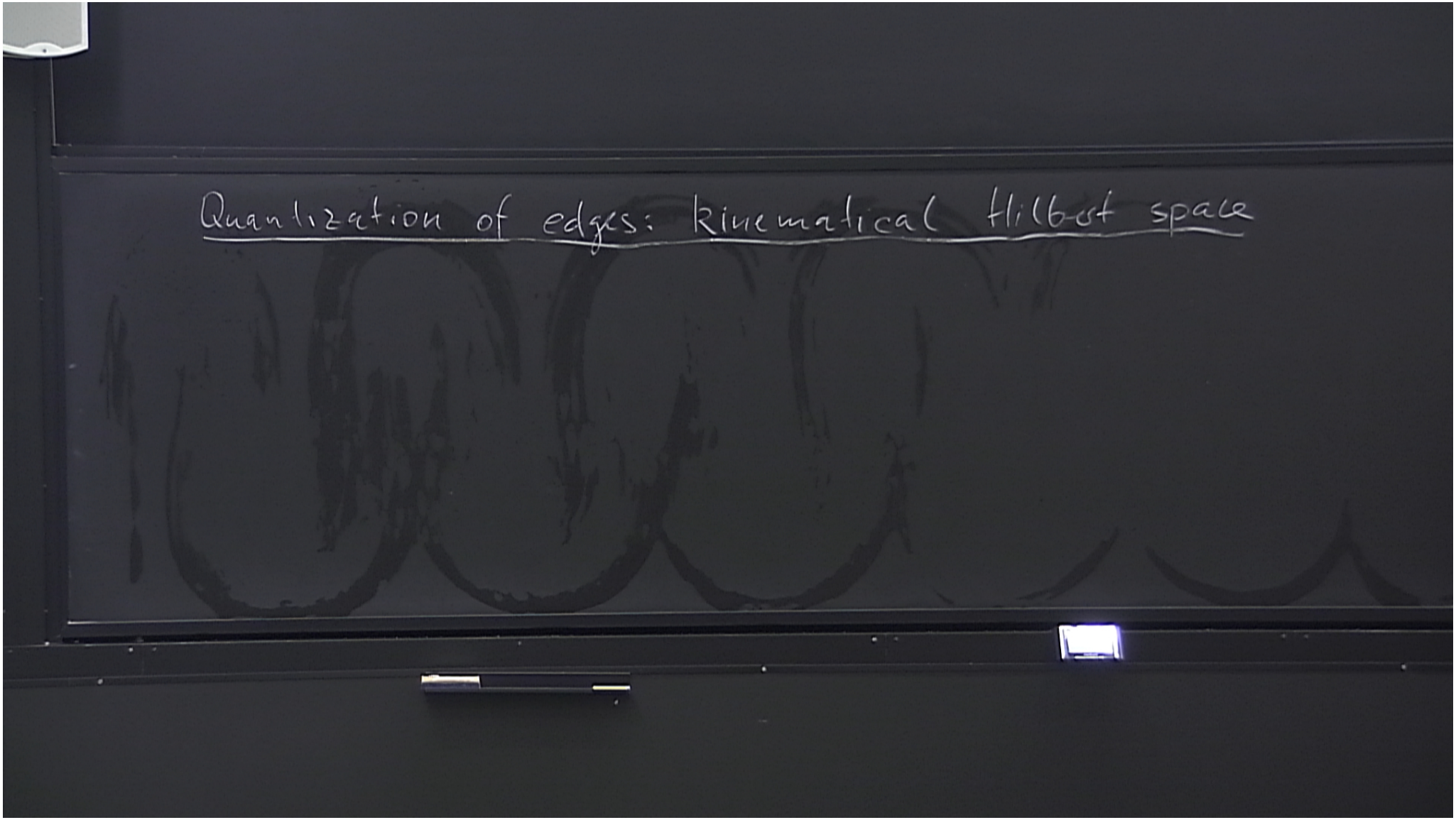
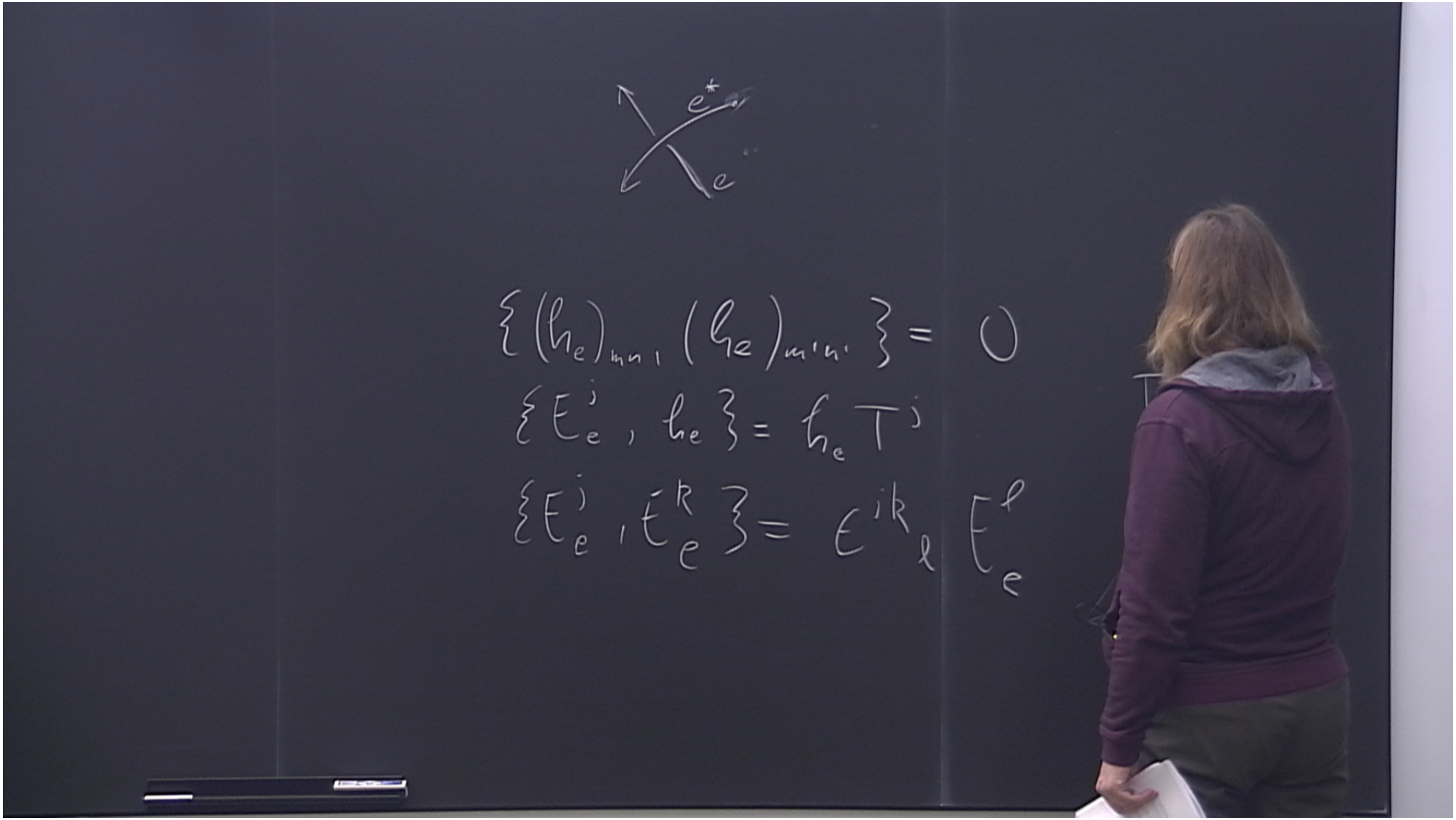
Title: Quantum Gravity Review-9

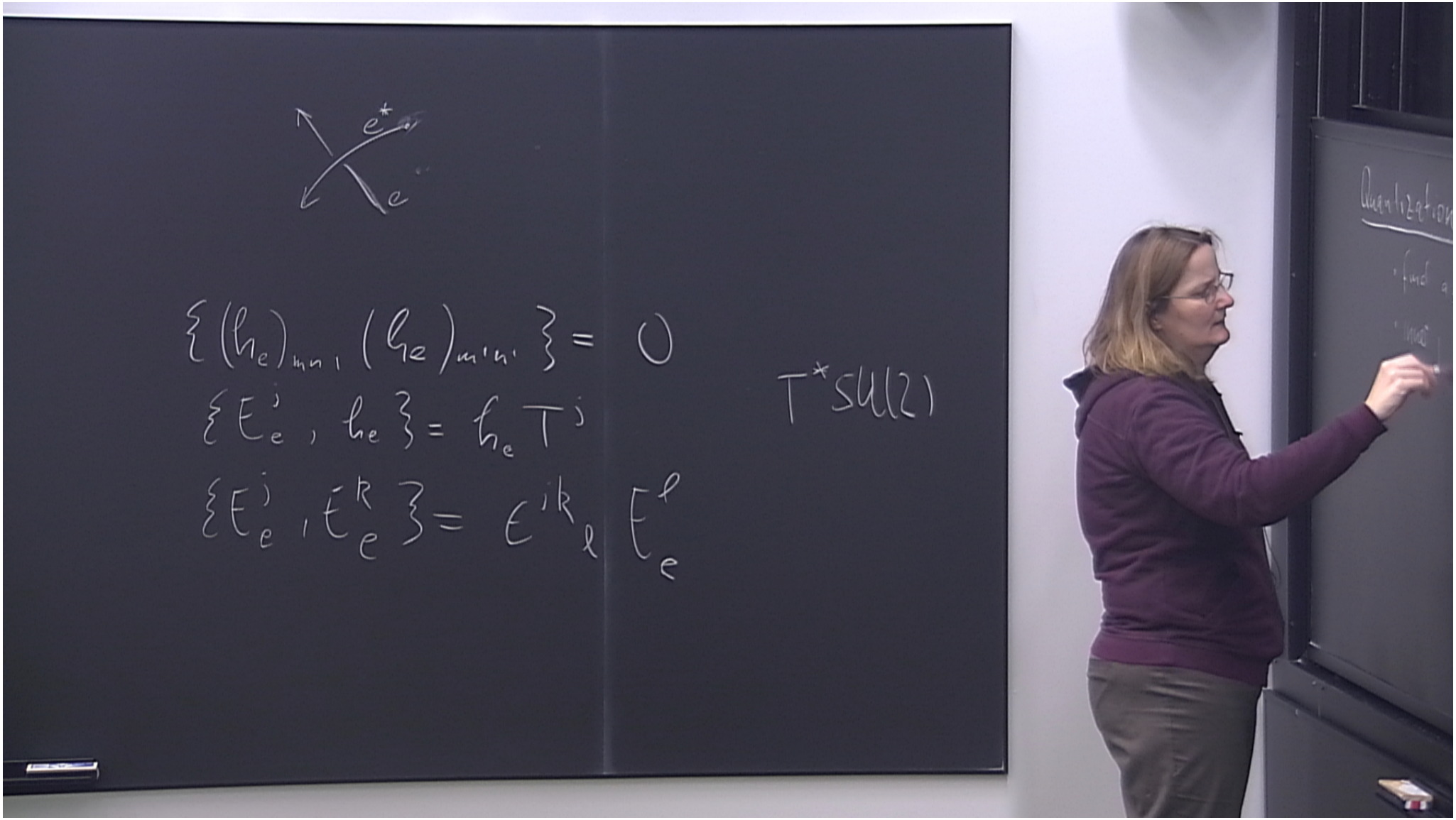
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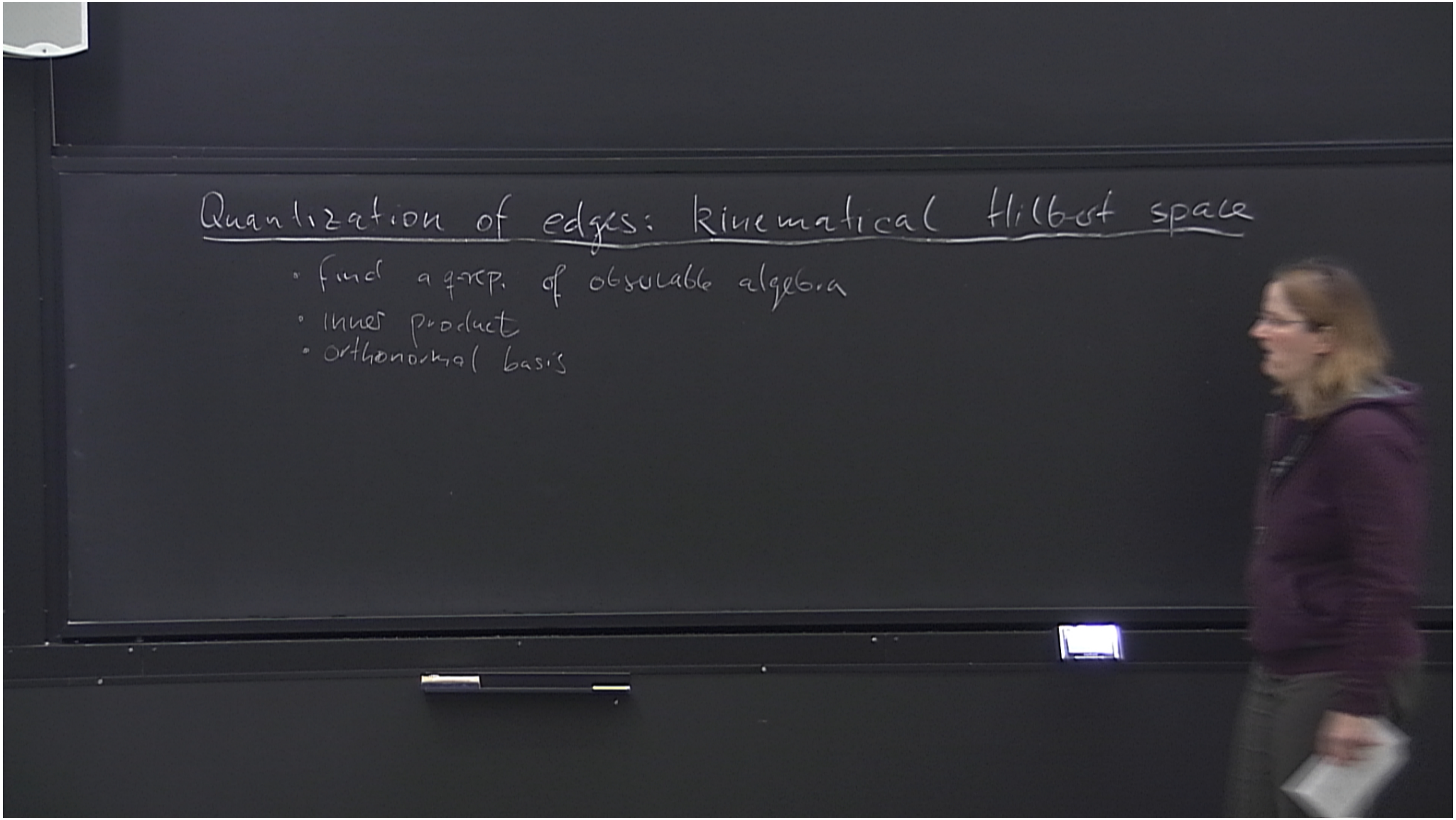
URL: http://www.pirsa.org/15020060

Abstract:


Quantization of edges: Kinematical Hilbert space


e*
e
$\{(h_e)_{mn}, (h_e)_{m'n'}\} = 0$
$\{E^j_e, h_e\} = h_e T^j$
$\{E^j_e, E^k_e\} = \epsilon^{jk}{}_l E^l_e$


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$\{(h_e)_{mn}, (h_e)_{m'n'}\} = 0$
$\{E^i_e, h_e\} = h_e T^i$
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$T^*SU(2)$
Quantization
• find a


Quantization of edges: Kinematical Hilbert space
• find a grp. of observable algebra
• inner product
• orthonormal basis

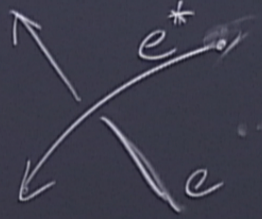
Quantization of edges: kinematical Hilbert space

- find a q-rep. of observable algebra
- inner product
- orthonormal basis

- functions of holonomy: $f : G \to \mathbb{C}$

$\hat{f}\, T(g) = f(g)\, T(g)$

- multiplication of commute

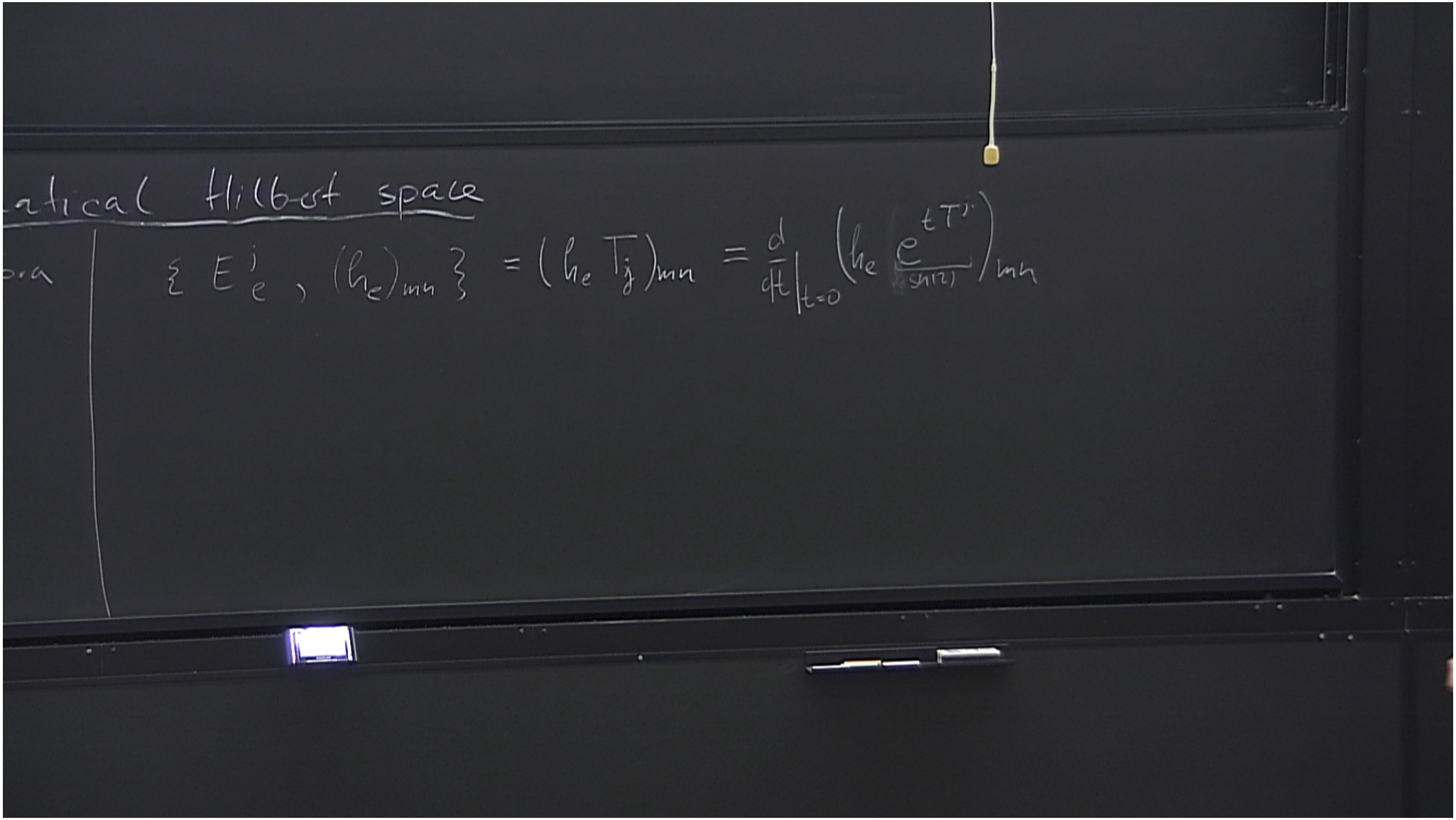


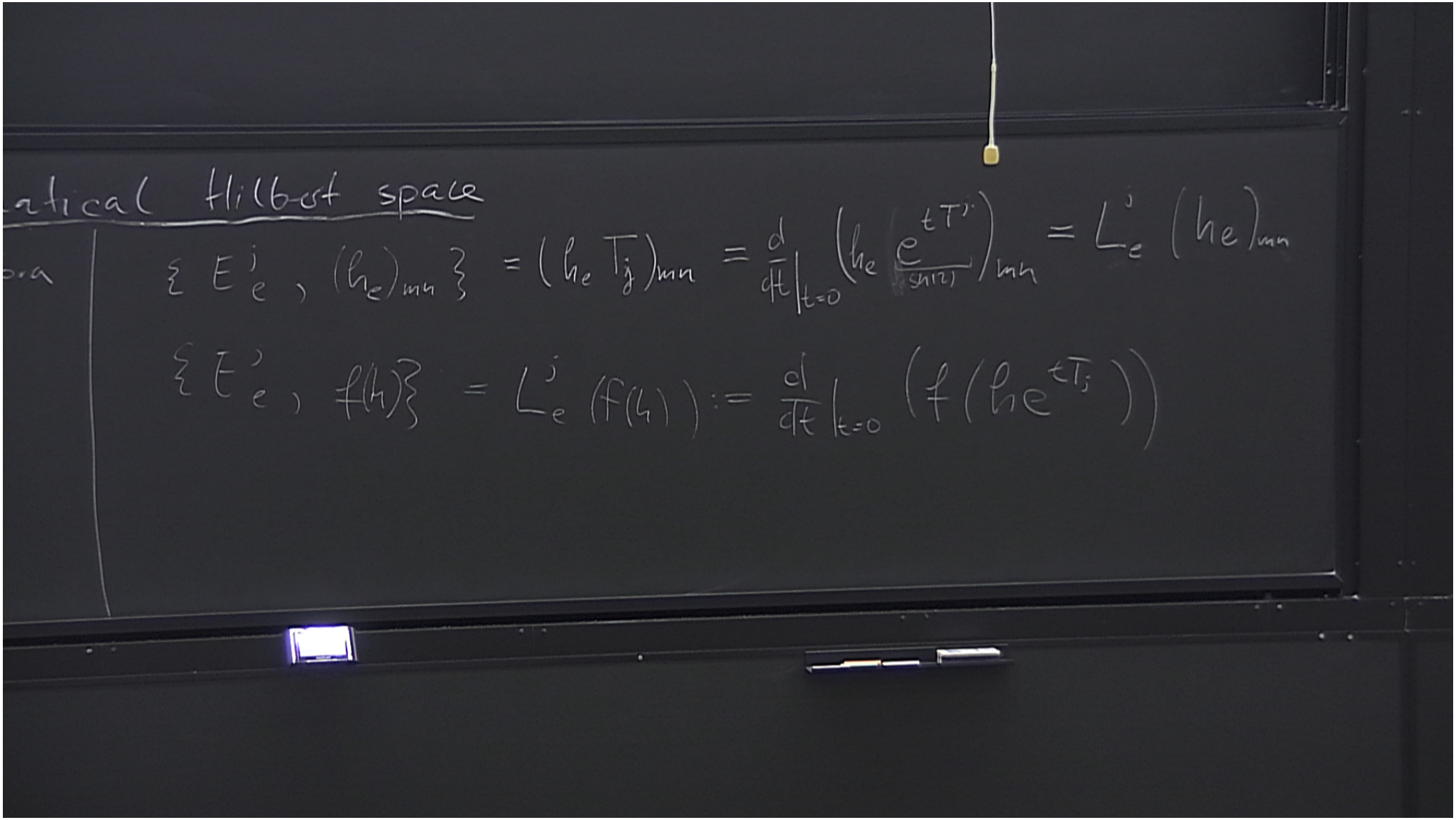
$$\{(h_e)_{mn}, (h_e)_{m'n'}\} = 0$$

$$\{E^j_e, h_e\} = h_e T^j$$

$$\{E^j_e, E^k_e\} = \epsilon^{jk}{}_l E^l_e$$

- $T^*\mathbb{R} \to L^2(\mathbb{R})$
  $x \qquad \psi(x)$
  $p \to -i\hbar \frac{d}{dx}$
- $T^*SU(2)$


atical Hilbert space
$\{E^j_e, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \left(h_e e^{tT^j}\right)_{mn}$


atical Hilbert space
$\{E^j_e, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \left(h_e e^{tT^j}\right)_{mn} = L^j_e (h_e)_{mn}$
$\{E^j_e, f(h)\} = L^j_e (f(h)) := \frac{d}{dt}\Big|_{t=0} \left(f(h e^{tT_j})\right)$

cal Hilbert space

$$\{E^j_e, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \left(h_e \underbrace{e^{tT^j}}_{SU(2)}\right)_{mn} = L^j_e (h_e)_{mn}$$

$$\{E^j_e, f(h)\} = L^j_e (f(h)) := \frac{d}{dt}\Big|_{t=0} \left(f\left(h_e e^{tT_j}\right)\right)$$

tical Hilbert space

$$\{E^j_e, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \left(h_e \frac{e^{tT^j}}{sh(2)}\right)_{mn} = L^j_e (h_e)_{mn}$$

$$\{E^j_e, f(h)\} = L^j_e (f(h)) := \frac{d}{dt}\Big|_{t=0} \left(f(h_e e^{tT_j})\right)$$

$$i\hbar \widehat{(L^j_e f_e)}\,|1\rangle = i\hbar \widehat{\{E^j, f_e(h_e)\}}\,|1\rangle \overset{\text{want}}{=} [\hat{E}^j, \hat{f}_e]\,|1\rangle$$

$$= \hat{E}^j |f_e\rangle - \hat{f}_e \hat{E}^j |1\rangle$$

$f(g) \equiv 1$

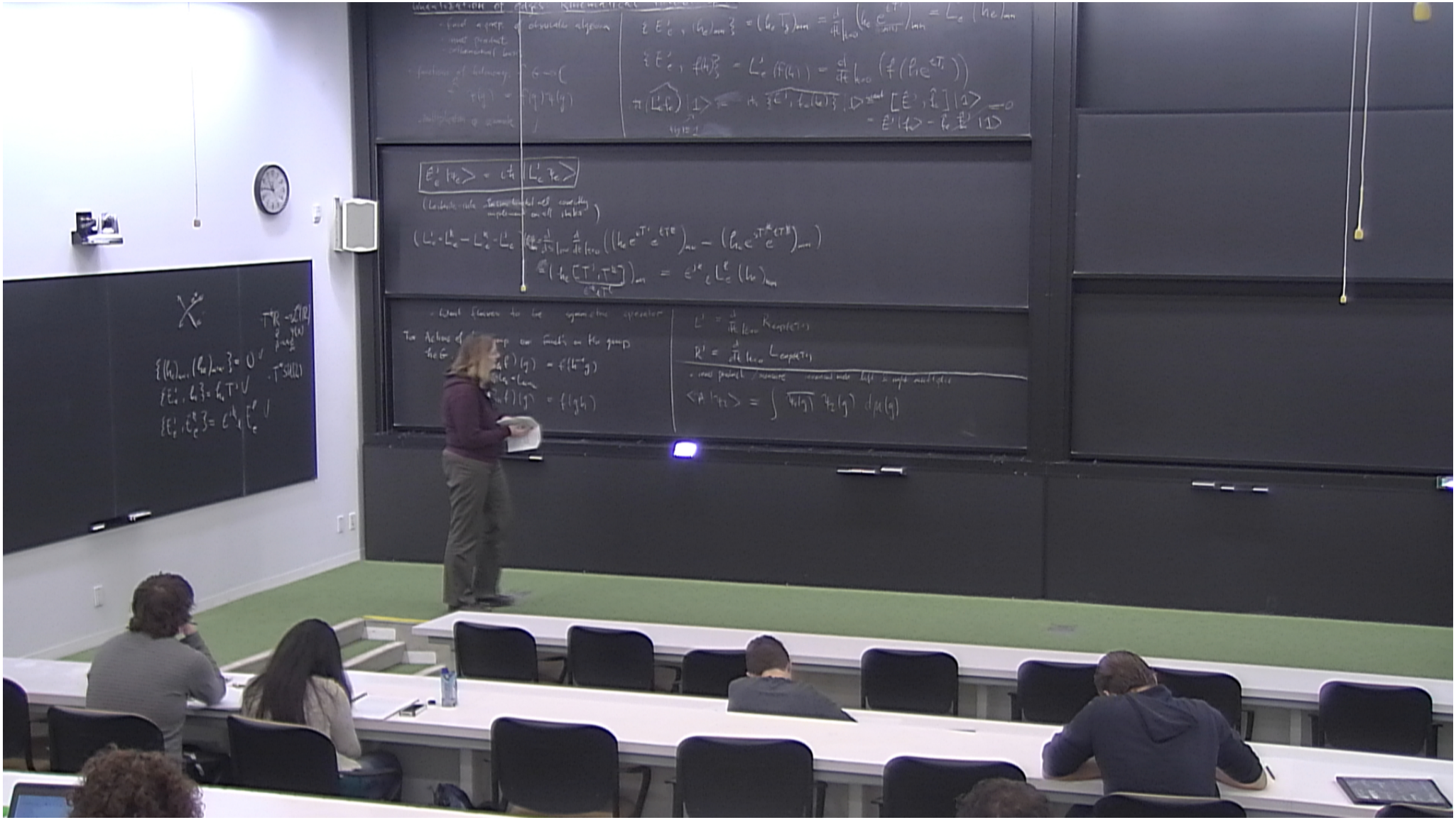
$\chi(g) \equiv 1$

$$\hat{E}^j_e \, |\psi_e\rangle = i\hbar \, |L^j_e \psi_e\rangle$$

(Leibnitz-rule: Poisson bracket rel correctly implement on all states)

$$\left(L^j_e \circ L^k_e - L^k_e \circ L^j_e\right)(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \left(\left(h_e e^{sT^j} e^{tT^k}\right)_{mn} - \left(h_e e^{sT^k} e^{tT^j}\right)_{mn} \right)$$

$$= \left(h_e \underbrace{[T^j, T^k]}_{\epsilon^{jk}{}_l T^l}\right)_{mn} = \epsilon^{jk}{}_l \, L^l_e \, (h_e)_{mn}$$



$$(R_h f)(g) = f(gh)$$

Haar measure : $SU(2) \longrightarrow S^3$

g: α, $\vec{n}$ = normalized vector $\exp(\alpha T \cdot \vec{n}) = g$

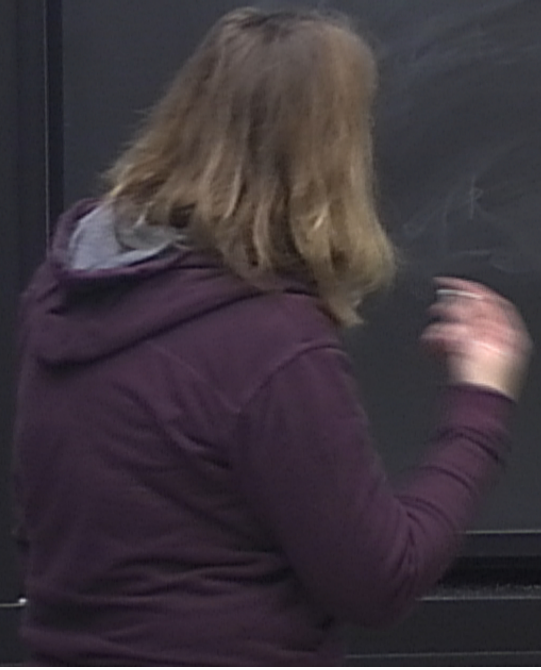
$[0, 2\pi]$

$n^1 = \sin\psi \cos\phi$

$n^2 = \cos\psi$

$n^3 = \sin\psi \sin\phi$

$$dg = \frac{1}{4\pi^2}$$



$$(R_h f)(g) = f(gh)$$

Haar measure: $SU(2) \to S^3$

g: α, n = normalized vector $\exp(\alpha T \cdot \hat{n}) = g$

$[0, 2\pi]$

$$n^1 = \sin\chi \cos\phi$$
$$n^2 = \cos\chi$$
$$n^3 = \sin\chi \sin\phi$$

$$dg = \frac{1}{4\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin(\chi)\, d\alpha\, d\alpha\, d\chi\, d\phi$$

$$\int_G 1\, dg = 1$$

Haar measure

$d_H g = d_H(hg) = d_H(gh) = d_H g^{-1}$

$\mathcal{H}_e = \mathcal{L}^2(G, d_H g) \Rightarrow$ ONB ?

$\mathcal{L}^2(S^1, d_H g)$, $\psi_k = e^{ik\varphi}$, $k \in \mathbb{Z}$ ONB

↳ 1 dim Rep. labelled by k

$e^{ik\varphi_1} \cdot e^{ik\varphi_2} = e^{ik(\varphi_1+\varphi_2)}$



$$dg = \frac{1}{4\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin(\psi)\, d\alpha\, d\varphi\, d\psi\, d\phi$$

$$\int_G dg = 1$$

- irreduc. rep. R

- compact semi-simple Lie-group. Peter-Weyl thm:

  - (normal) matrix el. of irreps form an orthonormal basis.

$\left(\frac{\alpha}{2}\right) \sin(\gamma)\, d\alpha\, d\gamma\, d\phi$

↳ 1 dim Rep labelled by k

$e^{ikq_1} \cdot e^{ikq_2} = e^{ik(q_1+q_2)}$

- irreduc. rep. k ⟶ matrix dim. of irrep k

group. Peter – Weyl thm:

of irreps form an orthonormal basis.

$\langle g | \underbrace{j,m,n}_{\text{basis states}} \rangle = \psi_{j,m,n}(g) = \sqrt{d_j}\, D^j_{mn}(g)$

(d_j: dim of V_j; j: label of rep)

$$dg = \frac{1}{4\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin(\gamma)\, d\alpha\, d\alpha\, d\gamma\, d\phi$$

$$\int_G 1\, dg = 1$$

- irreduc. ND. R

- compact semi-simple Lie-group. Peter-Weyl thm:

* (normal) matrix el. of irreps form an orthonormal basis.

$$\int_G \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j}\, \delta_{jj'}\, \delta_{mm'}\, \delta_{nn'}$$

$$\sum_{j \in \mathbb{N}/2} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |j,m,n\rangle\langle j,m,n| = \mathbb{1}_{L^2(G, dg)}$$

(4) $d\alpha\, d\gamma\, d\phi$

↳ 1dim Rep. labelled by k

$e^{ik\varphi_1} \cdot e^{ik\varphi_2} = e^{ik(\varphi_1+\varphi_2)}$

- irreduc. rep. $k \longrightarrow$ matrix dim. of irrep k

ter – Weyl thm:

form an orthonormal basis. $\langle g | j,m,n \rangle = \psi_{j,m,n}(g) = \sqrt{d_j}\, D^j_{mn}(g)$

dim of V_j · label of rep · basis states

$\delta_{jj'} \delta_{mm'} \delta_{nn'}$

$|j,m,n\rangle \simeq \mathbb{1}_{L^2(G, d_{Haar})}$

SU(2)-Rep: $j \in \mathbb{N}/2$, $\dim V_j = 2j+1 = d_j$

D^j_{mn}

$-j \le j, m, n \le +j$

(7) $\alpha\, d\alpha\, d\gamma\, d\phi$

↳ 1dim Rep. labelled by k

$e^{ikq_1} \cdot e^{ikq_2} = e^{ik(q_1+q_2)}$

• irreduc. rep. $k \longrightarrow$ matrix dim. of rep k

ter – Weyl thm:

form an orthonormal basis: $\langle g | \underbrace{j,m,n}_{\text{basis states}} \rangle = \psi_{j,mn}(g) = \sqrt{d_j}\, D^j_{mn}(g)$

(dim of V_j; label of rep)

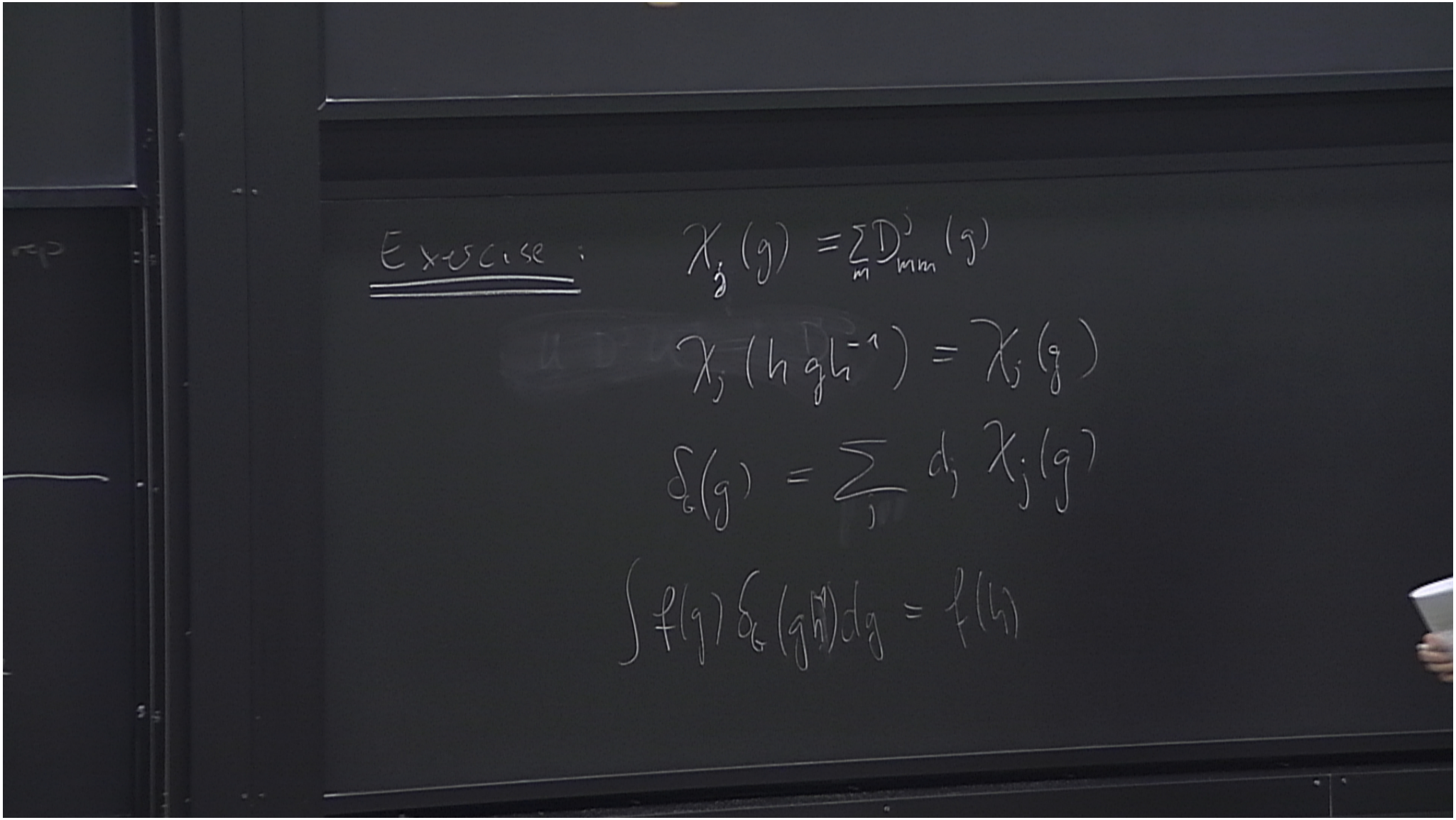
$\delta_{jj'} \delta_{mm'} \delta_{nn'}$

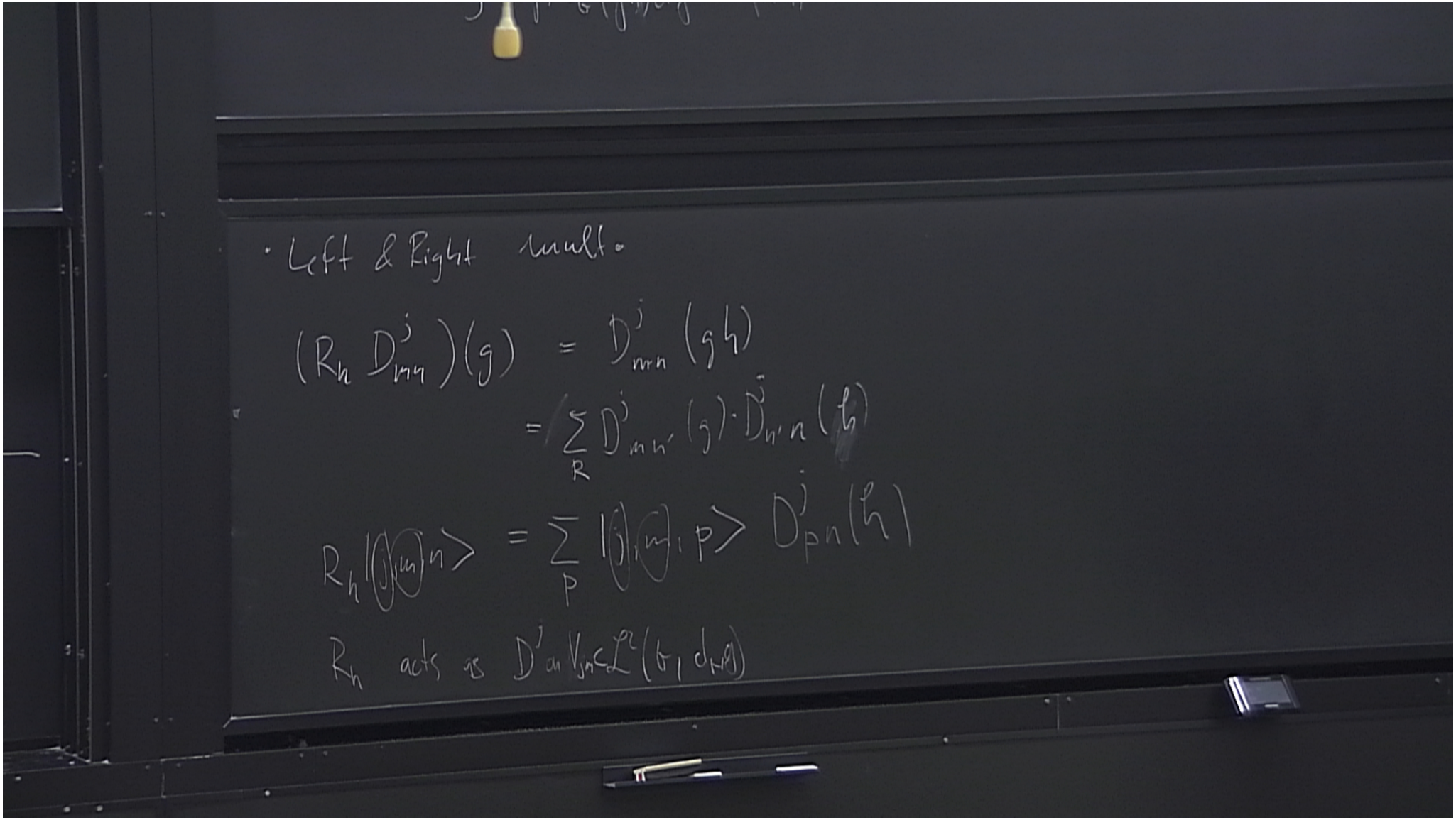
$|j,m,n\rangle \simeq \mathbb{1}_{L^2(G, d_H g)}$

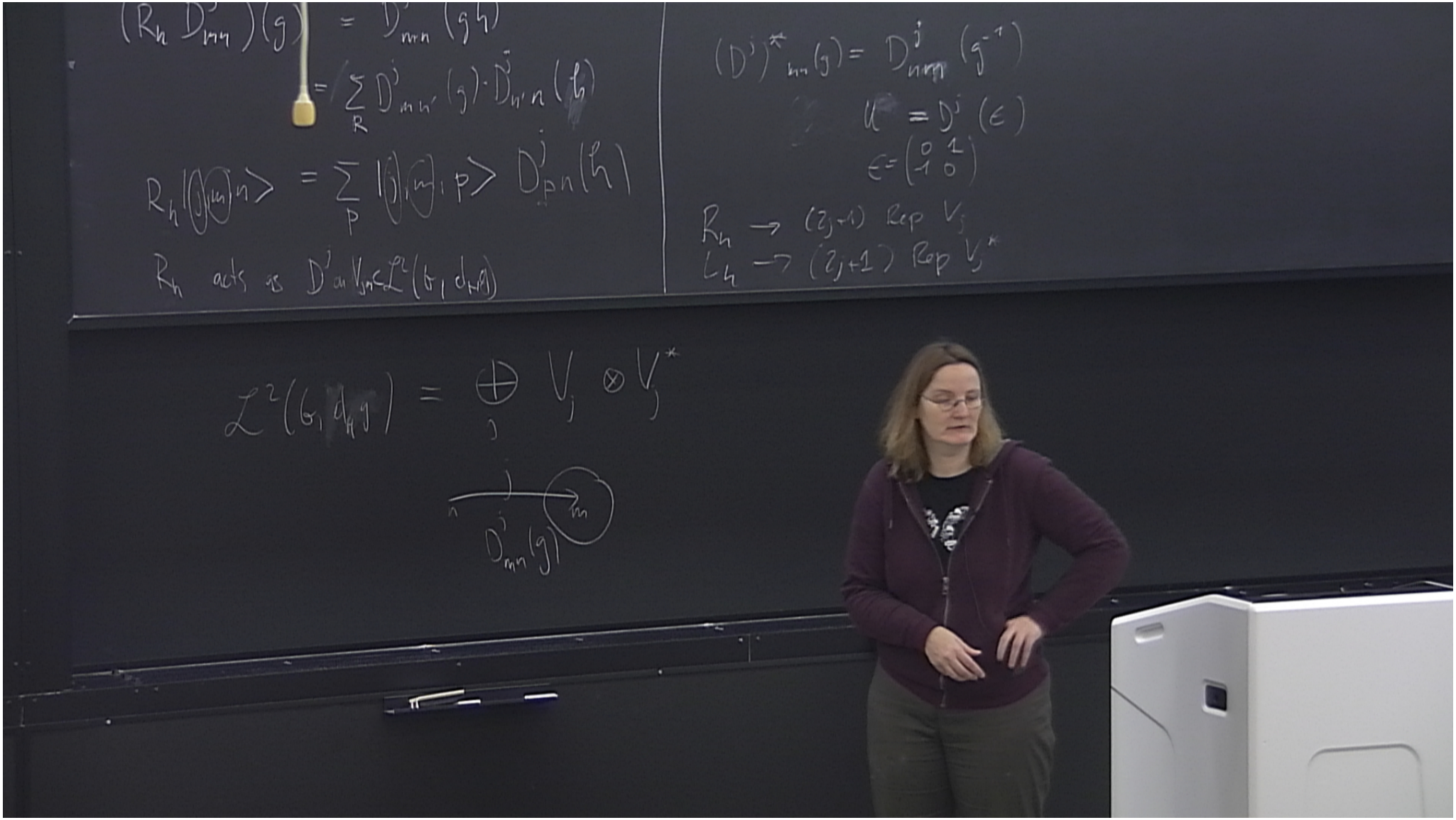
SU(2)-Rep: $j \in \mathbb{N}/2$, $\dim V_j = 2j+1 = d_j$

$$D^j_{mn}(g) = \sqrt{\binom{2j}{j+m}} \sqrt{\binom{2j}{j+n}} \sum_{P(2j)} g_{M_{\sigma(1)} N_1} g_{M_{\sigma(2)} N_2} \cdots g_{M_{\sigma(2j)} N_{2j}}$$

$m, n = -j, -j+1, \ldots, j-1, j$


Exercise :
$\chi_j(g) = \sum_m D^j_{mm}(g)$
$\chi_j(hgh^{-1}) = \chi_j(g)$
$\delta_G(g) = \sum_j d_j \chi_j(g)$
$\int f(g)\,\delta_G(gh^{-1})\,dg = f(h)$


· Left & Right mult.
$(R_h D^j_{mn})(g) = D^j_{mn}(gh)$
$= \sum_R D^j_{mn'}(g) \cdot D^j_{n'n}(h)$
$R_h |(j,m),n\rangle = \sum_p |(j,m),p\rangle D^j_{pn}(h)$
R_h acts as D^j on $V_{jn} \subset L^2(G, d_{HAR})$


$(R_h D^j_{mn})(g) = D^j_{mn}(gh)$
$= \sum_R D^j_{mm'}(g) \cdot D^j_{m'n}(h)$
$R_h |j,m,n\rangle = \sum_p |j,m,p\rangle D^j_{pn}(h)$
R_h acts as D^j on $V_{jn} \subset L^2(G, d_H g)$
$(D^j)^*_{mn}(g) = D^j_{nm}(g^{-1})$
$U = D^j(\epsilon)$
$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$R_h \to (2j+1)$ Rep V_j
$L_h \to (2j+1)$ Rep V_j^*
$L^2(G, d_H g) = \bigoplus_j V_j \otimes V_j^*$
$D^j_{mn}(g)$