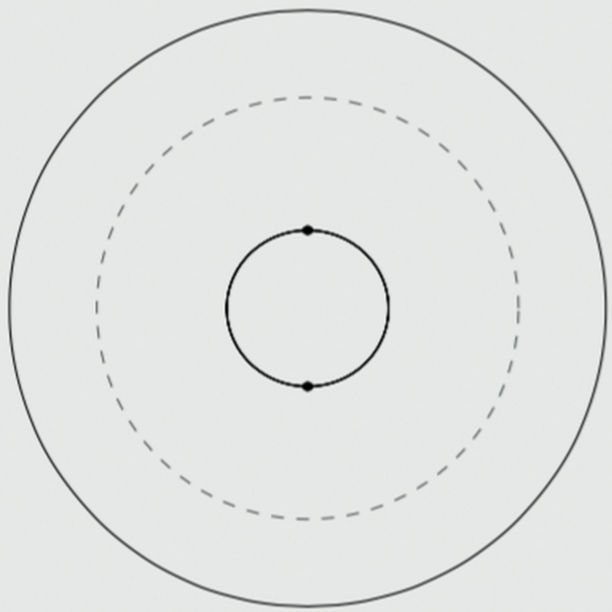

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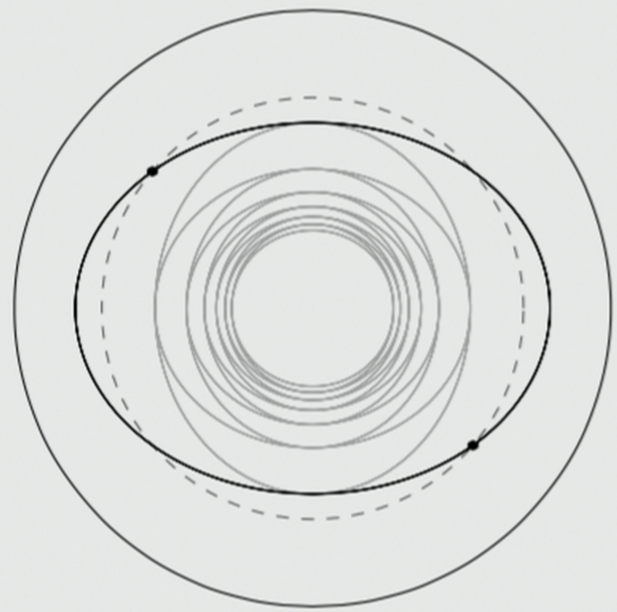

CONSEQUENCE: KOCHEN-SPECKER

Kochen-Specker Theorem

Pick **exactly** one element in each maximal orthogonal family of vectors

Kochen-Specker	117 vectors in dimension 3
Peres	33 vectors in dimension 3
Cabello	17 vectors in dimension 4

Sasaki Filters

Pick **at most** one element in each maximal orthogonal family of vectors

2 vectors in dimension 3

CONSEQUENCE: ONTOLOGICAL MODELS

Theorem

If $\dim \mathcal{H} \geq 3$,

weakly noncontextual ontological model of $\mathcal{H}$

CONSEQUENCE: POSITION AND MOMENTUM

Heisenberg's Uncertainty Relation

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

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There is no model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_{L(\mathcal{H})}$ with an element $a \in A$ such that

$$a \blacktriangleright_{\mathfrak{G}} [x = x_0] \qquad \text{and} \qquad a \blacktriangleright_{\mathfrak{G}} [p = p_0]$$

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A REPRESENTATION THEOREM IN FINITE DIMENSION

THE EVOLUTION OF VERIFIED PROPERTIES

Given a model $\mathfrak{G} = (A, M)$, from

$$a \blacktriangleright_{\mathfrak{G}} p \text{ and } M(a, q, b) \implies b \blacktriangleright_{\mathfrak{G}} p \,\&\, q$$

we deduce

$$M(a, p, b) \implies [\![a]\!]_{\mathfrak{G}} \,\&^{\#}\, p \subseteq [\![b]\!]_{\mathfrak{G}}$$

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Proposition

Given $a \in A$ and $p \in L$,

$$\neg\left(\exists\, b \in A \colon [\![a]\!]_{\mathfrak{G}} \,\&^{\#}\, p \subseteq [\![b]\!]_{\mathfrak{G}}\right) \implies p^{\perp} \in [\![a]\!]_{\mathfrak{G}}$$

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Theorem

If $3 \leq \dim \mathcal{H} < \infty$ and $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$, then

$$\forall\, a \in A, \exists\, e(a) \in L(\mathcal{H}) : [\![a]\!]_{\mathfrak{G}} = e(a)^{\uparrow}$$

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THE METAPHYSICAL DISASTER

[is] the error of ***equating properties*** *of a physical system on the one hand* ***with experimentally testable propositions*** *about the system on the other hand.*

Unfortunately, this is precisely what is done in conventional Hilbert-space based quantum mechanics where both properties and experimentally testable propositions are represented by projection operators.

(David Foulis, private communication)

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SEQUENCE OF OUTCOMES

A **word** on L is a **finite sequence of elements** of L

Definition

Given a model $\mathfrak{G} = (A, M)$ of $\mathcal{T}_L$, a word $\mathbf{p} = p_1 p_2 \cdots p_n$ is in $\ell(\mathfrak{G})$ iff

$$\exists\, a_0, a_1, \ldots, a_n \in A:$$

$$M(a_0, p_1, a_1) \text{ and } \cdots \text{ and } M(a_{n-1}, p_n, a_n)$$

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If $3 \leq \dim \mathcal{H} < \infty$ and $\mathfrak{G} = (A, M)$ is a model of $\mathcal{T}_{L(\mathcal{H})}$, then

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SKETCH OF PROOF, $\ell(\mathfrak{G}) \subseteq \ell(\mathfrak{L}_{L(\mathcal{H})})$

$$a_0 \xrightarrow{p_1} a_1 \xrightarrow{p_2} a_2 \xrightarrow{p_3} \cdots \xrightarrow{p_{n-1}} a_{n-1} \xrightarrow{p_n} a_n$$

REPRESENTATION THEOREM: CONCLUSION

Theorem

If $\mathcal{H}$ is such that $3 \leq \dim \mathcal{H} < \infty$, then for all model $\mathfrak{G}$ of $\mathcal{T}_{L(\mathcal{H})}$,

$$\ell(\mathfrak{G}) = \ell(\mathfrak{L}_{L(\mathcal{H})}) = \ell(\mathfrak{H}_H) = \ell(\mathcal{H})$$

Given a word $p_1 p_2 \cdots p_n$ on $L(\mathcal{H})$,

$$\begin{aligned} p_1 p_2 \cdots p_n \in \ell(\mathcal{H}) &\iff p_1 \,\&\, p_2 \,\&\, \cdots \,\&\, p_n \neq \bot \\ &\iff \Pi_{p_n} \Pi_{p_{n-1}} \cdots \Pi_{p_1} \neq 0 \end{aligned}$$

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Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

Yes, as regards possibilities.

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FUTURE DEVELOPMENTS

Model of $\mathcal{T}_L$ associated to an **ontological model**

Representation theorem in **infinite dimension**

From **possibilities** to **probabilities**

Compound systems and **entanglement**

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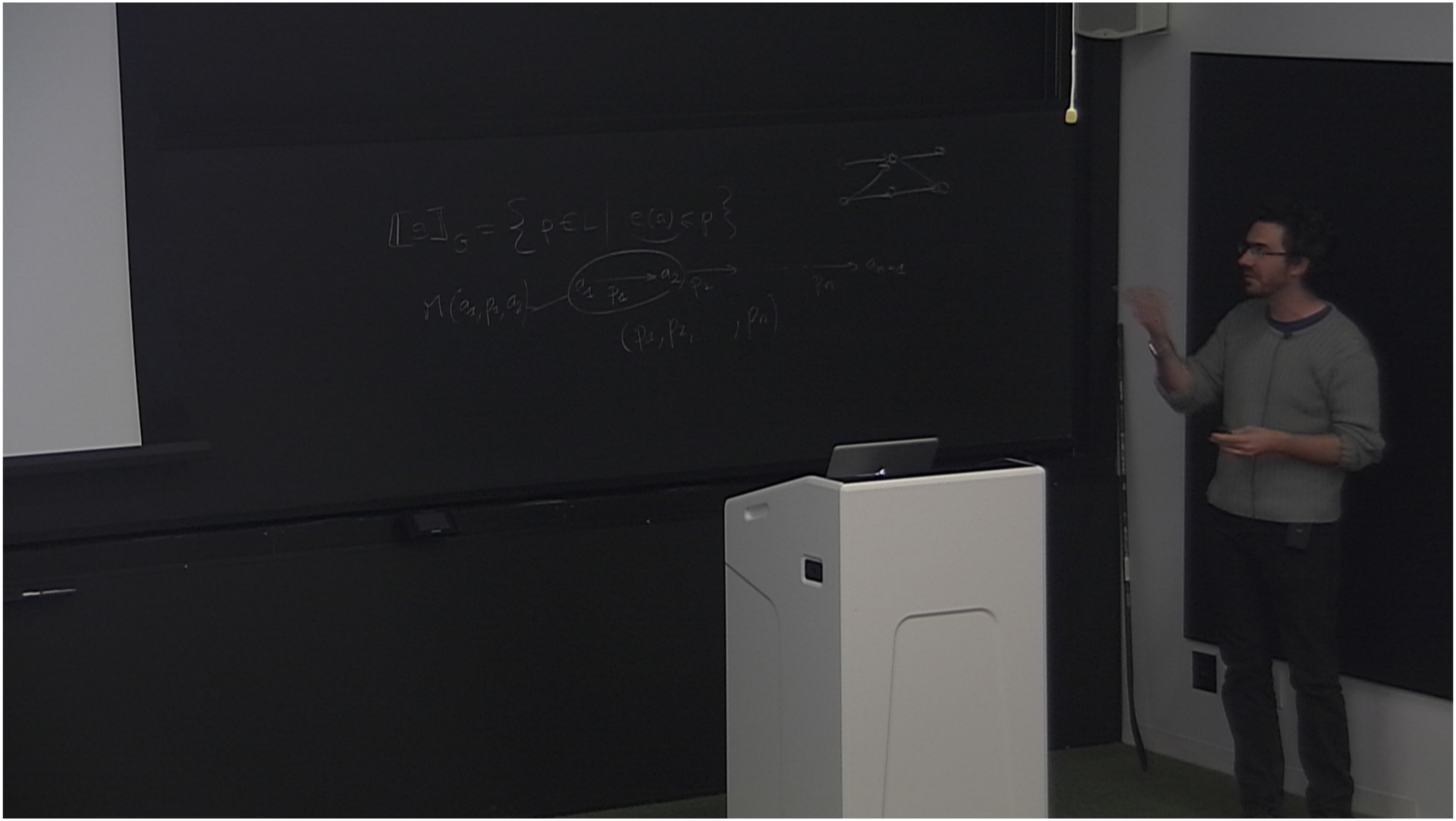

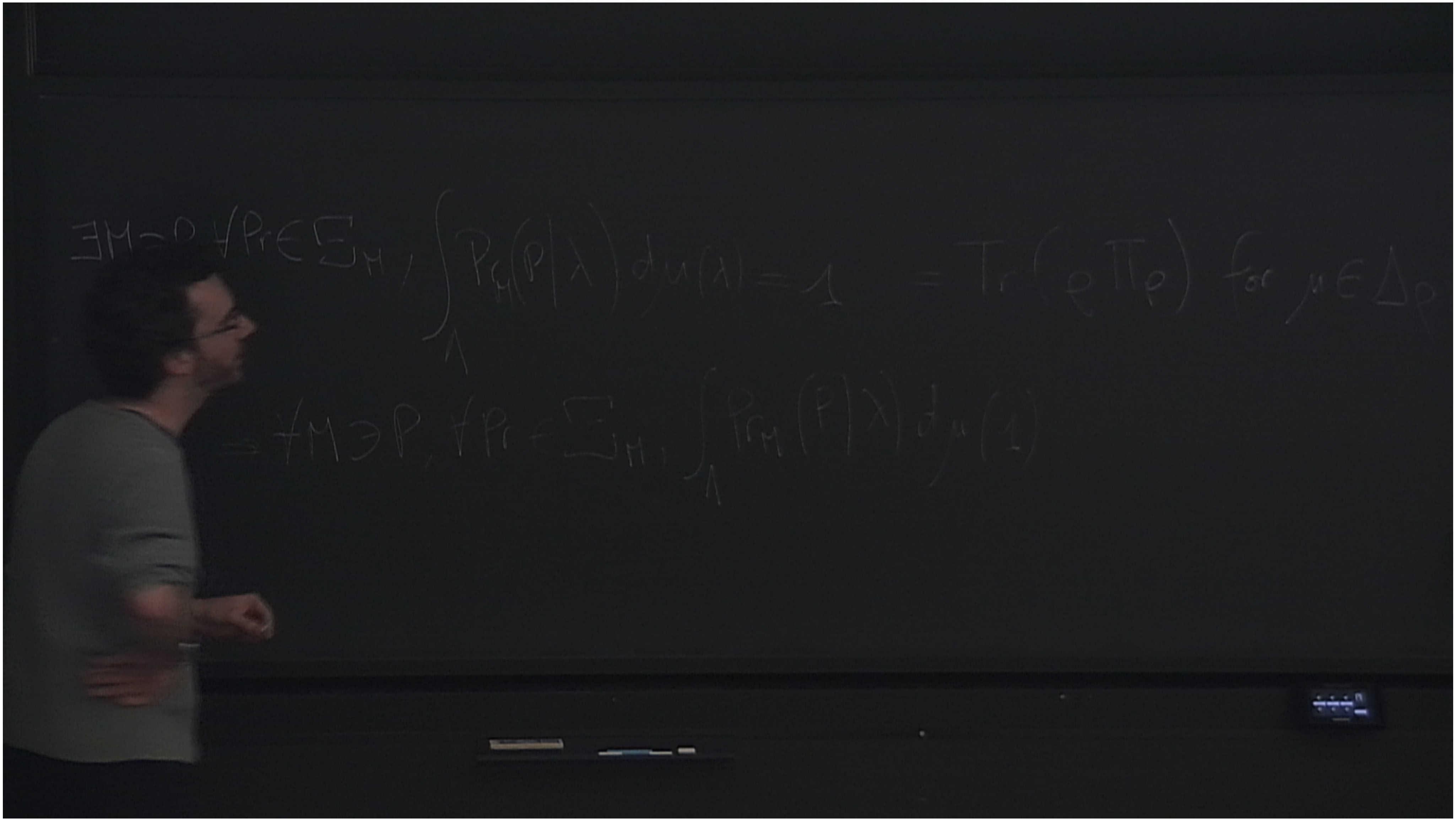
∃M→P ∀Pr∈Ξ_M, ∫_Λ Pr_M(P|λ) dμ(λ) = 1 = Tr(ρ Π_P) for μ∈Δ_ρ
⇒ ∀M∃P, ∀Pr∈Ξ_M, ∫_Λ Pr_M(P|λ) dμ(λ)