Title: Quantum Gravity Review-7

Date: Feb 25, 2015 10:15 AM

URL: http://www.pirsa.org/15020058

Abstract:

Quantum Geometry

Find a "quantization" of (kinematical) phase space.

$A_a^k(x)$, $E_j^a(x)$ $\qquad \{A(x), E(y)\} = \dots \delta(x,y)$

$\downarrow$

$\hat{A}$, $\hat{E}$ $\qquad \{\cdot,\cdot\} \longmapsto [\cdot,\cdot]$

$\mathcal{H}_{kin}$, operators

Quantum Geometry

Find a "quantization" of (kinematical) phase space.

$$A^k_a(x),\ E^a_j(x) \qquad \{A(x), E(y)\} = \dots\ \delta(x,y)$$

$$\downarrow \qquad \text{"Algebra Homom."}$$

$$\hat{A},\ \hat{E} \qquad \{\cdot,\cdot\} \longmapsto \frac{1}{i\hbar}[\cdot,\cdot]$$

- $\mathcal{H}_{kin}$, operators
- select a (sub) algebra of basic ph

geometry

on" of (kinematical) phase space

$\{ A(x), E(y) \} = \dots \delta(x,y)$

"Algebra Homom."

$$\{\cdot,\cdot\} \longmapsto \frac{1}{i\hbar}[\cdot,\cdot]$$

- select a (sub) algebra of basic phase space function

- (quantum) field theory

$\hat{\phi}(x)$, $\hat{\pi}(y)$ are <u>not</u> operators

(operator valued distributions)

$\{ \phi(x), \pi(y) \} = \delta(x,y)$

$\int f(y)\delta(x,y)\,dy = f(x)$

- smear phase space fcts ⇝ (operator valued distr.)

$\{ \phi[f], \pi[f'] \} = \int f f' \, d^n x$

$\phi(f) = \int \phi f \, d^n x$

- (quantum) field theory

$\hat{\phi}(x)$, $\hat{\pi}(y)$ are <u>not</u> operators

(operator valued distributions)

$\{\phi(x), \pi(y)\} = \delta(x,y)$

$\int f(y)\delta(x,y)\,dy = f(x)$

- smear phase space fcts $\rightsquigarrow$ (operator valued distr.)

$\{\phi[f], \pi[f']\} = \int \sqrt{g} f f' \, d^n x$

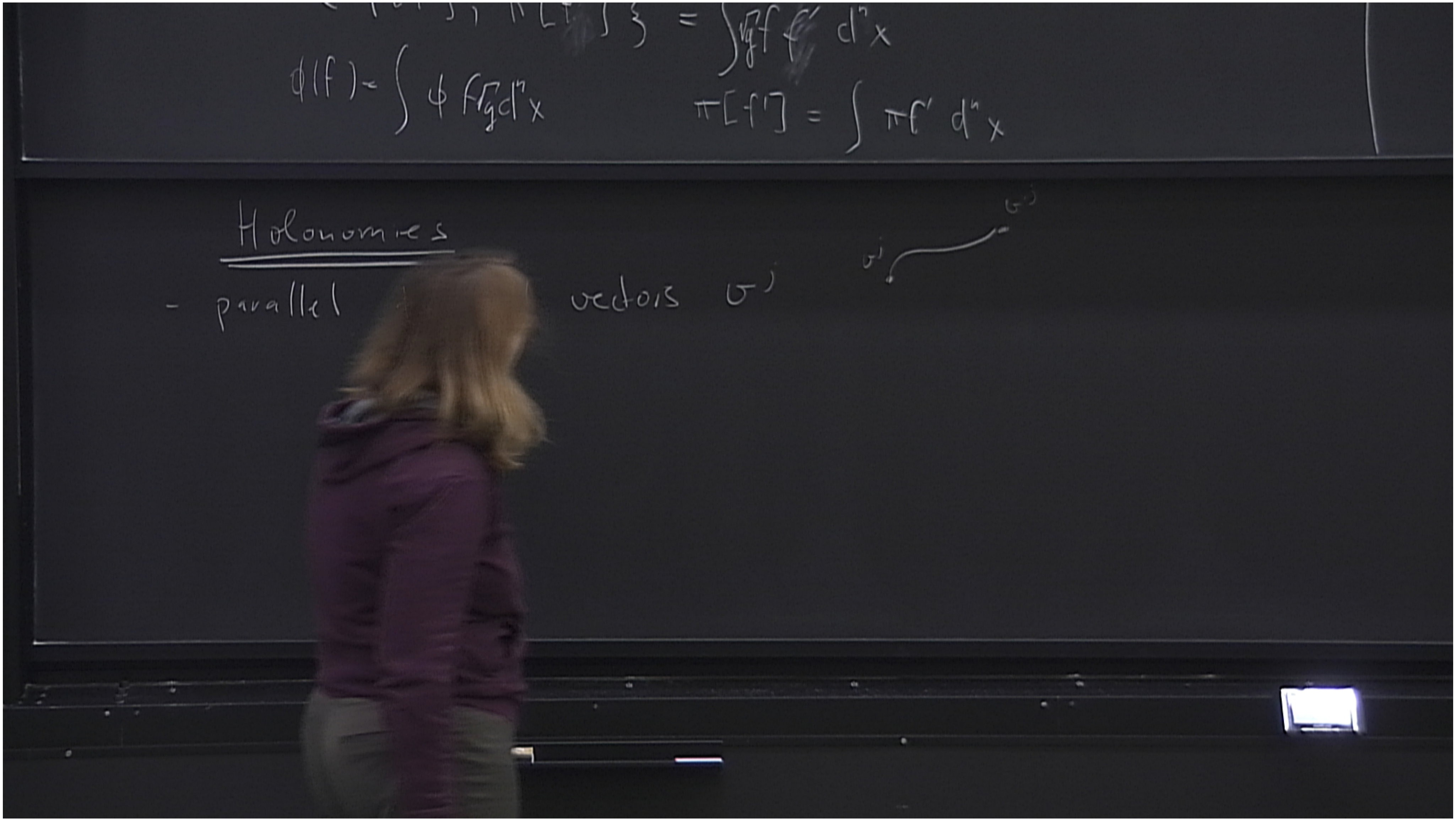
$\phi(f) = \int \phi f \sqrt{g} \, d^n x$ $\qquad$ $\pi[f'] = \int \pi f' \, d^n x$

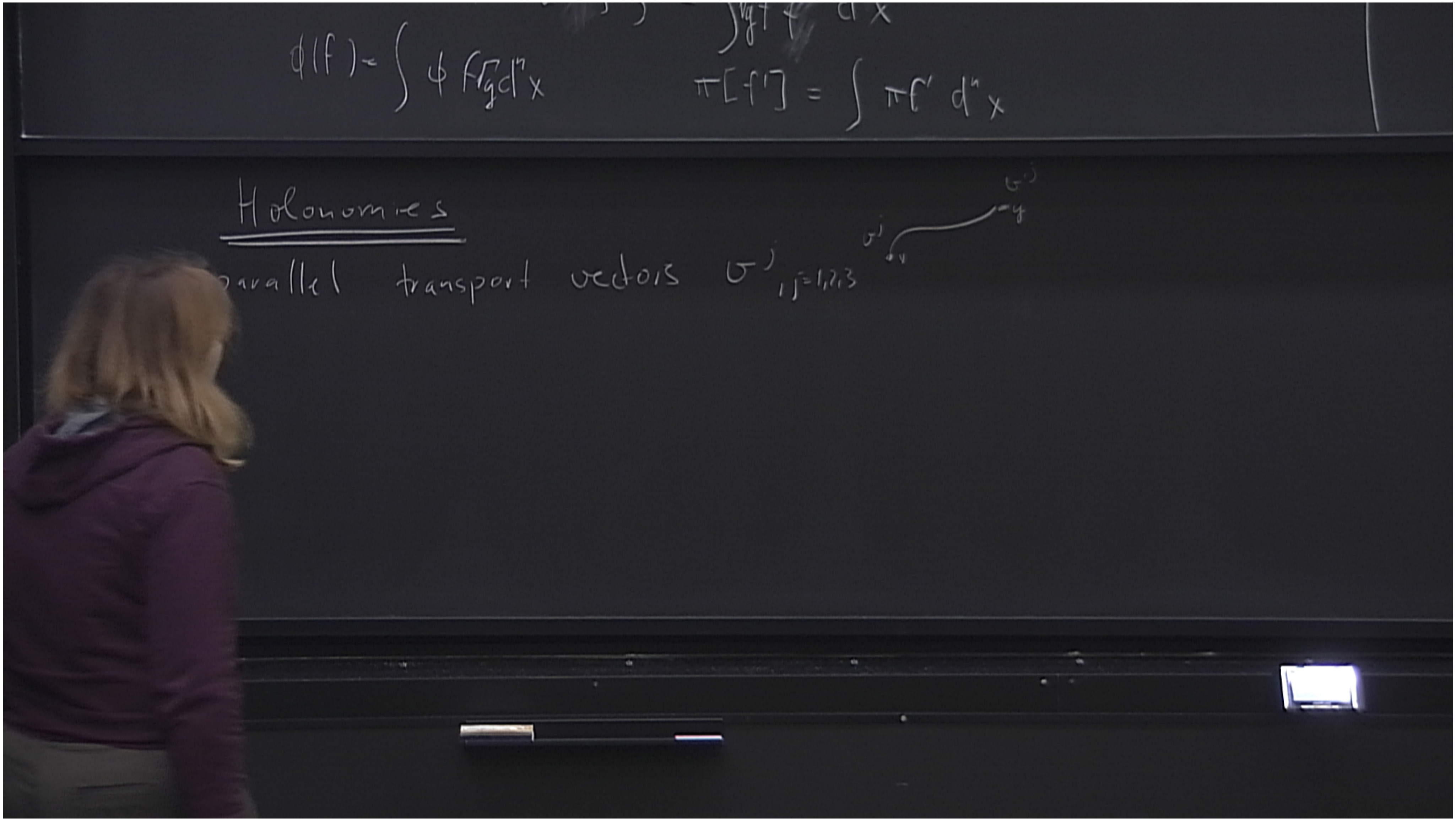
- nice transformation property wrt gauge symmetries of the system

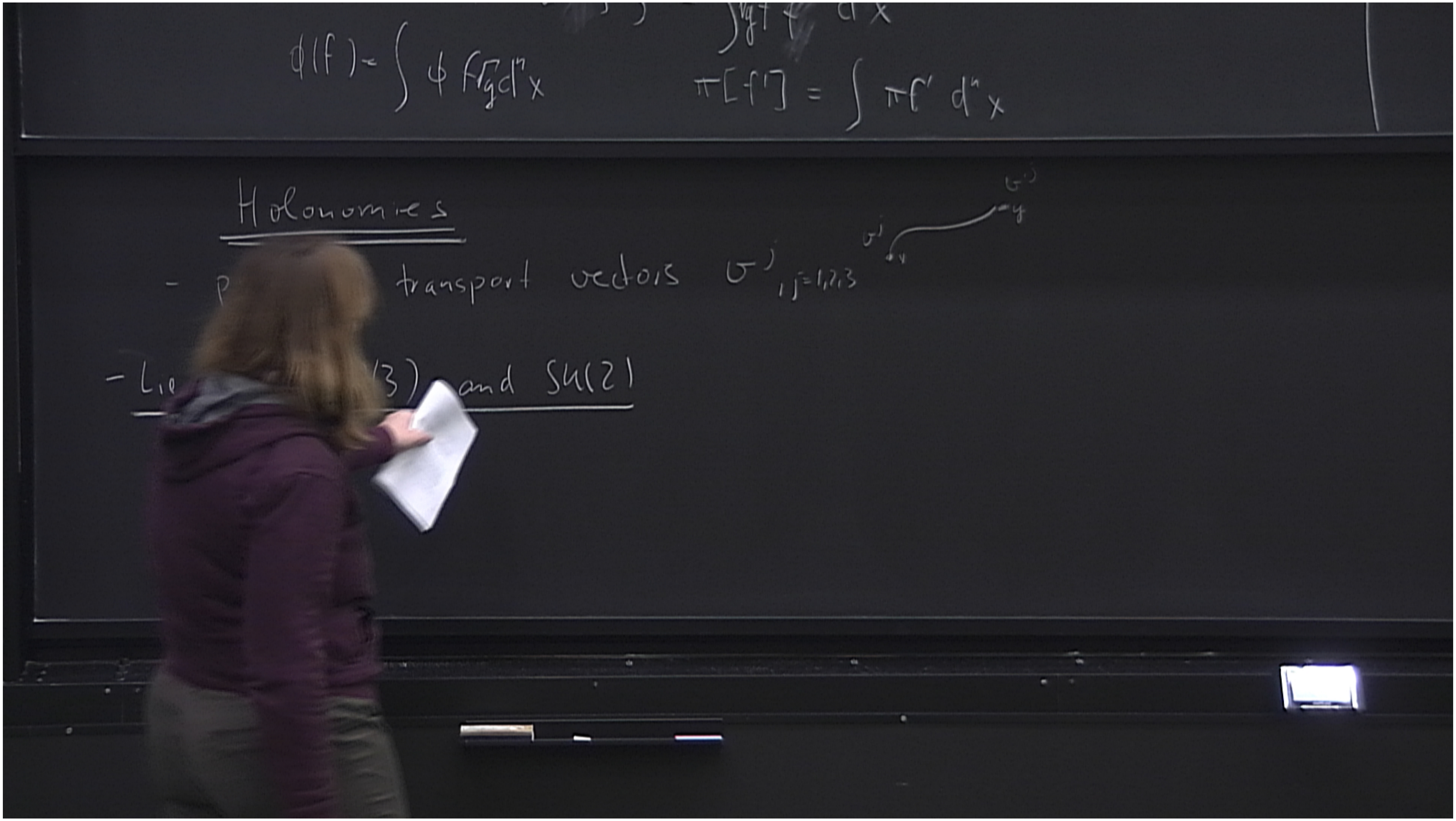
— connections one-forms → integrate over 1-dim objects

— fluxes → $(d-1)$ → ...

$$\{\text{hol}, \text{flux}\} = \begin{matrix} 1 \\ 0 \end{matrix}$$


$\phi(f) = \int \phi f \sqrt{g}\, d^n x$
$\pi[f'] = \int \pi f'\, d^n x$
Holonomies
- parallel
vectors v^j


ϕ(f) = ∫ ϕ f √g dⁿx
π[f'] = ∫ π f' dⁿx
Holonomies
parallel transport vectors v^j , j=1,2,3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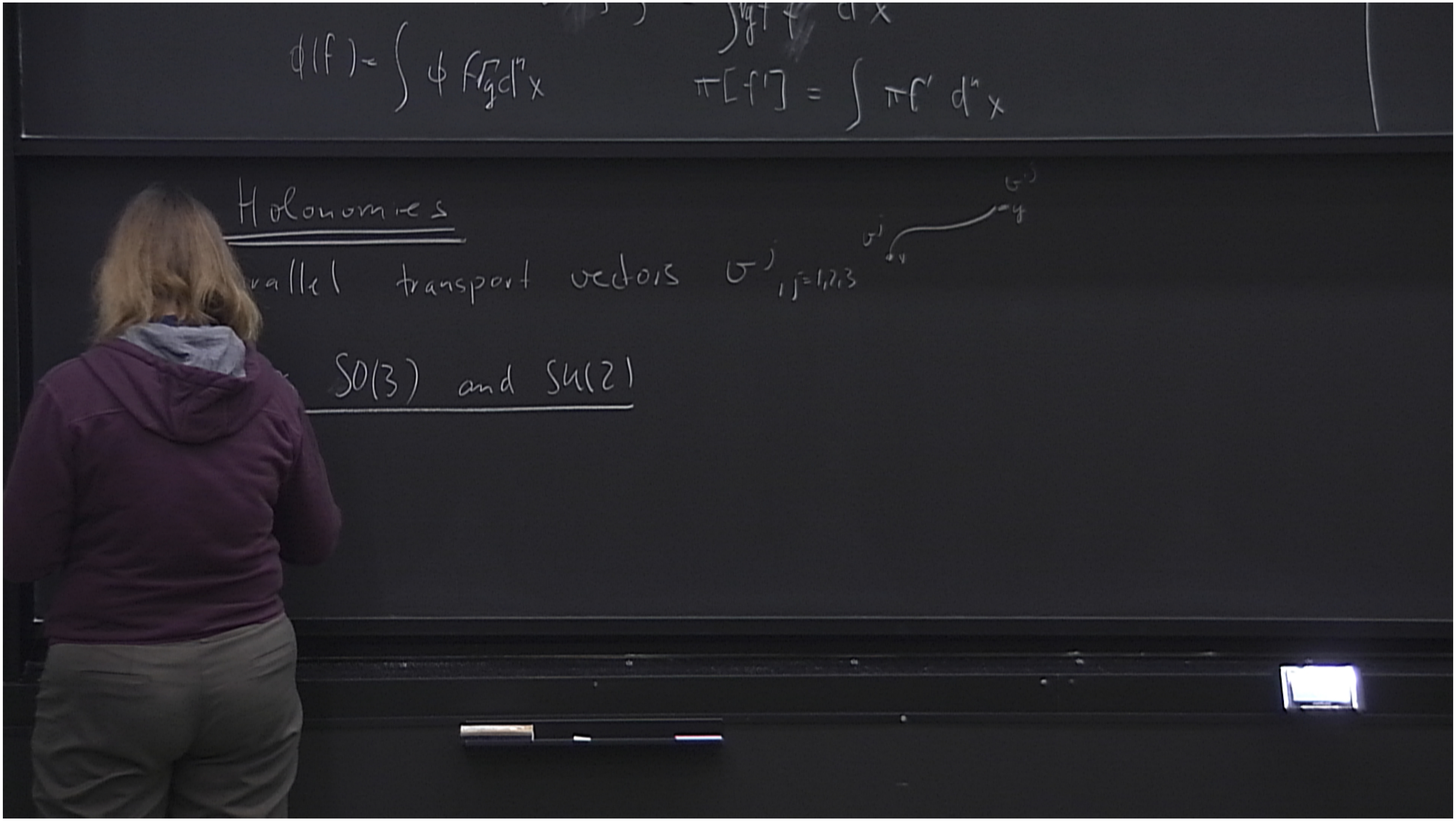
Holonomies
transport vectors v^j, $j = 1,2,3$
and SU(2)


$\phi(f) = \int \phi f \sqrt{g} d^n x$
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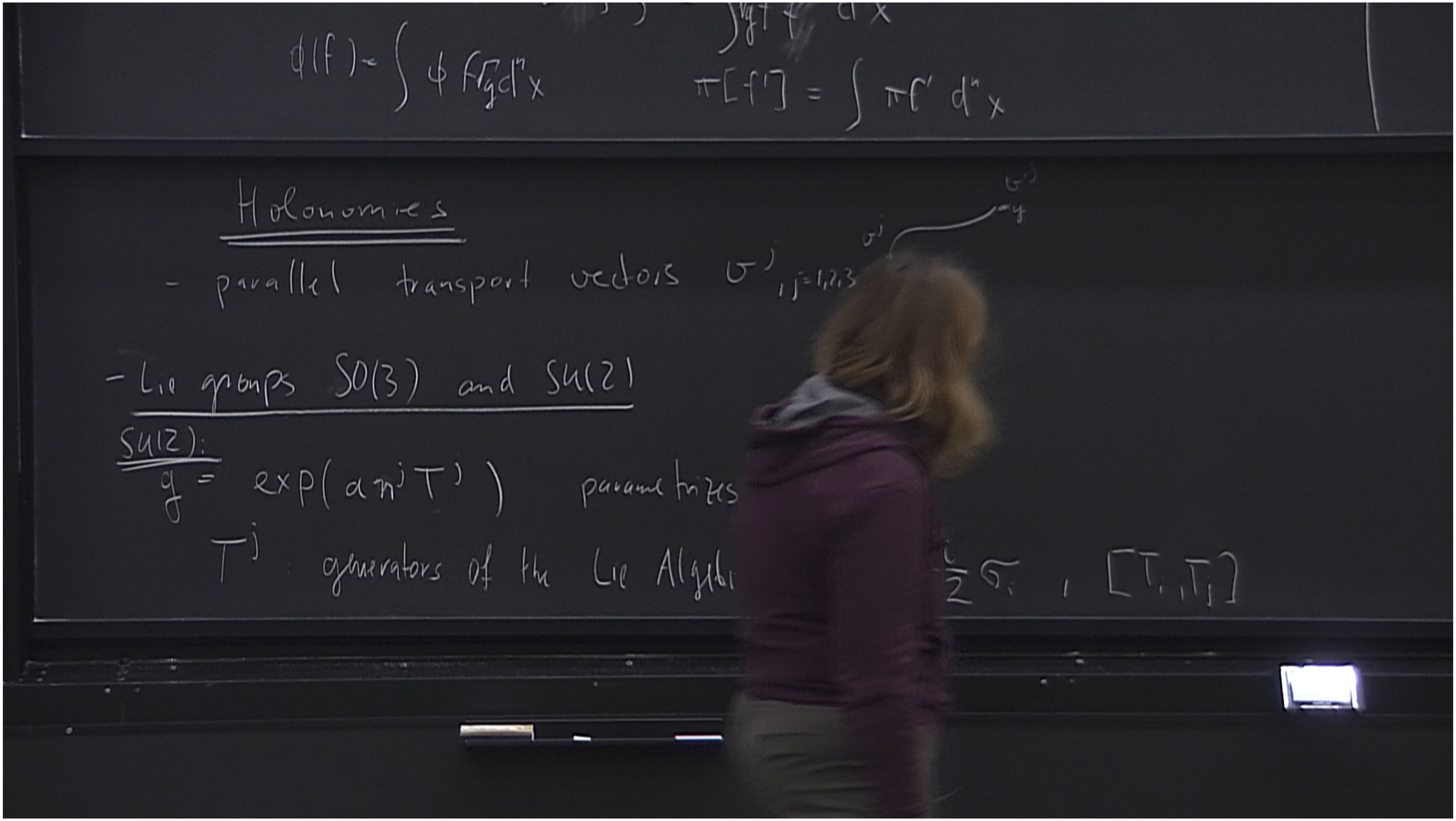
Holonomies
rallel transport vectors v^j, $j=1,2,3$
SO(3) and SU(2)


ϕ(f) = ∫ ϕ f √g dⁿx
π[f'] = ∫ π f' dⁿx
Holonomies
- parallel transport vectors σ^j, j=1,2,3
- Lie groups SO(3) and SU(2)
SU(2):
g = exp(a n^j T^j) parametrizes
T^j : generators of the Lie Algebr
σ_i , [T_i, T_j]

$$\phi(f) = \int \phi f \sqrt{q}\, d^n x \qquad \pi[f'] = \int \pi f' \, d^n x$$

Holonomies

- parallel transport vectors v^j, $j = 1,2,3$

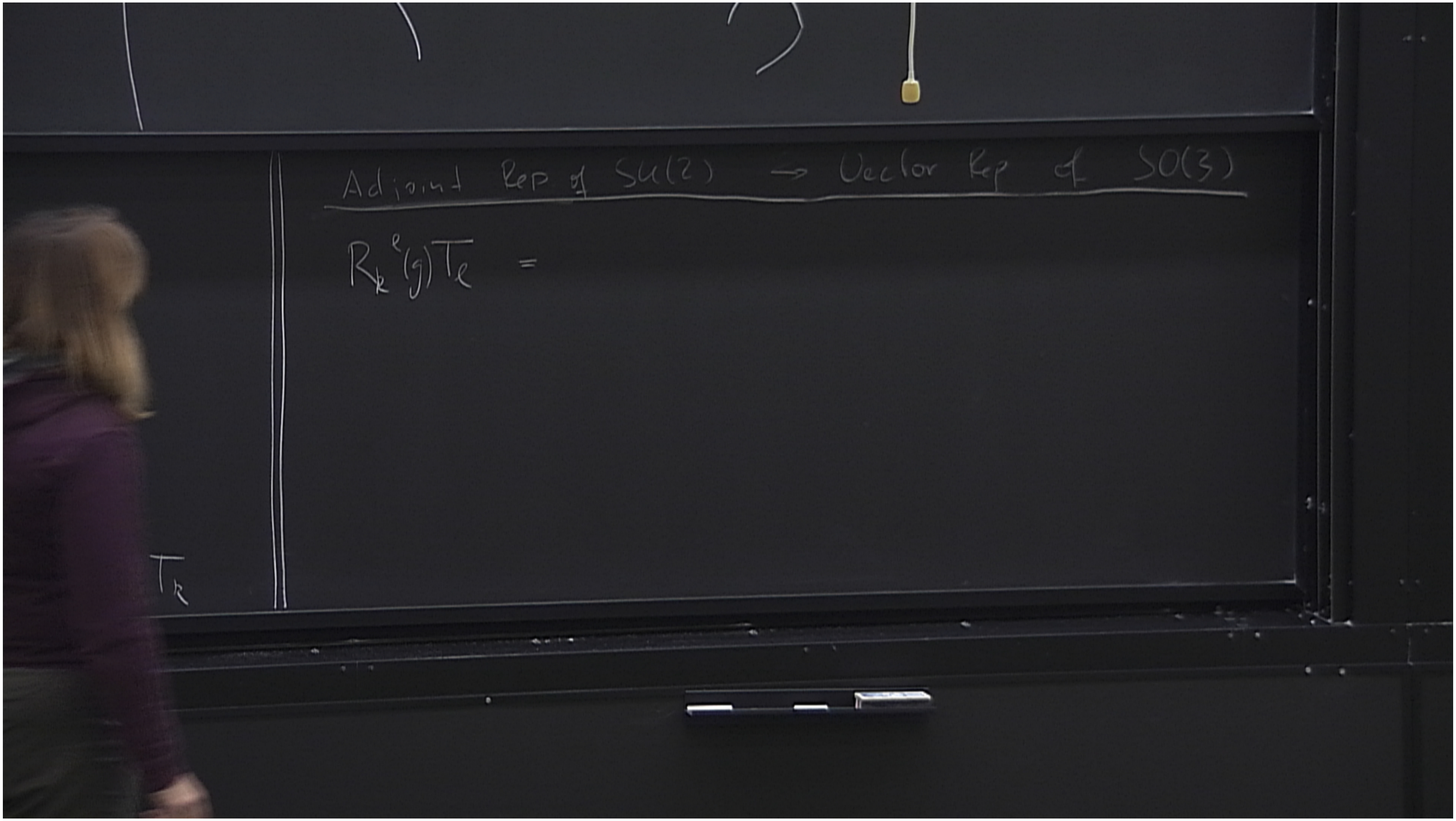
v^j x $\qquad$ v^j y

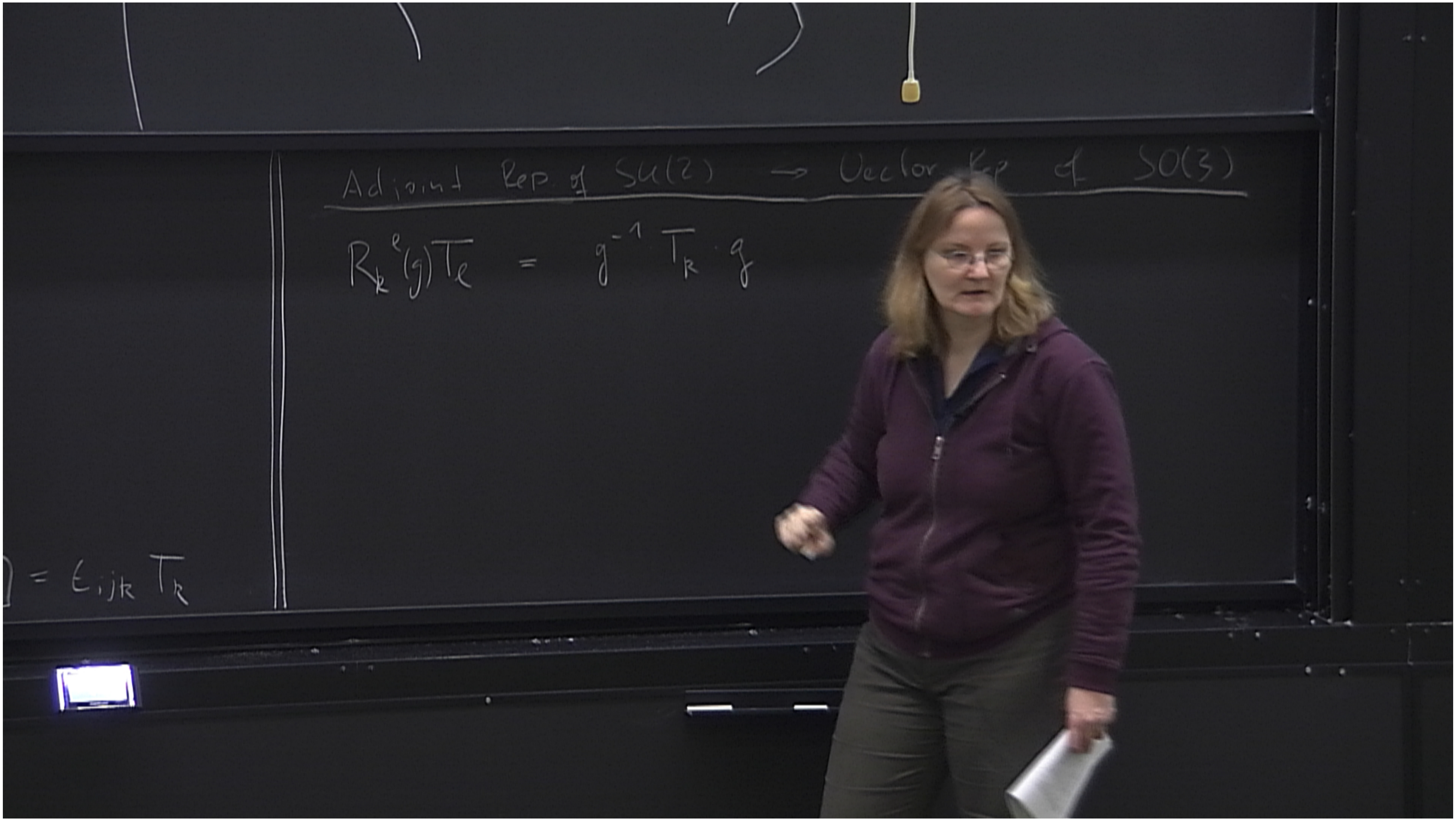
- Lie groups $SO(3)$ and $SU(2)$

SU(2):

$g = \exp(a n^j T^j)$ parametrizes $SU(2)$

T^j : generators of the Lie Algebra, $T_i = -\frac{i}{2}\sigma_i$, $[T_i, T_j] = \epsilon_{ijk} T_k$

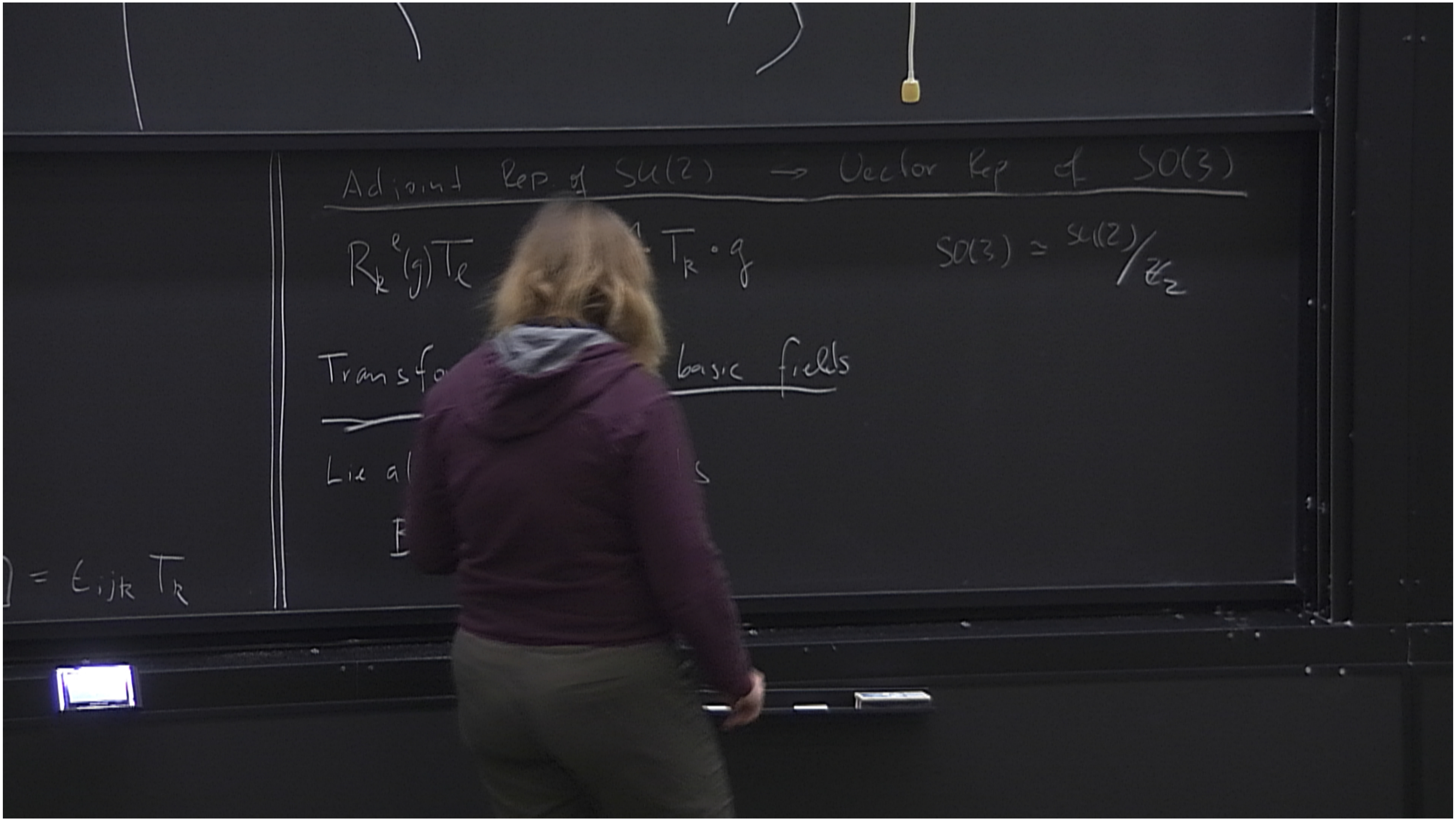

Adjoint Rep of SU(2) → Vector Rep of SO(3)
R_k^l(g) T_l =
T_k


Adjoint Rep. of SU(2) → Vector Rep of SO(3)
$R_k{}^l(g)\, T_l = g^{-1} \cdot T_k \cdot g$
$= \epsilon_{ijk} T_k$

Adjoint Rep of $SU(2)$ $\rightarrow$ Vector Rep of $SO(3)$

$R_k{}^{\ell}(g)\, T_\ell \;=\; g^{-1} \cdot T_k \cdot g$ $\qquad SO(3) \simeq SU(2)/\mathbb{Z}_2$

$] = \epsilon_{ijk} T_k$


Adjoint Rep of SU(2) → Vector Rep of SO(3)
SO(3) ≃ SU(2)/Z2
basic fields
= εijk Tk

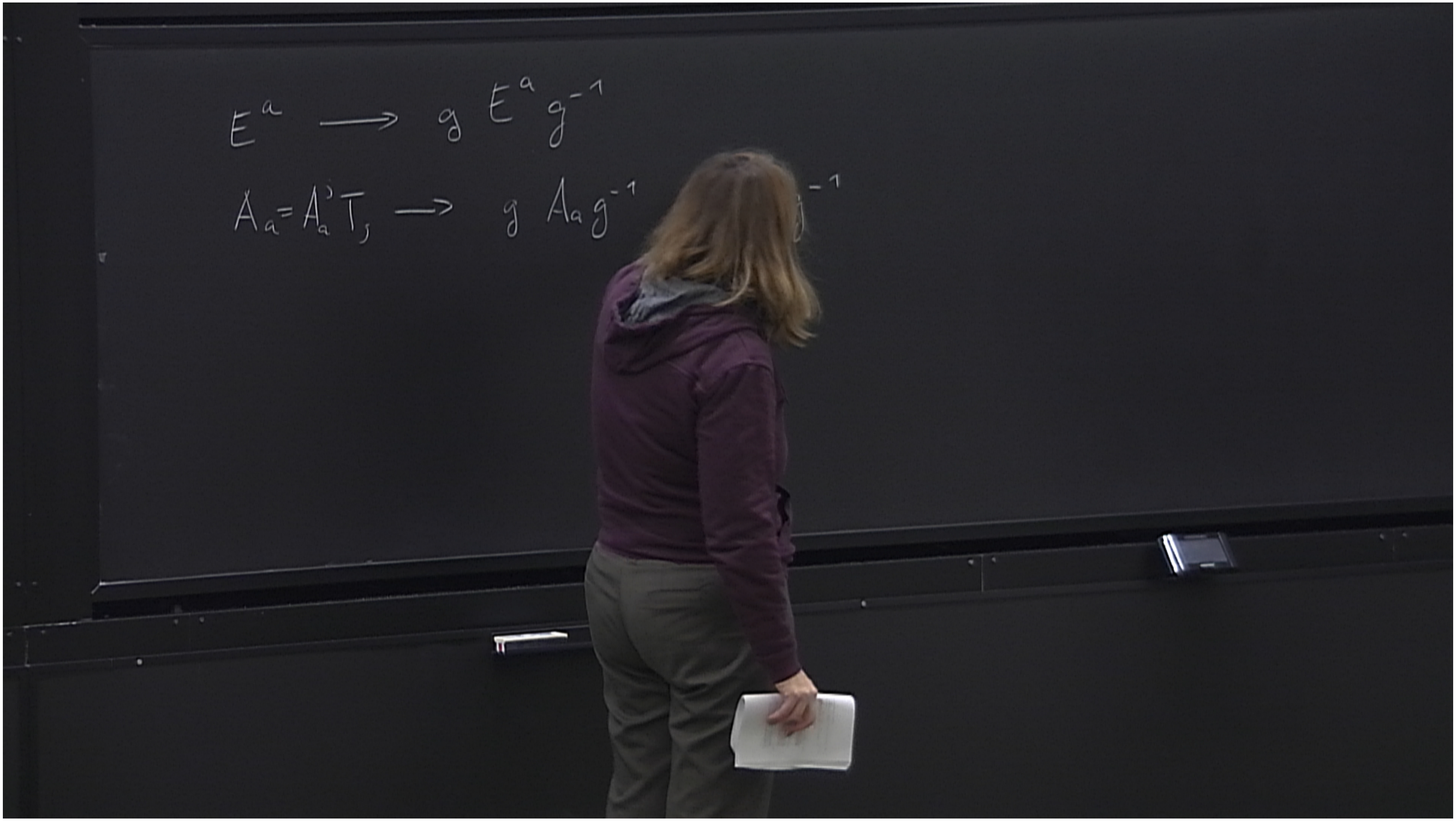
Adjoint Rep of SU(2) $\rightarrow$ Vector Rep of SO(3)

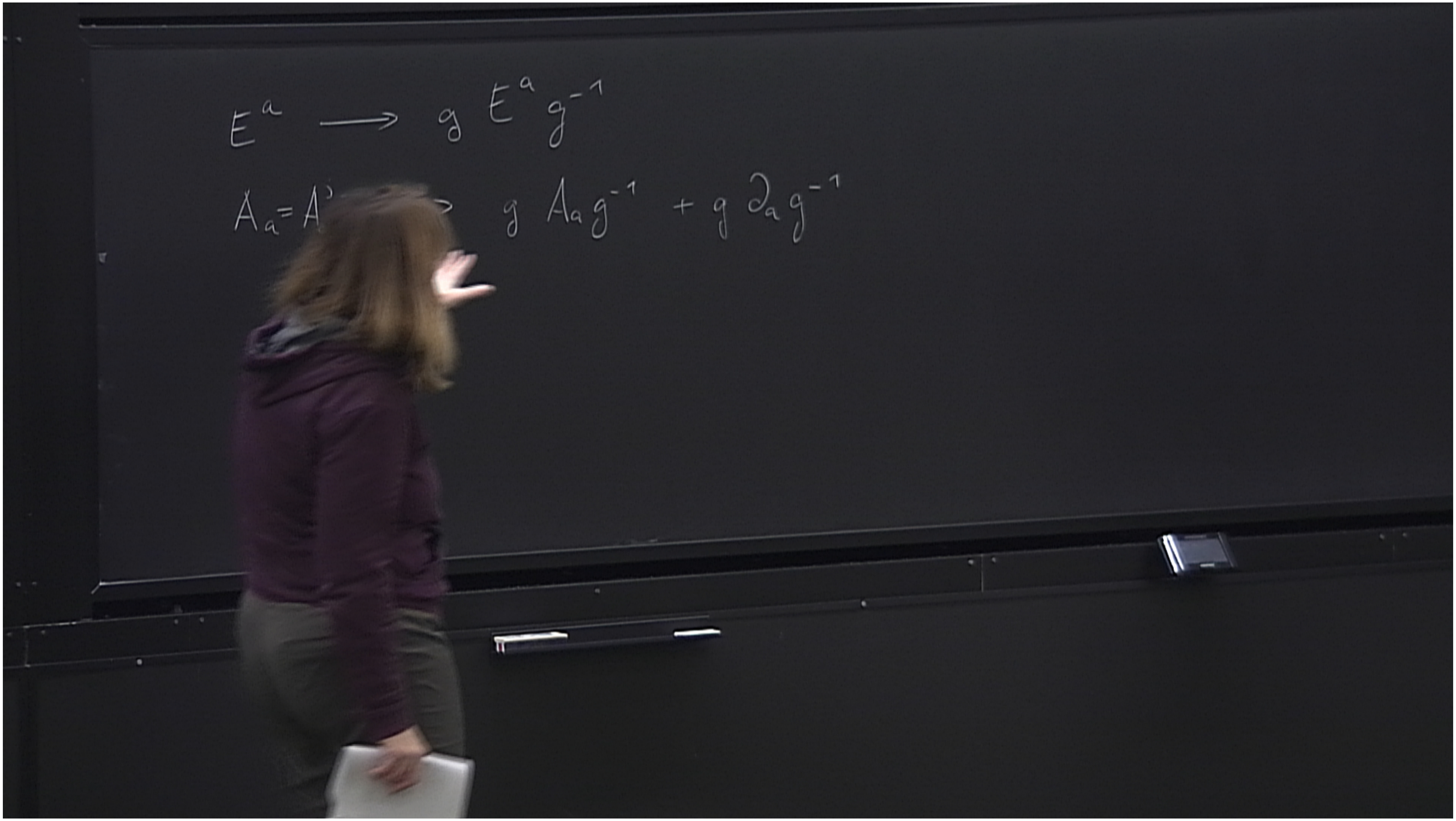
$$R_k{}^l(g)\,T_l = g^{-1}\cdot T_k \cdot g \qquad SO(3) = SU(2)/\mathbb{Z}_2$$

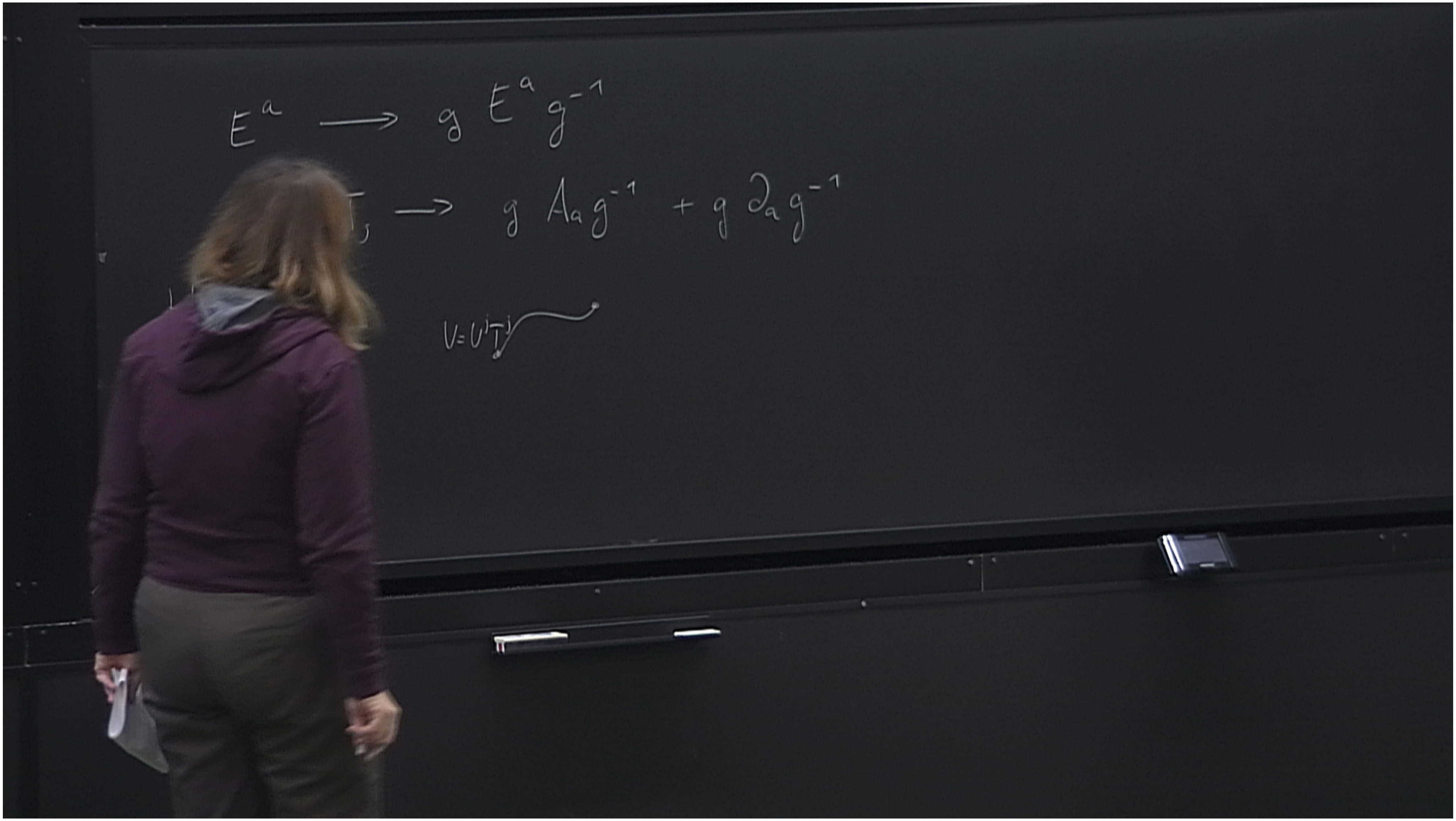
Transformation of basic fields

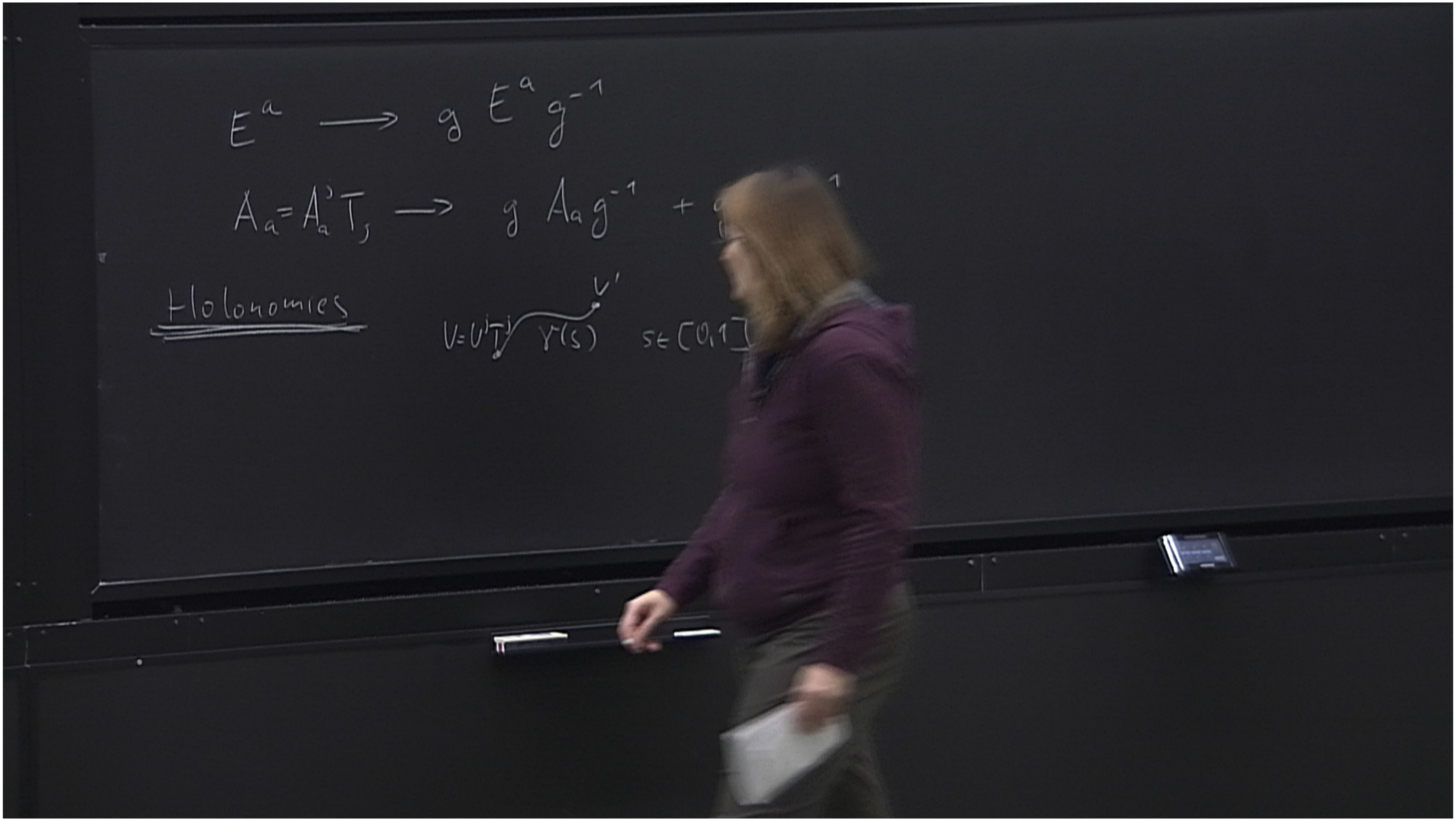
Lie algebra valued objects

$$B = B^j T_j \quad ; \quad E^a = E^{aj}\, T_j$$


$E^a \longrightarrow g E^a g^{-1}$
$A_a = A_a^j T_j \longrightarrow g A_a g^{-1}$


$E^a \longrightarrow g E^a g^{-1}$
$g A_a g^{-1} + g \partial_a g^{-1}$


$E^a \longrightarrow g\, E^a g^{-1}$
$\longrightarrow g\, A_a g^{-1} + g\, \partial_a g^{-1}$


$E^a \longrightarrow g E^a g^{-1}$
$A_a = A_a^j T_j \longrightarrow g A_a g^{-1} +$
Holonomies
V'
$V = V^j T_j$ $\gamma(s)$ $s \in [0,1]$

$$E^a \longrightarrow g\, E^a g^{-1}$$

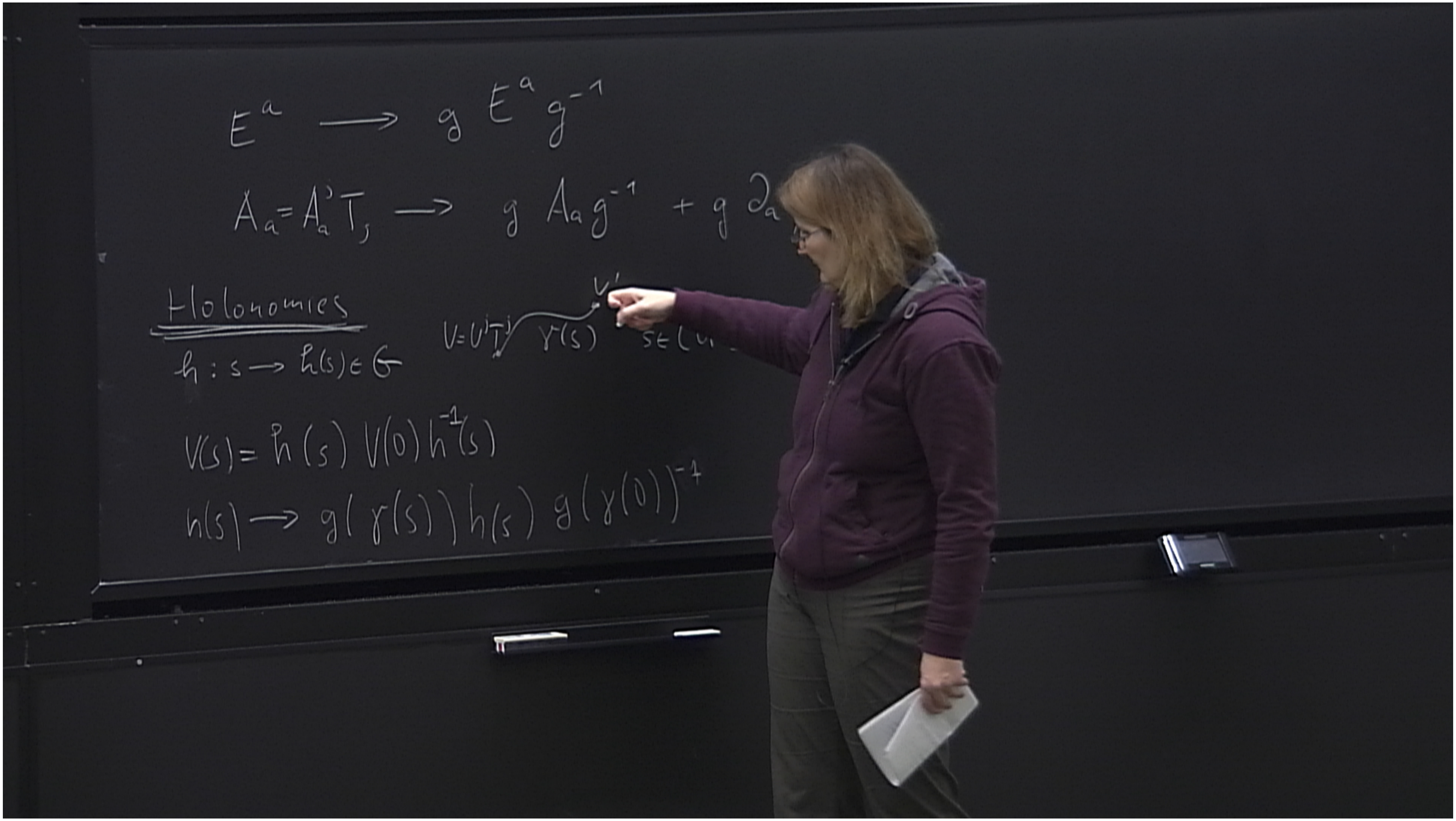
$$A_a = A_a^i T_i \longrightarrow g\, A_a g^{-1} + g\, \partial_a g^{-1}$$

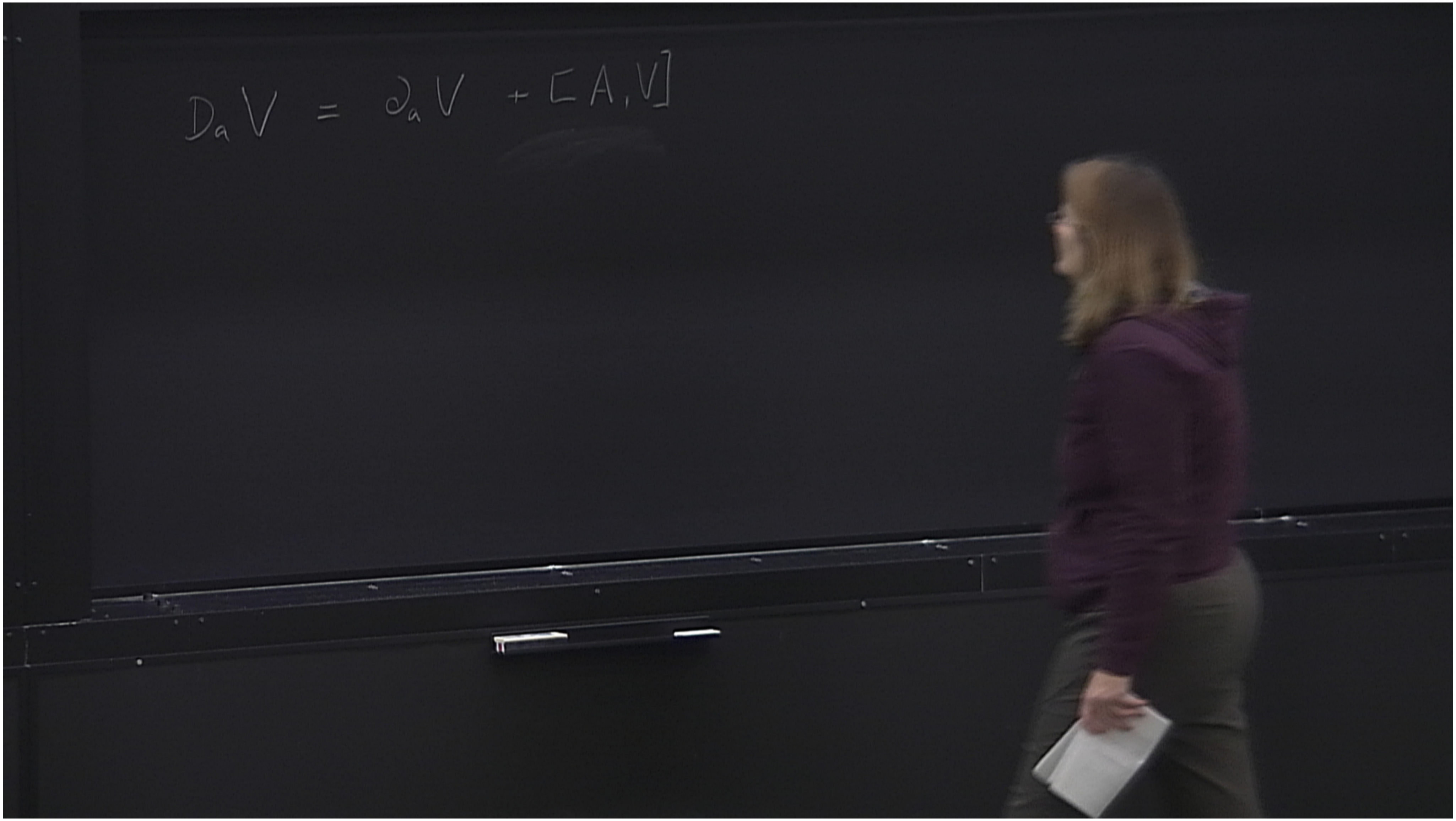
<u>Holonomies</u>

$h : s \longrightarrow h(s) \in G$

$v = v^i T_i$ $\quad \gamma(s) \quad$ v' $\quad s \in [0,1]$

$$V(s) = h(s)\, V(0)\, h^{-1}(s)$$


E^a ⟶ g E^a g^-1
A_a = A_a^j T_j ⟶ g A_a g^-1 + g ∂_a
Holonomies
h : s ⟶ h(s) ∈ G
V = V^j T_j
γ(s)
V'
s ∈
V(s) = h(s) V(0) h^-1(s)
h(s) ⟶ g(γ(s)) h(s) g(γ(0))^-1


$D_a V = \partial_a V + [A, V]$

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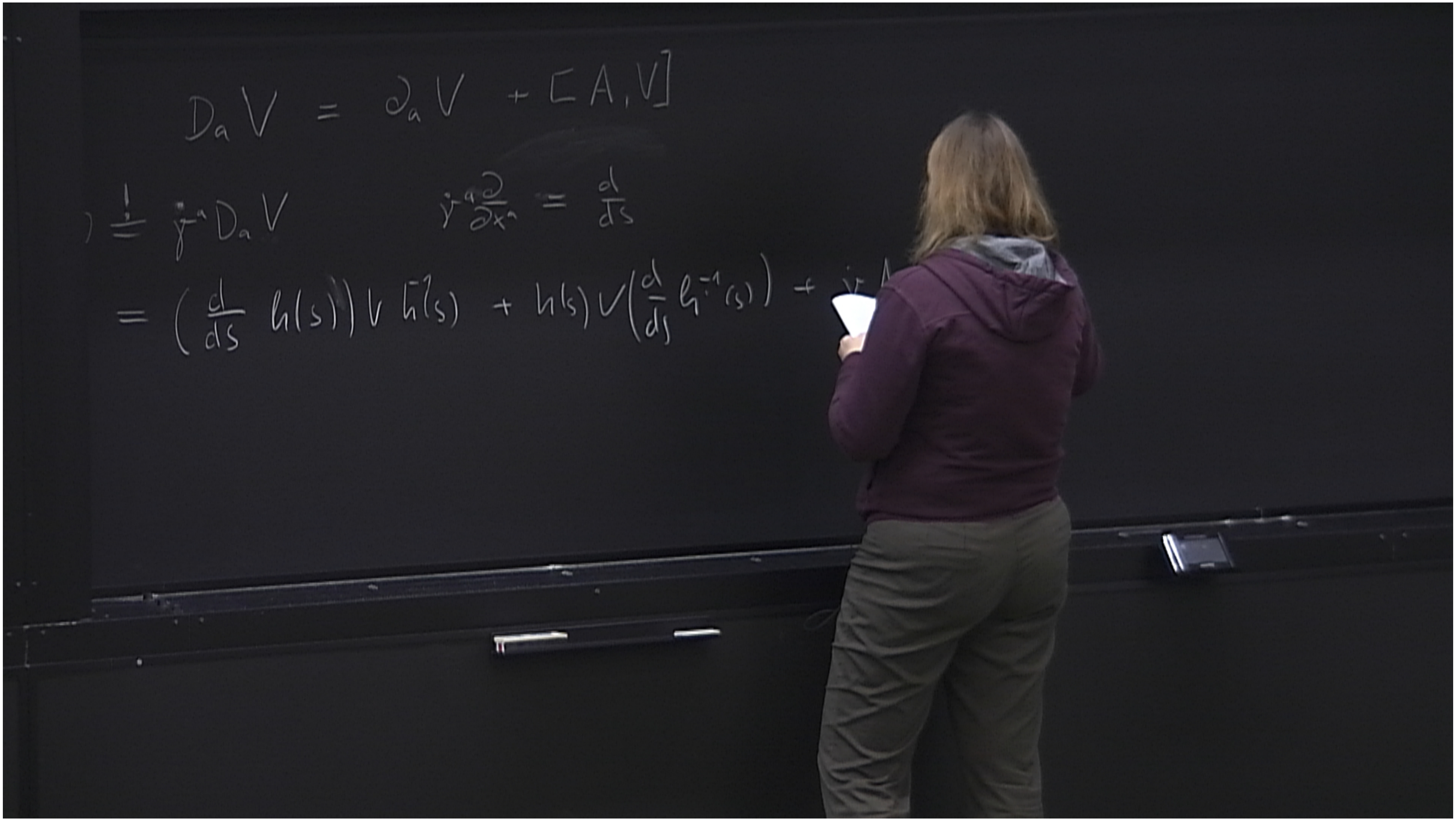
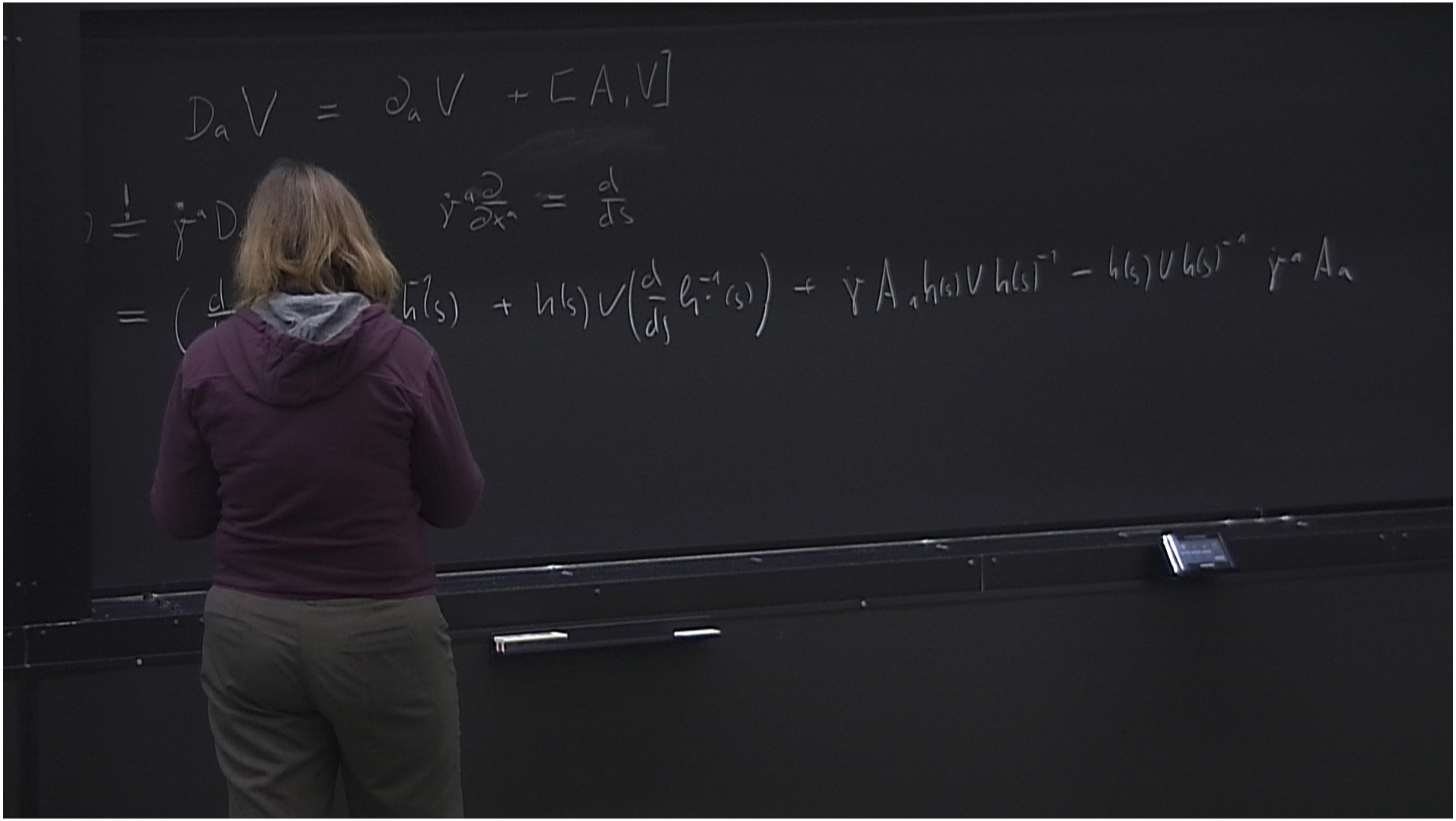
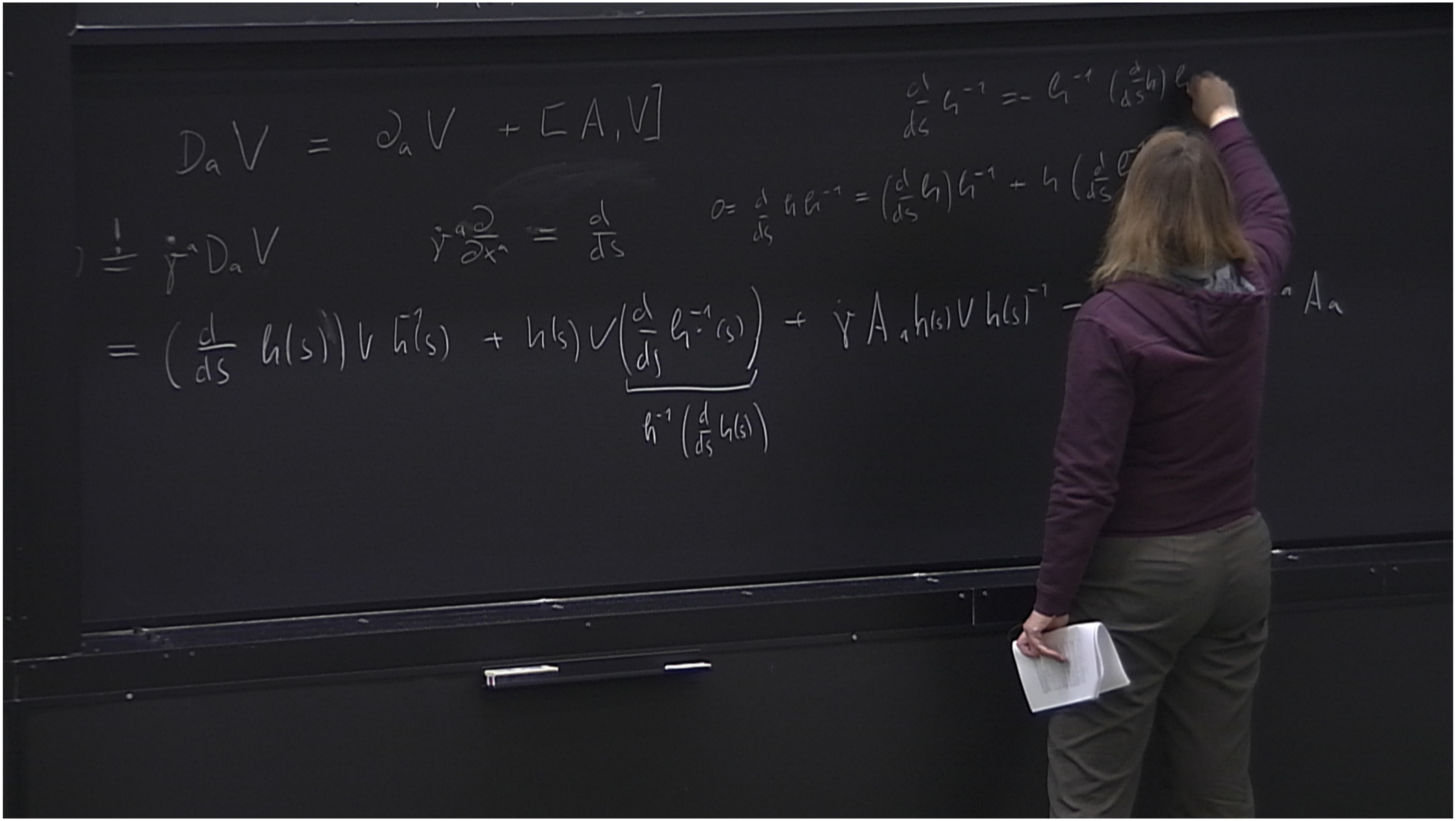
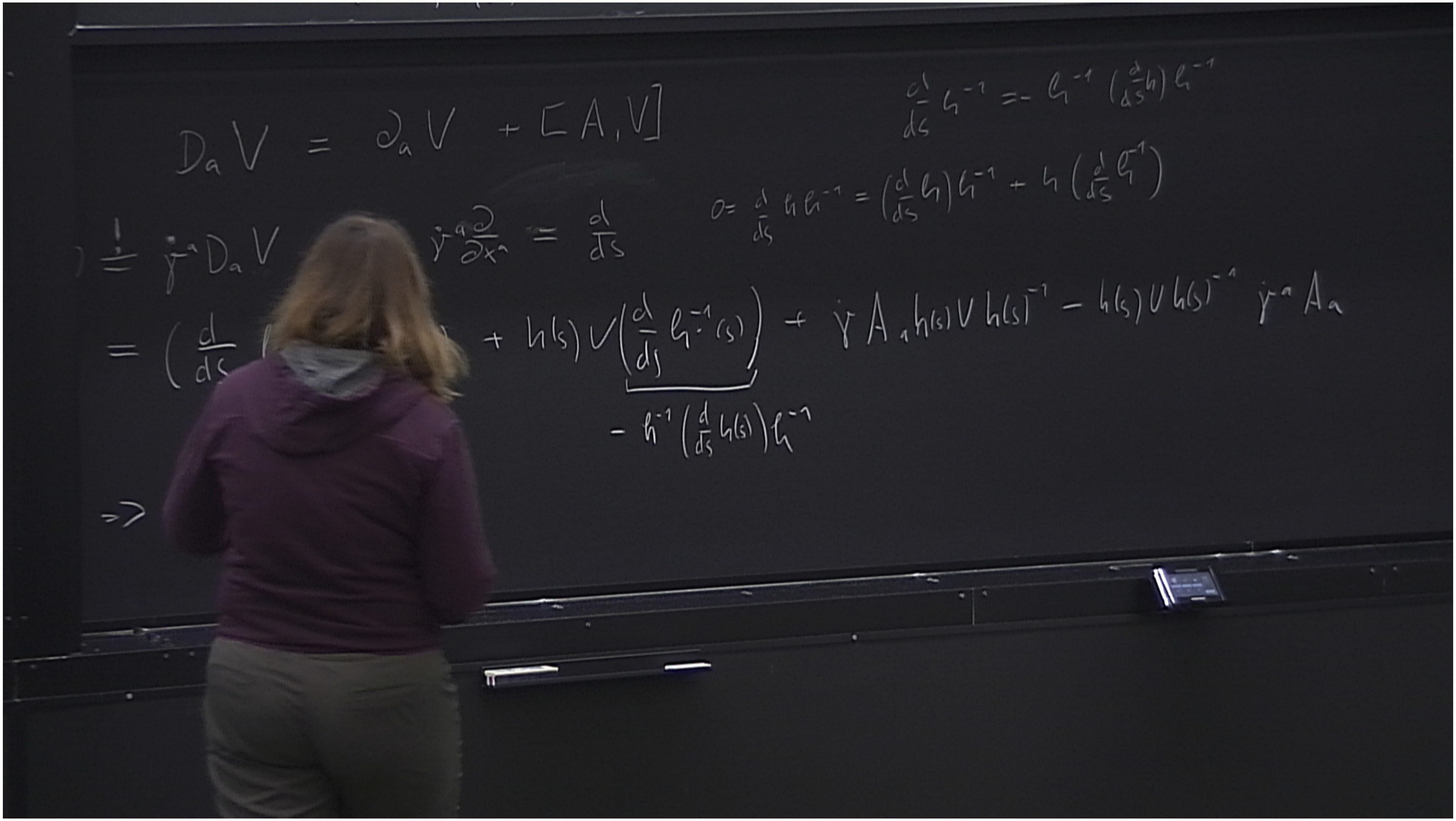
$$\frac{1}{=} \gamma^a D_a V$$



$D_a V = \partial_a V + [A, V]$
$\dot{y}^a D_a V$
$\dot{y}^a \frac{\partial}{\partial x^a} = \frac{d}{ds}$
$= \left(\frac{d}{ds} h(s)\right) V h^{-1}(s) + h(s) V \left(\frac{d}{ds} h^{-1}(s)\right) +$


DaV = ∂aV + [A,V]


$D_a V = \partial_a V + [A, V]$
$\frac{d}{ds} h^{-1} = - h^{-1} \left(\frac{d}{ds} h\right) h$
$0 = \frac{d}{ds} h h^{-1} = \left(\frac{d}{ds} h\right) h^{-1} + h \left(\frac{d}{ds}\right.$
$\overset{!}{=} \dot\gamma^a D_a V$
$\dot\gamma^a \frac{\partial}{\partial x^a} = \frac{d}{ds}$
$= \left(\frac{d}{ds} h(s)\right) V h^{-1}(s) + h(s) V \underbrace{\left(\frac{d}{ds} h^{-1}(s)\right)}_{h^{-1}\left(\frac{d}{ds} h(s)\right)} + \dot\gamma A_a h(s) V h(s)^{-1}$
A_a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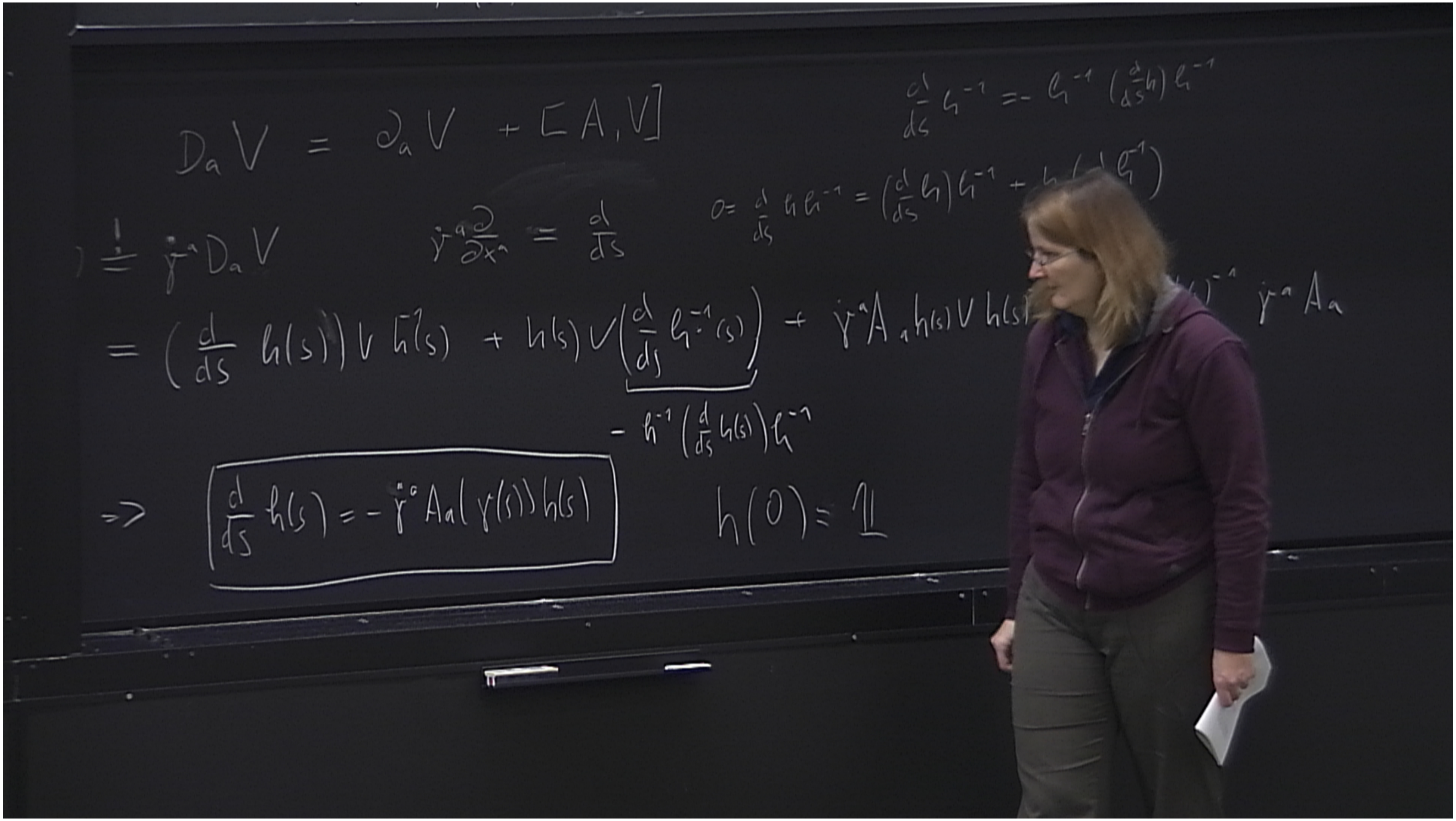
$0 = \frac{d}{ds} h h^{-1} = \left(\frac{d}{ds} h\right) h^{-1} + h \left(\frac{d}{ds} h^{-1}\right)$
$\dot{y}^a \frac{\partial}{\partial x^a} = \frac{d}{ds}$
$\stackrel{!}{=} \dot{y}^a D_a V$
$= \left(\frac{d}{ds} \right. \ldots + h(s) V \underbrace{\left(\frac{d}{ds} h^{-1}(s)\right)}_{- h^{-1}\left(\frac{d}{ds} h(s)\right) h^{-1}} + \dot{y} A_a h(s) V h(s)^{-1} - h(s) V h(s)^{-1} \dot{y}^a A_a$
$\Rightarrow$



$D_a V = \partial_a V + [A, V]$
$\frac{d}{ds} h^{-1} = - h^{-1} \left(\frac{d}{ds} h\right) h^{-1}$
$0 = \frac{d}{ds} h h^{-1} = \left(\frac{d}{ds} h\right) h^{-1} + h \left(\frac{d}{ds} h^{-1}\right)$
$\dot{\gamma}^a \frac{\partial}{\partial x^a} = \frac{d}{ds}$
$\overset{!}{=} \dot{\gamma}^a D_a V$
$= \left(\frac{d}{ds} h(s)\right) V h^{-1}(s) + h(s) V \underbrace{\left(\frac{d}{ds} h^{-1}(s)\right)}_{- h^{-1} \left(\frac{d}{ds} h(s)\right) h^{-1}} + \dot{\gamma}^a A_a h(s) V h(s)^{-1} \quad \dot{\gamma}^a A_a$
$\Rightarrow \boxed{\frac{d}{ds} h(s) = - \dot{\gamma}^a A_a(\gamma(s)) h(s)} \quad h(0) = \mathbb{1}$

$$\frac{d}{ds} h^{-1} = - h^{-1} \left(\frac{d}{ds} h\right) h^{-1}$$

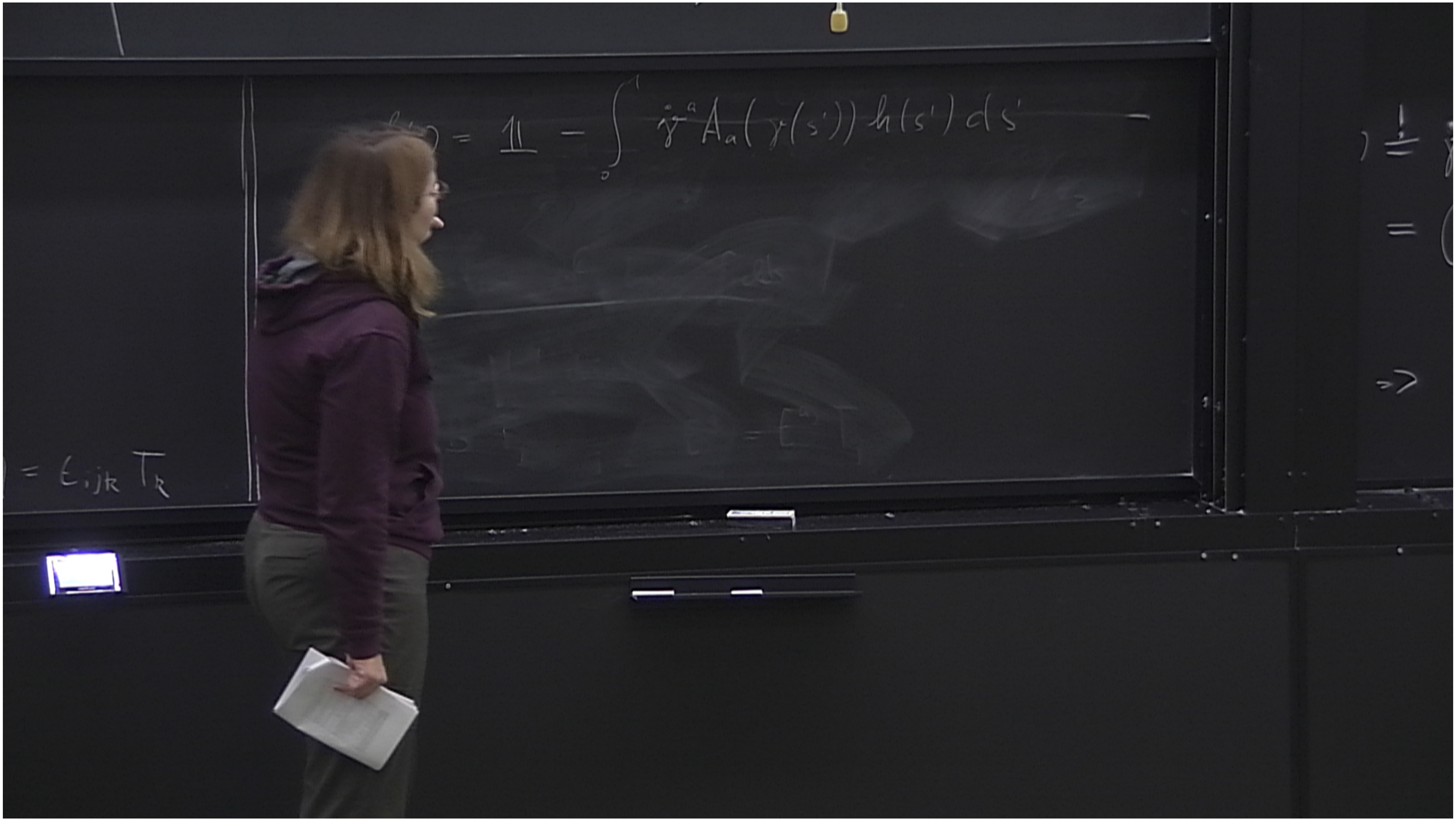
$$D_a V = \partial_a V + [A, V]$$

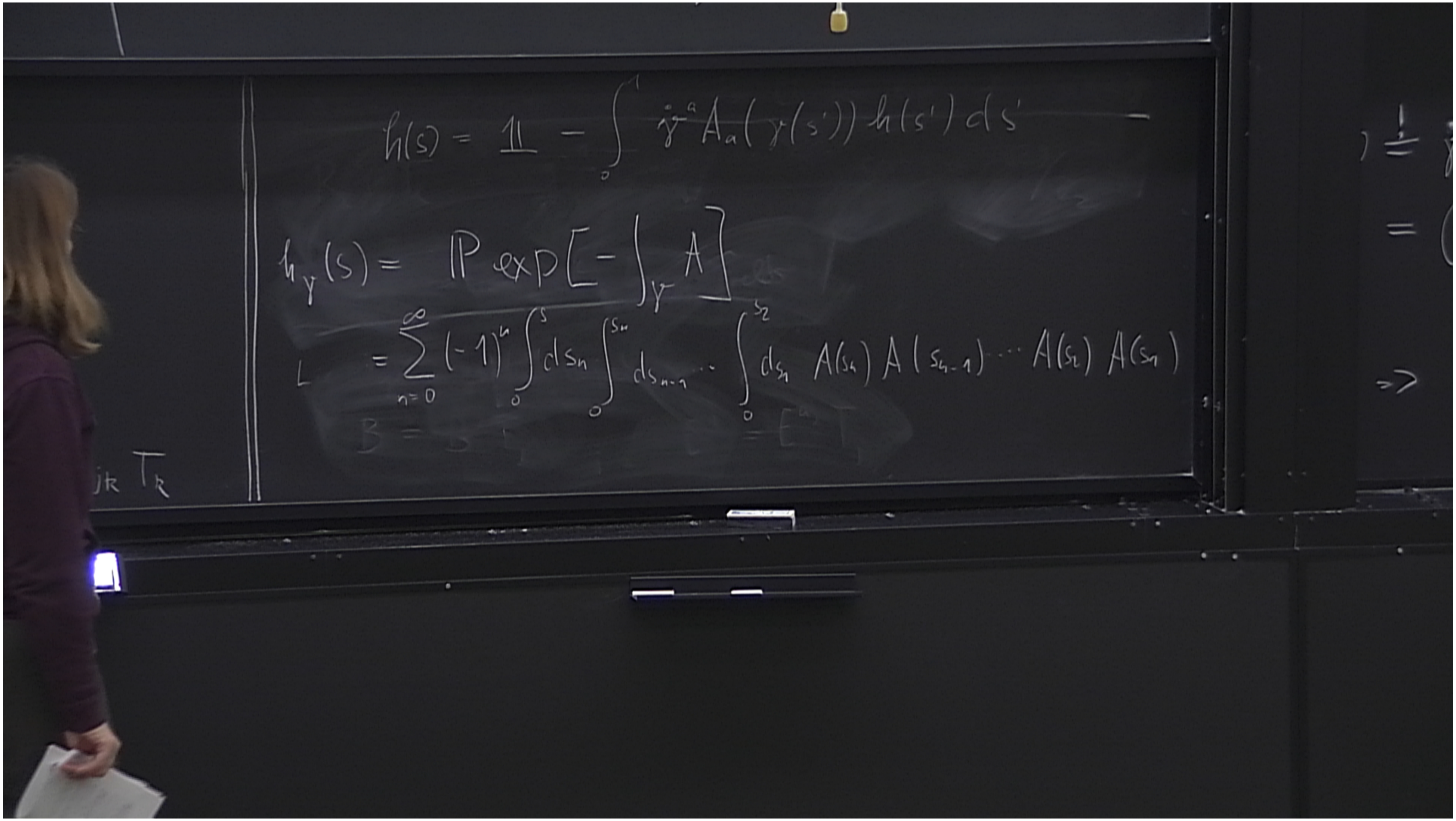
$$0 = \frac{d}{ds} h h^{-1} = \left(\frac{d}{ds} h\right) h^{-1} + h \left(\frac{d}{ds} h^{-1}\right)$$

$$0 \stackrel{!}{=} \dot\gamma^a D_a V \qquad \dot\gamma^a \frac{\partial}{\partial x^a} = \frac{d}{ds}$$

$$= \left(\frac{d}{ds} h(s)\right) V h^{-1}(s) + h(s) V \underbrace{\left(\frac{d}{ds} h^{-1}(s)\right)}_{- h^{-1} \left(\frac{d}{ds} h(s)\right) h^{-1}} + \dot\gamma^a A_a h(s) V h(s)^{-1} - h(s) V h(s)^{-1} \dot\gamma^a A_a$$

$$\Rightarrow \boxed{\frac{d}{ds} h(s) = - \dot\gamma^a A_a(\gamma(s)) h(s)} \qquad h(0) = \mathbb{1}$$


= 1 − ∫ γ̇ᵃ Aₐ(γ(s')) h(s') ds'
= εᵢⱼₖ Tₖ


$h(s) = 1 - \int_0^1 \dot{\gamma}^a A_a(\gamma(s')) h(s') ds'$
$h_\gamma(s) = \mathbb{P} \exp[-\int_\gamma A]$
$= \sum_{n=0}^{\infty} (-1)^n \int_0^s ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \, A(s_n) A(s_{n-1}) \cdots A(s_2) A(s_1)$

$$h(s) = \mathbb{1} - \int_0^{s} \dot{\gamma}^a A_a(\gamma(s'))\, h(s')\, ds'$$

$$h_\gamma(s) = \mathbb{P}\exp\left[-\int_\gamma A\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{s} ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1\, A(s_n)\, A(s_{n-1}) \cdots A(s_2)\, A(s_1)$$

$$= \epsilon_{ijk} T_k$$