Title: Upper Bounds on Query Complexity Inspired by the Elitzur-Vaidman Bomb Tester

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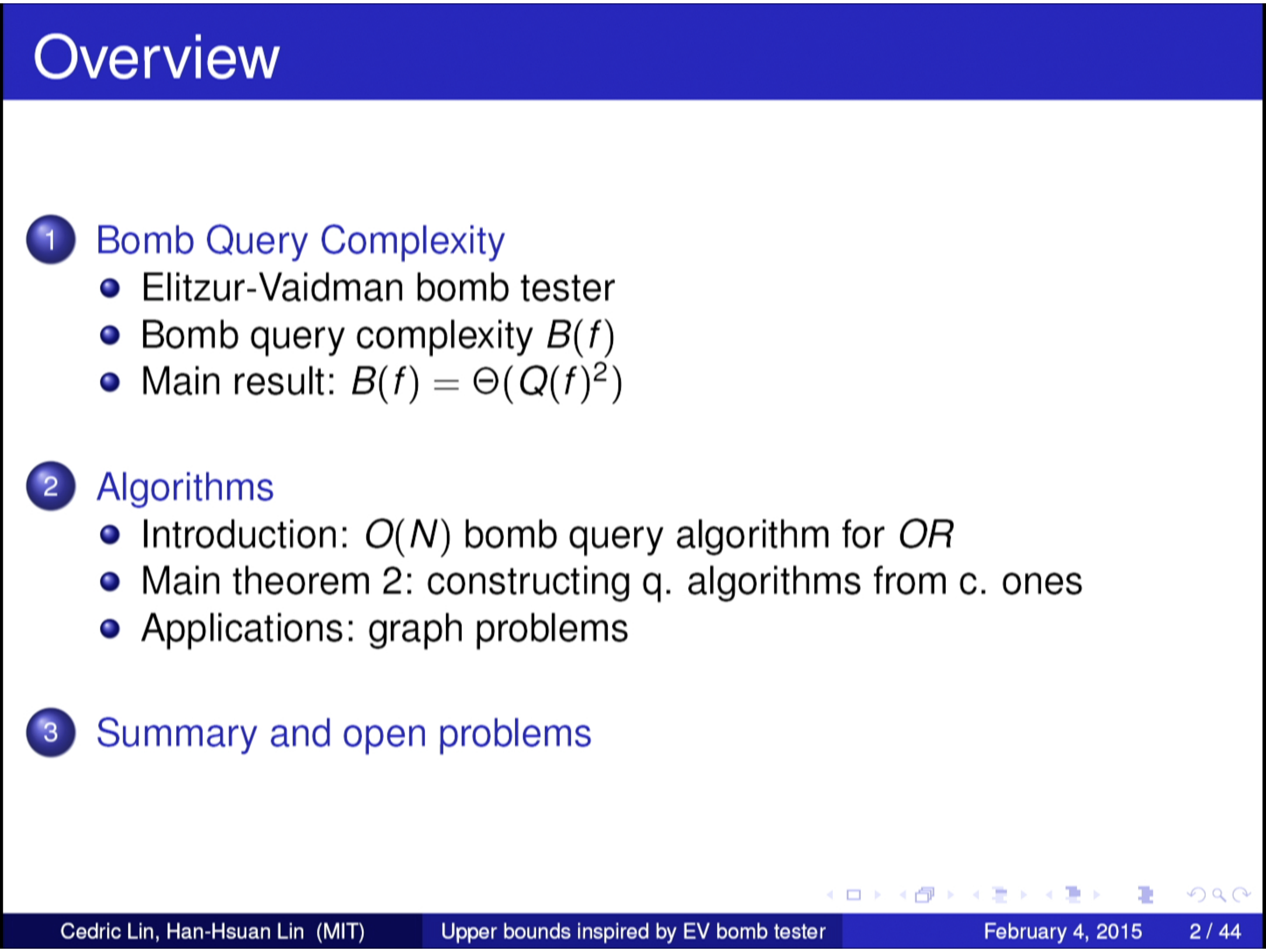
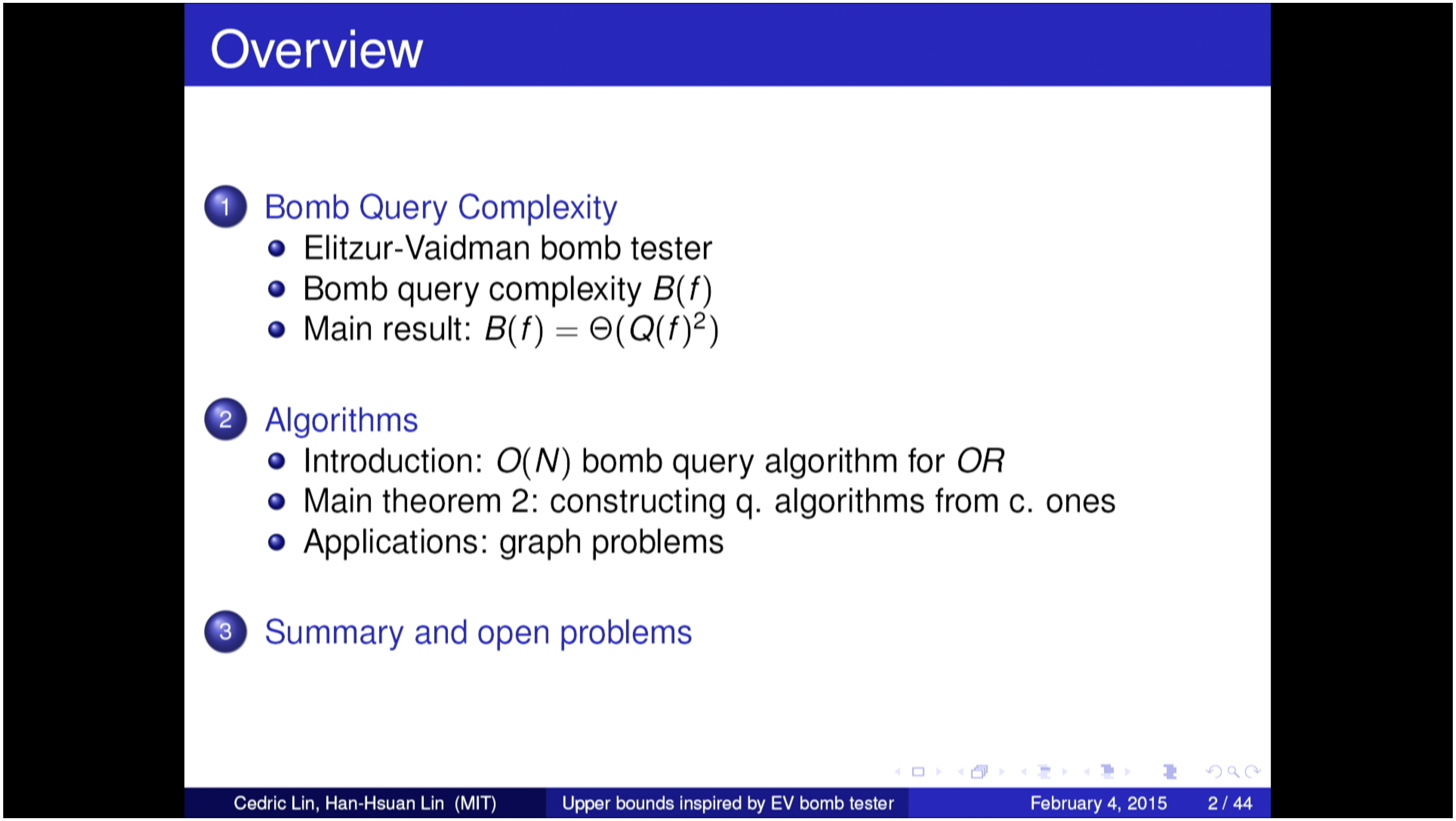
URL: http://pirsa.org/15020109

Abstract: <p>The Elitzur-Vaidman bomb tester allows the detection of a photon-triggered bomb with a photon, without setting the bomb off. This seemingly impossible task can be tackled using the quantum Zeno effect. Inspired by the EV bomb tester, we define the notion of "bomb query complexity". This model modifies the standard quantum query model by measuring each query immediately after its application, and ends the algorithm if a 1 is measured.</p>

<p>We show that the bomb query complexity is asymptotically the square of the quantum query complexity. Moreover, we will show how this characterization inspires a general theorem relating classical and quantum query complexity, and derive new algorithms from this theorem.</p>


Overview

1. Bomb Query Complexity
   - Elitzur-Vaidman bomb tester
   - Bomb query complexity $B(f)$
   - Main result: $B(f) = \Theta(Q(f)^2)$
2. Algorithms
   - Introduction: $O(N)$ bomb query algorithm for OR
   - Main theorem 2: constructing q. algorithms from c. ones
   - Applications: graph problems
3. Summary and open problems



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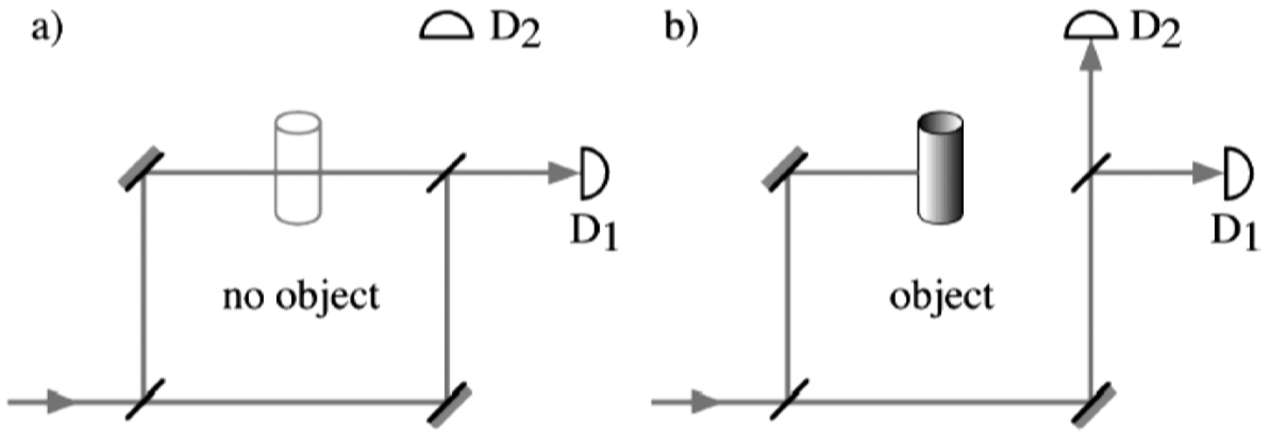
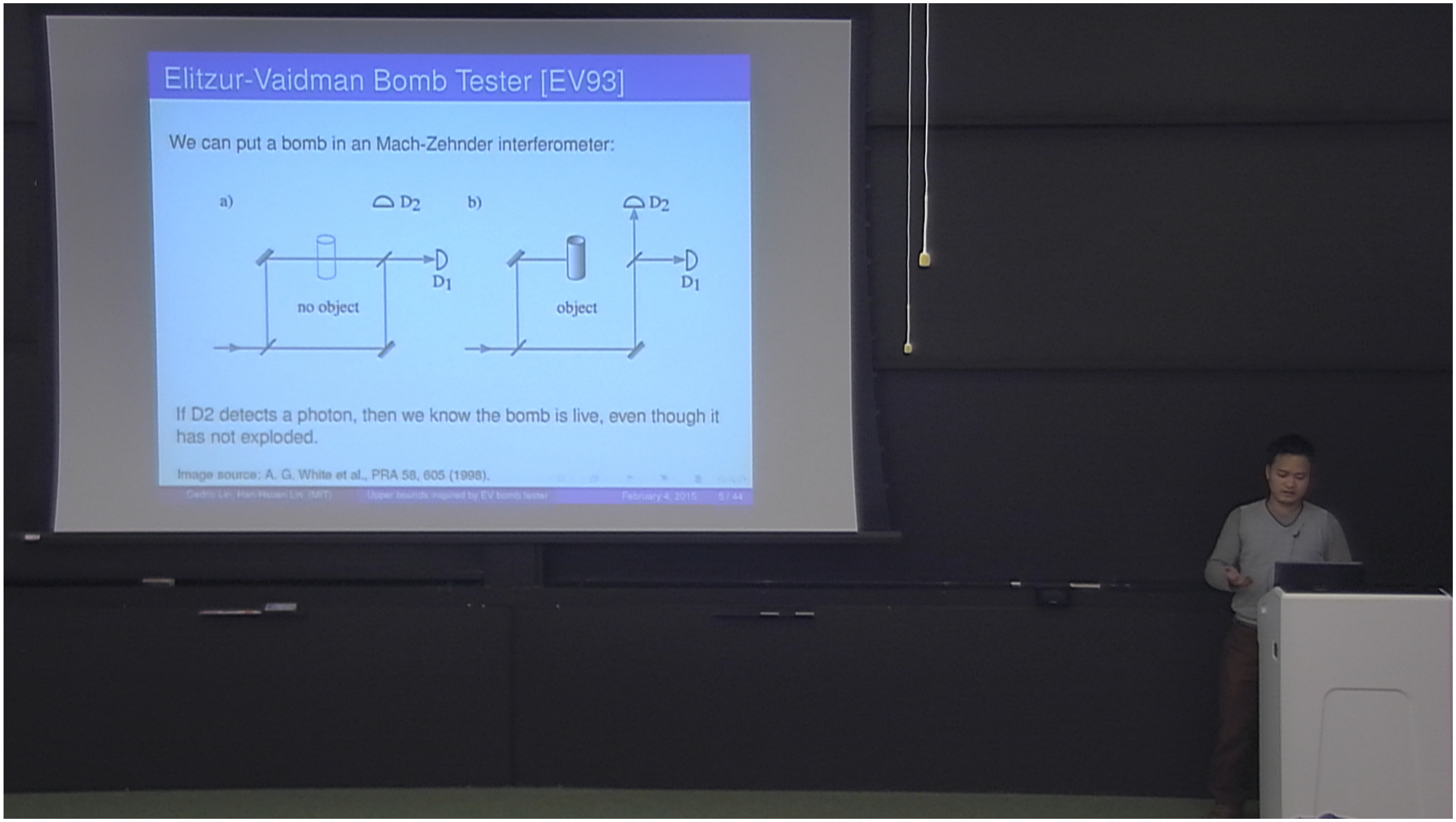
Elitzur-Vaidman Bomb Tester [EV93]

We can put a bomb in an Mach-Zehnder interferometer:



If D2 detects a photon, then we know the bomb is live, even though it has not exploded.

Image source: A. G. White et al., PRA 58, 605 (1998).



Elitzur-Vaidman Bomb Tester [EV93]

We can put a bomb in an Mach-Zehnder interferometer:

a)
D_2
D_1
no object
b)
D_2
D_1
object

If D2 detects a photon, then we know the bomb is live, even though it has not exploded.

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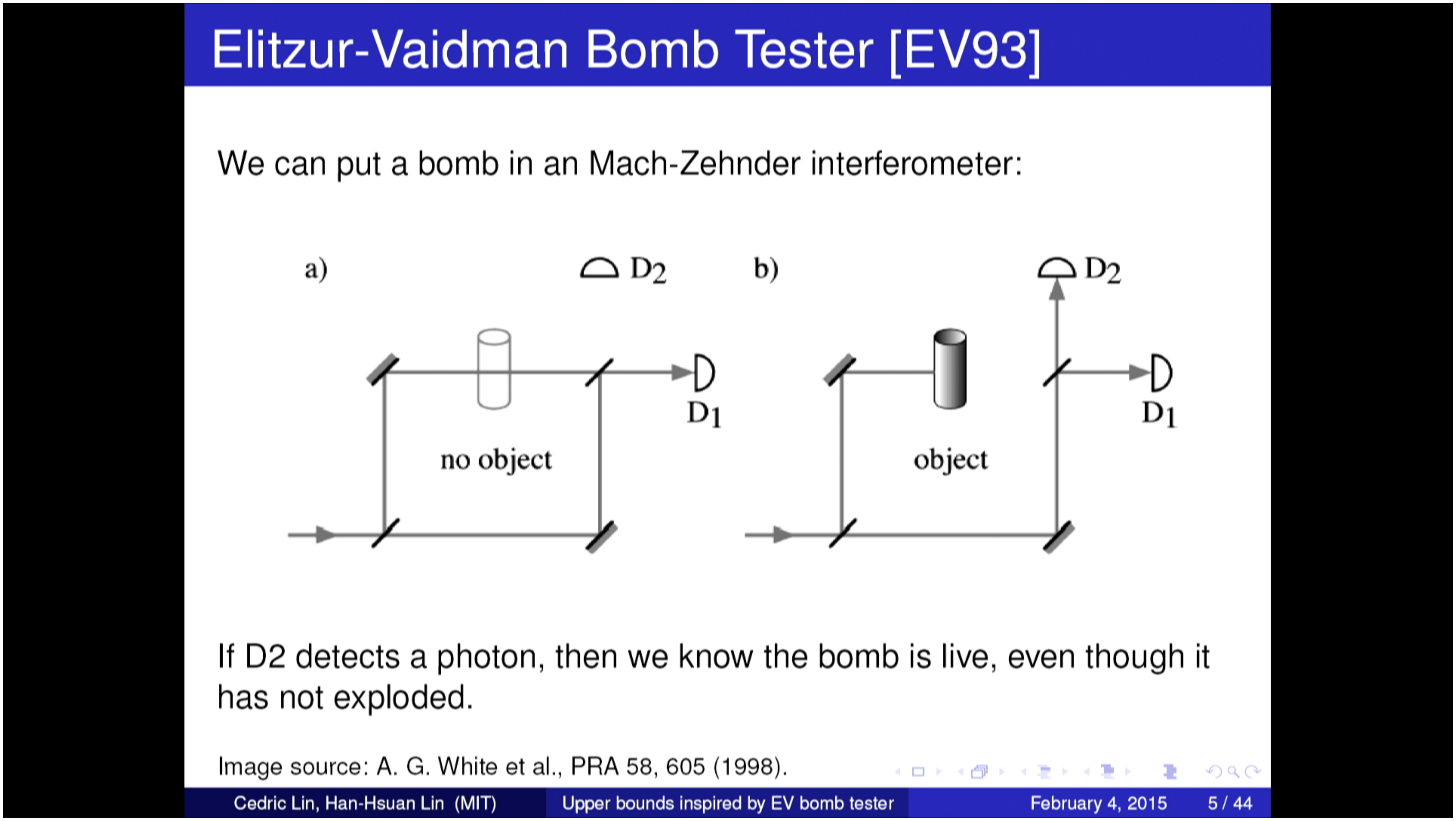
Cedric Lin, Han-Hsuan Lin (MIT) Upper bounds inspired by EV bomb tester February 4, 2015 5 / 44



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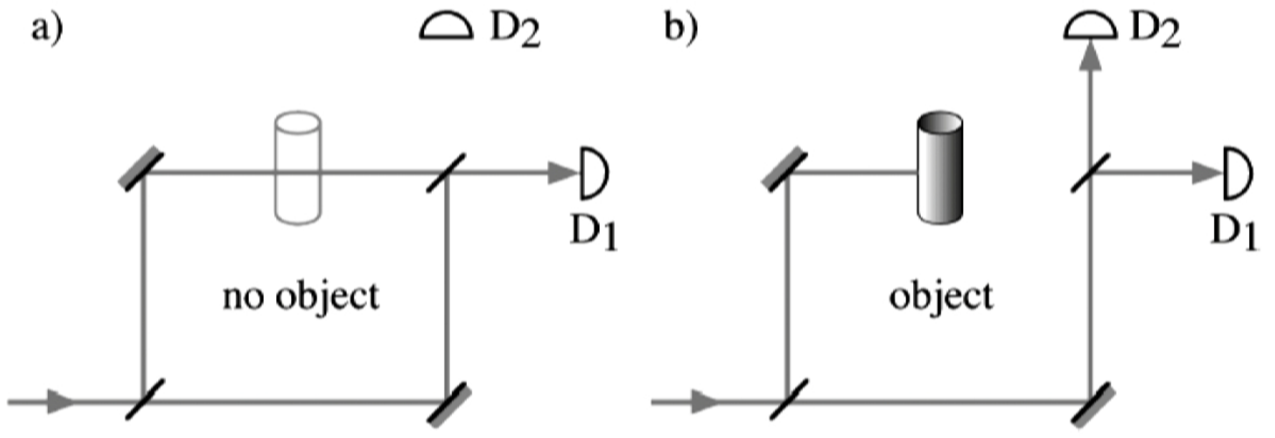

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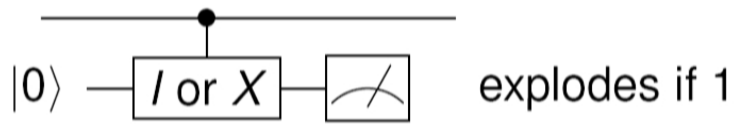
EV bomb in circuit model

We can rewrite the Elitzur-Vaidman bomb in the circuit model:

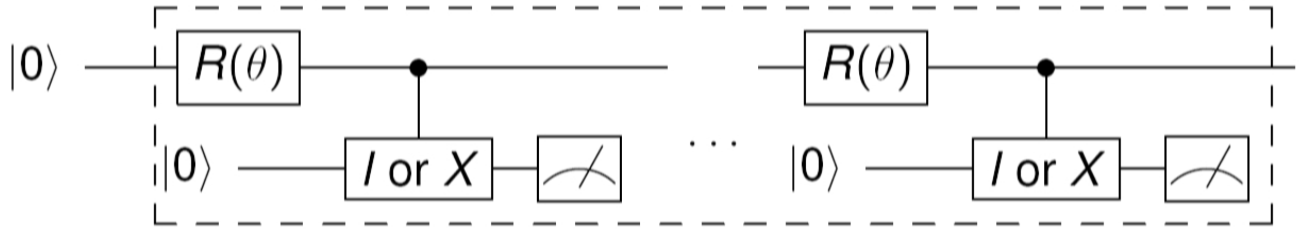


Live bomb: X in the above diagram

Dud: I in the above diagram

Quantum Zeno Effect [KWH+95]

Let $R(\theta) = \exp(i\theta X) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.



$\pi/(2\theta)$ times in total

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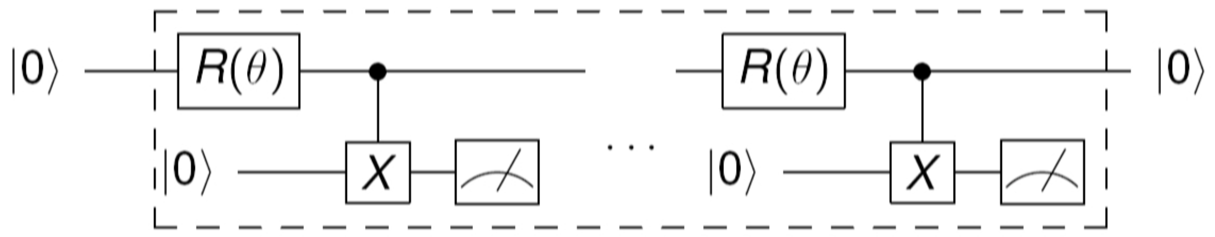


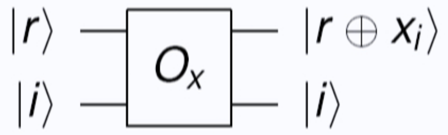
$\pi/(2\theta)$ times in total

If live: First register is projected back to $|0\rangle$ on each measurement; with probability $\sin^2(\theta) = \Theta(\theta^2)$ the bomb explodes.

Total probability of explosion: $\Theta(\theta^2) \times \Theta(1/\theta) = \Theta(\theta)$.

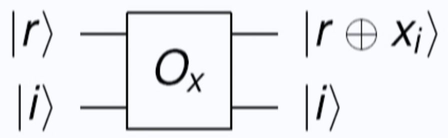
Quantum Query

Quantum query

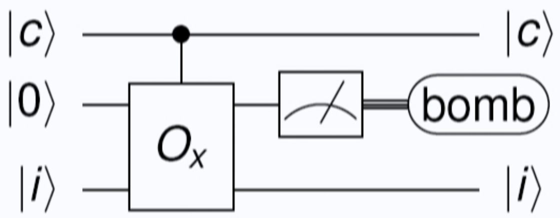


Quantum Query vs Bomb Query

Quantum query

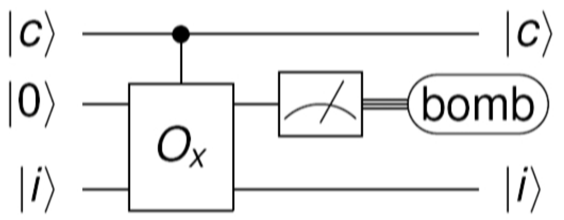


Bomb query



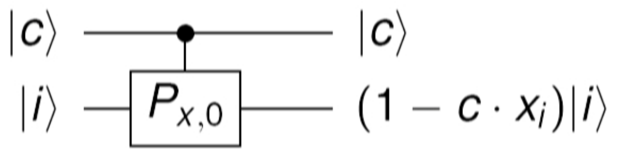
explodes if $c \cdot x_i = 1$

Bomb Query



explodes if $c \cdot x_i = 1$

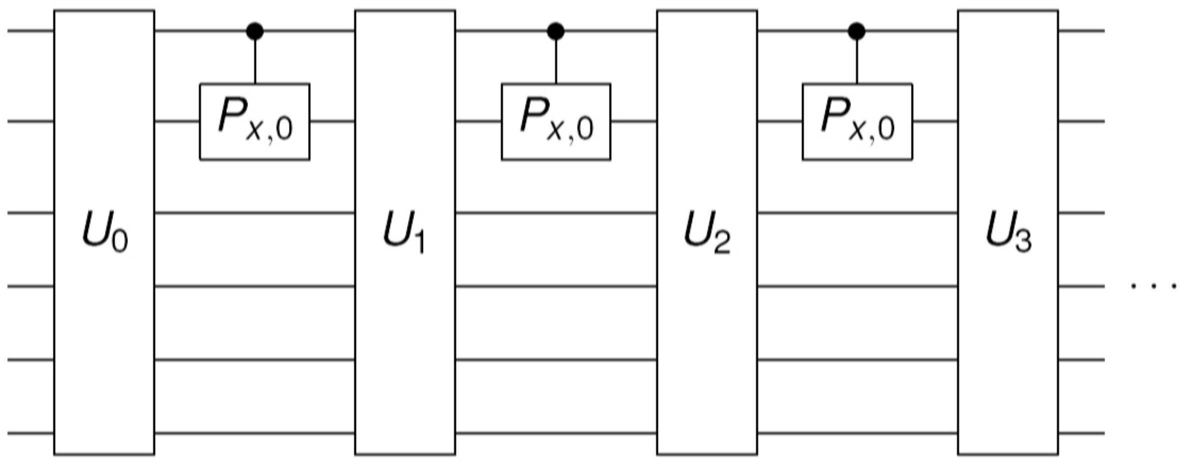
equivalent to



where

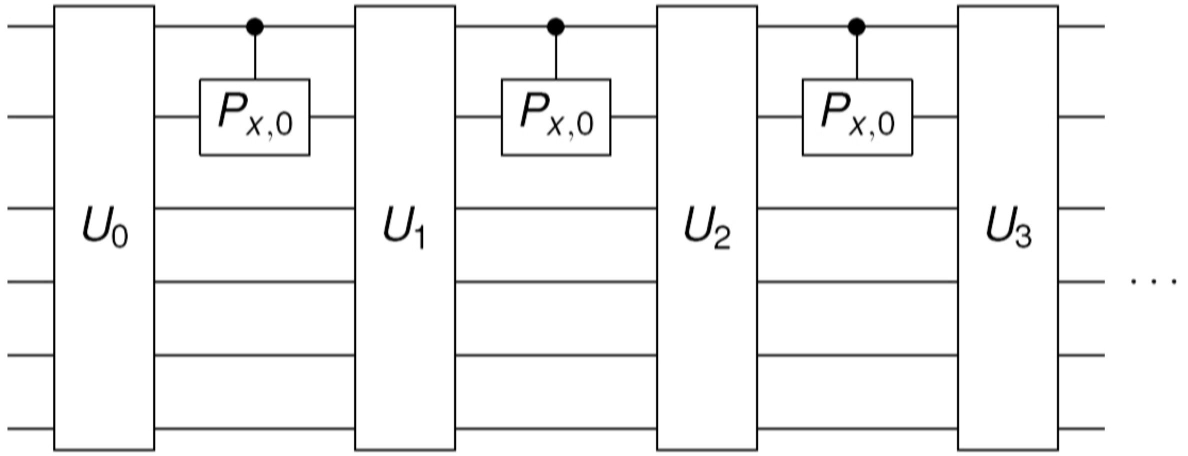
$$P_{x,0} = \sum_{x_i=0} |i\rangle\langle i|, \quad \mathrm{Ctrl} - P_{x,0} = I - \sum_{x_i=1} |1, i\rangle\langle 1, i|$$

Bomb Query Complexity



Call the minimum number of bomb queries needed to determine f with bounded error, with probability of explosion $\leq \epsilon$, the bomb query complexity $B_\epsilon(f)$.

Bomb Query Complexity



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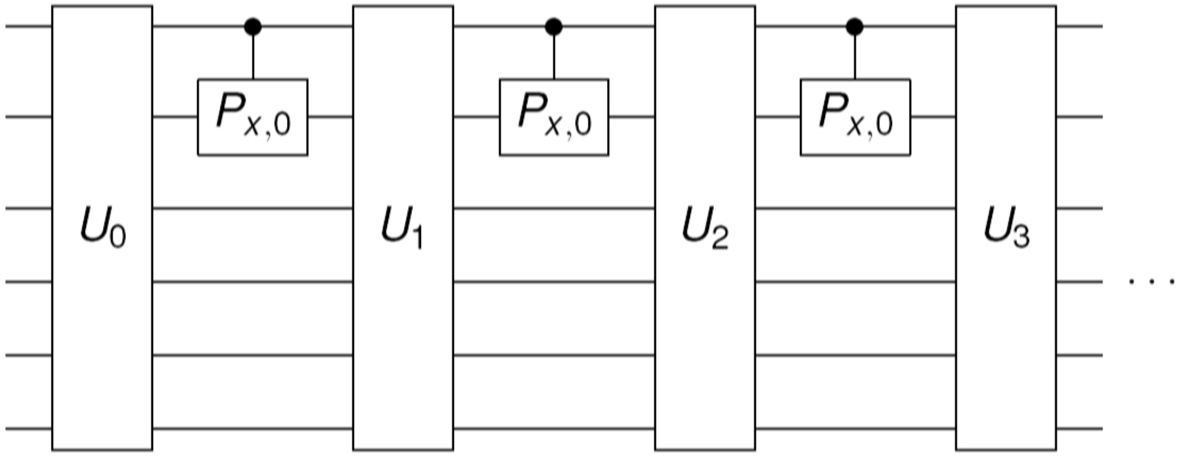
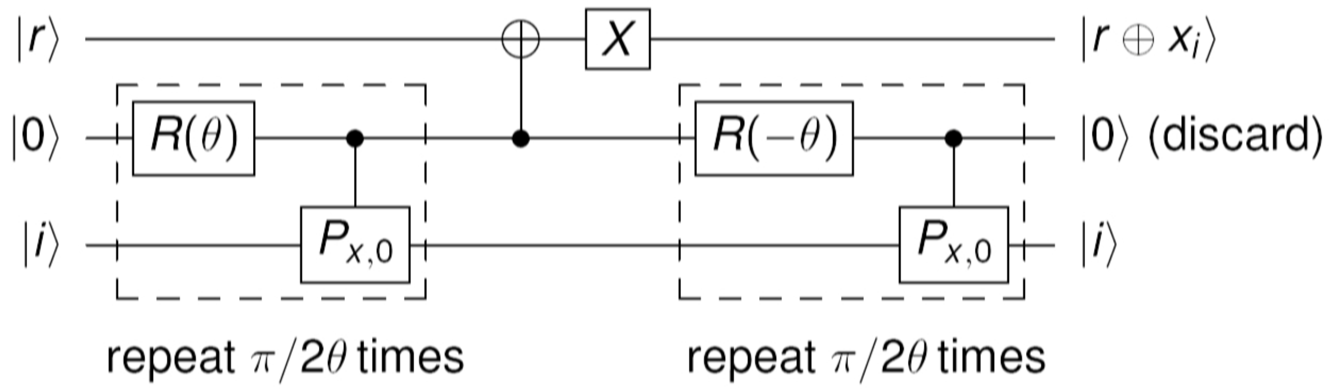


Call the minimum number of bomb queries needed to determine f with bounded error, with probability of explosion $\leq \epsilon$, the bomb query complexity $B_\epsilon(f)$.

$B_\epsilon(f) = O(Q(f)^2/\epsilon)$: Proof

We can simulate each quantum query using $\Theta(1/\theta)$ bomb queries:



Total probability of explosion: $\Theta(\theta) \cdot Q(f) = \Theta(\epsilon)$, if $\theta = \Theta(\epsilon/Q(f))$.

Total number of bomb queries: $\Theta(1/\theta) \cdot Q(f) = O(Q(f)^2/\epsilon)$.

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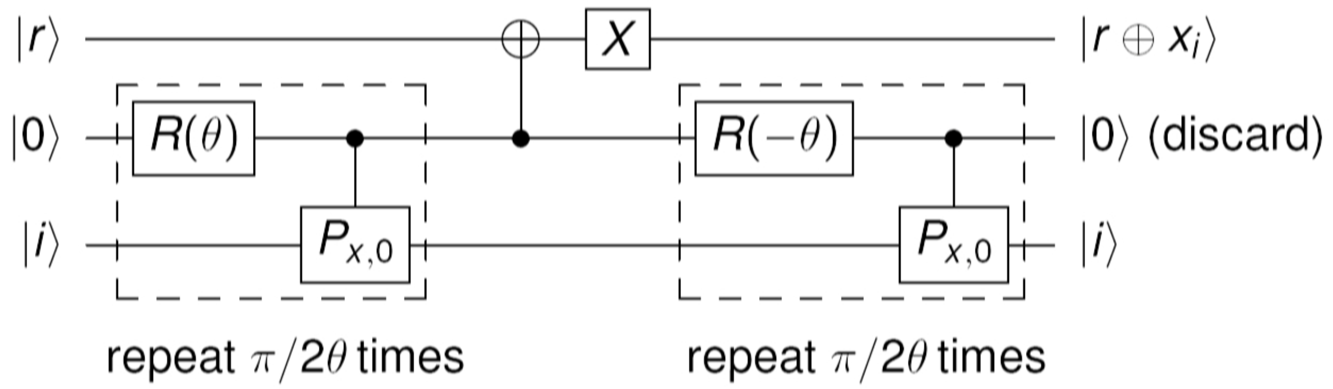

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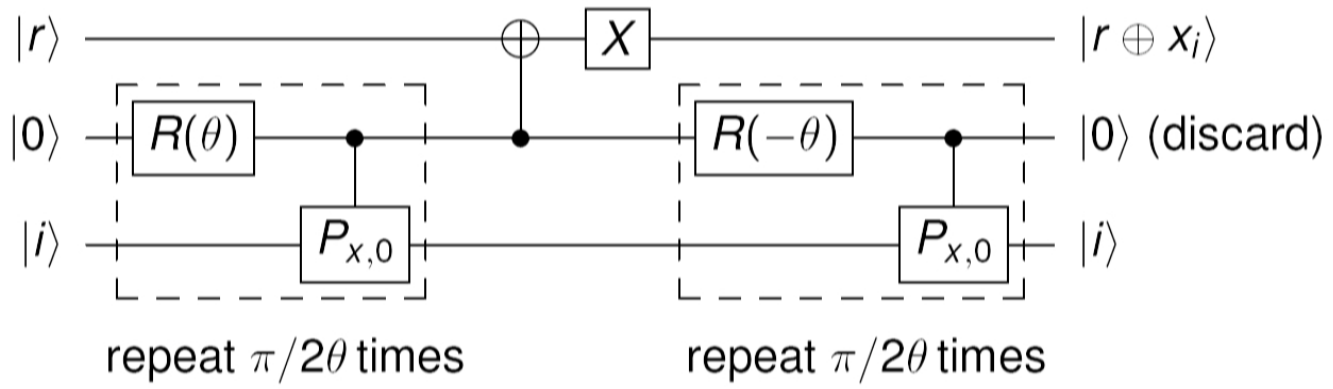

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$B_\epsilon(f) = \Omega(Q(f)^2/\epsilon)$: Proof

The proof uses the general-weight adversary method [HLS07].
We know [Rei09,Rei11,LMR+11] that the general-weight adversary bound tightly characterizes quantum query complexity:
$\mathrm{Adv}^{\pm}(f) = \Theta(Q(f))$.

By modifying the proof of the general-weight adversary bound, we can show that $B_\epsilon(f) = \Omega(\mathrm{Adv}^{\pm}(f)^2/\epsilon)$.

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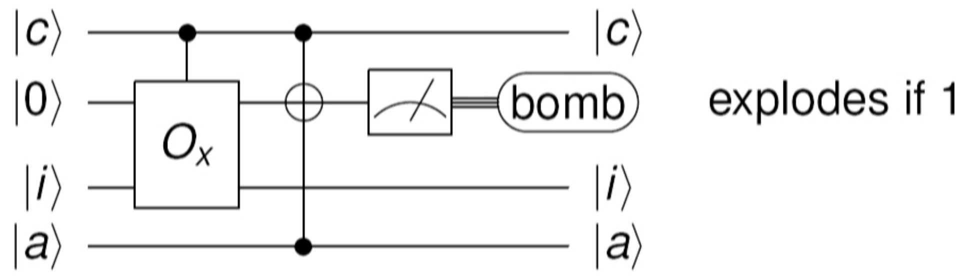
Since $B(OR) = O(N)$, $Q(OR) = O(\sqrt{N})$.

This is a nonconstructive proof of the existence of Grover's algorithm!

Can we generalize this further?

Interlude: symmetric variant of bomb model

Consider using the following construction as our bomb oracle instead:



Here we allow an extra register a to hold a *guess* for the query result.
The bomb explodes only if both $c = 1$ and $x_i = a$.

Let the number of queries required to evaluate f be $\tilde{B}_\epsilon(f)$.
It can be shown (using Main Theorem) that $B_\epsilon(f) = \Theta(\tilde{B}_\epsilon(f))$.

Main Theorem 2

Theorem

Suppose there is a classical randomized algorithm $\mathcal{A}$ that computes $f(x)$ using at most T queries. Moreover, suppose there is an algorithm $\mathcal{G}$ that predicts the results of each query $\mathcal{A}$ makes (0 or 1), making at most an expected G mistakes.

Then $B(f) = O(TG)$, and $Q(f) = O(\sqrt{TG})$.

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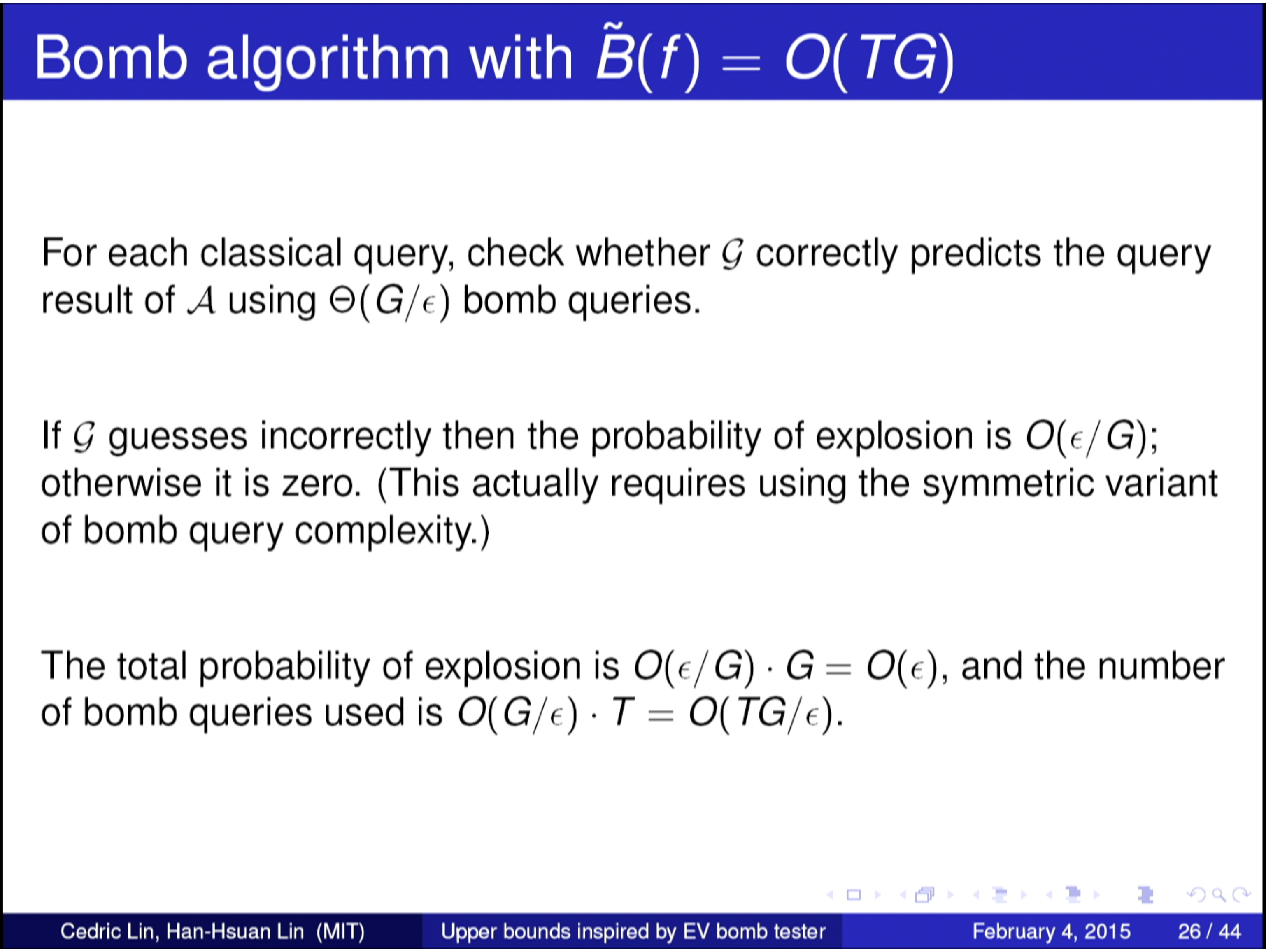
For example, for OR we have $T = N$ and $G = 1$, so $Q(f) = O(\sqrt{N})$.

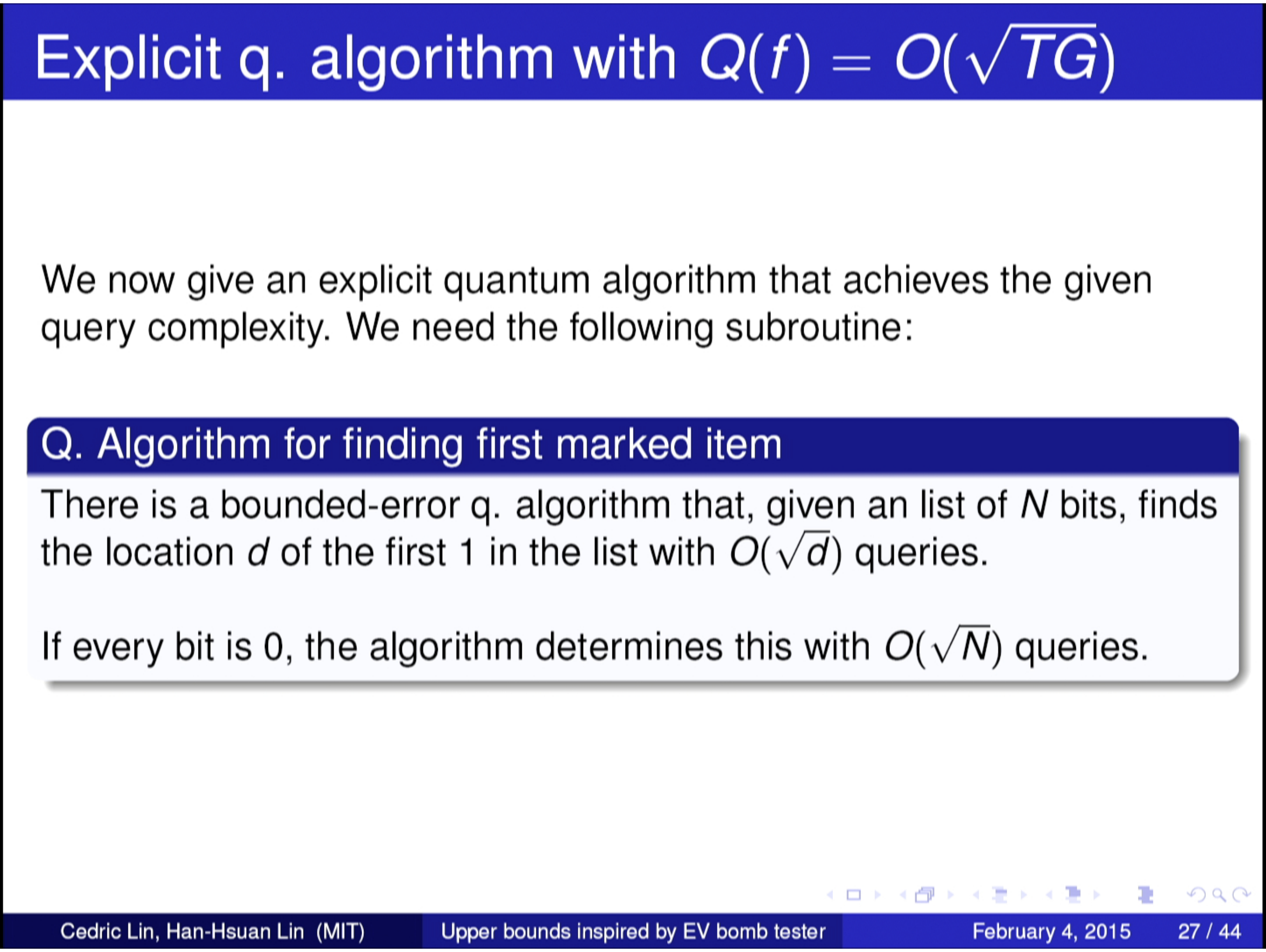


Bomb algorithm with $\tilde{B}(f) = O(TG)$

For each classical query, check whether $\mathcal{G}$ correctly predicts the query result of $\mathcal{A}$ using $\Theta(G/\epsilon)$ bomb queries.

If $\mathcal{G}$ guesses incorrectly then the probability of explosion is $O(\epsilon/G)$; otherwise it is zero. (This actually requires using the symmetric variant of bomb query complexity.)

The total probability of explosion is $O(\epsilon/G) \cdot G = O(\epsilon)$, and the number of bomb queries used is $O(G/\epsilon) \cdot T = O(TG/\epsilon)$.



Explicit q. algorithm with $Q(f) = O(\sqrt{TG})$

We now give an explicit quantum algorithm that achieves the given query complexity. We need the following subroutine:

Q. Algorithm for finding first marked item

There is a bounded-error q. algorithm that, given an list of N bits, finds the location d of the first 1 in the list with $O(\sqrt{d})$ queries.

If every bit is 0, the algorithm determines this with $O(\sqrt{N})$ queries.

Explicit q. algorithm with $Q(f) = O(\sqrt{TG})$

- Repeat until all queries of $\mathcal{A}$ are determined:
    1. Use $\mathcal{G}$ to predict all remaining queries of $\mathcal{A}$, under assumption it makes no mistakes.
    2. Search for the location d_j of first mistake, using $O(\sqrt{d_j - d_{j-1}})$ quantum queries.
    3. This determines the actual query results up to the d_j-th query that $\mathcal{A}$ would have made.

Kothari's algorithm for oracle identification [Kot14] actually already uses these steps above.

Explicit q. algorithm with $Q(f) = O(\sqrt{TG})$

- Repeat until all queries of $\mathcal{A}$ are determined:
  1. Use $\mathcal{G}$ to predict all remaining queries of $\mathcal{A}$, under assumption it makes no mistakes.
  2. Find the location d_j of first mistake, using $O(\sqrt{d_j - d_{j-1}})$ queries to the black box.
  3. This determines the actual query results up to the d_j-th query that $\mathcal{A}$ would have made.

i	2	5	4	3	12	7	6	9	10
x_i	0	1	1	0	1	1	0	0	1

Explicit q. algorithm with $Q(f) = O(\sqrt{TG})$

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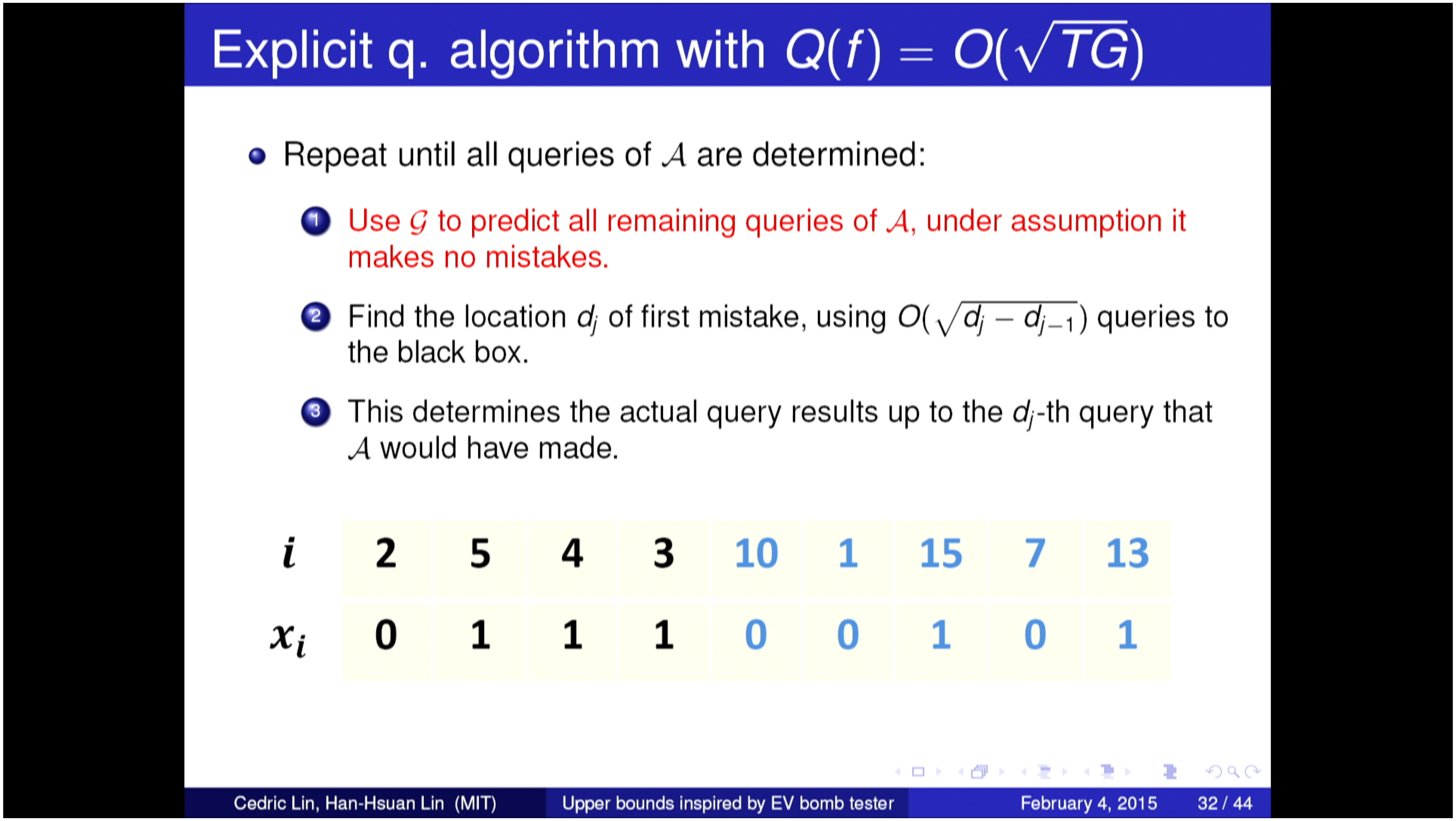

Explicit q. algorithm with $Q(f) = O(\sqrt{TG})$

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i	2	5	4	3	10	1	15	7	13
x_i	0	1	1	1	0	0	1	0	1

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Query complexity: $O(G) \cdot O(\sqrt{T/G}) = O(\sqrt{TG})$.
It looks like error reduction may give extra log factors, but [Kot14] showed that the log factors can be removed using span programs.

Applications: Breadth First Search

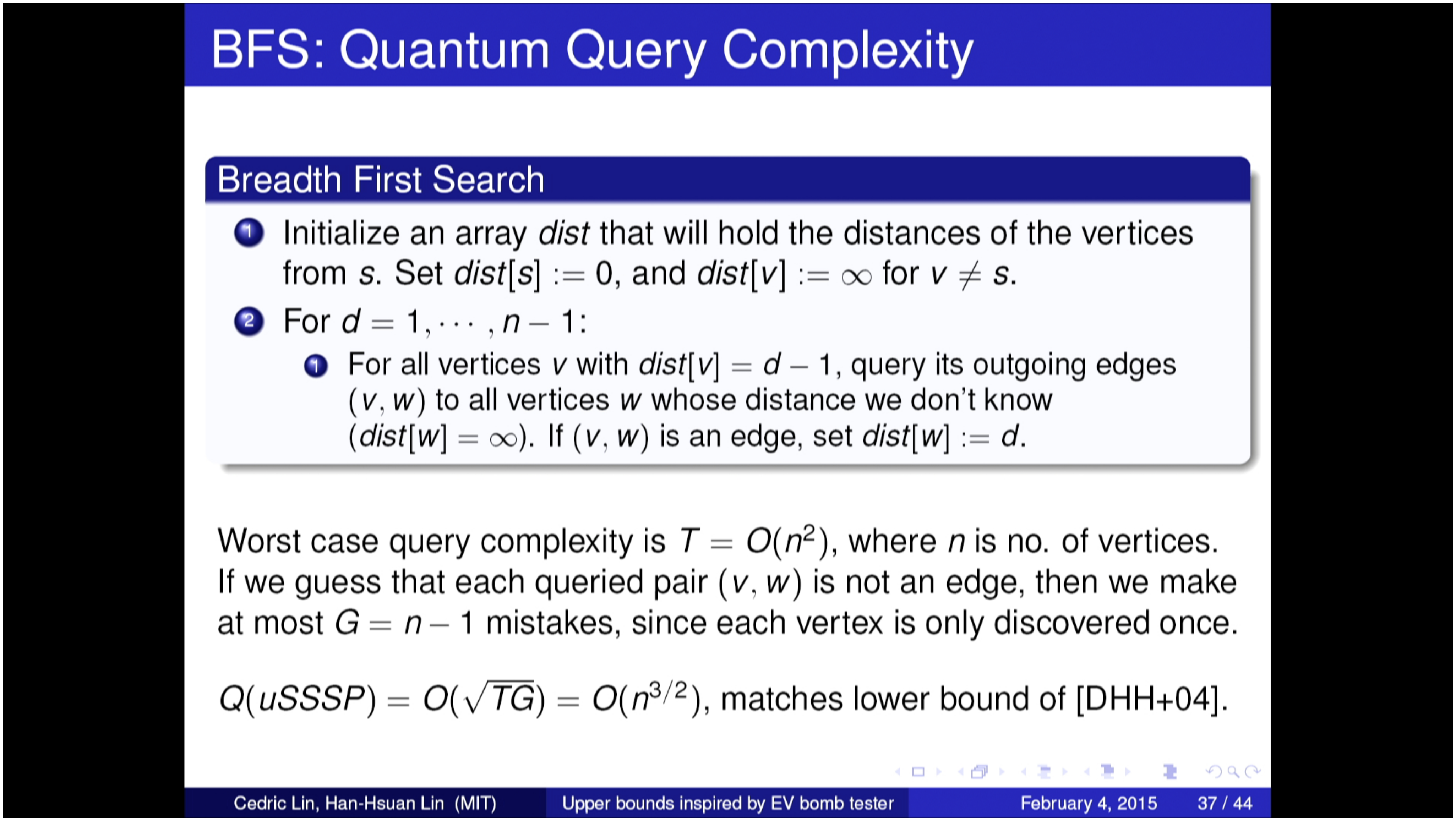
Problem: Unweighted Single-Source Shortest Paths

Given the adjacency matrix of an unweighted graph as a black box, find the distances from a vertex s to all other vertices.

Classical algorithm: *Breadth First Search*.

Breadth First Search

1. Initialize an array *dist* that will hold the distances of the vertices from s. Set $dist[s] := 0$, and $dist[v] := \infty$ for $v \neq s$.
2. For $d = 1, \cdots, n-1$:
    1. For all vertices v with $dist[v] = d - 1$, query its outgoing edges (v, w) to all vertices w whose distance we don't know $(dist[w] = \infty)$. If (v, w) is an edge, set $dist[w] := d$.



BFS: Quantum Query Complexity

Breadth First Search

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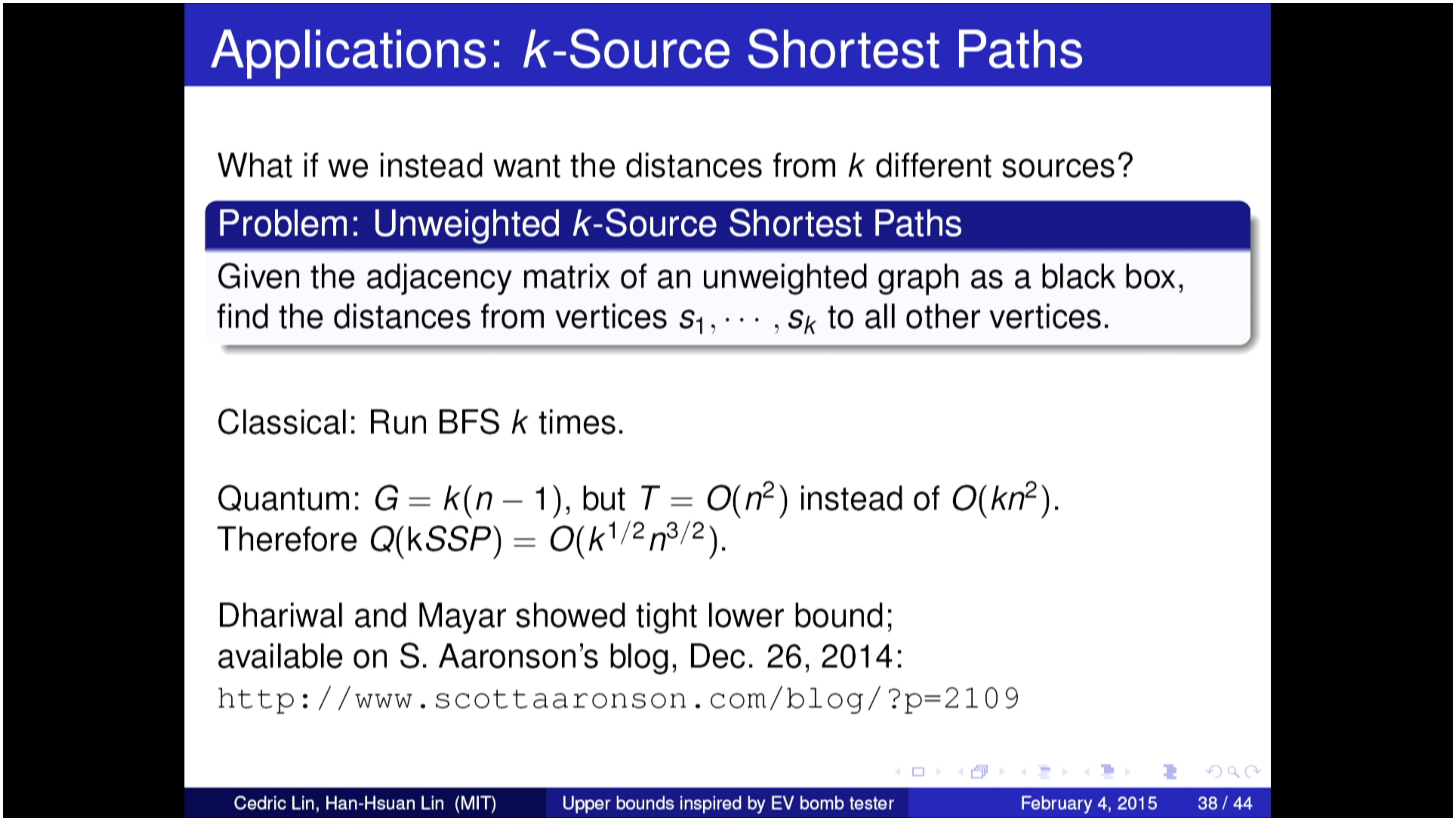
2. For $d = 1, \cdots, n-1$:
   1. For all vertices v with $dist[v] = d-1$, query its outgoing edges (v, w) to all vertices w whose distance we don't know ($dist[w] = \infty$). If (v, w) is an edge, set $dist[w] := d$.

Worst case query complexity is $T = O(n^2)$, where n is no. of vertices. If we guess that each queried pair (v, w) is not an edge, then we make at most $G = n-1$ mistakes, since each vertex is only discovered once.

$Q(uSSSP) = O(\sqrt{TG}) = O(n^{3/2})$, matches lower bound of [DHH+04].



Applications: k-Source Shortest Paths

What if we instead want the distances from k different sources?

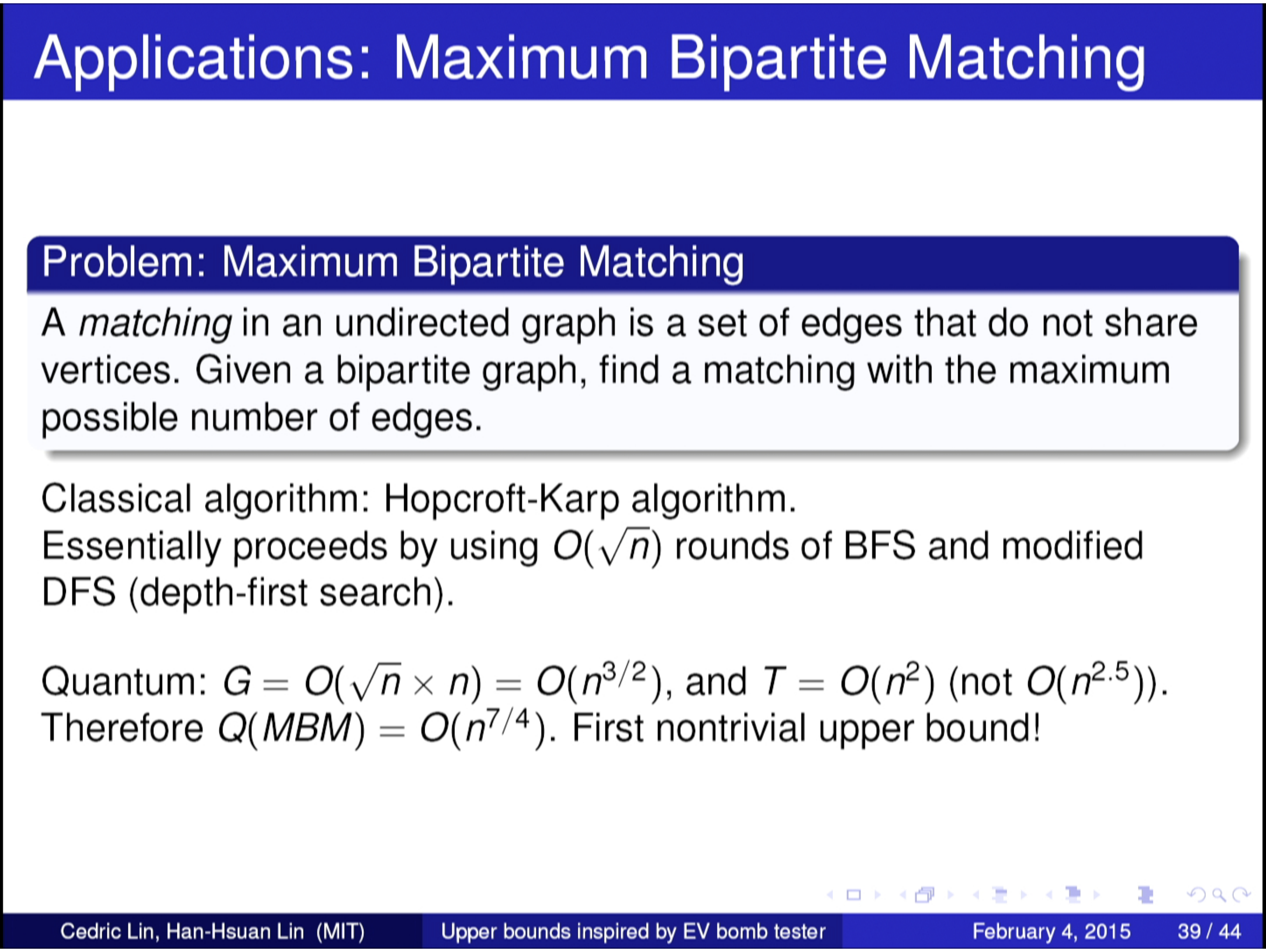
Problem: Unweighted k-Source Shortest Paths

Given the adjacency matrix of an unweighted graph as a black box, find the distances from vertices $s_1, \cdots, s_k$ to all other vertices.

Classical: Run BFS k times.

Quantum: $G = k(n-1)$, but $T = O(n^2)$ instead of $O(kn^2)$.
Therefore $Q(\mathrm{k}SSP) = O(k^{1/2}n^{3/2})$.

Dhariwal and Mayar showed tight lower bound;
available on S. Aaronson's blog, Dec. 26, 2014:
`http://www.scottaaronson.com/blog/?p=2109`



Applications: Maximum Bipartite Matching

Problem: Maximum Bipartite Matching

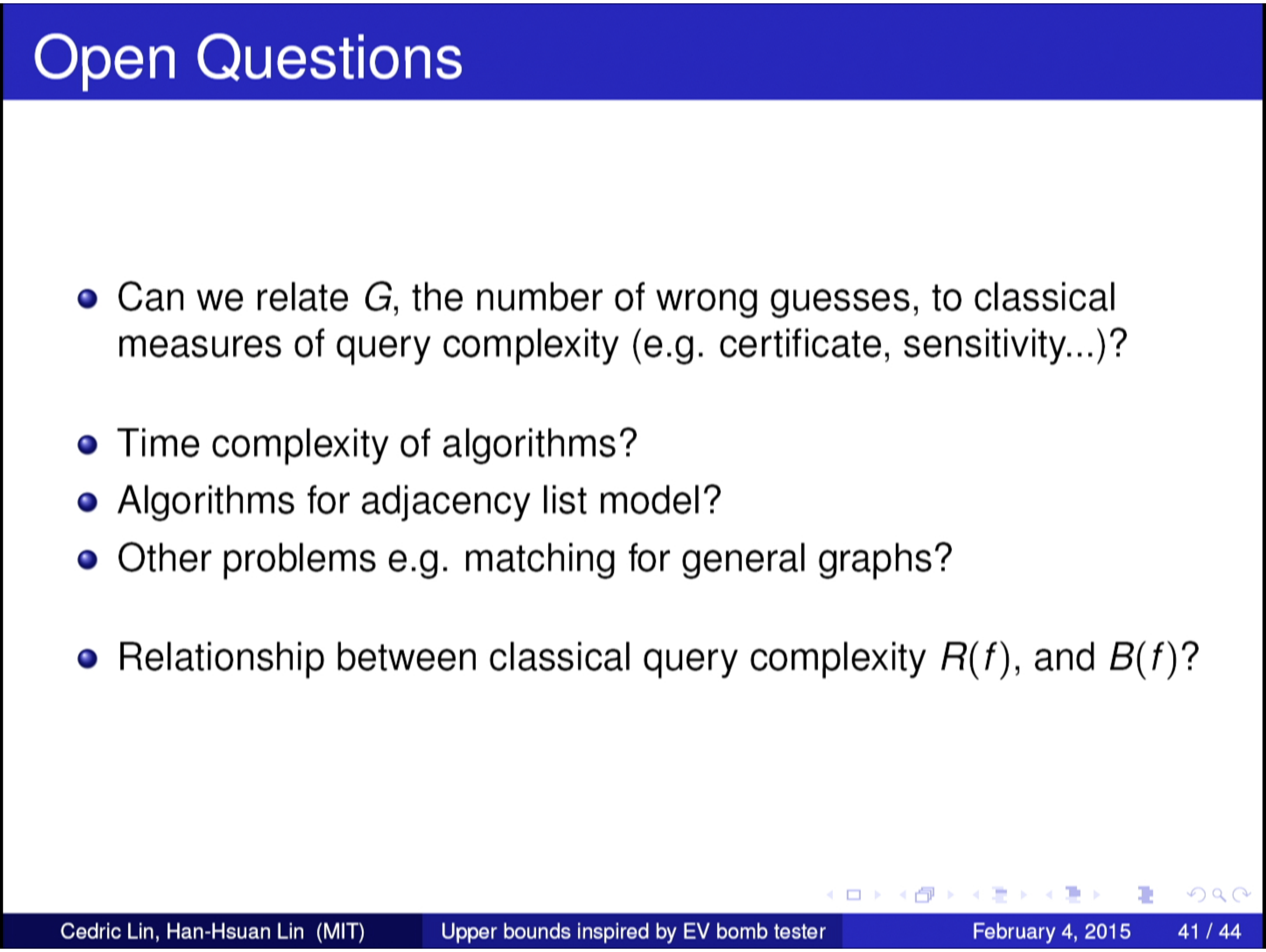
A *matching* in an undirected graph is a set of edges that do not share vertices. Given a bipartite graph, find a matching with the maximum possible number of edges.

Classical algorithm: Hopcroft-Karp algorithm.
Essentially proceeds by using $O(\sqrt{n})$ rounds of BFS and modified DFS (depth-first search).

Quantum: $G = O(\sqrt{n} \times n) = O(n^{3/2})$, and $T = O(n^2)$ (not $O(n^{2.5})$).
Therefore $Q(MBM) = O(n^{7/4})$. First nontrivial upper bound!

Summary

- Inspired by the EV bomb tester, we defined the notion of *bomb query complexity*, and showed the relation $B(f) = \Theta(Q(f)^2)$.
- Bomb query complexity further lead us to a general construction of quantum query algorithms from classical algorithms, giving us an $O(n^{1.75})$ quantum query algorithm for maximum bipartite matching.



Open Questions

- Can we relate G, the number of wrong guesses, to classical measures of query complexity (e.g. certificate, sensitivity...)?
- Time complexity of algorithms?
- Algorithms for adjacency list model?
- Other problems e.g. matching for general graphs?
- Relationship between classical query complexity $R(f)$, and $B(f)$?

Relationship between $R(f)$ and $B(f)$?

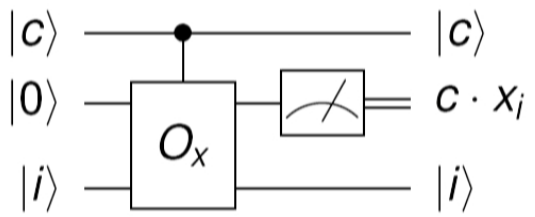
For total functions the largest known separation between $R(f)$ and $Q(f)$ is quadratic (for the OR function).
It is conjectured this is the extreme case, $R(f) = O(Q(f)^2)$.
We only know that [BBC+98] $R(f) = O(Q(f)^6)$.

We know that $B(f) = \Theta(Q(f)^2)$.
Therefore the conjecture is equivalent to $R(f) = O(B(f))$.

We give some motivation for why this conjecture might be true...

Projective Query Complexity, $P(f)$

Aaronson (unpublished, 2002) considered allowing access to the black box only with the following:



We call the number of queries required the *projective query complexity*, $P(f)$. Note the algorithm does *not* end on measuring a 1.

Straightforwardly $Q(f) \leq P(f) \leq R(f)$ and $P(f) \leq B(f)$.
Regev and Schiff [RS08]: $P(OR) = \Omega(N)$.

Open question: Does $P(f) = \Theta(R(f))$ for all total functions?
If this is true, implies $R(f) = O(B(f)) = O(Q(f)^2)$.