Title: Entanglement Spectroscopy of Quantum Matter - Andreas Lauchli

Date: Feb 03, 2015 03:30 PM

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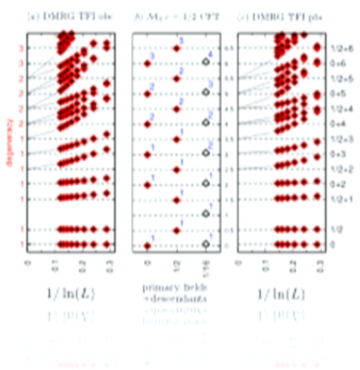
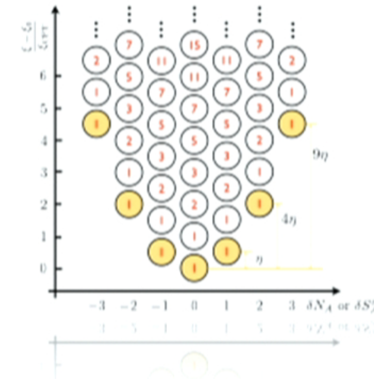
Abstract: <p>The entanglement spectrum, i.e. the logarithm of the eigenvalues of reduced density matrices of
 quantum many body wave functions, has been the focus of a rapidly expanding research endeavor recently.
 Initially introduced by Li & Haldane in the context of the fractional quantum Hall effect, its usefulness has been
 shown to extend to many more fields, such as topological insulators, fractional Chern insulators, spin liquids,
 continuous symmetry breaking states, etc.
After a general introduction to the field we review some of our own contributions to the field, in particular the
 perturbative structure of the entanglement spectrum in gapped phases, the entanglement spectrum across the
 Mott-insulator transition in the Bose-Hubbard model, and the relation of the entanglement spectrum of
 (1+1) dimensional quantum critical systems to the operator content of their underlying CFT.</p>





Entanglement Spectroscopy of Quantum Matter

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Support: FWF SFB FoQuS Foundations and Applications of Quantum Science

AML, arXiv:1303:0741
L. Bonnes, & AML, in preparation

Seminar, PI, 2/3/2015 PERIMETER PI INSTITUTE FOR THEORETICAL PHYSICS

Outline of this talk

- Introduction to entanglement and entanglement spectra (ES)
- CFT in a nutshell
- CFT operator content of the ES in 1+1D critical systems

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- iMPS versus finite size DMRG/MPS operator content

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- Conclusion / Outlook

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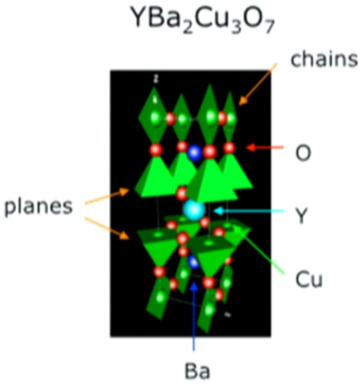
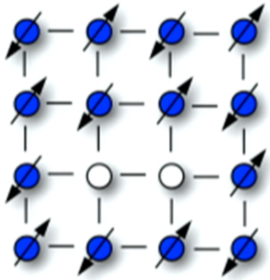
Quantum Many Body Systems
Strongly correlated electrons in solids

- High Tc superconductors & Quantum Magnets

Bednorz & Müller, 1987





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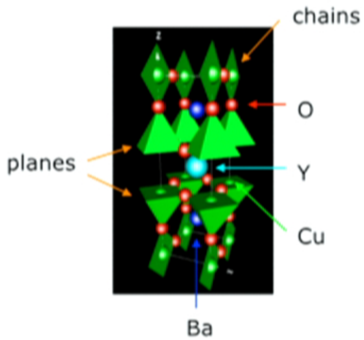
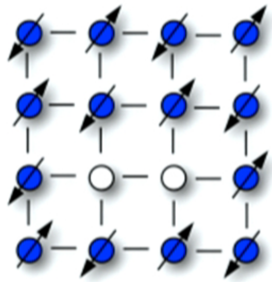
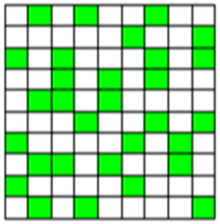
$YBa_2Cu_3O_7$

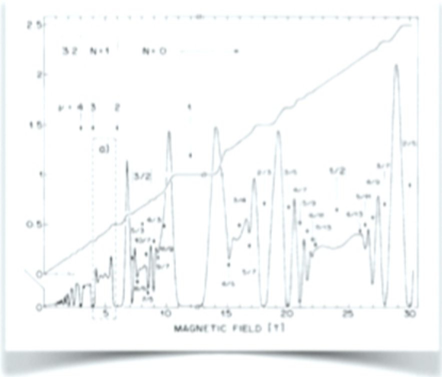




- Fractional Quantum Hall Effect

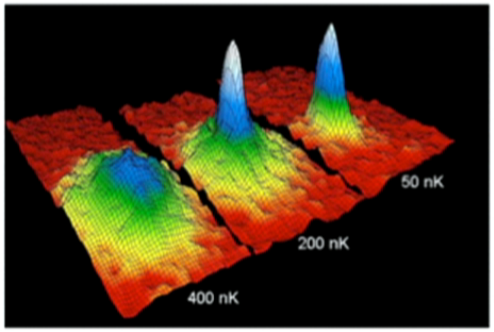
$\nu = 1/3$





Stoermer, Tsui, Laughlin, 1998

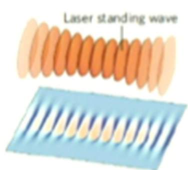
Quantum Many Body Systems
Ultracold atomic gases



Cornell, Ketterle, Wiemann, 2001

From weakly interacting Bose gases

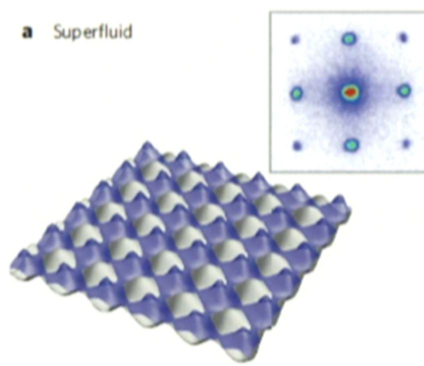
to strongly interacting gases in optical lattices

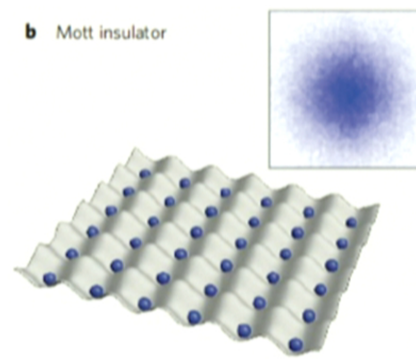


$$H = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i n_i (n_i - 1)$$

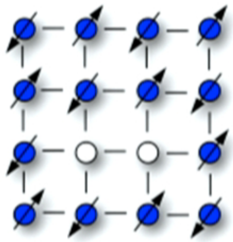
D. Jaksch et al., PRL (1998)

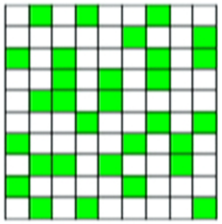
M. Greiner et al., Nature (2002)

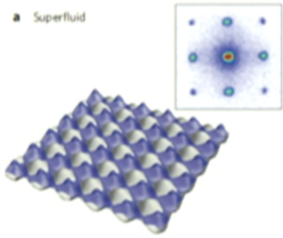




Quantum Many Body Systems







- Common to all these problems is the desire to calculate ground states of certain Hamiltonians and then to characterize these states.

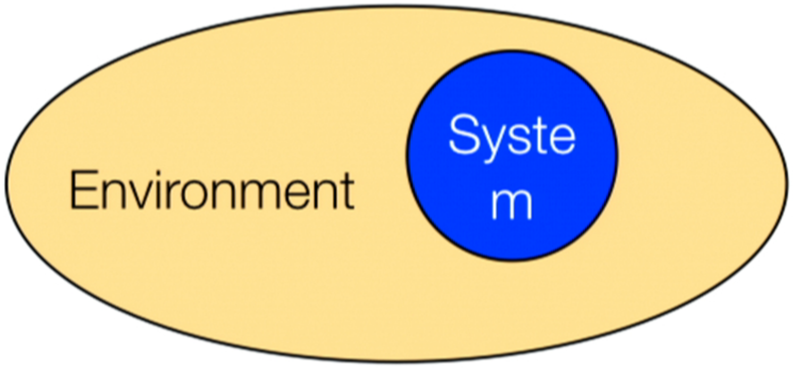
- If one knows what to look for, then correlation functions of appropriate operators are very useful.

- Are there ways to characterize the system without too much prior knowledge ?

- Let us look at entanglement quantities !

Entanglement Entropy

- Let us look at reduced density matrices, and their entanglement entropies (EE)

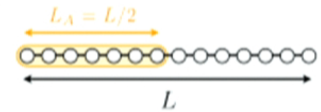


$$\rho = \mathrm{Tr}_E |\psi\rangle\langle\psi|$$

$$S(\rho) = \mathrm{Tr}[-\rho \log \rho]$$

- How does this (von Neumann) entropy depend on the size of the subsystem ?
  - For a random state: entropy is proportional to volume of subsystem
  - For ground state of a gapped local Hamiltonian: entropy is proportional to perimeter (celebrated “area” law)
  - Logarithmic corrections to the area law: for a CFT in 1+1D and for free fermions with Fermi surface in any D.

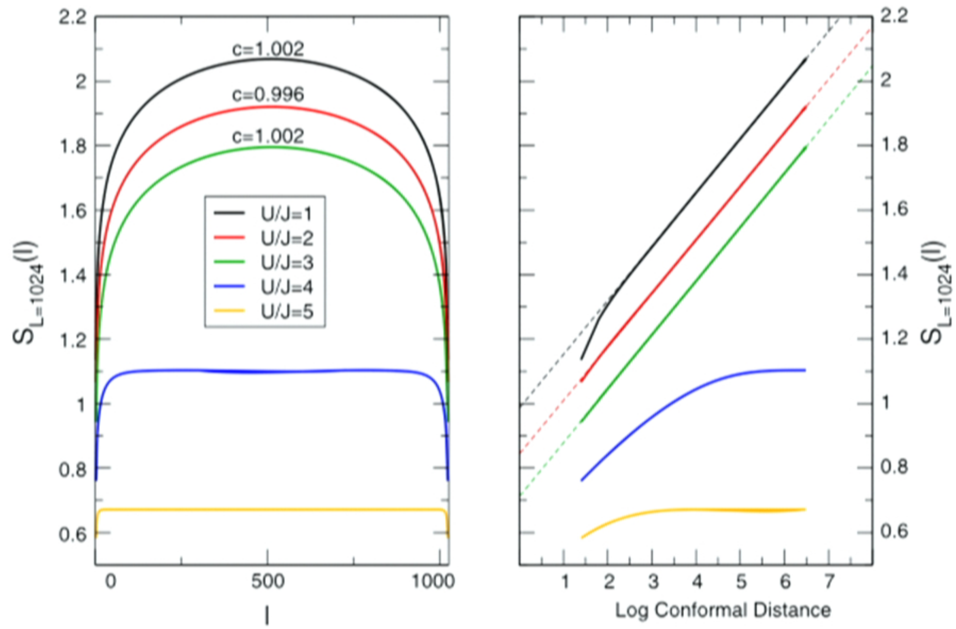
Locate a quantum phase transition based on EE: The Bose-Hubbard chain at unit filling



- In DMRG simulations, the Entanglement entropy (EE) is readily accessible.

- Use formula by Calabrese & Cardy (2004) to determine central charge c

$$S_L(l) = \frac{c}{6}\log\left[\frac{2L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + \log g + s_1/2.$$



AML & C. Kollath, JSTAT '08

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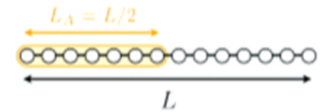
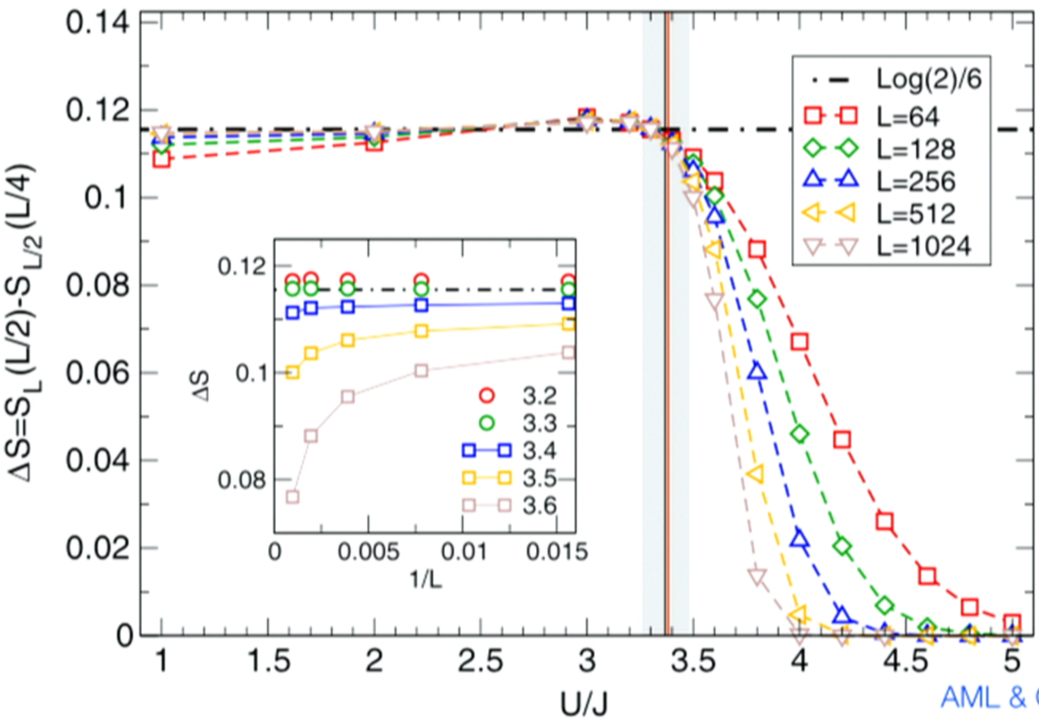


- Investigate EE increment upon doubling the system size
- Constant region ln(2)/6~0.116 indicates gapless region with c=1

$$S_L(l) = \frac{c}{6}\log\left[\frac{2L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right] + \log g + s_1/2.$$



AML & C. Kollath, JSTAT '08

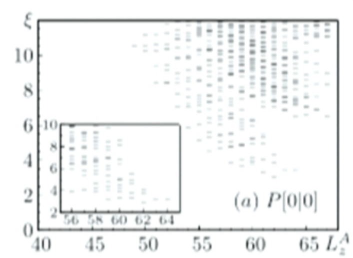
Entanglement spectrum

- So far we only considered a single number - the von Neumann entropy - distilled out of the reduced density matrix.

- In a seminal paper Li & Haldane, PRL 2008 asked the question whether the eigenvalues of a reduced density matrix might contain even more information than the von Neumann entropy alone.

$$\{\lambda\} = \sigma(\rho_A) \quad \Rightarrow \quad \rho_A = e^{-\mathcal{H}_A} \quad \Rightarrow \quad \boxed{\sigma(\mathcal{H}_A) \equiv -\log(\{\lambda\})}$$

- They studied fractional quantum Hall states on the sphere and found that the entanglement spectrum showed structures very similar to the physical edge energy spectrum (described by CFT) !



Entanglement Spectra studies: some examples

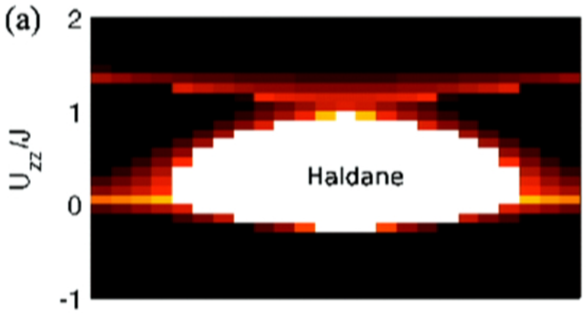
- Many further studies of FQH states on the sphere/torus and also in real space as opposed to orbital space. Also very useful in Fraction Chern insulator studies.

- Early studies on "non-interacting" systems
  E.g.: M.-C. Chung and I. Peschel, Phys. Rev. B **64**, 064412 (2001)

- Applied to many other systems (Topological insulators, dimer models, Kitaev models, Bose condensates, spin chains, continuous symmetry breaking)

- Different techniques:
  Exact Diagonalization
  Free (quadratic) Fermions/Bosons
  DMRG/MPS (almost for free)
  PEPS
  Perturbation theory

- Twofold degenerate ES in the Haldane phase - complementary to the 'string order parameter'
  F. Pollmann et al., Phys. Rev. B **81**, 064439 (2010)



Entanglement spectrum in 1+1D CFT

- How much information is contained in the EE/ES ?

- Entanglement Entropy:
  Logarithmic scaling, depends only on central charge

- Finite size effects show some algebraic corrections, which depend on scaling dimensions.

- Calabrese & Lefevre, PRA 08:
  The density of states of the ES only depends on the central charge.

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  The density of states of the ES only depends on the central charge.
- Is this really all ?
- We believe the entanglement spectrum of a single interval in a system described by a CFT is related to the energy spectrum of the CFT with particular boundary conditions, and can thus in principle be used to reveal operator content of the CFT.

Useful CFT References

- J. Cardy, *Conformal Invariance and Statistical Mechanics*, Les Houches 1988 lecture notes, available in electronic form on his website.
- P. Ginsparg, *Applied Conformal Field Theory*, Les Houches 1988 lecture notes, available on arXiv.
- P. Di Francesco, P. Matthieu, D. Senechal, *Conformal Field Theory* (informally: "The yellow book"), Springer

- G. Mussardo: *Statistical Field Theory*, Oxford University Press

- ...

What is Conformal Field Theory (CFT)

- It is expected that scale invariant systems (e.g. systems at a [quantum] critical point) also exhibit conformal invariance.
  This means invariance under locally angle-preserving maps.

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- In 1+1D or 2D the conformal group is surprisingly large (~ analytic functions), leading to a theory with a very rich structure, and where many aspects are understood.
- In higher dimensions the restrictions on the theory by the conformal symmetry are less severe, so much less is known about the operator content of such theories.

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- Many 2D statistical mechanics models at their critical point or 1+1D quantum many body system at a quantum critical point exhibit critical behavior described by CFT.
- Important CFTs in this context:
  Ising, multicritical Ising, 3-state Potts, Luttinger liquid, WZWN models,....

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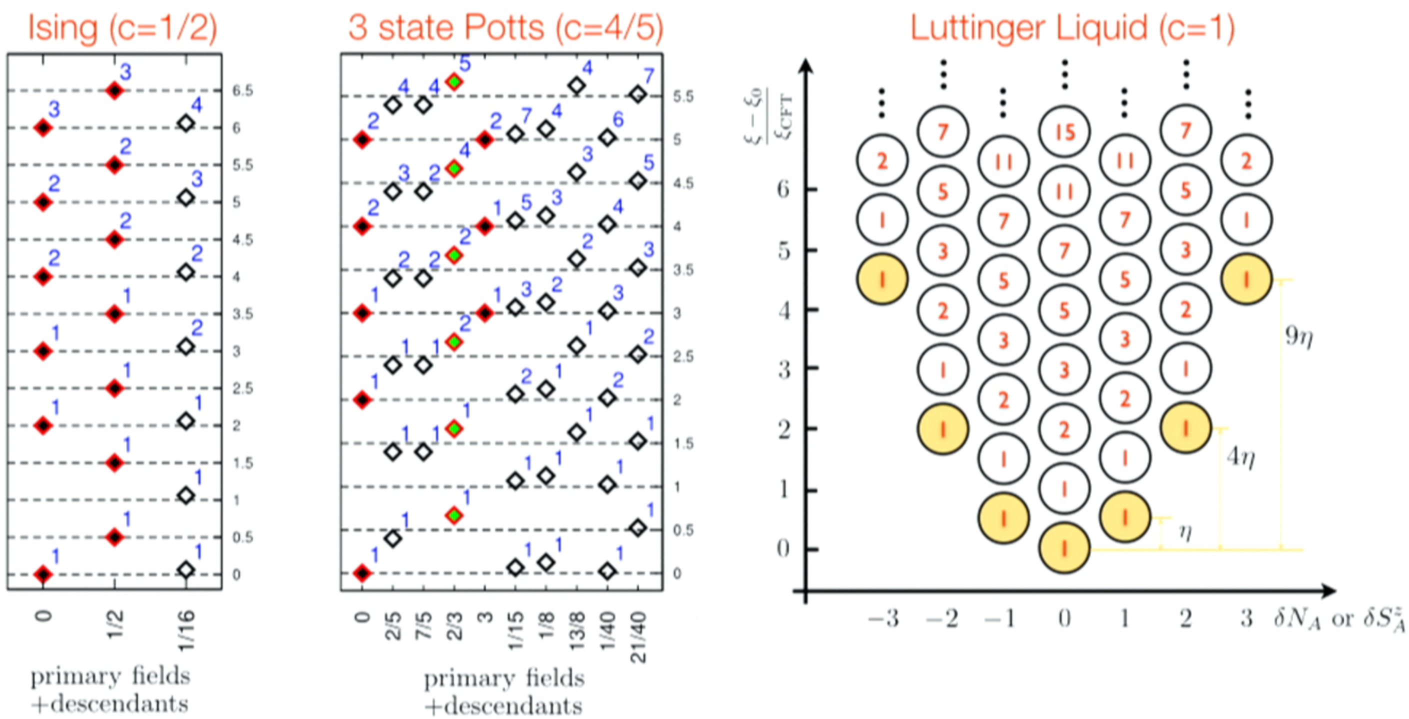
- These CFTs are described by their central charge (available from entanglement entropies as shown before), but also their operator content, operator product expansion, boundary condition dependence,...

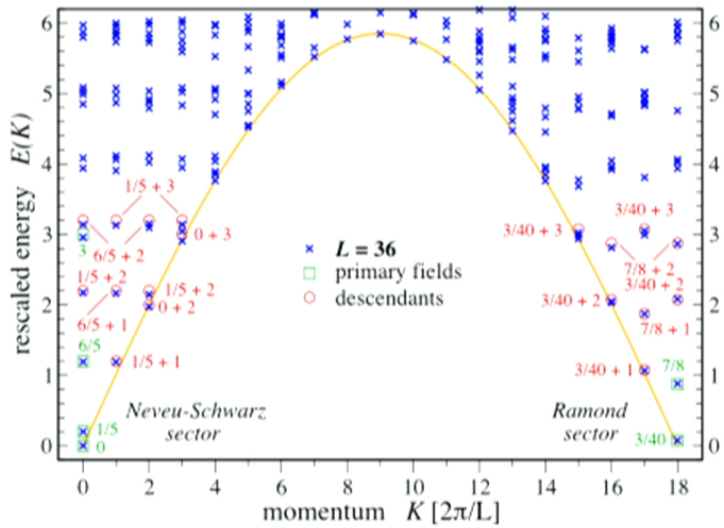
What is Conformal Field Theory (CFT)

- Conformal towers contain scaling dimensions of primary fields and their descendants. They control critical exponents, corrections to scaling,... (much more info than c alone)



CFT energy spectra

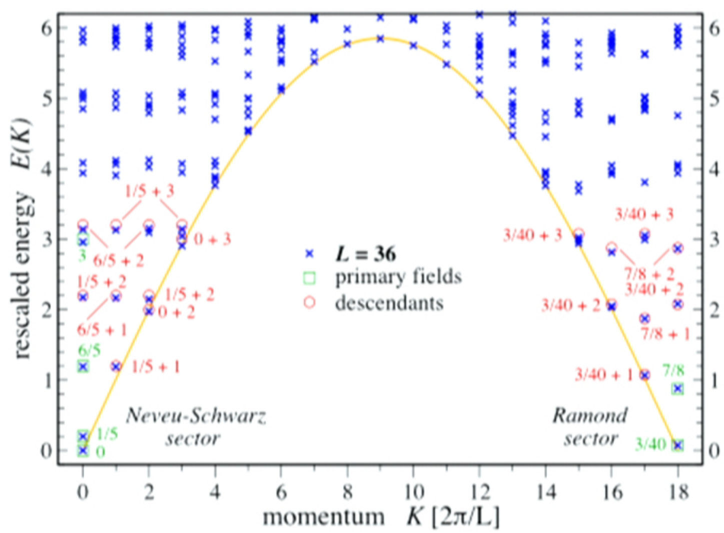
- Energy spectra of finite size systems arrange into conformal towers !



A. Feiguin et al. PRL 2007
tricritical Ising CFT Spectrum in anyon chains

CFT energy spectra

- Energy spectra of finite size systems arrange into conformal towers !
- Simple to obtain with Exact Diagonalization (but on small systems only)
- Hard to get in DMRG, due to lack of spatial quantum numbers, and DMRG has difficulty calculating many excited states (entanglement issues)



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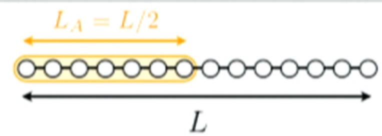
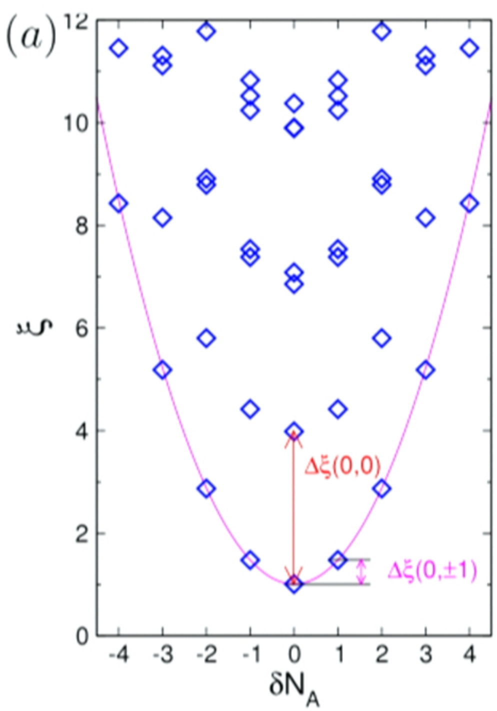
Entanglement Spectrum of the Bose Hubbard chain

- Open boundaries, U/J=2 (superfluid/LL phase)





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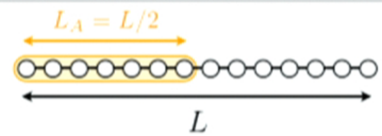
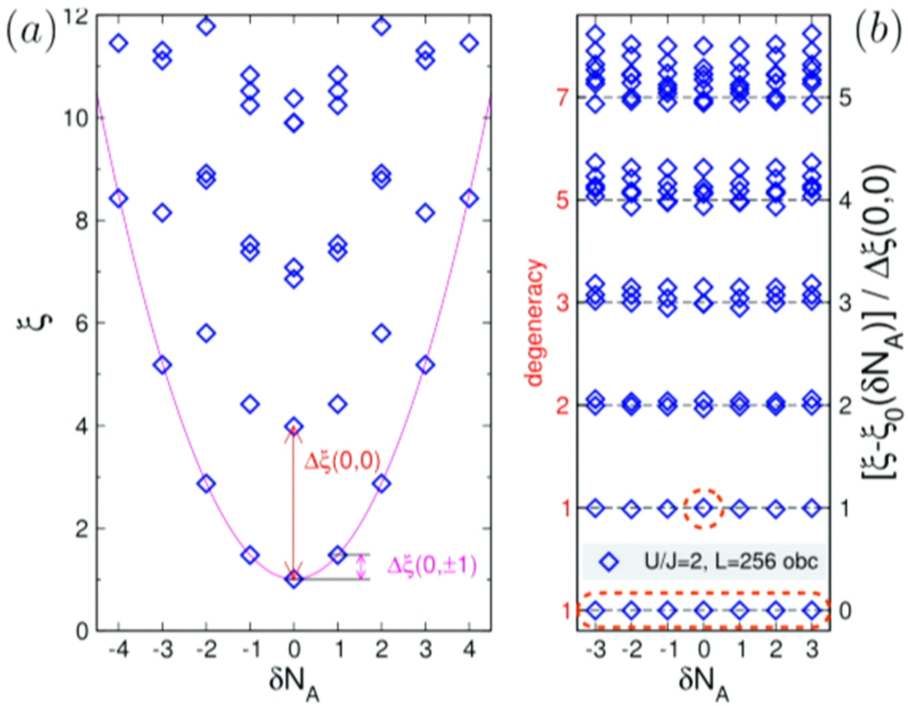
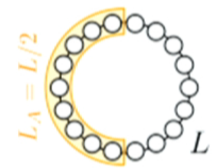
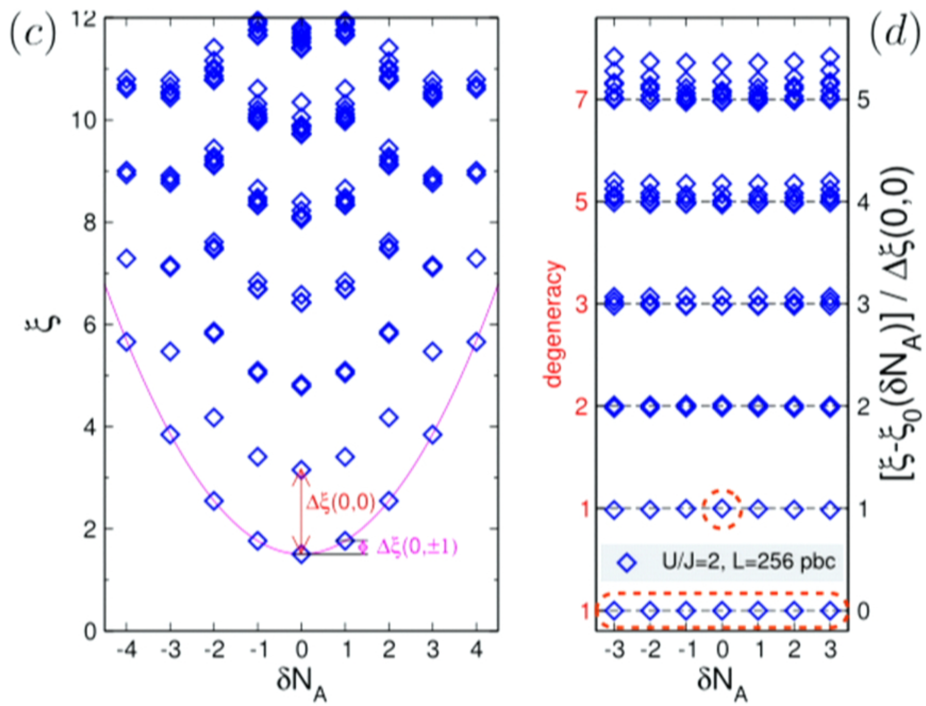




Entanglement Spectrum of the Bose Hubbard chain

- periodic boundaries, U/J=2 (superfluid/LL phase)

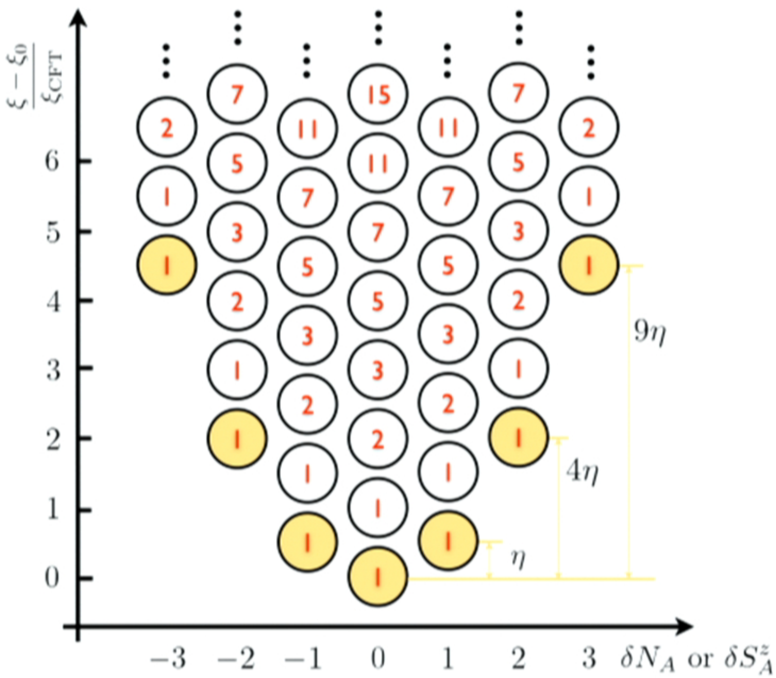




- isostructural to the OBC case, but spacing halved (to account for doubled entropy in PBC case)

AML, arXiv:1303:0741

Interpretation ?



- The structure revealed both in the OBC and PBC ES is the energy spectrum of a compactified boson with free boundary conditions.
- The envelope encodes the compactification radius/ Luttinger parameter/ correlation exponent in appropriate units.
- Can we use this estimator to detect the phase transition to the Mott insulator ?

AML, arXiv:1303:0741

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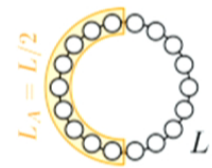

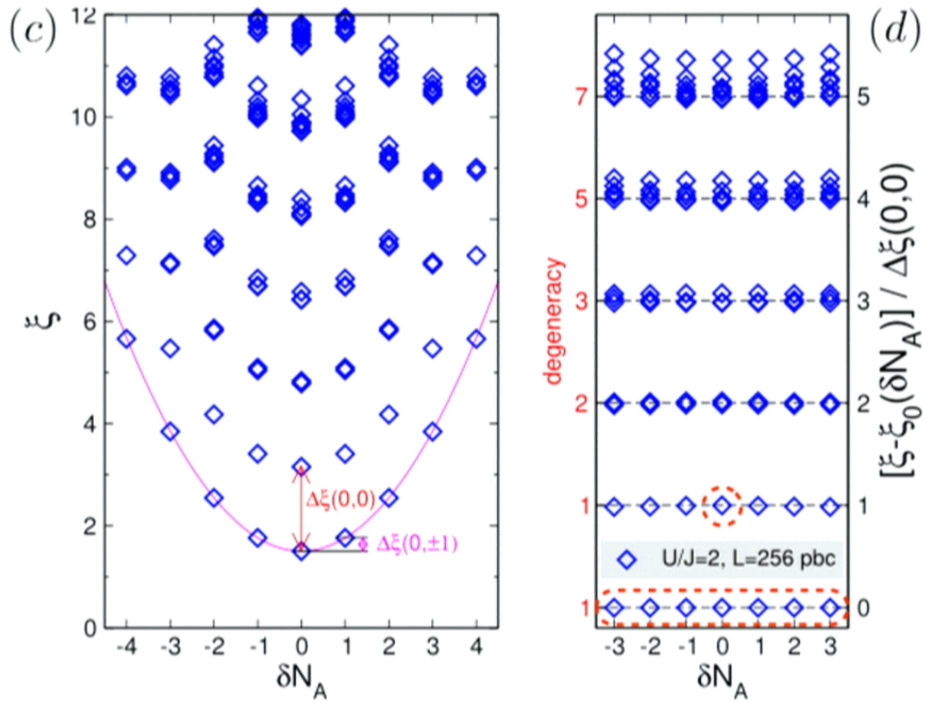

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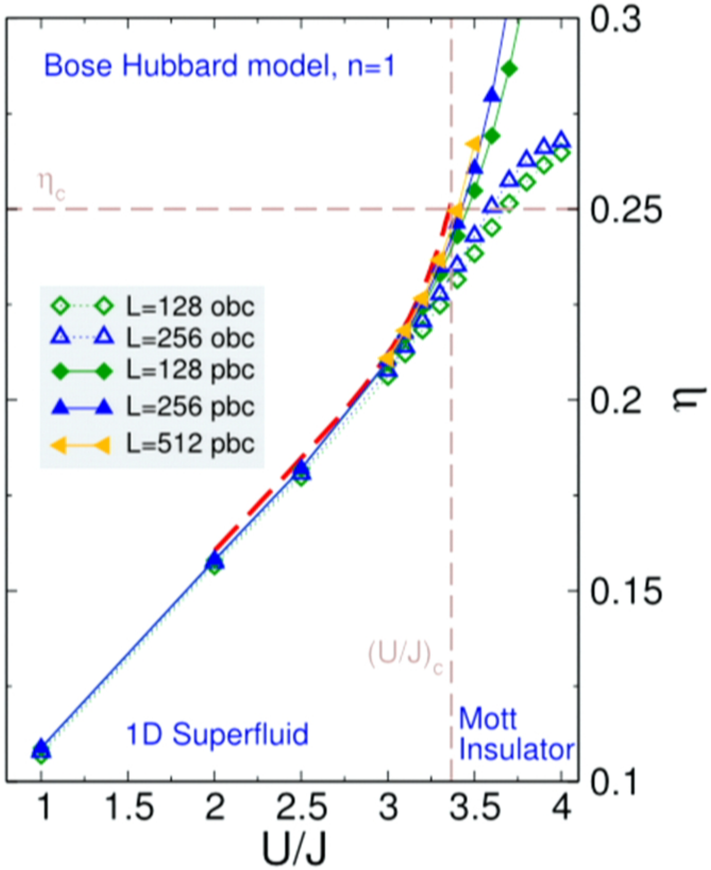
ES based phase transition locator

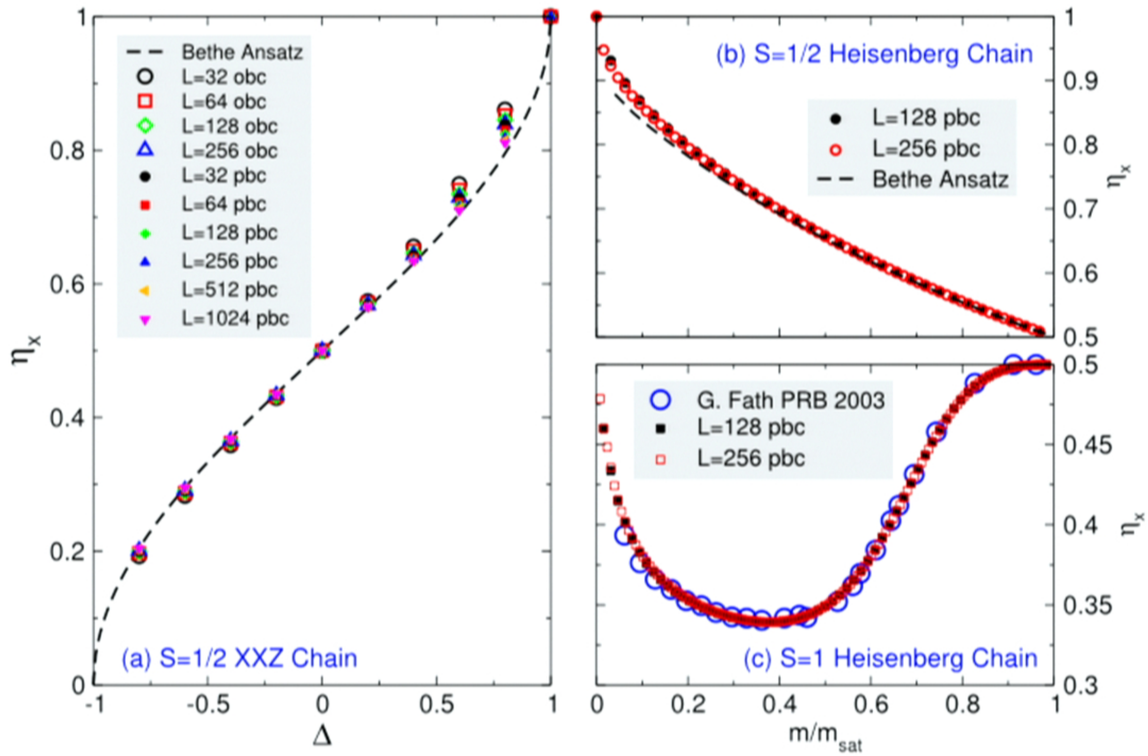


- In Sine-Gordon theory the transition is triggered at η=1/4.
- Quite accurate values for η in the 1D superfluid phase.
- Finite size effects become more important at the transition. Is this due to a marginal operator ?

AML, arXiv:1303:0741

Spin chain examples:

- Accurate at modest effort, and does not require fitting of correlation functions !



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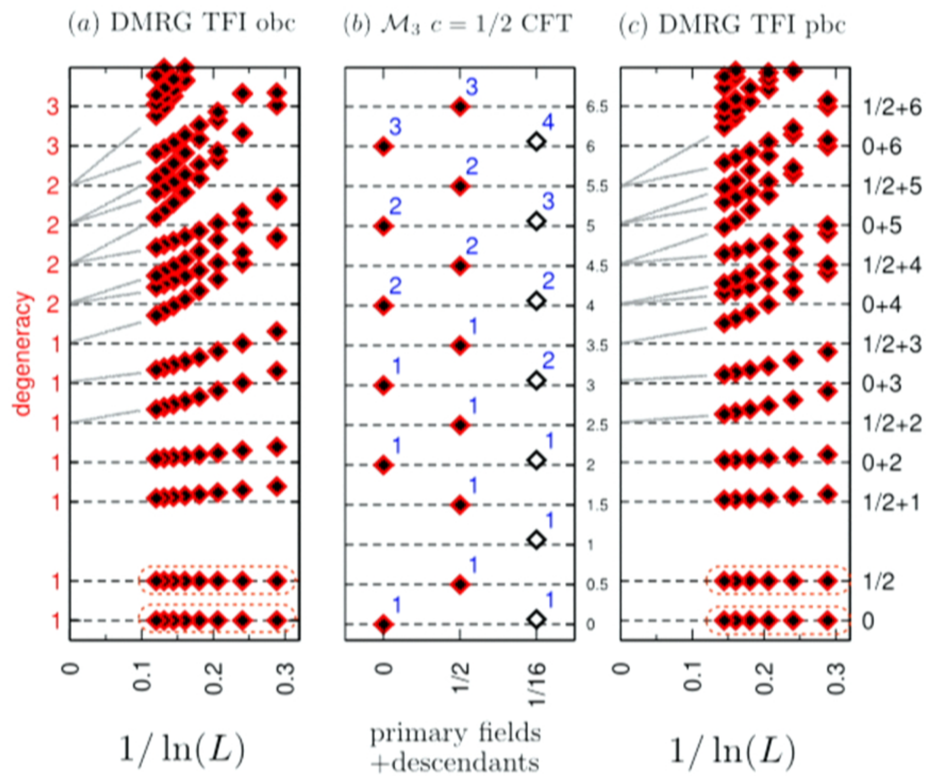
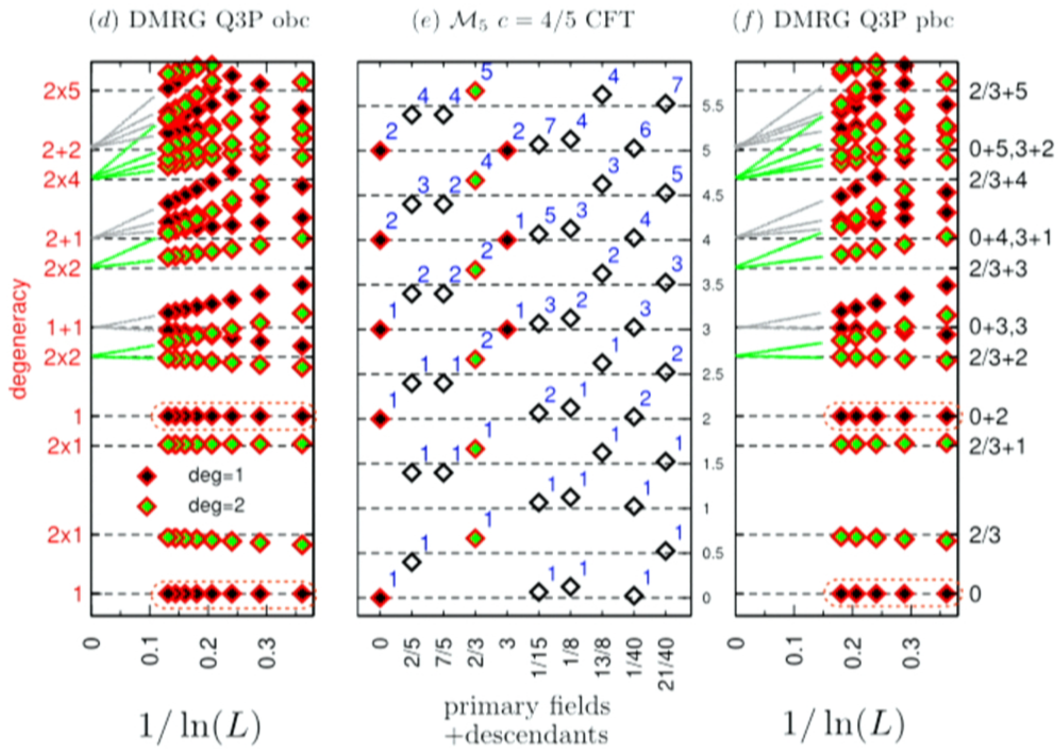
Other examples:
Transverse Field Ising Model



- ES contains 0+1/2
- Operator content as in the CFT partition function on the strip with free boundaries ! Cardy

AML, arXiv:1303:0741

Other examples:
Quantum Three State Potts Model

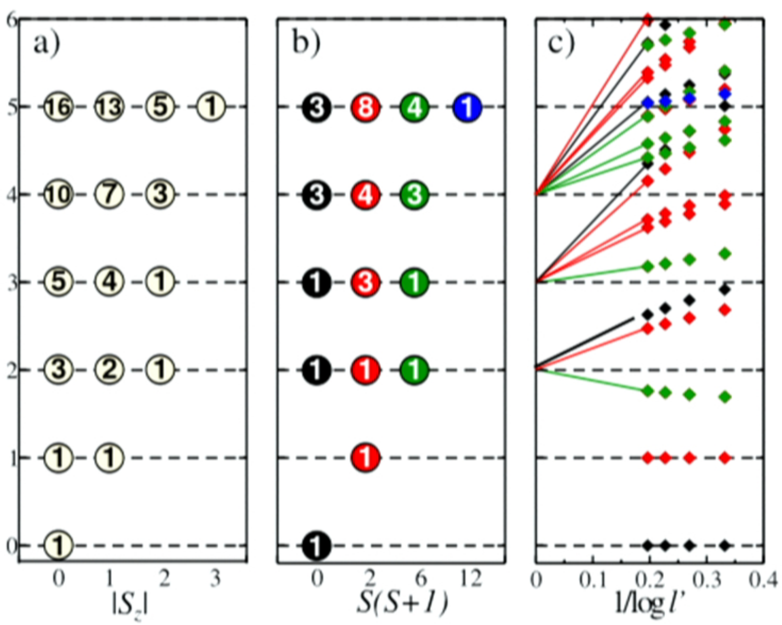


- ES contains 0+2/3(x2)+3
- Operator content as in the partition function on the strip with free boundaries ! Bauer&Saleur/Cardy

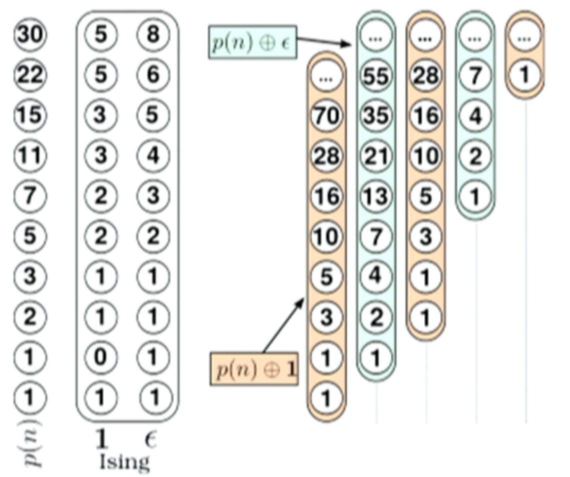
AML, arXiv:1303:0741

c>1 example:
S=1 Bilinear Biquadratic Spin Chain

- S=1 Bilinear Biquadratic Chain at the integrable Takhtajan-Babudjan point: WZWN $SU(2)_2$ (c=3/2) CFT



L. Bonnes & AML, in preparation



c.f. X.G. Wen, Phys. Rev. Lett. **70** 355 (1993)

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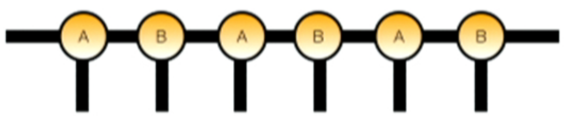
iMPS results

- So far the results were obtained using DMRG on finite size system in a regime where the number of states is large enough to get a very accurate wave function (Finite size scaling regime Pirvu et al., PRB 2012). In this regime we are probing the genuine properties of the wave function irrespective of the method.

- iMPS (i.e. iTEBD, iDMRG) however works directly for infinite systems, but the finite bond dimension introduces a finite correlation length (Finite entanglement scaling regime Pirvu et al., PRB 2012).

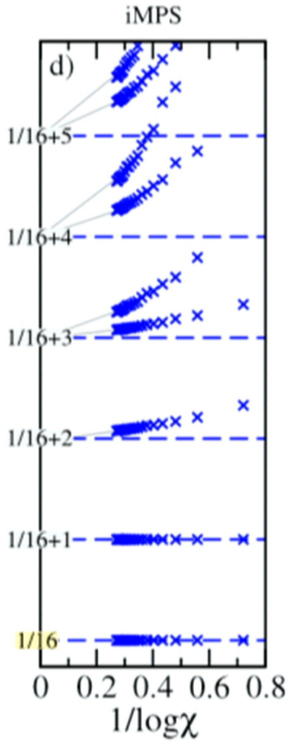


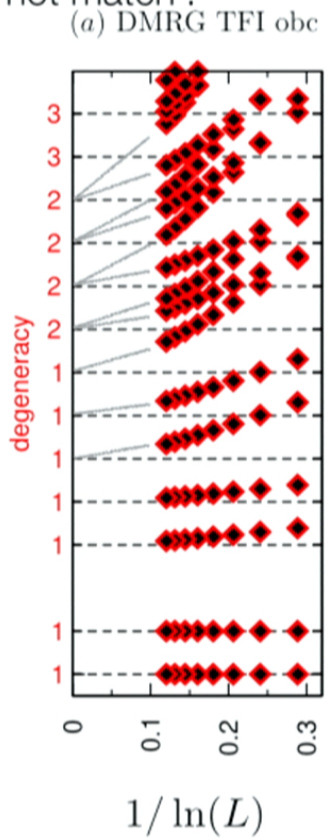
- Is there also CFT operator content visible ? (yes)

- If so, is it identical to the finite-size DMRG operator content (sometimes) ?

iMPS results: Transverse Field Ising model

- iMPS has interesting structure, but the results do not match !

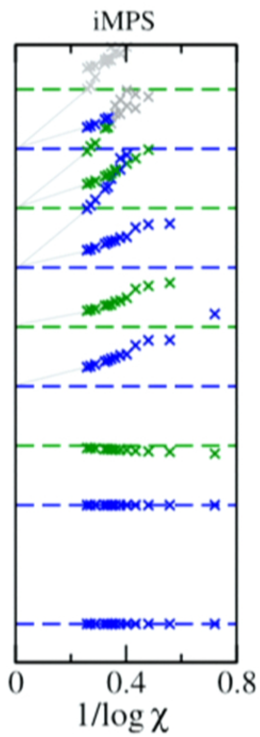


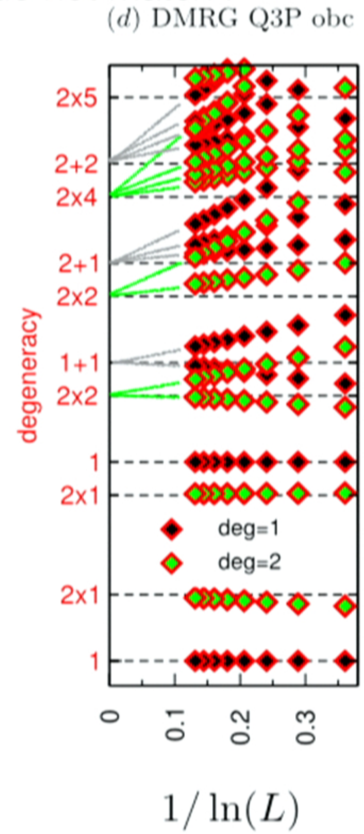


L. Bonnes & AML, in preparation

iMPS results: Three state Potts model

- iMPS has interesting structure, but the results do not match !





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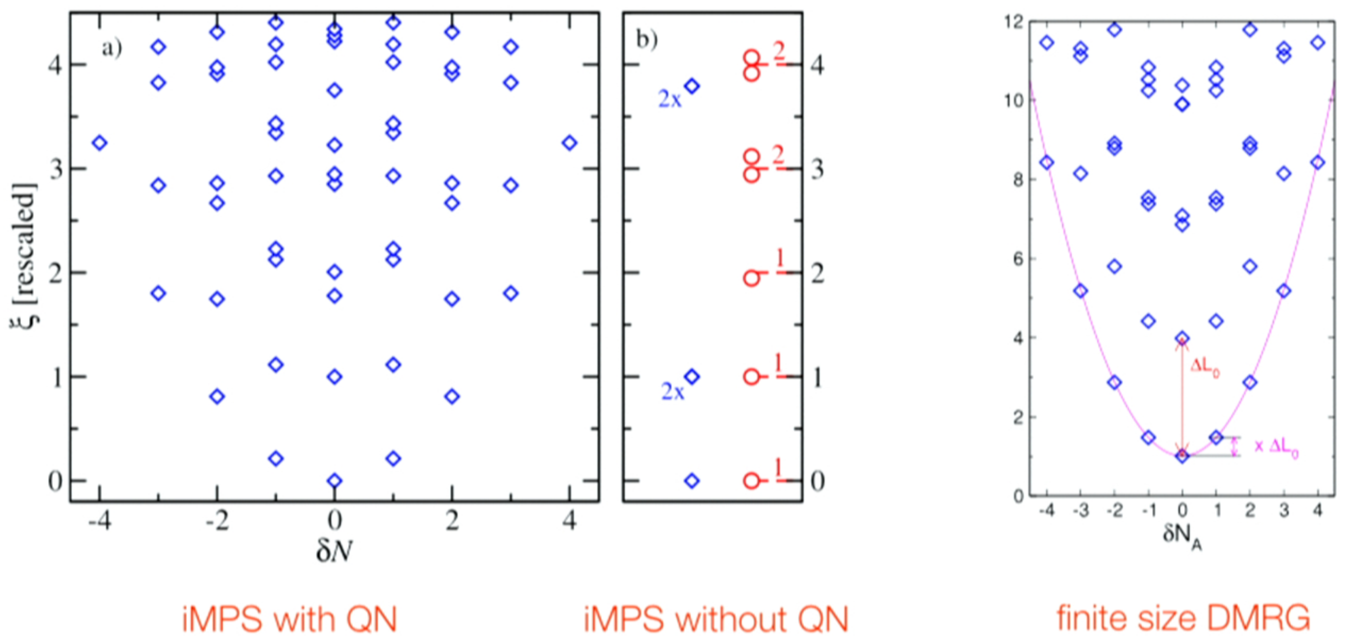
iMPS results: Bose Hubbard model

- iMPS results with QN are identical to finite size DMRG, but iMPS w/o QN are not !



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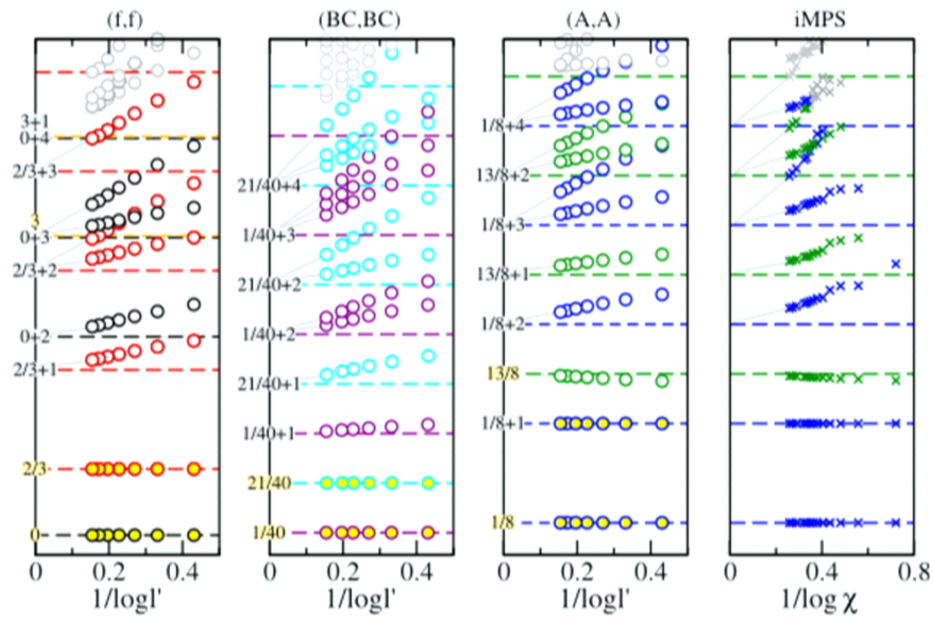
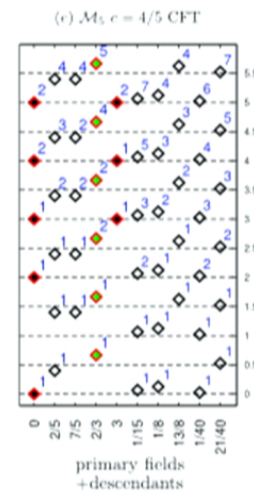
Consistent interpretation: different CFT boundary conditions

- Boundary conditions in CFT have a direct impact on the visible operator content of a theory (subfield of CFT termed “boundary CFT”).
- let us go back to finite size DMRG and apply different boundary conditions to the OBC system (free, longitudinal field, ...)
- let us focus on the 3 state Potts model:

Three state Potts: Results for different boundary conditions

- Operator content depends on the boundary conditions.
- iMPS operator content corresponds to one particular case
  (paradox: infinite system MPS feels some non-trivial boundary conditions!)





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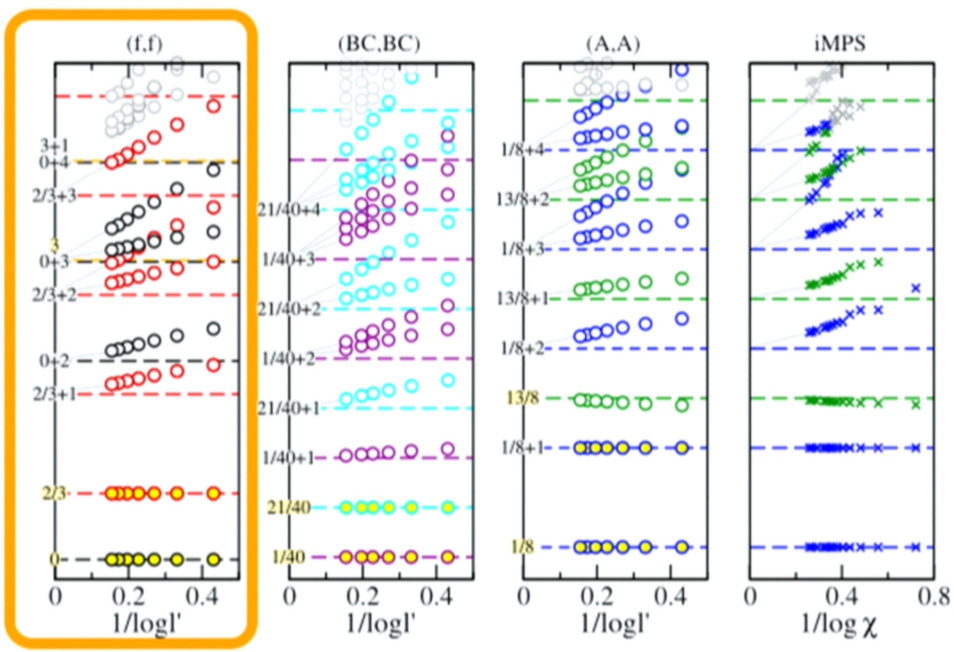

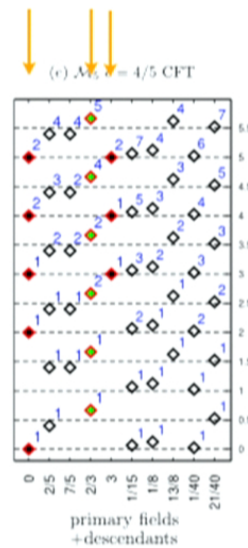

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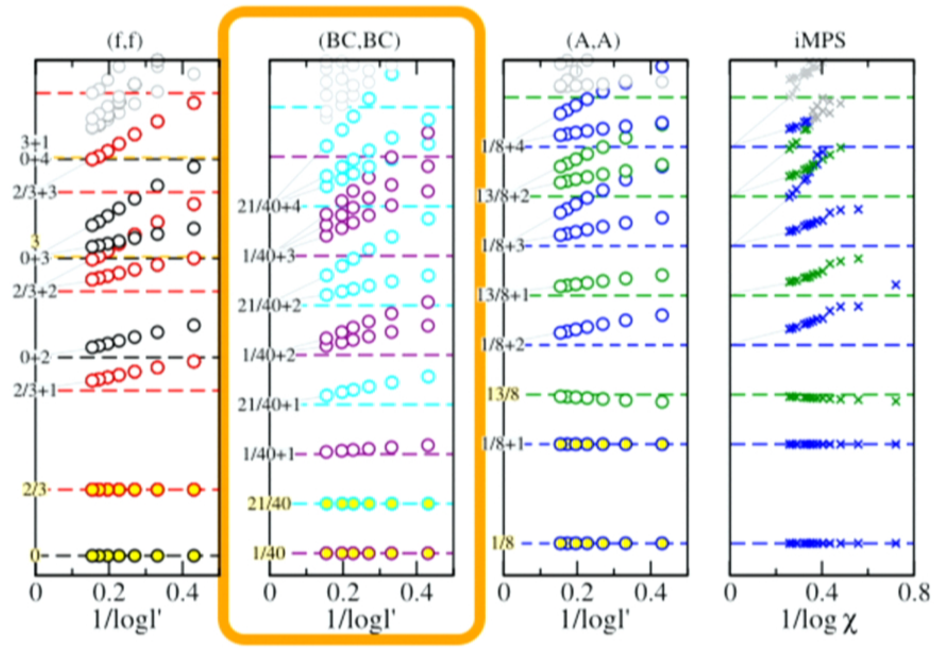

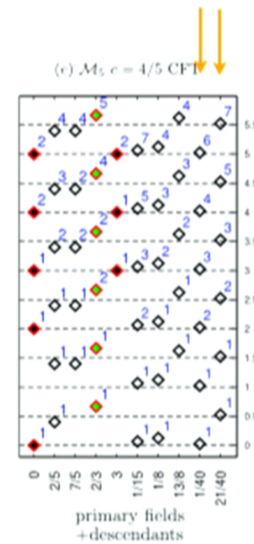

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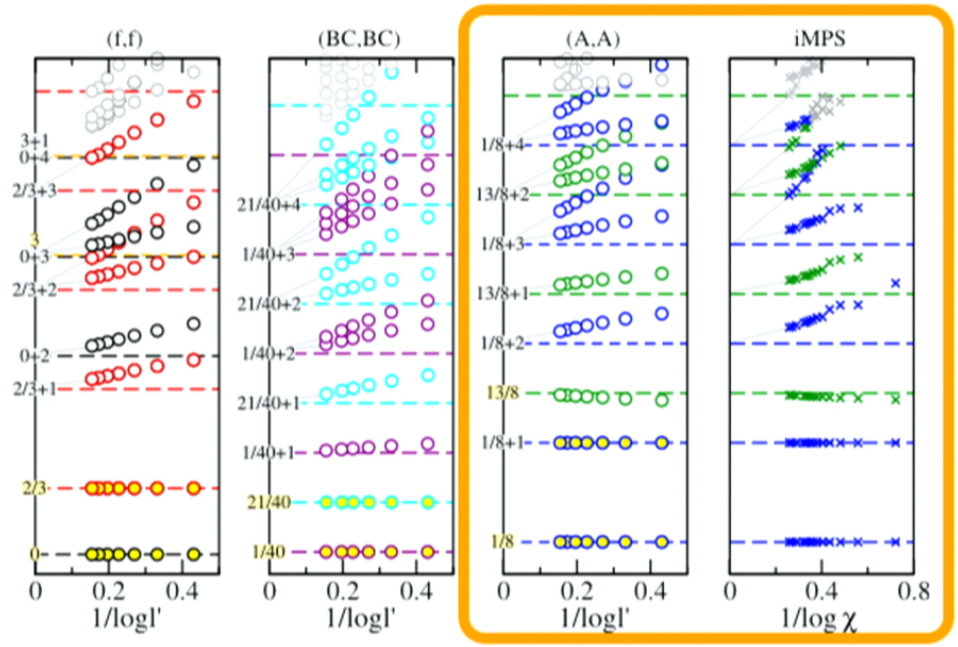

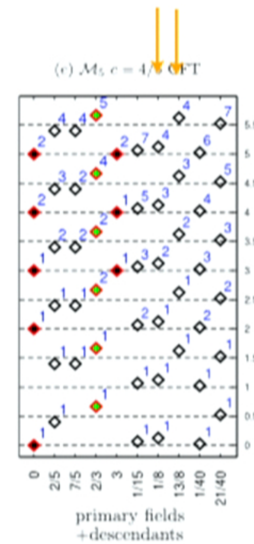

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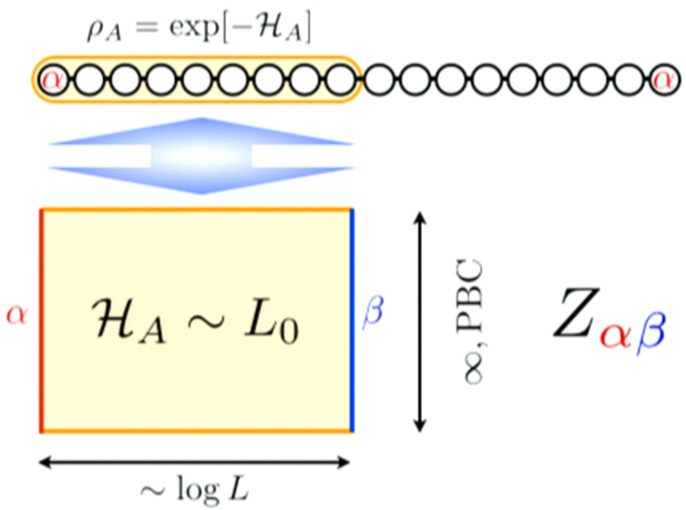
Boundary CFT interpretation:
Illustration of the relation to the strip geometry

- Operator content correspondence between the density matrix and the partition function on a strip geometry. Can explain all numerical observations so far.



α Physical boundary conditions of the quantum chain

β Boundary conditions at the entanglement cut (free in all cases we studied so far)

Operator content of $Z_{\alpha\beta}$ is known for many (but not all) CFTs

Relation to more abstract previous work

Towards a derivation of holographic entanglement entropy

Horacio Casini,[a] Marina Huerta[a] and Robert C. Myers[b]

[a] *Centro Atómico Bariloche and Instituto Balseiro, 8400-S.C. de Bariloche, Río Negro, Argentina*
[b] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*

JHEP 05 036 (2011)

2.2 Thermal behaviour in $R \times H^{d-1}$

Next we would like to extend this approach of transplanting modular flows to relate the density matrix of a CFT on $\mathcal{D}$ to that on a new geometry $R \times H^{d-1}$, which we will denote as $\mathcal{H}$ in the following. In particular, we will show that beginning with the Minkowski vacuum for an arbitrary CFT, the density matrix on $\mathcal{D}$ becomes a thermal density matrix on $\mathcal{H}$. In fact, our result is a generalization of previous observations made in ref. [19]. There the generation of a thermal state by conformal mappings was observed for free conformal field theories in $d = 4$.

- Entanglement spectrum of a half-space (Lorentz) => thermal density matrix in Rindler space
- Entanglement spectrum of a ball (CFT) => thermal density matrix in hyperbolic space

arXiv:1304.6402

Structure of entanglement in regulated Lorentz invariant field theories

Brian Swingle
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Dated: April 25, 2013)

Consider first the half-space $x^0 = t = 0$, $x^1 > 0$, and $x^i \in \mathbb{R}$ for $i = 2, ..., d$. As shown in Refs. [13–15, 17] the density matrix for this subsystem is given by

$$\rho_{x^1>0} = \exp(-2\pi J) \qquad (3.1)$$

with J a certain (regulated) boost generator

$$J = \int_{x^1>a} x^1 T^{00}. \qquad (3.2)$$

T^{00} is a component of the stress tensor and a is a UV cutoff. Thus in the notation introduced above we have $K_{x^1>0} = 2\pi J$. This expression is quite powerful because it gives us the spectrum of the reduced density matrix of a special subsystem in terms of a simple geometric problem: the spectrum of the field theory on Rindler space $ds^2 = -\rho^2 d\eta^2 + d\rho^2 + dx_i^2$ ($i = 2, ..., d$). Here the Rindler coordinates η and ρ are related to the usual coordinates $x^0 = t$ and x^1 via

$$t \pm x^1 = \pm \rho e^{\pm\eta}. \qquad (3.3)$$

Rindler space enters precisely because J is the generator of time translations in Rindler space, i.e. J generates $\eta \to \eta + \delta\eta$, so the half-space density matrix is a thermal state of the time translation generator in Rindler space at a certain temperature.

Entanglement Spectrum in Rindler space ?

- Energy / Entanglement spectrum of 2+1D Dirac fermions on the honeycomb lattice

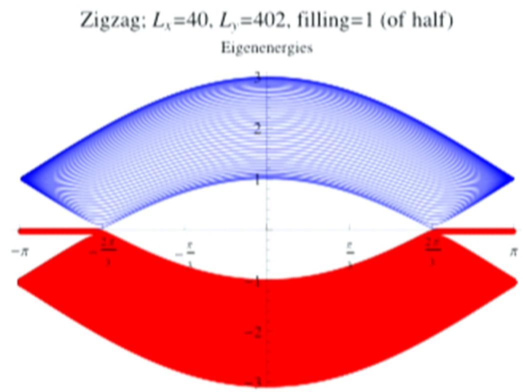
- Energy spectrum soft at two Dirac points

- Cylinder setup, divided into two equal halves

- Momentum along boundary conserved (ky)

- Entanglement spectrum has a 1/log(Lx) gap closing feature at the same Dirac points.

- Interacting systems ?



Entropy perspective previously in :
H Ju, AB Kallin, P Fendley, MB Hastings, RG Melko, PRB 2012

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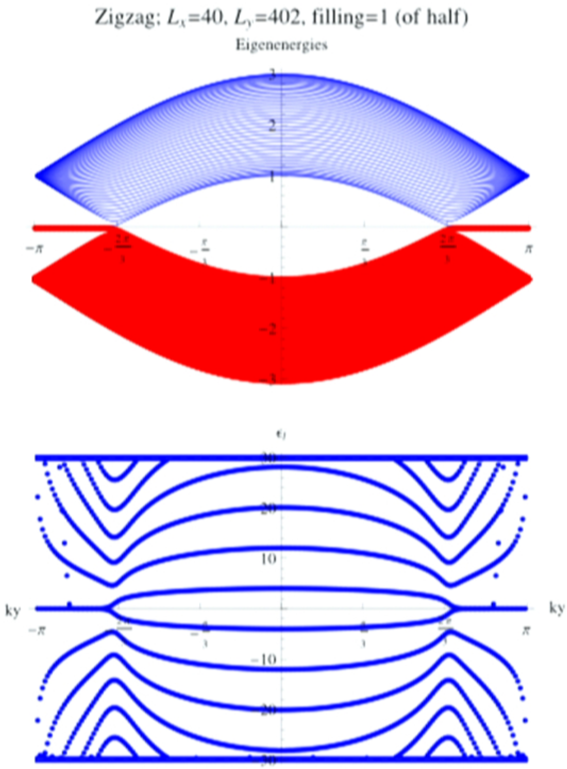

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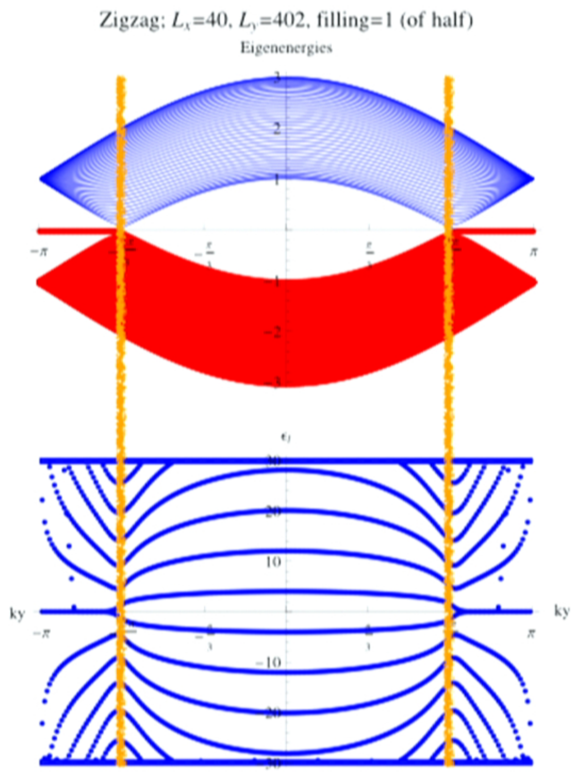
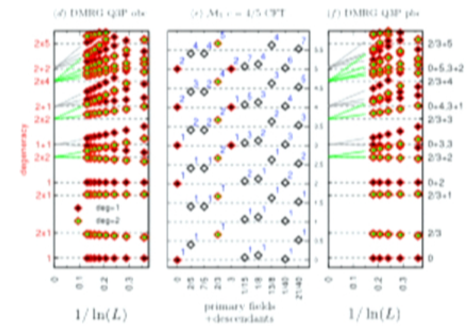
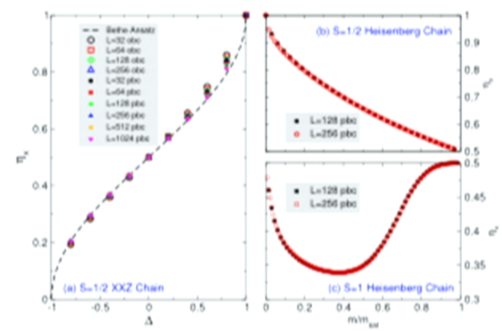


Conclusions

- ES allows to inspect a subset of the CFT operator content based on ground state wave functions.
- Operator content is the one of the CFT with (free) boundary conditions.
- Has tremendous practical value, because it allows to extract Luttinger parameter(s) from the ES, which is easily available in DMRG/MPS.



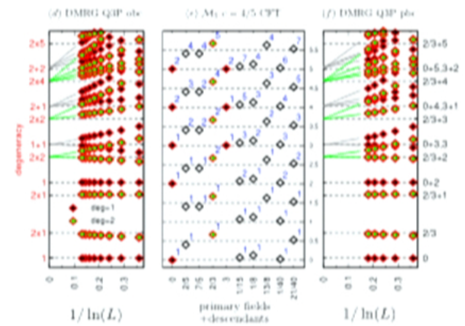


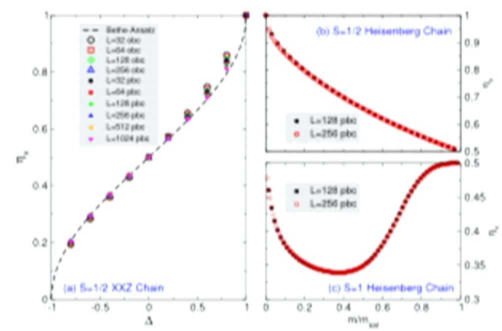
AML, arXiv:1303:0741
L. Bonnes & AML, in preparation

Outlook

- Is conceptually important in order to write down infinite MPS trial wave functions based on CFT à la Cirac & Sierra, PRB 2010

- Open Questions:
  * Finite size corrections ?
  * How general are free boundary conditions at the entanglement cut ?
  * Entanglement spectrum in excited states ?
  * Relation to holography ? / Higher dimensions ?





AML, arXiv:1303:0741

L. Bonnes & AML, in preparation

Collaborators

Lars Bonnes

+ inspiring discussions with M. Metlitski (KITP) and J. Dubail (Yale)