Title: Quantum Gravity Review-2

Date: Feb 18, 2015 10:15 AM

URL: http://pirsa.org/15020051

Abstract:

Triads and Spin connection

co-triads

$$g_{\mu\nu} = e^{i}_{\mu} e^{j}_{\nu} \delta_{ij}$$

$$g^{\mu\nu} e^{i}_{\nu} \delta_{ij} = e^{\mu}_{j}$$: frame · triads

$$e^{\mu}_{j} e^{i}_{\mu} = e^{\mu}_{j} e^{i\nu}_{\mu} \quad g_{\mu\nu} = \delta^{i}_{j}$$

- objects w/

- objects w/ space time ind.ces $\mu, \nu, \ldots$ ← space time tensor
  & internal indices $i, j, \ldots$ ← transformation rule w/ internal rotations
- covariant derivative ∇_μ (wrt $\Gamma^\rho_{\mu\nu}$ L-C - connection)
  ↳ w/ internal indices. D_μ !! $\omega^i{}_{\mu k}$ spin-connection

$$D_\mu \phi^i = \partial_\mu \phi^i + \omega^i{}_{\mu k}\, \phi^k$$

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega^k{}_{\mu j}\, \phi_k$$

$$D_\mu v^i_\nu = \partial_\mu v^i_\nu - \Gamma^\rho_{\mu\nu}\, v^i_\rho + \omega^i{}_{\mu k}\, v^k_\nu$$

- objects w/ space time ind.ces $\mu, \nu, \ldots$ ← space time tensor
  & internal indices $i, j, \ldots$ ← transformation rule w/ internal rotations
- covariant derivative ∇_μ (wrt $\Gamma^\rho_{\mu\nu}$ L-C - connection)
  ↳ w/ internal indices. D_μ !! $\omega^i{}_{\mu k}$ spin-connection

$$D_\mu \phi^i = \partial_\mu \phi^i + \omega^i{}_{\mu k}\, \phi^k$$

$$D_\mu \phi_i = \partial_\mu \phi_i - \omega^k{}_{\mu i}\, \phi_k$$

$$D_\mu v^i_\nu = \partial_\mu v^i_\nu - \Gamma^\rho_{\mu\nu}\, v^i_\rho + \omega^i{}_{\mu k}\, v^k_\nu$$

time indices $\mu, \nu, \ldots$ ← space time tensor

al indices $i, j, \ldots$ ← transformation rule w/ internal rotations

∇_μ (wrt $\Gamma^\rho_{\mu\nu}$ L-C-connection)

$\omega^{ij}_{\mu k}$ spin-connection

$\omega' \sim$ inhom.

$(\partial_\mu U^{ij}) \sim$ inhom.

$R \partial_\mu R$

$+ \omega^{ij}_{\mu k} \phi^k$

$- \omega^{ik}_{\mu j} \phi_k$

$- \Gamma^\rho_{\mu\nu} U^{ij}_\rho + \omega^{i}_{\mu k} U^{kj}$

Compatib condition

$$D_\mu e^j_\nu = 0$$

$$\partial_\mu e^j_\nu - \Gamma^\rho_{\mu\nu} e^j_\rho + \omega^j_{\mu k} e^k_\nu \overset{!}{=} 0$$

$$\boxed{\omega^j_{\mu k} = -e^\nu_k D_\mu e^j_\nu}$$

spin-connection

$$\tilde{\Gamma}^{s}_{\mu\nu} = e^{s}_{j}\,\omega^{j}_{\mu k}\,e^{k}_{\nu} + e^{s}_{j}\,\partial_{\mu} e^{j}_{\nu}$$

When is $\tilde{\Gamma} = \Gamma$?

$$0 = \nabla_{\mu}\left(\underbrace{e^{\nu}_{k} e^{s}_{j}}_{\delta_{kj}}\, g_{\nu s}\right) = -\,\omega_{\mu jk} - \omega_{\mu kj}$$

$$\omega_{\mu jk} = \omega^{\ell}_{\mu k}\,\delta_{\ell j}$$

metric compatible $\Leftrightarrow$ antisymm of ω

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = e^{\rho}_{j}\,\omega^{j}_{\mu k}\,e^{k}_{\nu} + e^{\rho}_{j}\,\partial_{\mu}e^{j}_{\nu}$$

When is $\tilde{\Gamma} = \Gamma$?

$$= \nabla_{\mu}\big(\underbrace{e^{\nu}_{k}e^{\rho}_{j}}_{\delta_{kj}}\,g_{\nu\rho}\big) = -\omega_{\mu jk} - \omega_{\mu kj}$$

$$\omega_{\mu jk} = \omega^{\ell}_{\mu k}\,\delta_{\ell j}$$

metric compatib of Γ $\Leftrightarrow$ antisymm of ω

$\nabla_{\mu}(\delta_{jk}) = 0$

triads

$e^{\mu}_{j}\, e^{i}_{\mu} = e^{\mu}_{j}\, e^{i}_{\mu}$ $\quad g_{\mu\nu} = \delta^{i}_{j}$

$D_\mu \phi^j = \partial_\mu \phi^j + \omega^j_{\mu k}\, \phi^k$

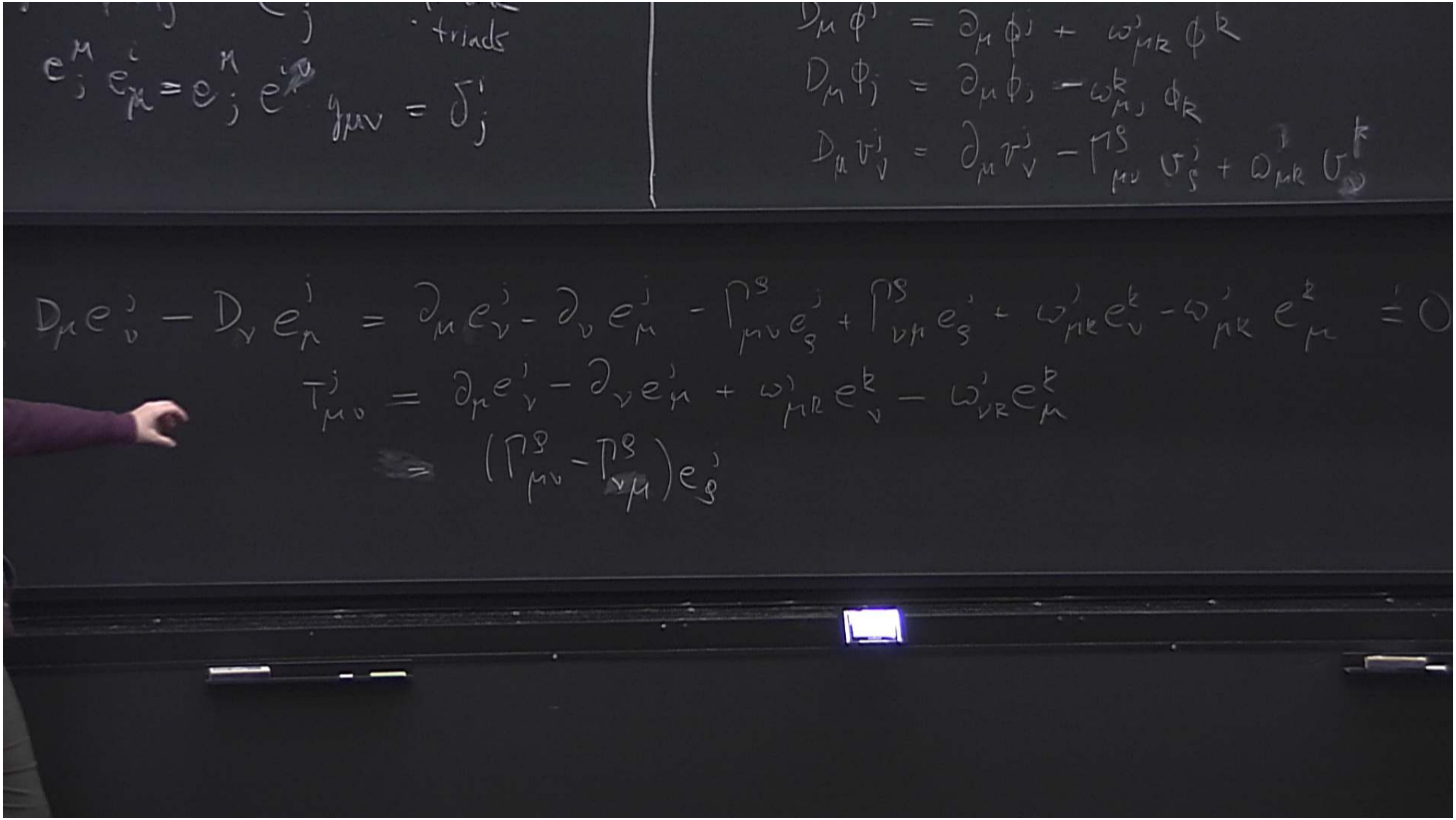
$D_\mu \phi_j = \partial_\mu \phi_j - \omega^k_{\mu j}\, \phi_k$

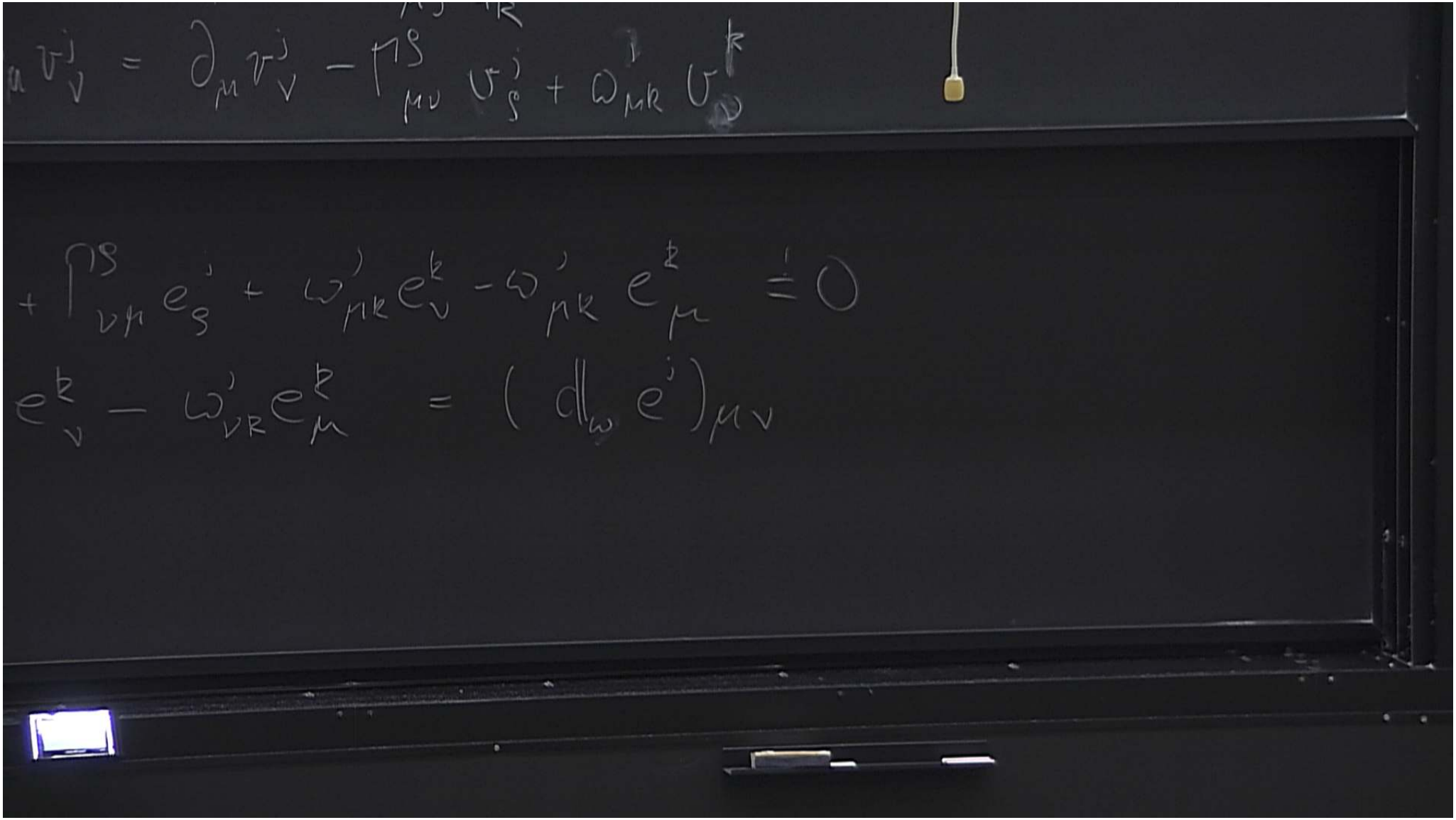
$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^\rho_{\mu\nu}\, v^j_\rho + \omega^j_{\mu k}\, v^k_\nu$

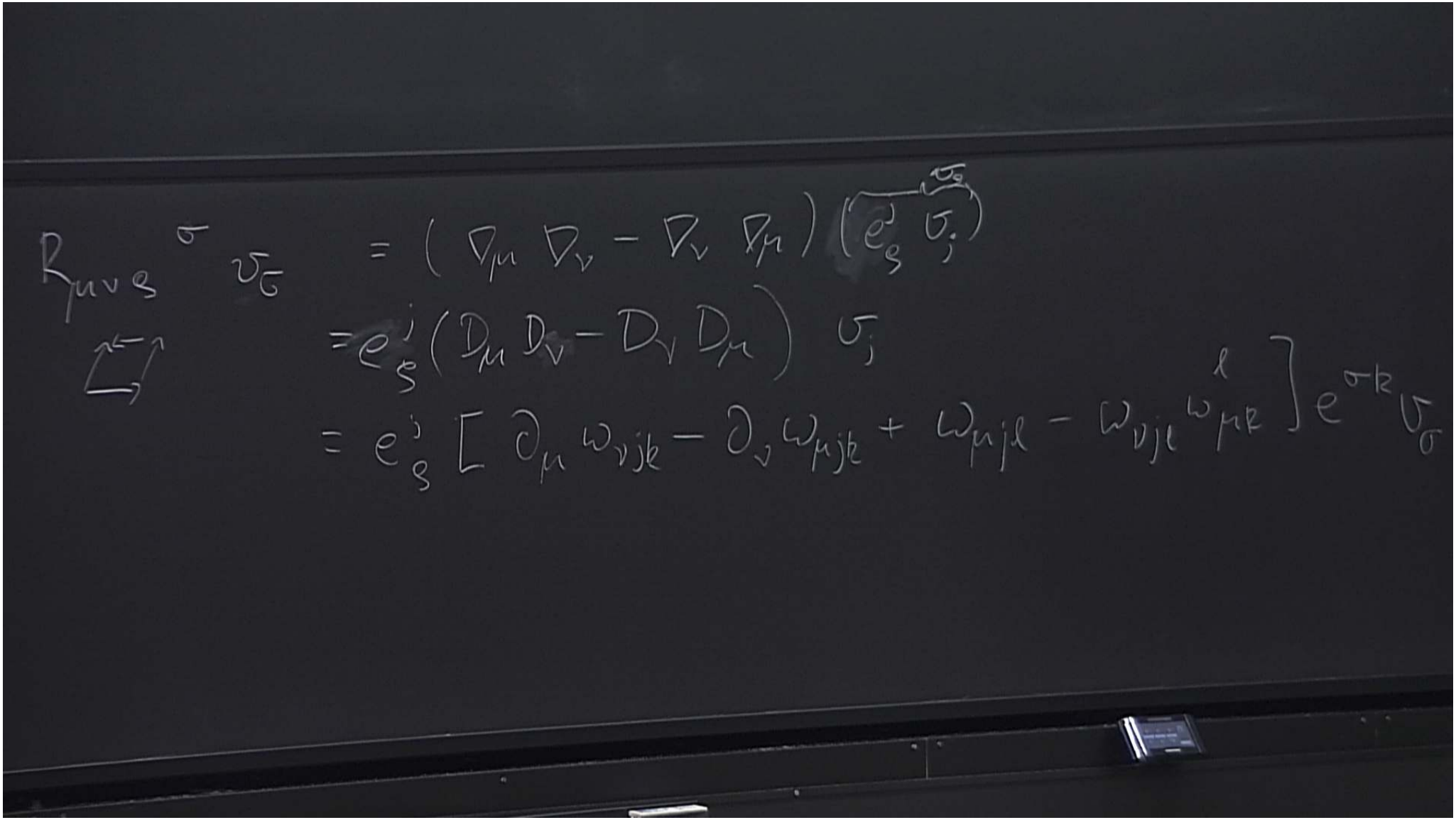
$$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu \stackrel{!}{=} 0$$

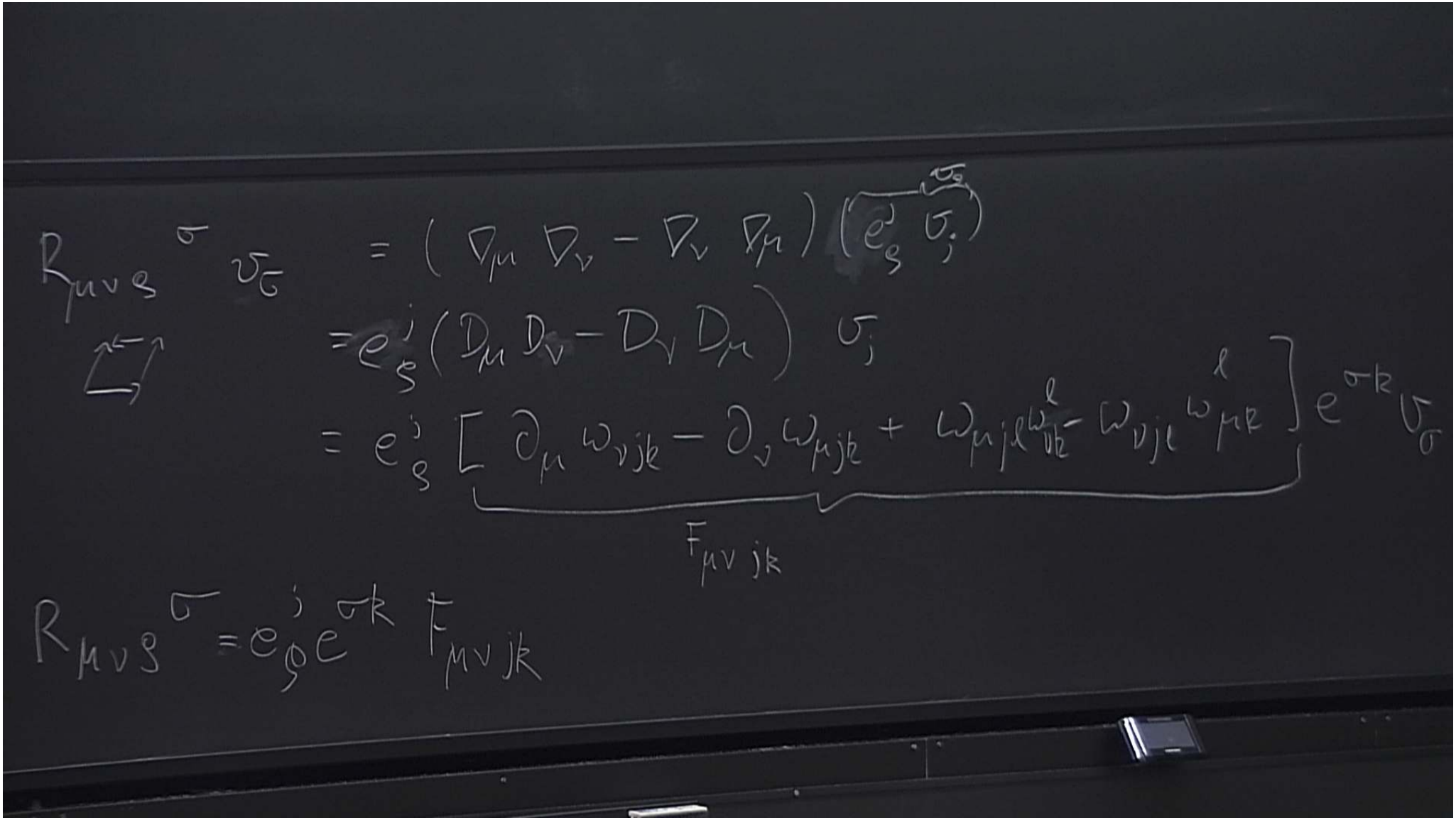
$$\partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$$

$$= \left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right) e^j_\rho$$


triads
$e^\mu_j e^i_\mu = e^\mu_j e^i_\nu \quad g_{\mu\nu} = \delta^i_j$
$D_\mu \phi^j = \partial_\mu \phi^j + \omega^j_{\mu k} \phi^k$
$D_\mu \phi_j = \partial_\mu \phi_j - \omega^k_{\mu j} \phi_k$
$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^\rho_{\mu\nu} v^j_\rho + \omega^j_{\mu k} v^k_\nu$
$D_\mu e^j_\nu - D_\nu e^j_\mu = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho + \omega^j_{\mu k} e^k_\nu - \omega^j_{\mu k} e^k_\mu \stackrel{!}{=} 0$
$T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$
$= (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}) e^j_\rho$


$\nabla_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^\rho_{\mu\nu} v^j_\rho + \omega^j_{\mu k} v^k_\nu$
$+ \Gamma^\rho_{\nu\mu} e^j_\rho + \omega^j_{\mu k} e^k_\nu - \omega^j_{\mu k} e^k_\mu = 0$
$e^k_\nu - \omega^j_{\nu k} e^k_\mu = (d_\omega e^j)_{\mu\nu}$


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(e_\rho^j v_j)$
$= e_\rho^j (D_\mu D_\nu - D_\nu D_\mu) v_j$
$= e_\rho^j [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j\ell} - \omega_{\nu j\ell} \omega_{\mu k}^{\ell}] e^{\sigma k} v_\sigma$


$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)(e_\rho^i v_i)$
$= e_\rho^i (D_\mu D_\nu - D_\nu D_\mu) v_i$
$= e_\rho^j [\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j\ell}\omega^{\ell}_{\nu k} - \omega_{\nu j\ell}\omega^{\ell}_{\mu k}] e^{\sigma k} v_\sigma$
$F_{\mu\nu jk}$
$R_{\mu\nu\rho}{}^{\sigma} = e_\rho^j e^{\sigma k} F_{\mu\nu jk}$

$$R_{\mu\nu\rho}{}^{\sigma} = e_{\rho}^{\ j} e^{\sigma k} F_{\mu\nu jk}$$

$F_{\mu\nu jk}$

$$R = R_{\mu\nu} g^{\mu\nu} = e_j^{\mu} e_k^{\sigma} F_{\mu\nu}^{\ \ jk}(\omega)$$

$$\sqrt{g} = \cdots |\det(e)|$$

$e^{\mu}_{i}\, e_{\mu}^{j} = e^{\mu}_{i}\, e^{\nu}_{j}\, g_{\mu\nu} = \delta^{j}_{i}$

$D_\mu \phi_j = \partial_\mu \phi_j - \omega^k_\mu$

$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^{\rho}_{\mu}$

3D first order action

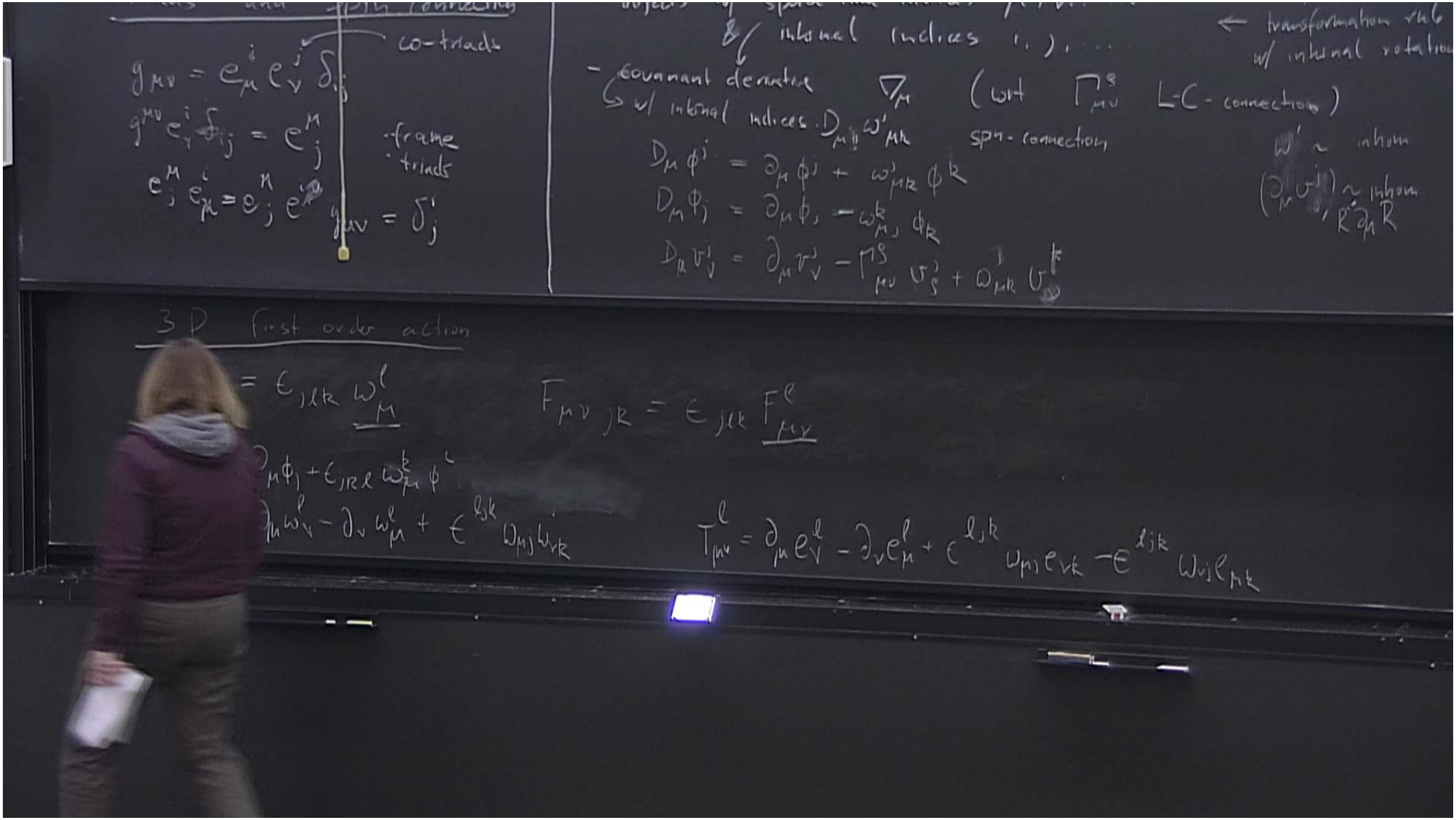
$\omega_{\mu jk} = \epsilon_{jlk}\, \omega^{l}_{\mu}$

$F_{\mu\nu\, jk} = \epsilon_{jlk}\, F^{l}_{\mu\nu}$

$D_\mu \phi_j = \partial_\mu \phi_j + \epsilon_{jkl}\, \omega^{k}_{\mu}\, \phi^{l}$

$F^{l}_{\mu\nu} = \partial_\mu \omega^{l}_{\nu} - \partial_\nu \omega^{l}_{\mu} + \epsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$

$T^{l}_{\mu\nu} = \partial_\mu e^{l}_{\nu} - \partial_\nu e^{l}_{\mu}$


co-triads
frame triads
3D first order action

$$R_{\mu\nu\rho}{}^{\sigma} = e_{\rho}{}^{j} e^{\sigma k} F_{\mu\nu jk}$$

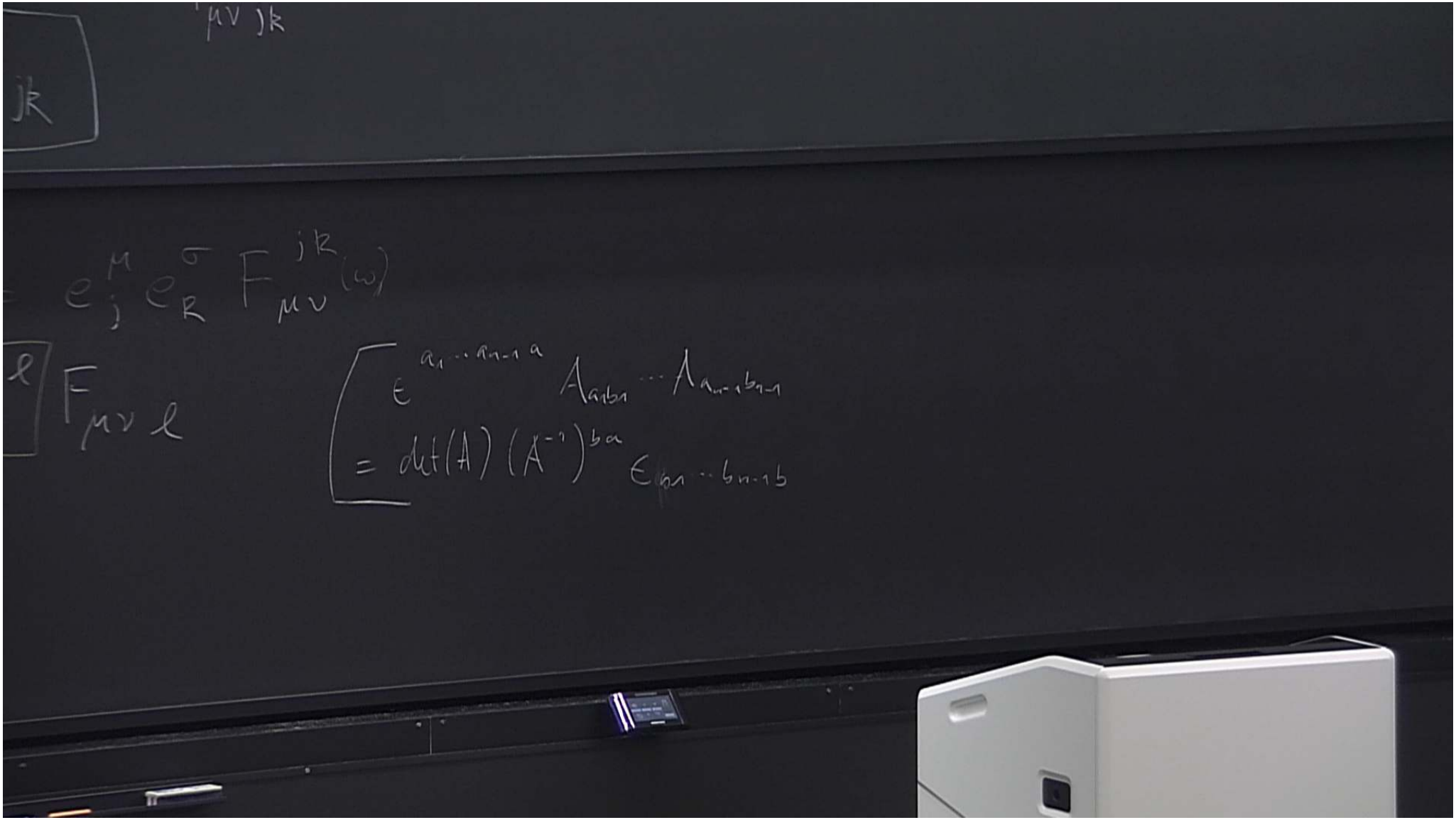
$$R = R_{\mu\nu} g^{\mu\nu} = e_{j}^{\mu} e_{k}^{\nu} F_{\mu\nu}^{jk}(\omega)$$

$$R = e_{j}^{\mu} e_{k}^{\nu} \epsilon^{jkl} F_{\mu\nu l}$$

$$R_{\mu\nu\rho}{}^{\sigma} = e_{\rho} e^{\sigma k} F_{\mu\nu jk}$$

$$R = R_{\mu\nu} g^{\mu\nu} = e_j^{\mu} e_k^{\nu} F_{\mu\nu}^{jk}(\omega)$$

$$R = \boxed{e_j^{\mu} e_k^{\nu} \epsilon^{jklm}} F_{\mu\nu lm}$$


$\mu\nu\, jk$
jk
$= e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\nu}(\omega)$
ℓ
$F_{\mu\nu\ell}$
$\epsilon^{a_1 \cdots a_{n-1} a} A_{a_1 b_1} \cdots A_{a_{n-1} b_{n-1}}$
$= \det(A)\, (A^{-1})^{ba}\, \epsilon_{b_1 \cdots b_{n-1} b}$

$$R_{\mu\nu\rho}{}^{\sigma} = e_\rho^{\ j} e^{\sigma k} F_{\mu\nu jk}$$

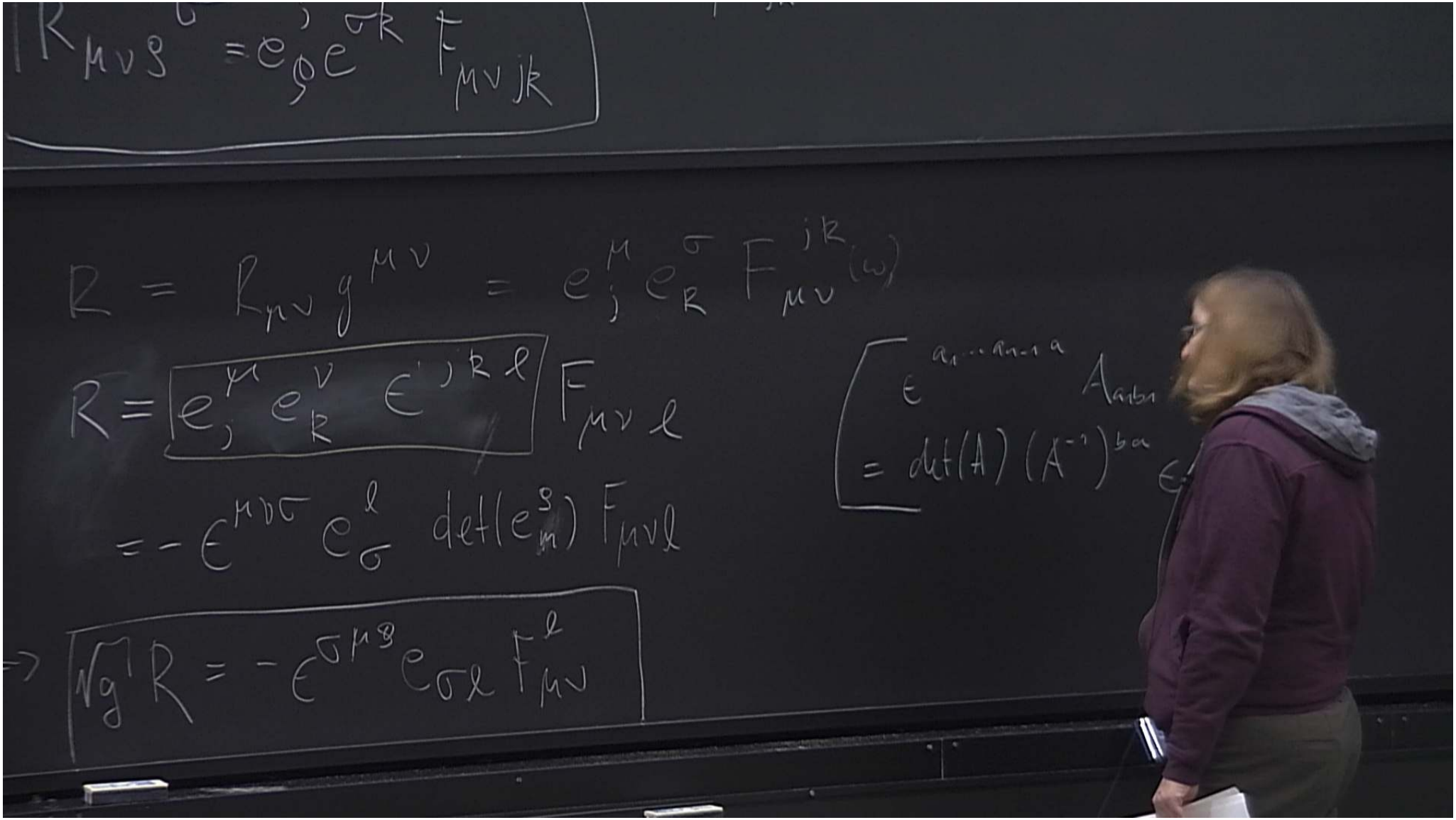
$$R = R_{\mu\nu} g^{\mu\nu} = e_j^{\mu} e_k^{\sigma} F_{\mu\nu}^{\ \ jk}(\omega)$$

$$R = \boxed{e_j^{\mu} e_k^{\nu} \epsilon^{jkl}} F_{\mu\nu l}$$

$$= -\epsilon^{\mu\nu\sigma} e_\sigma^{l} \det(e_m^{s}) F_{\mu\nu l}$$

$$\Rightarrow \sqrt{g}\, R$$

$$\epsilon^{a_1 \cdots a_{n-1} a} \Lambda_{a a_1} \cdots \Lambda_{a_{n-1} b_{n-1}}$$
$$= \det(\Lambda) (\Lambda^{-1})^{b a} \epsilon_{a_1 \cdots b_{n-1} b}$$


$R_{\mu\nu\rho}{}^{\sigma} = e_\rho^j e^{\sigma k} F_{\mu\nu jk}$
$R = R_{\mu\nu} g^{\mu\nu} = e_j^\mu e_k^\nu F_{\mu\nu}^{jk}(\omega)$
$R = e_j^\mu e_k^\nu \epsilon^{jkl} F_{\mu\nu l}$
$= -\epsilon^{\mu\nu\sigma} e_\sigma^l \det(e_m^s) F_{\mu\nu l}$
$\Rightarrow \sqrt{g} R = -\epsilon^{\sigma\mu\rho} e_{\sigma l} F^l_{\mu\nu}$
$\epsilon^{a_1 \dots a_{n-1} a} A_{a a_1 b_1}$
$= \det(A) (A^{-1})^{b a}$

$$R_{\mu\nu\rho}{}^{\sigma} = e_{\rho}{}^{\tau} e^{\sigma k} F_{\mu\nu jk}$$

$$\int R\sqrt{g}\, d^3x = -\int e_{\tau a} F^{a}_{\mu\nu} \epsilon^{\tau\mu\nu} d^3x$$

EOM

$$\delta e^{k}_{\tau}: \quad F = 0$$

$$\boxed{R_{\mu\nu\rho}{}^{\sigma} = e^{j}_{\rho} e^{\sigma k} F_{\mu\nu jk}}$$

$$\int R\sqrt{g}\, d^3x = -\int e_{\sigma l} F^{l}_{\mu\nu}(\omega)\, \epsilon^{\sigma\mu\nu}\, d^3x$$

EOM

δe^{k}_{σ}: $\quad F = 0$

$\delta\omega$: $\quad T^{j}_{\mu\nu} = 0 \quad \Rightarrow \quad \omega^{l}_{\mu}$ leads to L-C connection