Title: Topological Strings from quantum mechanics

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Abstract: <p>In this talk I will propose a general correspondence which associates a non-perturbative quantum mechanical operator to a toric Calabi-Yau manifold, and I will propose a conjectural expression for its spectral determinant. As a consequence of these results, I will derive an exact quantization condition for the operator spectrum. I will give a concrete illustration of this conjecture by focusing on the example of local P2. This approach also provides a non-perturbative Fermi gas picture of topological strings on toric background and suggests the existence of an underlying theory of M2 branes behind this formulation.</p>

Topological Strings from Quantum Mechanics

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Mostly based on: A.G. , Y. Hatsuda, M. Mariño, 1410.3382

Outline

Toric Calabi- Yau X → Quantum operator : $\hat{\rho}_X(\hat{x}, \hat{p})$

The information on the spectrum can be encoded in the spectral determinant.

We conjecture an explicit expression for the spectral determinant in terms of Gopakumar-Vafa invariants of X.

This relates spectral theory and enumerative geometry in a novel way. This proposal is testable!

Interpretation of $\hat{\rho}_X(\hat{x}, \hat{p})$ as the density matrix of an ideal Fermi gas

→ Fermi gas formulation of topological strings on X

→ Non-perturbative, background independent, formulation of topological strings

Topological strings

Topological string is a 2 dimensional conformal field theory coupled to gravity

Topological sigma model on target space X

We will study the Free energy of topological strings on a Calabi-Yau X .

topological string coupling

Kähler parameter, it is related to the size of X

$$F^{\text{top}}(t, g_s) = \sum_g g_s^{2g-2} F_g(t) = \sum_m d_m(g_s) e^{-mt}$$

Determined by Gopakumar-Vafa invariant on X , e.g.

$$d_1(g_s) = \sum_{g \geq 0} n_1^g \left(2 \sin \frac{g_s}{2}\right)^{2g-2}$$

Refined Topological strings

It is possible to refine topological strings by introducing an additional coupling

Two couplings: ϵ_1, ϵ_2

The Free energy: $F^{\mathrm{RT}}(\epsilon_1, \epsilon_2, t)$

$\epsilon_1 = i\hbar, \quad \epsilon_2 \to 0$

Nekrasov–Shatashvili (NS) limit

$$F^{\mathrm{NS}}(\hbar, t) = \sum_m c_m(\hbar) e^{-mt}$$

Determined by refined GV , e.g.

$$c_1(\hbar) = \sum_{j_L, j_R} N^1_{j_L, j_R} \frac{\sin(\hbar(2j_L+1))\sin(\hbar(2j_R+1))}{2\sin\left(\frac{\hbar}{2}\right)\sin^2(\hbar)}$$

$\epsilon_1 = -\epsilon_2 = g_s$

Standard Topological strings

$F^{\mathrm{top}}(t, g_s)$

Mirror symmetry

Topological string is built on a topological sigma model

Two different types of sigma models can be used

A model | **B model**

Starting point for the construction of the spectral determinant

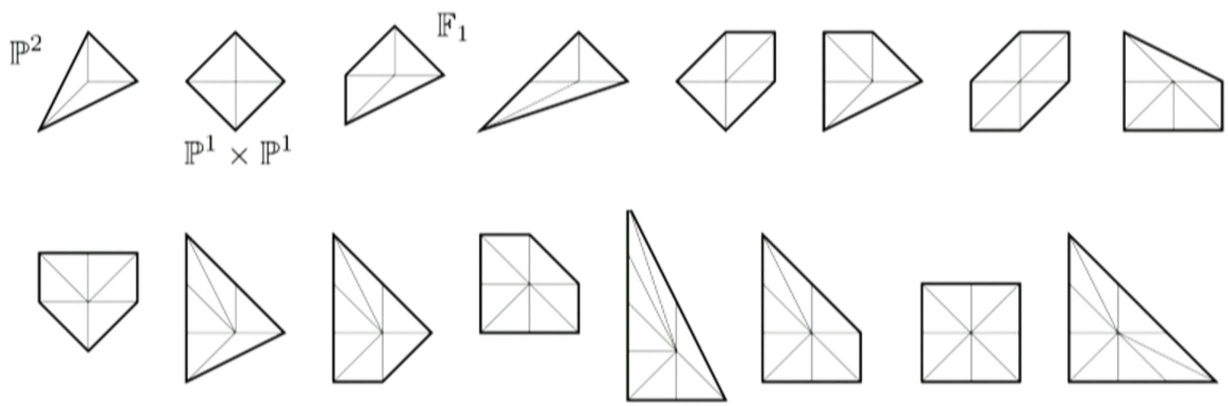
Mirror Symmetry

Given a Calabi-Yau X there is another Calabi-Yau $\tilde{X}$ such the A model on X is equivalent to the B model on $\tilde{X}$. The manifold $\tilde{X}$ is called the mirror of X.

Starting point for the construction of our quantum operator

The geometry

Our target space X is a toric del Pezzo Calabi-Yau. These are local Calabi-Yau which can be classified by polyhedra:



Batyrev, Klemm et al

We can read off the geometrical information we need out of these drawings.

The geometry

The mirror $\tilde{X}$ **of a toric del Pezzo Calabi-Yau X is described by the curve**

Hori-Vafa, Batyrev,
Katz-Klemm-Vafa, . . .

$$W_X(e^x, e^p) = vw$$

Polynomial in e^x, e^p

All the information needed to compute $F_g(t)$ **on X is encoded in**

$$W_X(e^p, e^x) = 0$$

Bouchard- Klemm,
Mariño-Pasquetti

Riemann surface of genus one

Example: Local $\mathbb{P}^2$

$$W_{\mathbb{P}^2}(e^x, e^p) = \underbrace{e^x + e^p + e^{-x-p}}_{= O_{\mathbb{P}^2}(x,p)} - \tilde{u}$$

Complex modulus

The geometry

Dijkgraaf et al,
Mironov-Morozov,
Aganagic et al

$$W_X(e^x, e^p) = O_X(x,p) - \tilde{u} = 0$$

We quantize this curve, i.e. we promote x and p to operators s.t $[\hat{x},\hat{p}] = i\hbar.$

Moreover we identify the modulus with the exponentiated energy: $\tilde{u} = e^E$

$$W_X(e^p, e^x) = 0 \quad \rightarrow \quad \hat{O}_X(\hat{x},\hat{p})|\psi_n\rangle = e^{E_n}|\psi_n\rangle$$

We will use: $\hat{\rho}_X(\hat{x},\hat{p}) = \hat{O}_X^{-1}(\hat{x},\hat{p})$

Similar to Schrödinger equation:

$$\frac{p^2}{2} + V(X) - E = 0 \quad \rightarrow \quad \left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)|\psi_n\rangle = E_n|\psi_n\rangle$$

The geometry

Example: Local $\mathbb{P}^2$

$$W_{\mathbb{P}^2}(e^x, e^p) = e^x + e^p + e^{-x-p} - \tilde{u} = 0$$

↓ **quantization**

$$\hat{O}_X(\hat{x}, \hat{p})|\psi_n\rangle = \left(e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{x}-\hat{p}}\right)|\psi_n\rangle = e^{E_n}|\psi_n\rangle$$

$$e^x \psi_n(x) + \psi_n(x - i\hbar) + e^{-x-i\frac{\hbar}{2}}\psi_n(x + i\hbar) = e^{E_n}\psi_n(x)$$

Nekrasov-Shatashvili, Mironov-Morozov, Aganagic et al

Example: SU(2) quantum Toda chain

$$e^p + e^{-p} + x^2 = E$$

↓ **quantization**

Gaudin-Pasquier

$$\psi_n(x + i\hbar) + \psi_n(x - i\hbar) + x^2\psi_n(x) = E_n\ \psi_n(x)$$

The Operator

The operators $\hat{\rho}_X(\hat{x},\hat{p}) = \hat{O}_X^{-1}(\hat{x},\hat{p})$

are well defined trace class operators acting on $L^2(\mathbb{R})$ **. They have a positive discrete spectrum.**

Kashaev-Mariño

Example local $\mathbb{P}^2$ **: the kernel can be computed explicitly.**

$$(x|\hat{\rho}_{\mathbb{P}^2}|y) = \rho_{\mathbb{P}^2}(x,y) = \frac{\Phi_b(x+ib/3)}{\Phi_b(x-ib/3)} \frac{e^{\pi b(x+y)/3}}{2b\cosh\left(\pi\left(\frac{x-y}{b}+\frac{i}{6}\right)\right)} \qquad b^2 = \frac{3\hbar}{2\pi}$$

$\Phi_b(x)$ **: the quantum dilogarithm**

Question: can we compute the spectrum of these of operators?
We will see that the spectrum is encoded in the GV/refined GV invariants of the corresponding toric Calabi-Yau.

Old vs New

Mironov-Morozov,
Aganagic-Cheng-
Dijkgraaf-Krefl-Vafa

Already known:

The quantization of the mirror curve leads to a difference equation:

$$e^x \psi_n(x) + \psi_n(x - i\hbar) + e^{-x-i\frac{\hbar}{2}} \psi_n(x + i\hbar) = e^{E_n} \psi_n(x)$$

The perturbative WKB quantization condition for the spectrum of this equation is closely related to the NS limit.

New approach:

Behind the quantization of the mirror curve there is a well defined trace class quantum mechanical operator acting on $L^2(\mathbb{R})$:

$$\hat{\rho}_X(\hat{x}, \hat{p}) = \hat{O}_X^{-1}(\hat{x}, \hat{p}) \quad \textbf{inverse!}$$

In order to study its spectrum, the right object to look at is the spectral determinant. We found that the spectrum of this operator is determined both by standard and refined topological strings in NS limit.

↗

non perturbative corrections to WKB!

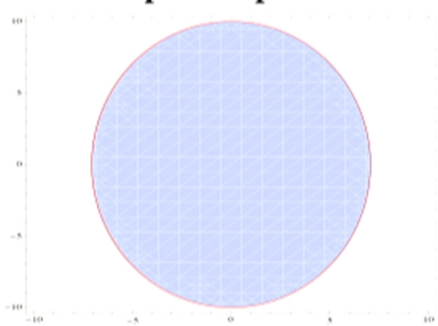
The Spectrum

From a physical point of view the discreteness of the spectrum can be understood from Bohr-Sommerfeld quantization condition:

Example: Harmonic oscillator

$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}\right)|\psi_n\rangle = E_n|\psi_n\rangle$$

phase space



Bohr-Sommerfeld: each cell of volume $2\pi\hbar$ in

$$\mathcal{R}(E) = \left\{(x,p) \in \mathbb{R}^2 |\ p^2 + x^2 \leq 2E\right\}$$

leads to a quantum state.

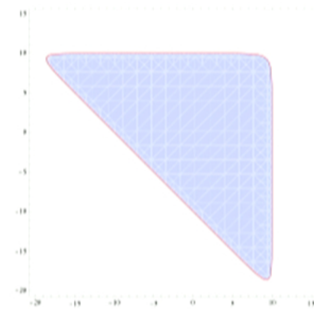
$$\mathrm{Vol}_{cl}(x,p) \approx 2\pi\hbar n \quad \longrightarrow \quad E_n \approx \hbar n$$

We expect a compact phase space region $\mathcal{R}(E)$ to lead to a positive discrete spectrum

The Spectrum

Example: Local $\mathbb{P}^2$

$$\mathcal{R} = \left\{ (x,p) \in \mathbb{R}^2 | O_{\mathbb{P}^2}(x,p) = e^x + e^p + e^{-x-p} \leq e^E \right\}$$



Bohr-Sommerfeld: **each cell of volume** $2\pi\hbar$ **in** $\mathcal{R}$ **leads to a quantum state.**

$$\mathrm{Vol}_{cl}(x,p) \approx 2\pi\hbar n \quad \rightarrow \quad E_n^2 \approx \frac{4\pi\hbar}{9} n$$

The quantum volume

Toric Calabi- Yau X → **Quantum operator :** $\hat{\rho}_X(\hat{x},\hat{p})$

↓ **Bohr-Sommerfeld**

$$\mathrm{Vol}_{cl}(E) = 2\pi\hbar(n+\frac{1}{2}), \quad n \geq 0$$

Widom

We expect quantum corrections of two types:

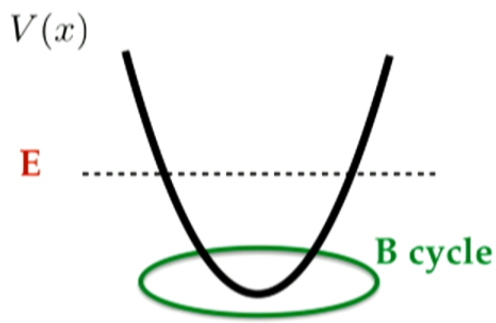
1) Perturbative in $\hbar$ **: all order WKB** Dunham

2) Non perturbative in $\hbar$ **:** $e^{-1/\hbar}$

$$\mathrm{Vol}_p(E) + \mathrm{Vol}_{np}(E) = 2\pi\hbar(n+\frac{1}{2}), \quad n \geq 0$$

The perturbative quantum volume

The classical volume: $\text{Vol}_{cl}(E) = \oint_B p(x)dx = \Pi_B(E)$ **: B period**



B cycle: represents oscillations around minima

p(x) determined by: $\frac{p^2}{2} + V(x) - E = 0$

Perturbative quantum volume: $\text{Vol}_p(E,\hbar) = \oint_B p(x,\hbar)dx = \Pi_B(E,\hbar)$

Quantum B period

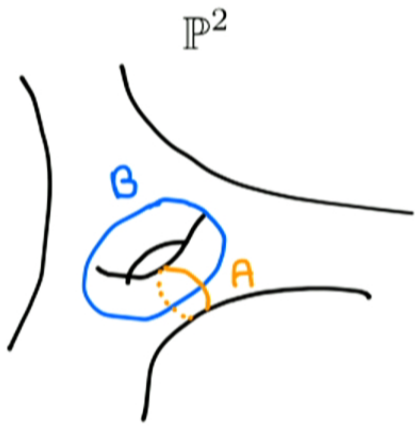
$p(x,\hbar)$ **determined by:** $\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x) - E\right) \exp\left[\frac{i}{\hbar}\int^x p(y,\hbar)dy\right] = 0$

all order WKB

The perturbative quantum volume

In our context: $W_X(e^p, e^x) = 0$ **is a genus one curve**

$\longrightarrow$ **it has A and B cycles**



$$\mathrm{Vol}_p(E,\hbar) = \oint_B p(x,\hbar)dx = \Pi_B(E,\hbar)$$

the quantum B period

$\downarrow$

Determined by refined topological strings

$$\partial_T F^{\mathrm{NS}} = \Pi_B(E,\hbar)$$

Nekrasov-Shatashvili,
Mironov-Mironov,
Aganagic et al

One also defines the quantum A period: $\oint_A p(x,\hbar)dx = \Pi_A(E,\hbar)$

The perturbative quantum volume

Perturbative quantization condition: $\mathrm{Vol}_p(E,\hbar) = \Pi_B(E,\hbar) = 2\pi\hbar(n+\frac{1}{2})$

Aganagic et al

Example: local $\mathbb{P}^2$

$$\mathrm{Vol}_p(E) = \frac{9}{2}E^2 - \frac{\pi^2}{2} - \frac{\hbar^2}{8} + \sum_\ell \left(Ea_\ell(\hbar) + b_\ell(\hbar)\right) e^{-3\ell E}$$

$$b_1(\hbar) = \frac{3}{2}\hbar \sin\left(\frac{3\hbar}{2}\right) \csc^2\left(\frac{\hbar}{2}\right)$$

! Diverge for infinitely many values $\hbar$

→ $\mathrm{Vol}_p(E)$ **needs extra corrections!** Källén, Mariño

The quantum volume

Including perturbative and non perturbative correction we found

$$\mathrm{Vol}_p(E) + \mathrm{Vol}_{np}(E) = \; CE^2 + 2\pi\hbar B(\hbar) - \frac{\pi^2}{3}C + \sum_{\ell\geq 1} E a_\ell(\hbar) e^{-r\ell E}$$

$$+ \sum_{m,n\geq 0} d_{m,n}(\hbar) e^{-r(n+2\pi m/\hbar)E}$$

well-defined finite quantity

$d_{0,n}(\hbar)$ **: refined topological strings in NS limit**

$d_{m,0}(\hbar)$ **: standard topological strings**

The operator $\hat{\rho}_X(\hat{x},\hat{p})$ contains information on both the standard and the NS limit of refined topological strings

We can derive all this in a single strike from our conjectural expression of spectral determinant associated to $\hat{\rho}_X(\hat{x},\hat{p})$

The quantum volume

$$\mathrm{Vol}_p(E,\hbar) + \mathrm{Vol}_{np}(E,\hbar) = 2\pi\hbar(n+\frac{1}{2}), \quad n \geq 0$$

Local $\mathbb{P}^2$ **with** $\hbar = 4\pi$

Order	E_0
$\mathrm{e}^{-3E/2}$	3.77764328326207137402
e^{-6E}	3.77770625855981285308
$\mathrm{e}^{-21E/2}$	3.77770625858220666231
e^{-12E}	3.77770625858220699760
$\mathrm{e}^{-27E/2}$	3.77770625858220699861
Numerical value	3.77770625858220699869

From a detailed numerical analysis of $\hat{O}_{\mathbb{P}^2}(\hat{x},\hat{p})$ Huang-Wang

This is a strong test of our conjectural expression for the spectral determinant !

The spectral determinant

Given an operator $\hat{\rho}$ the information about its spectrum e^{-E_n} can be encoded in the spectral determinant (or Fredholm determinant):

$$\Xi(\kappa) = \det(1 + \kappa\hat{\rho}) = \prod_n \left(1 + \kappa e^{-E_n}\right)$$

Once $\Xi(\kappa)$ is known, the spectrum can be computed by looking at its zeros:

$$\Xi(\kappa) = 0 \leftrightarrow \kappa = e^{E_n + i\pi}$$

The spectral determinant has an important property: is an entire function of κ.

The spectral determinant

We conjecture that the spectral determinant of $\hat{\rho}_X(\hat{x},\hat{p}) = \hat{O}_X(\hat{x},\hat{p})^{-1}$ is:

$$\Xi_X(\kappa,\hbar) = e^{J_X(\mu,\hbar)}\Theta_X(\mu,\hbar), \qquad \kappa = e^{\mu}$$

Grand Potential

Generalized Theta

The first ingredient is the chemical potential μ

$$t = r\mu_{\text{eff}} = r\mu - \sum_{\ell>0}(-1)^{r\ell} a_\ell(\hbar) \mathrm{e}^{-r\ell\mu}$$

Kähler parameter

Determined by the quantum A period (quantum mirror map) :

$$\Pi_A(\tilde{u},\hbar) = -r\log\tilde{u} + \sum_{m>0} a_\ell(\hbar)\tilde{u}^{-rm}$$

The spectral determinant

$$\Xi_X(\kappa,\hbar) = e^{J_X(\mu,\hbar)}\Theta_X(\mu,\hbar), \quad \kappa = e^{\mu}$$

Grand potential

At large μ

$$J_X(\mu,\hbar) = \frac{C(\hbar)}{3}\mu^3 + B(\hbar)\mu + A(\hbar) + J_{\rm M}(\mu,\hbar) + J_{{\rm W}S}(\mu_{\rm eff},\hbar)$$

Hatsuda,Mariño,
Moriyama,Okuyama

→ $J_{\rm WS}(\mu_{\rm eff},\hbar) = \sum_{m\geq 1} s_m(\hbar) e^{-2\pi m r \mu_{\rm eff}/\hbar}$ **is the total free energy of standard topological strings**

→ $J_{\rm M}(\mu,\hbar) = \sum_{m\geq 1} b_m(\mu,\hbar) e^{-mr\mu}$ **refined topological strings in NS limit**

→ **Cancel the poles and provide a non perturbative completion of topological string free energy**

The spectral determinant

$$\Xi_X(\kappa, \hbar) = e^{J_X(\mu,\hbar)} \Theta_X(\mu, \hbar), \quad \kappa = e^{\mu}$$

Generalized theta function

Similar to Eynard-Mariño

$$\Theta_X(\mu, \hbar) = \sum_{n \in \mathbb{Z}} \exp \left[J_X(\mu + 2\pi i n, \hbar) - J_X(\mu, \hbar) \right]$$

- **It is determined by standard/refined topological strings**
- **In some cases it becomes a standard theta function**
- **The zeros of the generalized theta function determine the quantum volume**

$$\Theta(\mu, \hbar) = 0 \leftrightarrow \mathrm{Vol}_p(E, \hbar) + \mathrm{Vol}_{np}(E, \hbar) = 2\pi\hbar(n + \frac{1}{2}), \quad n \geq 0$$

$$\mu = E + i\pi$$

The spectral determinant

For some value of $\hbar$ the expression for the spectral determinant is particularly elegant: these are the "maximally supersymmetric" cases Codesido, AG, Mariño

Example: $\hbar = 2\pi$ **standard Jacobi theta**

$$\Xi_X(\mu, 2\pi) = \mathrm{e}^{J_X(\mu,2\pi)}\vartheta_3\left(\xi + C, \frac{r^2\tau}{4}\right)$$

→ **has a nice and simple expression in term of genus zero and genus one topological string free energy**

$$\xi = \frac{r}{4\pi^2}\left(t\partial_t^2 F_0(t) - \partial_t F_0(t)\right)$$

$$\tau = \frac{2i}{\pi}\partial_t^2 F_0(t)$$

$$J_X(\mu, \hbar = 2\pi) = A(2\pi) + \frac{1}{4\pi^2}\left(F_0(t) - t\partial_t F_0(t) + \frac{t^2}{2}\partial_t^2 F_0(t)\right) + \frac{B(2\pi)}{r}t + F_1(t) + F_1^{\mathrm{NS}}(t)$$

The generalized theta function: an example

Example: Local $\mathbb{P}^2$ and $\hbar = 2\pi$

Generalized theta function: $\Theta_X(\mu, \hbar) = \vartheta_3\left(\xi - \frac{3}{8}, \frac{9\tau}{4}\right)$

↓ **the zeros are given by**

$$\xi(E) - \frac{1}{4} = s + \frac{1}{2}, \quad s = 0, 1, 2 \ldots$$

$$s = 0: \quad E_0 = 2.56264206862381937...$$

$$s = 1: \quad E_1 = 3.91821318829983977...$$

This agrees with detailed numerical analysis!

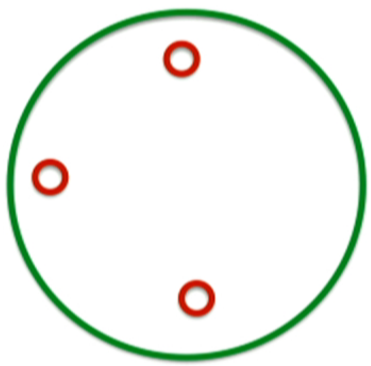
The moduli space

$\Xi(\kappa,\hbar) = e^{J_X(\mu,\hbar)}\Theta_X(\kappa,\hbar)$ **: is a entire function of the moduli space**

large radius: $\kappa = \infty$

conifold:

$\kappa = -\frac{1}{3}$



orbifold: $\kappa = 0$

$\kappa = e^{\mu}$

So far: **looking at** $\Xi(\kappa,\hbar)$ **close to the large radius we can derive the quantization condition and the eigenvalues of our operator.**

Next: **looking at** $\Xi(\kappa,\hbar)$ **near the orbifold we can compute the spectral traces of our operator**

The spectral traces

$$Z_\ell = \mathrm{Tr}\hat{\rho}_X^\ell = \sum_{n=0}^{\infty} e^{-\ell E_n}$$

From the spectral determinant:

a) We can compute them from the spectrum i.e. from the large radius expression:

local $\mathbb{P}^2$ **,** $\hbar = 2\pi$ **:** $\quad Z_1 = \frac{1}{9}, \quad Z_2 = \frac{1}{27} - \frac{1}{6\pi\sqrt{3}},$

$$Z_3 = \frac{1}{81} - \frac{1}{24\pi^2} - \frac{1}{24\pi\sqrt{3}}$$

b) We can also compute them by looking at $\Xi(\kappa, \hbar)$ **close to the orbifold point:**

$$\log \Xi(\kappa, \hbar) = J_X - \log \theta_3 \left(\xi - \frac{3}{8}, \frac{8\tau}{4} \right) = \sum_{\ell \geq 1} Z_\ell \frac{(-\kappa)^\ell}{\ell}$$

Agree with previous ones!

The spectral traces

$$Z_\ell = \mathrm{Tr}\hat{\rho}_X^\ell = \sum_{n=0}^{\infty} e^{-\ell E_n}$$

From the operator:

c) The spectral traces can also be computed directly from the operator by using the explicit expression for the kernel:

Kashaev-Mariño

$$\rho_{\mathbb{P}^2}(x,y) = \frac{\Phi_b(x+ib/3)}{\Phi_b(x-ib/3)} \frac{e^{\pi b(x+y)/3}}{2b\cosh\left(\pi\left(\frac{x-y}{b}+\frac{i}{6}\right)\right)}$$

$$\mathrm{Tr}\rho_{\mathbb{P}^2}^\ell = \int dx_1 \dots dx_L \prod_{i=1}^{\ell} \rho_{\mathbb{P}^2}(x_i, x_{i+1})$$

! Perfect matching with the spectral traces computed from our spectral determinant !

The Fermi gas picture

Toric del Pezzo Calabi- Yau X → **Quantum operator :** $\hat{\rho}_X(\hat{x},\hat{p})$

Density matrix of an ideal Fermi gas

In this picture $J_X(\mu,\hbar)$ **is the grand potential of the ideal Fermi gas**

We consider the **large N limit** **of this ideal Fermi gas. We have two types of large N limit:**

→ **'t Hooft limit:** $\hbar \to \infty, \quad \lambda = \dfrac{N}{\hbar}$ **fixed:** $\quad J_X \to F_X^{top}$

→ **Thermodynamic limit:** $N \to \infty, \quad \hbar$ **fixed, small:** $\quad J_X \to J_M \sim \text{NS limit}$

Cure the singularities of the 't Hooft expansion

The Fermi gas picture

- Reproduces the perturbative expansion of topological strings

$$F^{top}(t, g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(t) \qquad g_s = 4\pi^2/\hbar$$

- Includes non-perturbative effects in the coupling g_s: $J_M \sim e^{-1/g_s}$

This gives a nice non-perturbative construction for topological strings in which refined topological strings in the NS limit provide a non-perturbative completion of standard topological strings.

- Leads to testable predictions

- All the information on this Fermi gas can be encoded in the spectral determinant, which is an entire function of the moduli. We have a background independent formulation.

Testing the 't Hooft expansion

In the Fermi gas formalism there are two ways to compute the partition function:

a) From the spectral determinant

Grand potential, determined by GV invariants

$$Z(N) = \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi i} \mathrm{e}^{J_{\mathbb{P}^2}(\mu) - N\mu}$$

The 't Hooft expansion gives the genus expansion of topological strings on $\mathbb{P}^2$.

b) From the density matrix

It is given in terms of quantum dilogarithm

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int d^N x \prod_i \rho_{\mathbb{P}^2}(x_i, x_{\sigma(i)})$$

Mariño-Zakany

The 't Hooft expansion at weak coupling reproduce the weak coupling genus expansion of topological strings on $\mathbb{P}^2$. This is a non trivial check of our conjecture !

Conclusions

- We have proposed a correspondence between the spectral theory of certain quantum operators and enumerative geometry.

  ! This proposal is concrete and testable

- In the " maximally supersymmetric" cases we were able to give explicit closed formulae for spectral determinants

  ! This is extremely rare, even in standard QM.

- Physically, the meaning behind these results is a Fermi gas formulation of topological strings wich is:

  a) Non-perturbative, background independent and testable

  b) Such that standard topological strings emerge as a 't Hooft limit of the Fermi gas

  c) The NS limit emerge by looking at the thermodynamic limit of the Fermi gas.

Future directions

- Why such a conjecture should be true?
  So far many successful test ... a proof?
- Higher genus curve ?
- It would be interesting to see if we can relax the conditions set on the parameters of the spectral problem ($\hbar$ real, $\tilde{u}$ real,...)
- Connection to integrable systems