Title: Classifications of symmetry protected topological phases in interacting boson/fermion systems

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Abstract: Symmetry protected topological (SPT) states are bulk gapped states with gapless edge excitations. The SPT phases in free fermion systems, like topological insulators, can be classified by K-theory. However, it is not yet known what SPT phases exist in general interacting systems. In this talk, I will first present a systematic way to construct SPT phases in interacting bosonic systems, which allows us to identify many new SPT phases. Just as group theory allows us to construct 230 crystal structures in three dimensions, we find that group cohomology theory allows us to construct many interacting bosonic SPT phases. In my talk, I shall show how topological terms in the path integral description of the system can be constructed from nontrivial group cohomology classes, giving rise to exactly soluble Hamiltonians with explicit ground state wavefunctions. Next, I will discuss the generalization of the classifying scheme to interacting fermionic systems and a new mathematical framework â€“ group supercohomology theory, which predicts a fermionic SPT phase that can neither be realized in free fermionic nor interacting bosonic systems.

Finally, I will briefly mention the deep relationship between SPT phases and chiral anomalies in high energy physics.

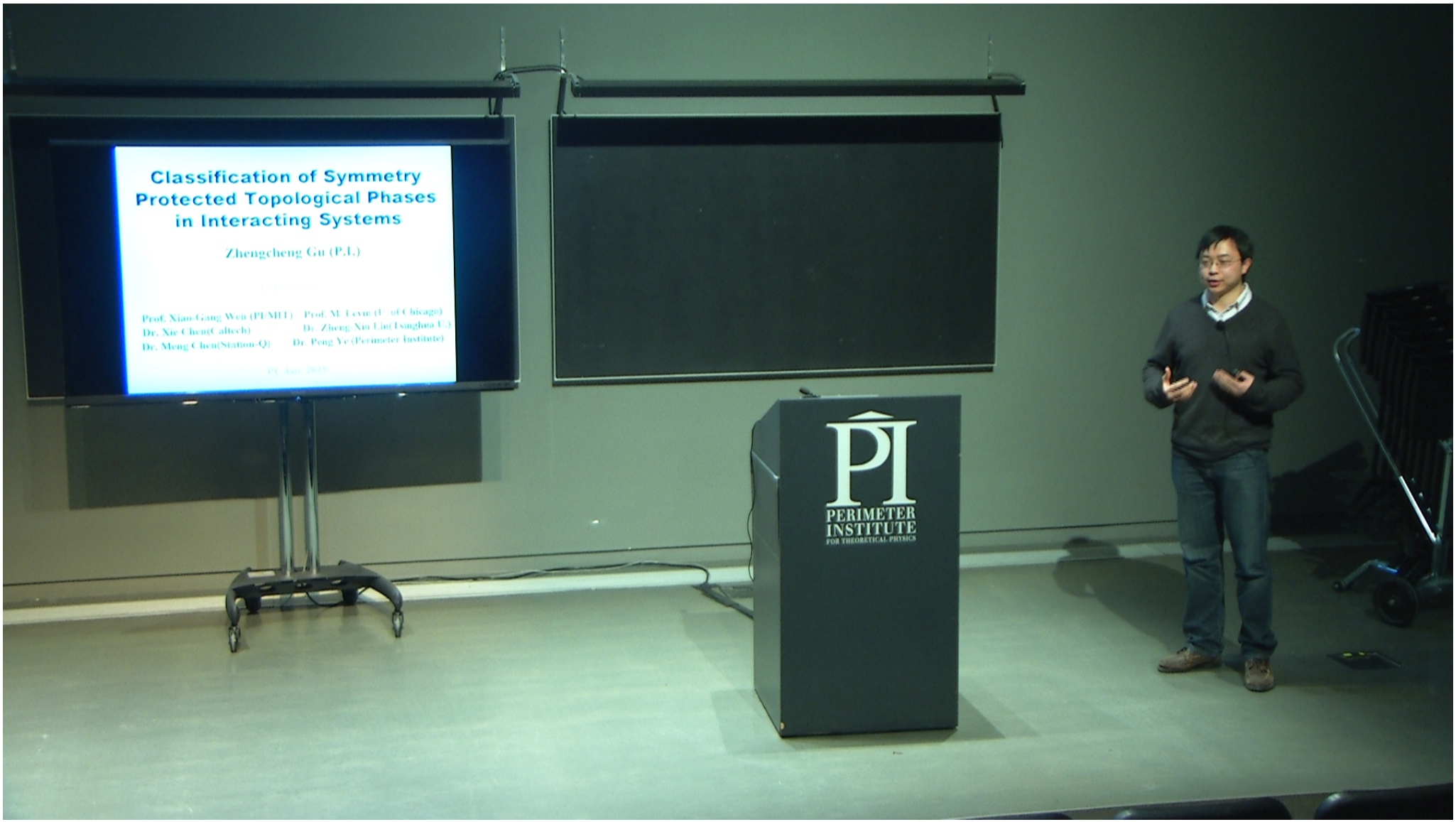
Classification of Symmetry Protected Topological Phases in Interacting Systems

Zhengcheng Gu (P.I.)

Collaborators:

Prof. Xiao-Gang Wen (PI/MIT)
Prof. M. Levin (U. of Chicago)
Dr. Xie Chen(Caltech)
Dr. Zheng-Xin Liu(Tsinghua U.)
Dr. Meng Chen(Station-Q)
Dr. Peng Ye (Perimeter Institute)

PI. Jan. 2015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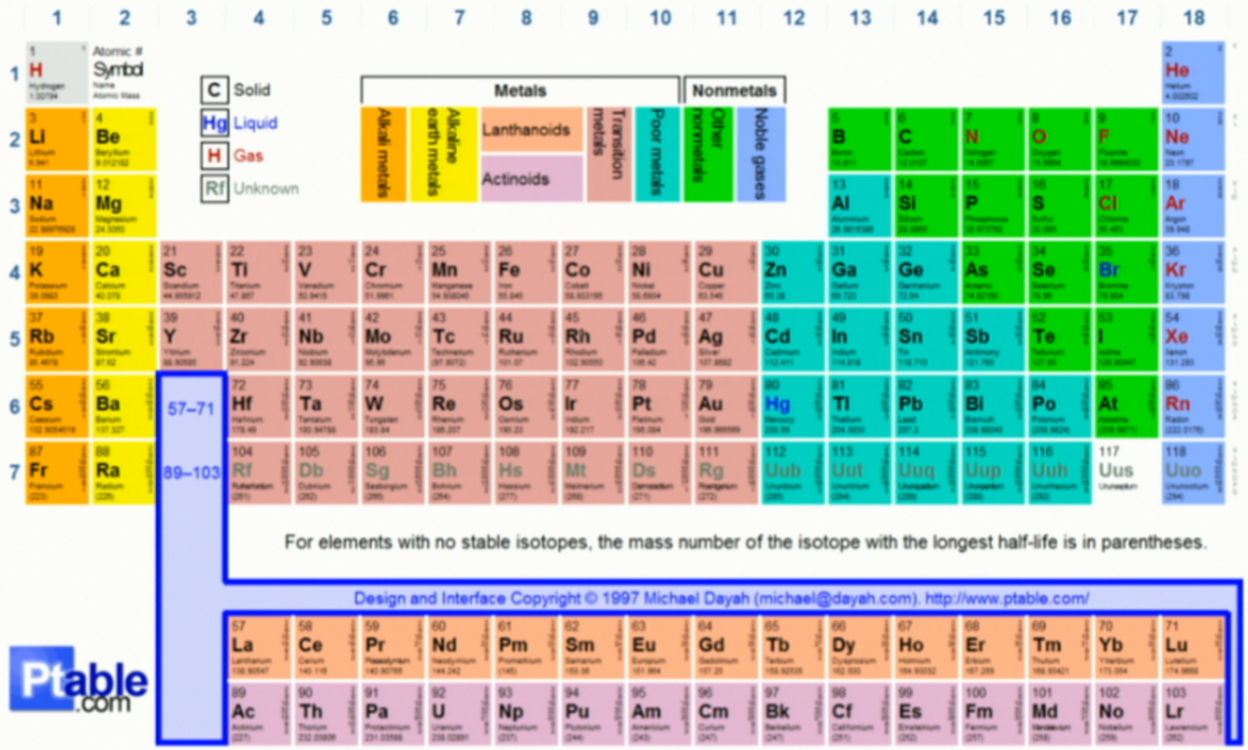
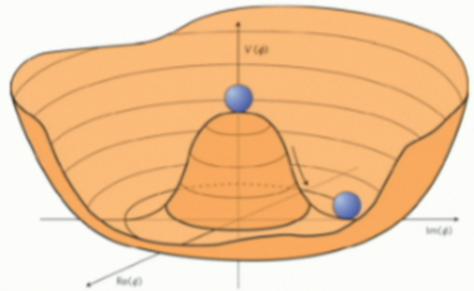
Why do we need a classification?

Periodic table in chemistry:

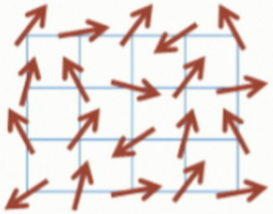


The Landau paradigm of phases and phase transitions -- Symmetry Breaking

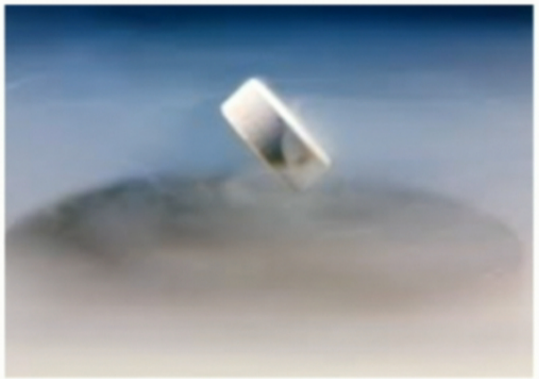
Symmetry breaking theory:



- Magnetic orders in spin systems



- Superconductivity



The underlying mathematical framework is group theory

Topological phases of quantum matter: beyond Landau's paradigm

What are topological phases of quantum matter?

Gapped quantum phases without symmetry breaking and long range correlation, but can not be adiabatically connected to a trivial disorder phase without phase transition.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Two basic classes of topological phases:
Intrinsic topological phases (long-range-entanglement)
• adiabatical paths with no symmetry
Symmetry protected topological (SPT) phases
• adiabatical paths with symmetry
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Topological phases of quantum matter: beyond Landau's paradigm

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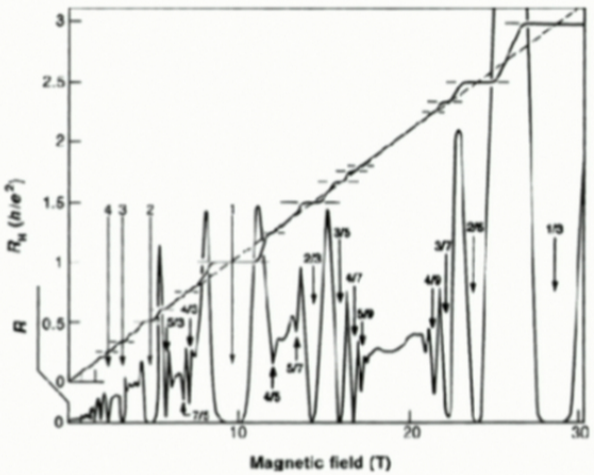
symmetry breaking Hamiltonians

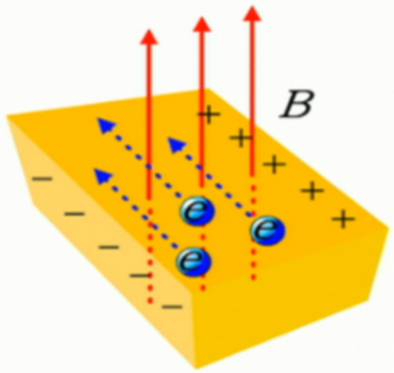
SPT phases $\Longleftrightarrow$ The trivial disorder phase

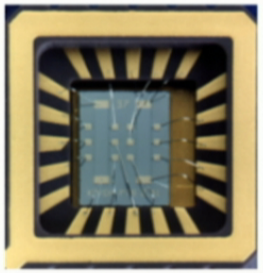
(Z C Gu, X G Wen 2009)

Examples of intrinsic topological phases in interacting systems (no symmetry)

Fractional Quantum Hall Effect(FQHE) D C Tsui, *et al* 1982







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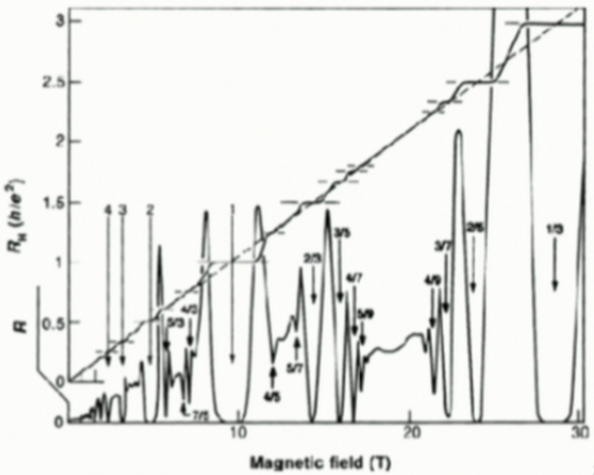
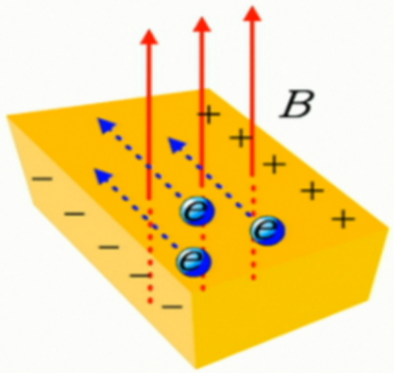
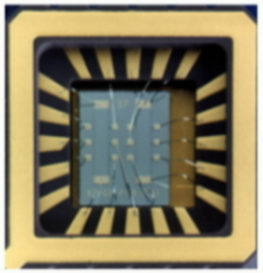


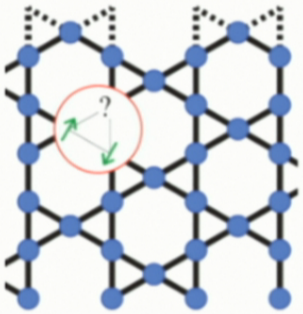




Spin liquid

- Frustrated magnets

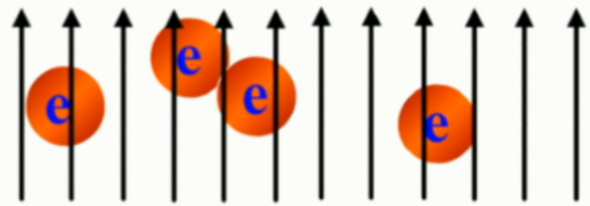
$$H = J_1 \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$



S Yan, D Huse and S White Science, 2011

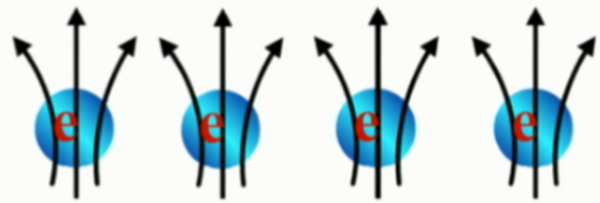
Topological terms for intrinsic topological phases (no symmetry)

FQHE $\Psi_3 = \prod_{i<j} (z_i - z_j)^3 e^{-\frac{1}{4}\sum_i |z_i|^2}$



$$\mathcal{L}_{\text{eff}} = \frac{2m+1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

R B Laughlin 1983
E Witten, 1989
S C Zhang, *et al* 1989
X G Wen, *et al* 1989



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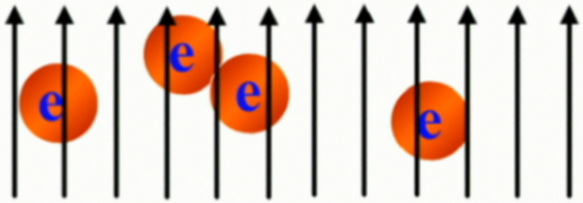

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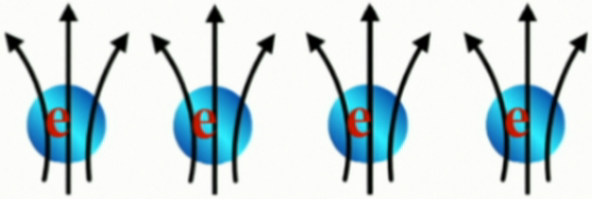
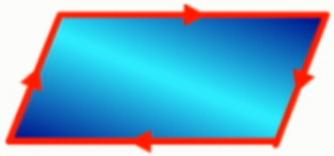


Braiding T, S matrices and chiral central charge as the universal data of topological order(no symmetry).

Ta: 

Sab: 

C: 

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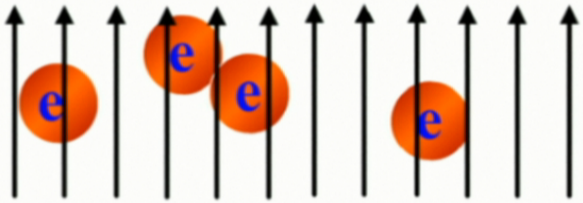

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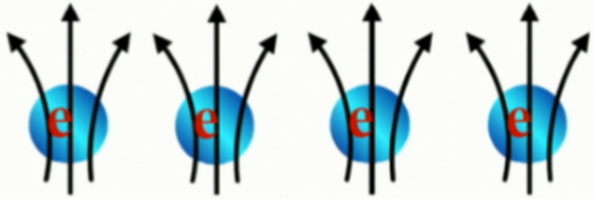

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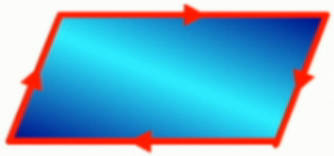
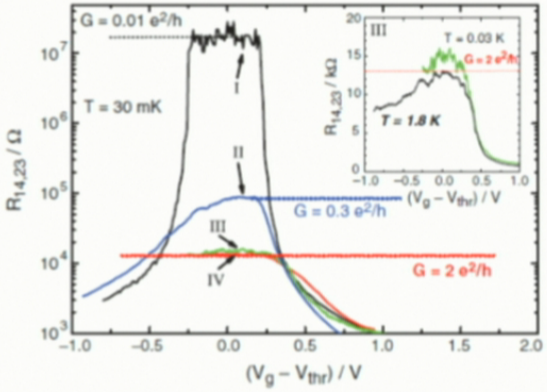
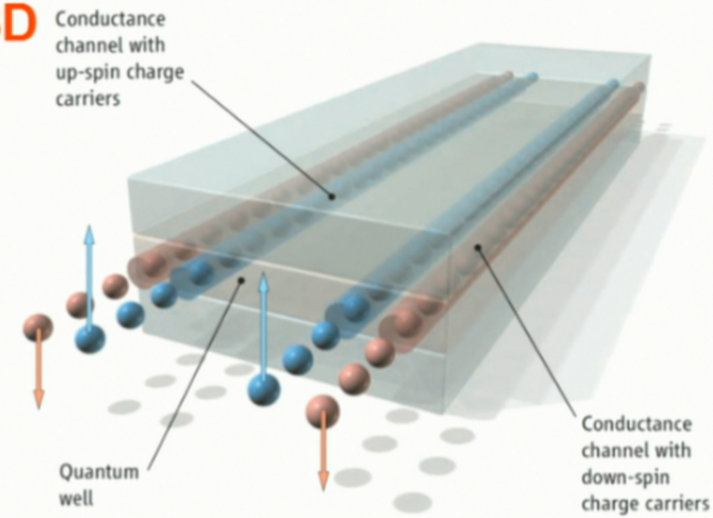
C: 

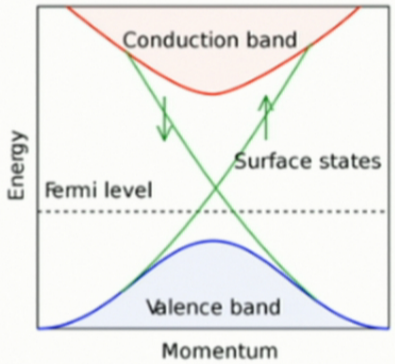
The mathematical framework for topological phases is known as unitary modular tensor category theory.

Examples of symmetry protected topological (SPT)phases in free fermion systems

Topological insulator in 2D/3D

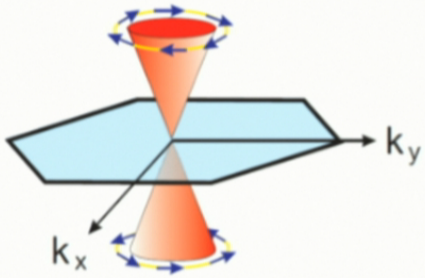






C L Kane, *et al*, 2005
B A Bernevig, et al 2006
W Molenkamp's group 2007
M Zahid Hasan, *et al*, 2008

from Wikipedia



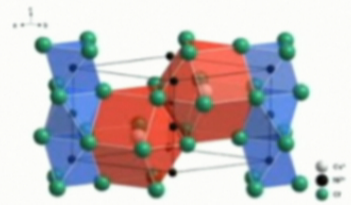
Examples of SPT phases in interacting models

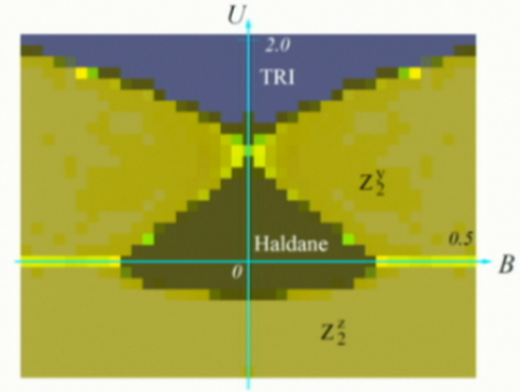
Spin one Haldane chain realizes 1D topological order

$$H = \sum_i \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2 + BS_i^x \right)$$

Uc~1(B=0)

$CsNiCl_3$(U~B~0)





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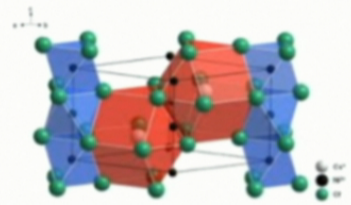
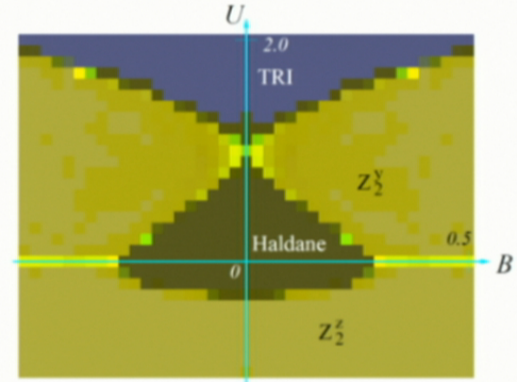


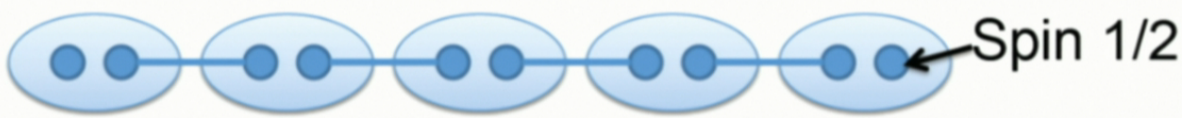


Haldane phase requires symmetry!

- Haldane phase can be protected by many kinds of symmetries: time reversal, spin rotation, etc...

Z C Gu and X G Wen, 2009, F Pollmann, *et al*, 2010

Fixed point wavefunction: spin-(1/2,1/2) dimer model



Projective representation and its (second) group cohomology classification

Projective representation on left/right dimer ends

$$u_L(g_1)u_L(g_2) = \omega(g_1,g_2)u_L(g_1,g_2); \quad u_R(g_1)u_R(g_2) = \omega(g_1,g_2)^{-1}u_R(g_1,g_2)$$

- Associativity and consistency -- the 2-cocycle group:

$$\mathcal{Z}^2[G,U(1)] = \{\omega \in U(1) | \omega(g_2,g_3)\omega(g_1,g_2g_3) = \omega(g_1,g_2)\omega(g_1g_2,g_3)\}$$

- Equivalent projective representation-- 2-coboundary group:

$$u_{L(R)}(g) \sim \beta_{L(R)}(g)u_{L(R)}(g); \quad \beta_{L(R)}(g) \in U(1)$$

$$\mathcal{B}^2[G,U(1)] = \{\omega \in U(1) | \omega(g_1,g_2) = \beta(g_1)\beta(g_2)/\beta(g_1g_2); \beta \in U(1)\}$$

Projective representation is classified by the quotient group -- the second group cohomology class

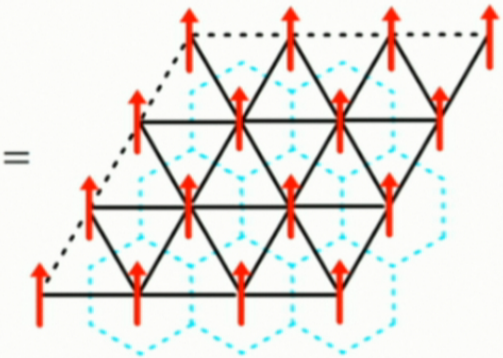
$$\mathcal{H}^2[G,U(1)] = \mathcal{Z}^2[G,U(1)]/\mathcal{B}^2[G,U(1)]$$

How to classify SPT phases in interacting spin/bosonic systems in higher dimensions with arbitrary (internal) symmetry group G?

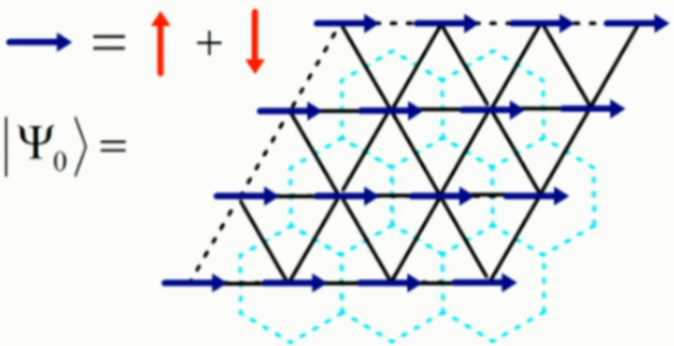
A revisit of transverse Ising model:

$$H = -\sum_{\langle pq \rangle} \sigma_p^z \sigma_q^z - t \sum_p \sigma_p^x$$

$|\Psi_{Ferro}\rangle =$



$|\Psi_0\rangle =$



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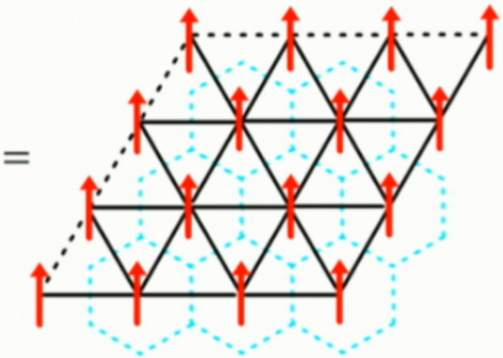
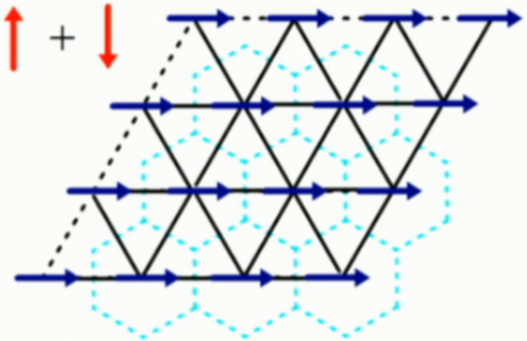


$\rightarrow = \uparrow + \downarrow$

$|\Psi_0\rangle =$



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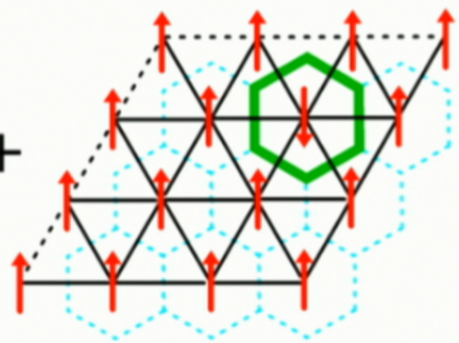
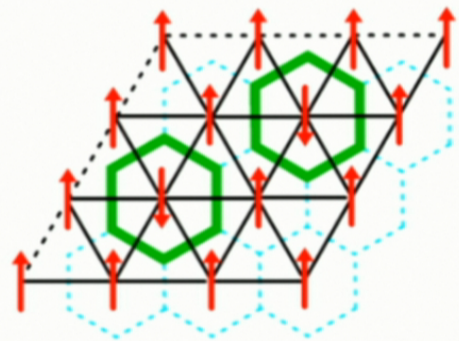


$+$



$+ \ldots$

An example of Ising SPT phase in 2D

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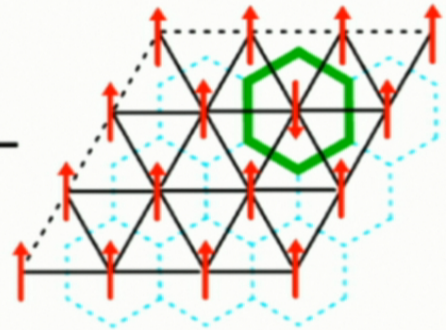
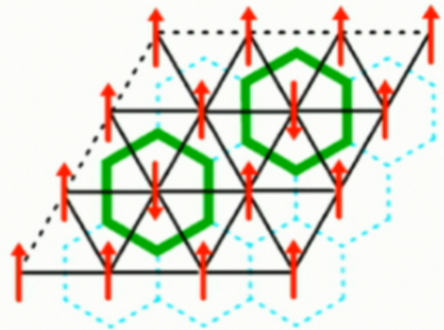
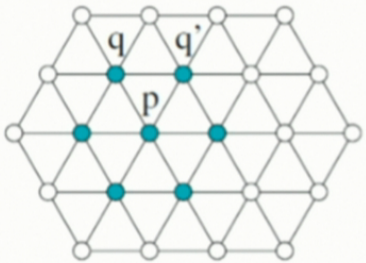
How many different paramagnetic phases? **Two!**

(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

$|\Psi_1\rangle = -$  $+$  $+ \ldots$



$$H_1 = -\sum_p B_p \ , \ B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}} \qquad [B_p, B_{p'}] = 0$$

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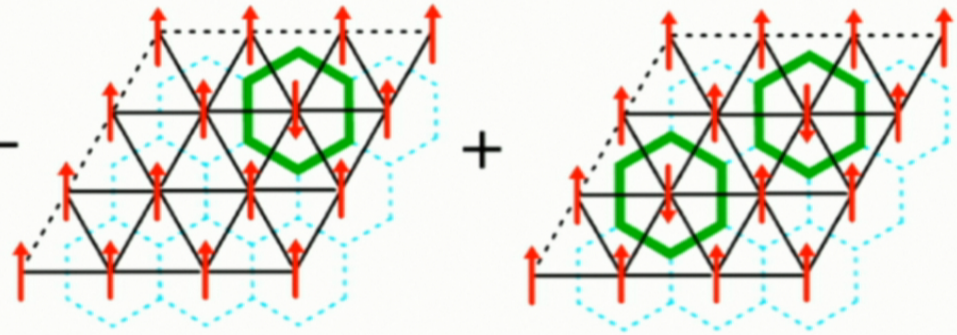
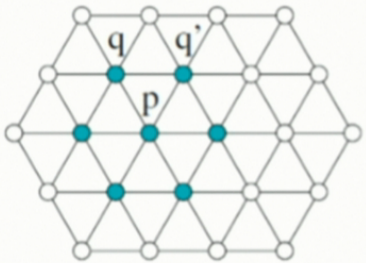
$|\Psi_1\rangle =$ 



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Domain deformation rule

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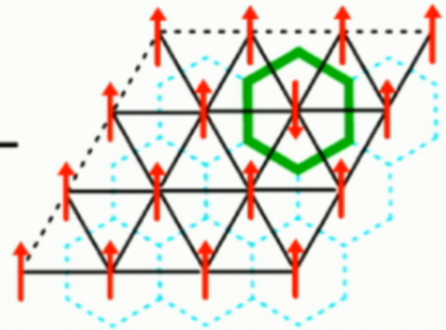
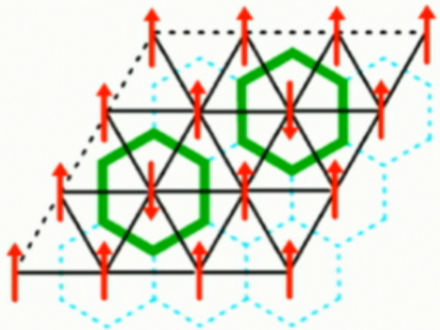
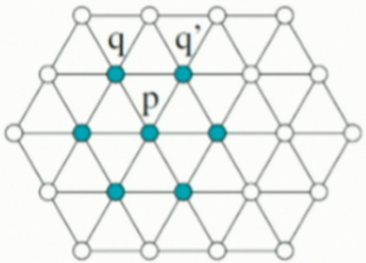
How many different paramagnetic phases? **Two!**

(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

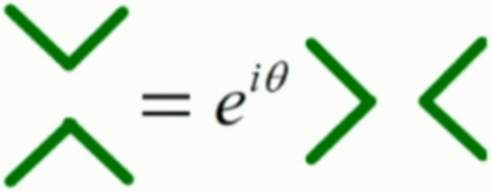
$|\Psi_1\rangle = -$  $+$  $+ \ldots$



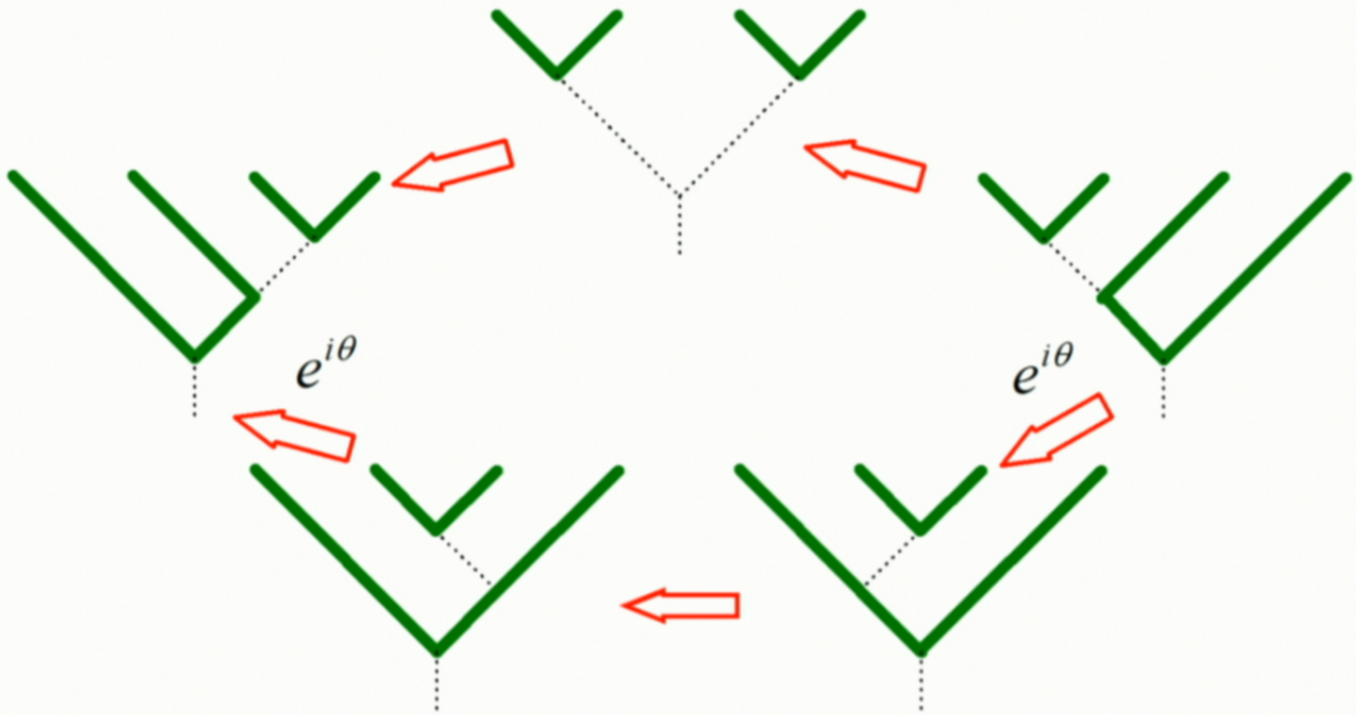
$$H_1 = -\sum_p B_p \ , \ B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}} \qquad [B_p, B_{p'}] = 0$$

Domain deformation rule

But why not?



Topologically consistent condition for fixed point wavefunction



$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

Bulk response and the nature of gapless edge

Assume that Ising spins carry Z_2 gauge charge and can couple to background Z_2 gauge field

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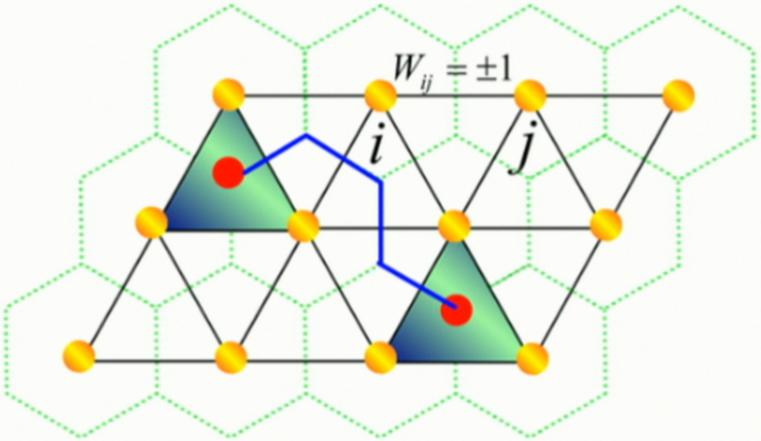

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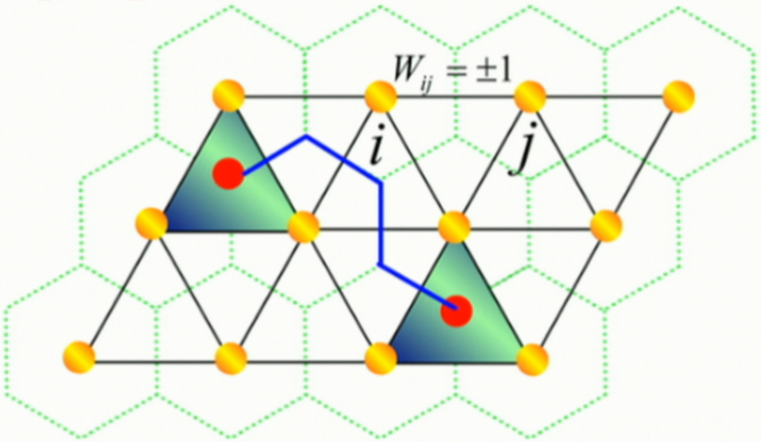

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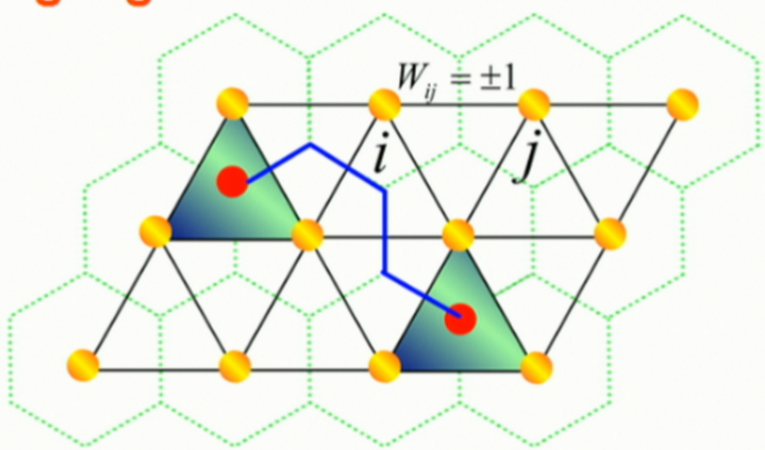

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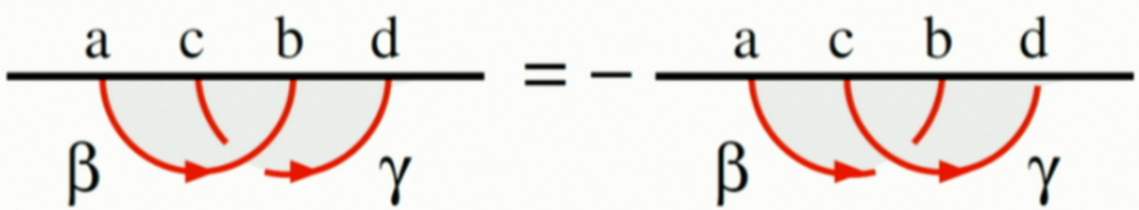
Non-trivial statistics of flux leads to degenerate edge states!



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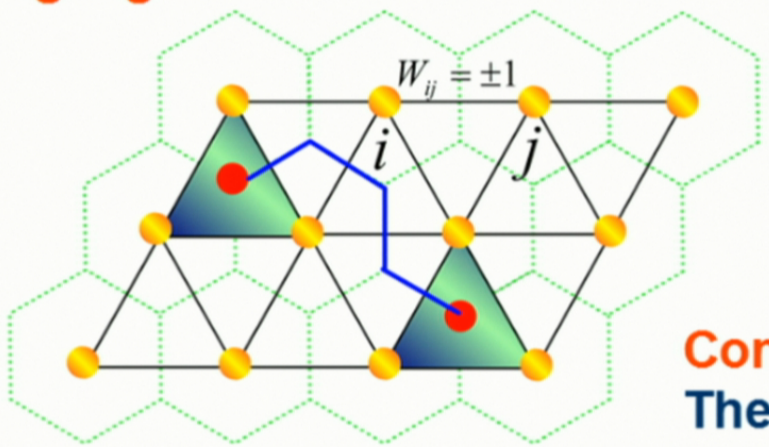

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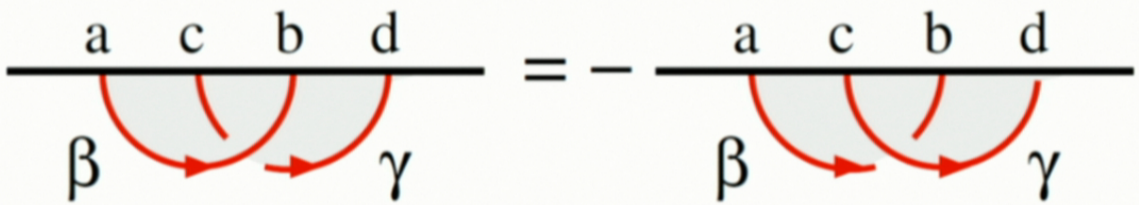
$$\tilde{W}_\beta \tilde{W}_\gamma = -\tilde{W}_\gamma \tilde{W}_\beta$$

Contradiction

There is No 1D representation!

Non-trivial statistics of flux leads to degenerate edge states!



Do we have a systematic way to classify bosonic SPT phases?

- In-equivalent projective representations are classified by second group cohomology class, which classifies all 1D bosonic SPT phases. (Xie Chen, Z C Gu, X G Wen PRB 83, 035107,2011)

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- **Conjecture:** Flux line statistics of G are classified by fourth group cohomology class, which classifies all 3D bosonic SPT phases. (C Wang and M Levin, PRL113, 080403 (2014))

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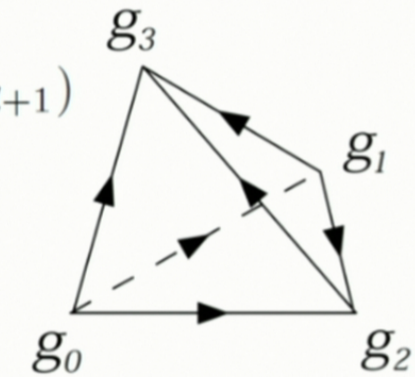
Topological nonlinear sigma model in discrete space-time to describe/classify SPT phases!

X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen (Science 338, 1604 (2012))

(bosonic) SPT phases in any dimensions with any symmetry

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_i\}} \prod_{d+1-\text{simplex}} \nu_{d+1}^{s_{01\cdots d}}(g_0, g_1, \cdots, g_{d+1})$$



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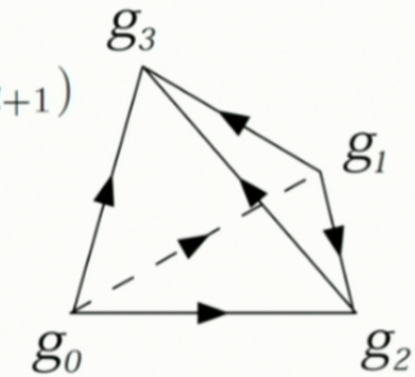

- Branched(vertex ordered) d+1-simplex

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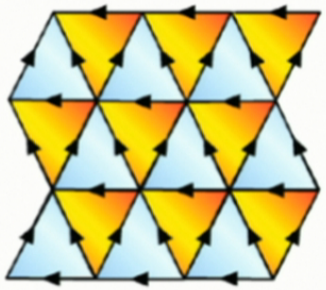
$$\nu_{d+1}(gg_0, gg_1, \cdots, gg_{d+1}) = \nu_{d+1}(g_0, g_1, \cdots, g_{d+1})$$

co-cycle condition: $$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{d+2}) = 1$$

An example of 1+1D case



$$Z = |G|^{-N_v} \sum_{\{g_i\}} \mathrm{e}^{-S(\{g_i\})}$$

$$\mathrm{e}^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$

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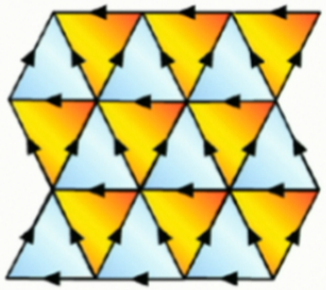

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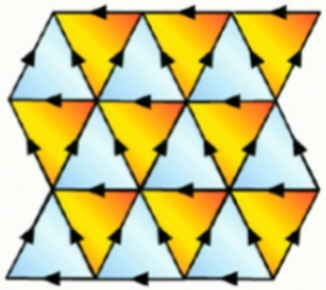

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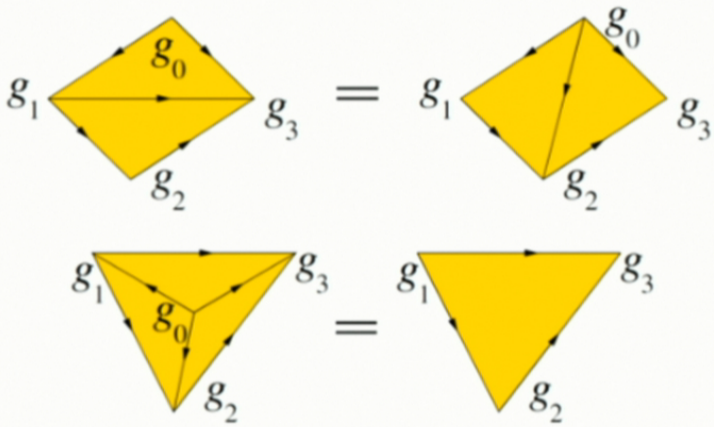
Topological invariant



$$\frac{\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3)\nu_2(g_0, g_1, g_2)} = 1$$

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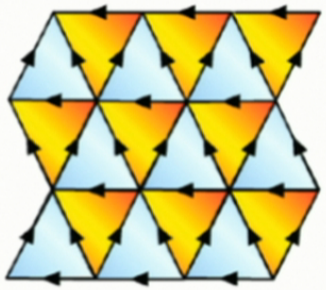

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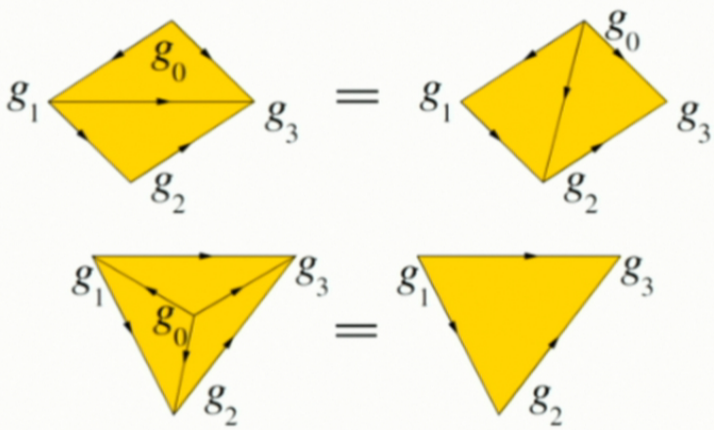

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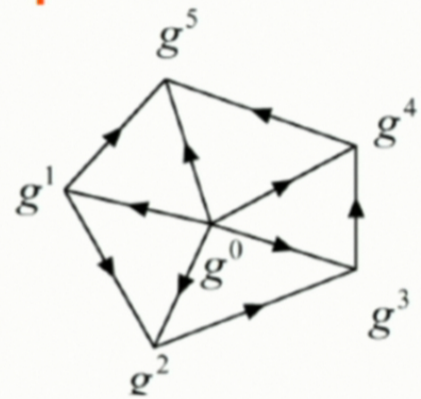

$$g : |\{g_i\}_M\rangle \to |\{g g_i\}_M\rangle, g \in G$$

$$\Psi(\{g_i\}_M) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$$

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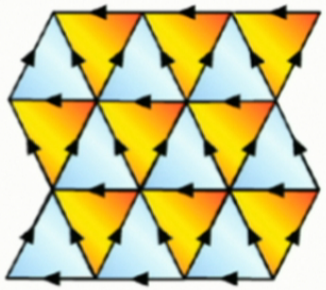

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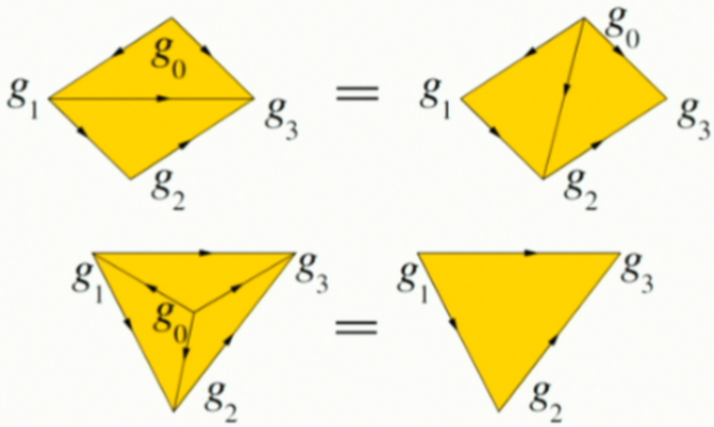

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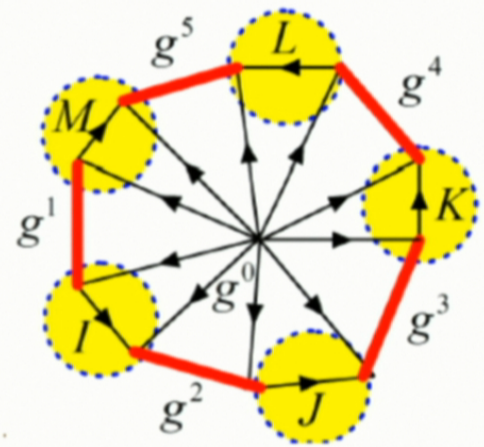
Fixed point wavefunction



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Classifications of bosonic SPT phases

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

Z_2^T means time reversal $T^2 = 1$

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An almost complete classification, except for time reversal symmetry or (non-local)one-form symmetry in 3D

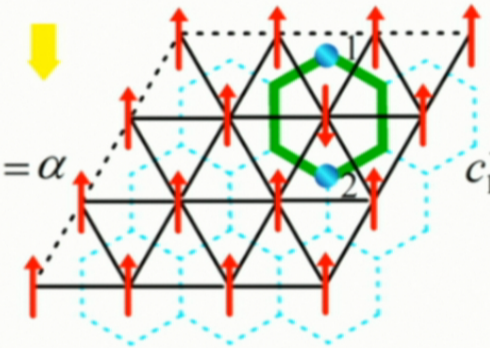
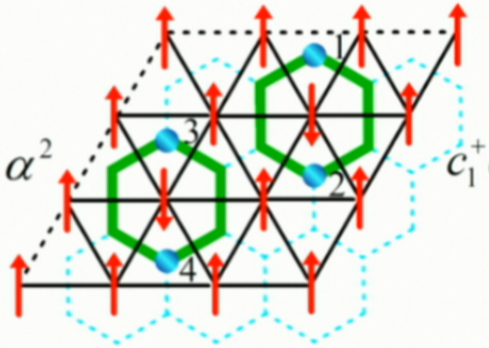
(Alexei Kitaev, private communications 2012,
Ashvin Vishwanath, T. Senthil, Phys. Rev. X 3, 011016 (2013)
Anton Kapustin, arXiv:1403.1467, arXiv:1404.6659
Peng Ye, Z C Gu, arXiv:1410.2594, Davide Gaiotto *et al,* arXiv:1412.5148)


Classifications of bosonic SPT phases
PERIMETER INSTITUTE

Basic concepts of classifying SPT phases in interacting fermion systems

- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry.
(Xie Chen, Z C Gu, X G Wen, Phys. Rev. B 84, 235128 (2011))

An example of intrinsic fermionic Ising SPT phase in 2D

$$\left|\Psi_f\right\rangle = \alpha$$  $$c_1^+ c_2^+ + \alpha^2$$  $$c_1^+ c_2^+ c_3^+ c_4^+ + \ldots$$

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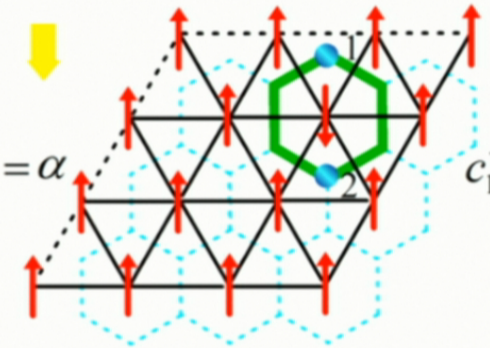
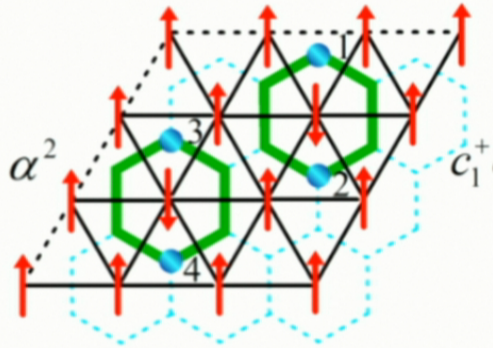
$|\Psi_f\rangle = \alpha$ 

$c_1^+ c_2^+ + \alpha^2$ 

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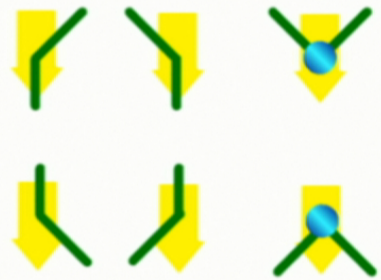
Domain decoration rule:



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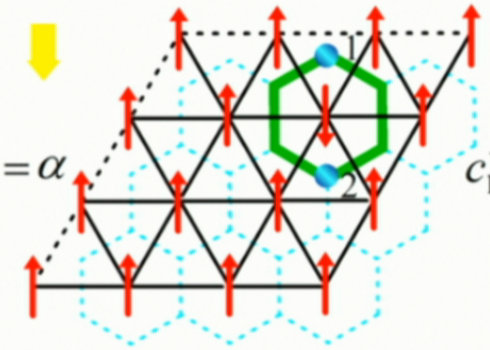
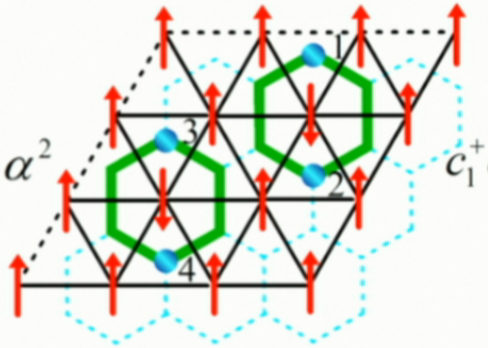
$$|\Psi_f\rangle = \alpha$$  $$c_1^+ c_2^+ + \alpha^2$$  $$c_1^+ c_2^+ c_3^+ c_4^+ + \ldots$$

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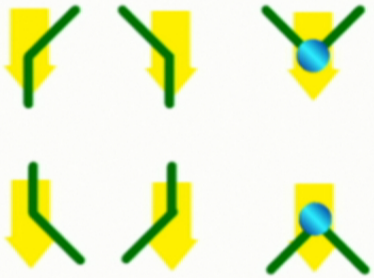
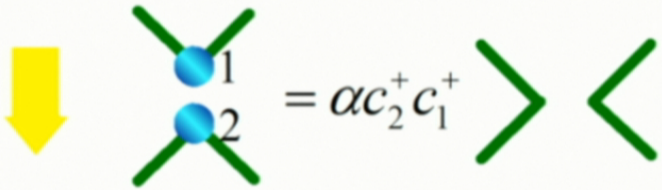
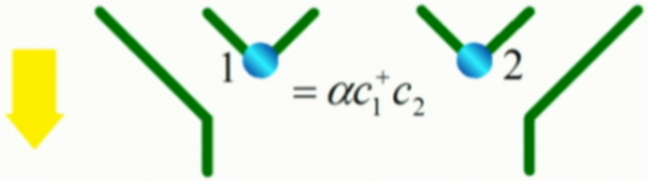


Domain deformation rule:



$$= \alpha c_2^+ c_1^+$$



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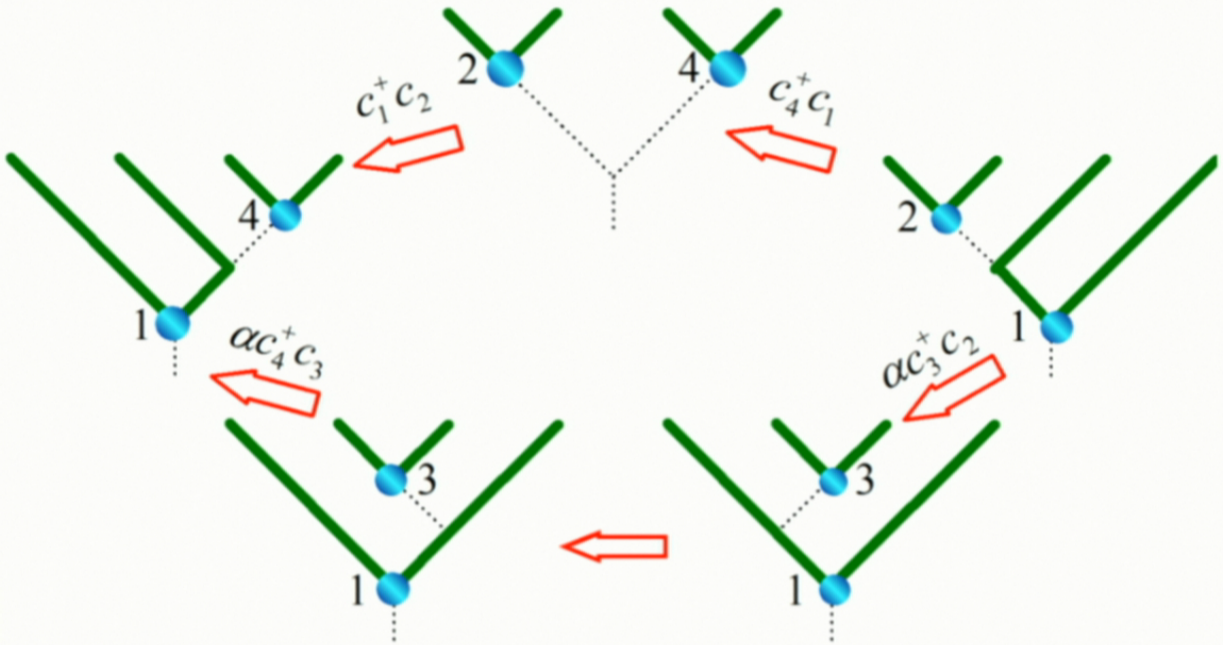
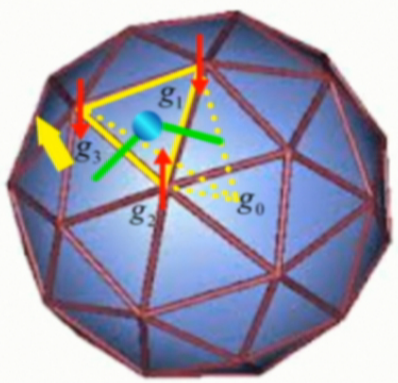
Topologically consistent condition for fixed point wavefunction



$$\alpha^2 c_4^+ c_3 c_3^+ c_2 = c_1^+ c_2 c_4^+ c_1 \Rightarrow \alpha^2 c_4^+ c_2 = c_2 c_4^+ \Rightarrow \alpha = \pm i$$

The concept of Grassmann valued topological nonlinear sigma model

The domain decoration picture for wavefunction implies Grassmann graded amplitude for partition function



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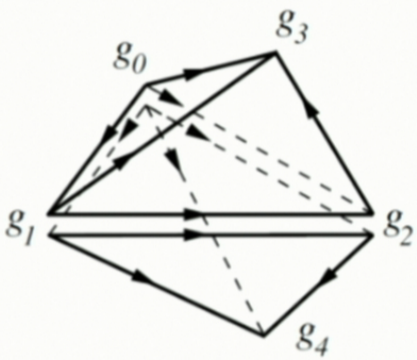

$$\mathcal{V}_3^+(g_0,g_1,g_2,g_3) = \nu_3^+(g_0,g_1,g_2,g_3)\times$$
$$\theta_{(1,2,3)}^{n_2(g_1,g_2,g_3)}\theta_{(0,1,3)}^{n_2(g_0,g_1,g_3)}\bar{\theta}_{(0,2,3)}^{n_2(g_0,g_2,g_3)}\bar{\theta}_{(0,1,2)}^{n_2(g_0,g_1,g_2)}$$

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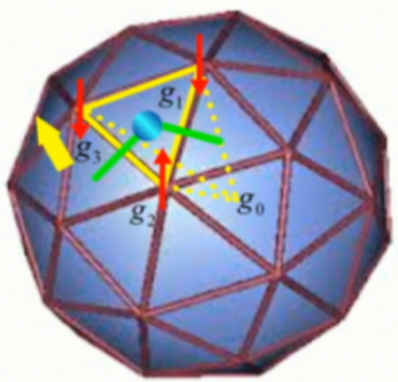

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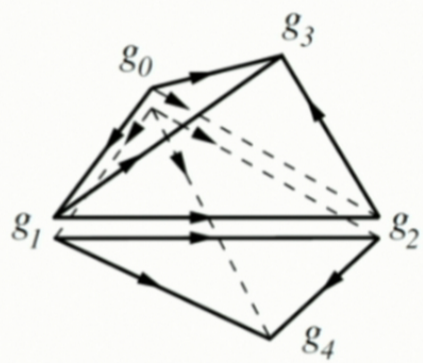

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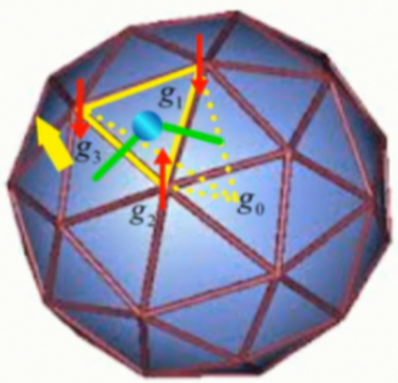

Total symmetry

$$G = G_b \otimes Z_2^f$$

Z_2 graded structure

$$n_{d-1}(g_i, g_j, ..., g_k) = 0, 1$$

$$\sum_{i=0}^{d} n_{d-1}(g_0, ..., \hat{g}_i, ..., g_d) = \text{ even}$$

Fermionic topological nonlinear sigma model

$$Z = \sum_{\{g_i\}} \int_{\mathrm{in}(\Sigma)} \prod_{[ab\ldots c]} \mathcal{V}_d^{s(a,b,\ldots,c)}$$

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Super co-cycle condition (consistent domain deformation rules)

Topological invariant conditions enforce $\nu_{d+1}^{\pm}$ can be expressed by m_{d-1} and ν_{d+1} that satisfies:

$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_{d+2}) = (-)^{f_{d+2}}$$

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Example in 2+1D:

$$\nu_3^+(g_0,g_1,g_2,g_3) = (-)^{m_1(g_0,g_2)}\nu_3(g_0,g_1,g_2,g_3),$$
$$\nu_3^-(g_0,g_1,g_2,g_3) = (-)^{m_1(g_1,g_3)}/\nu_3(g_0,g_1,g_2,g_3)$$

$$f_4(g_0,g_1,\ldots,g_4) = n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)$$

The inequivalent solution of n_d can be classified by $\mathcal{H}^d[G_b,\mathbb{Z}_2]$.

A (special) group super-cohomology theory

Compute group super-cohomology class by using short exact sequence

d_{sp}	short exact sequence
0	$0 \to \mathcal{H}^1[G_b, U_T(1)] \to \mathscr{H}^1[G_f, U_T(1)] \to \mathbb{Z}_2 \to 0$
1	$0 \to \mathcal{H}^2[G_b, U_T(1)] \to \mathscr{H}^2[G_f, U_T(1)] \to \mathcal{H}^1(G_b, \mathbb{Z}_2) \to 0$
2	$0 \to \mathcal{H}^3[G_b, U_T(1)] \to \mathscr{H}^3[G_f, U_T(1)] \to B\mathcal{H}^2(G_b, \mathbb{Z}_2) \to 0$
3	$0 \to \mathcal{H}^4_{\text{rigid}}[G_b, U_T(1)] \to \mathscr{H}^4[G_f, U_T(1)] \to B\mathcal{H}^3(G_b, \mathbb{Z}_2) \to 0$

$0 \to A \to B \to C \to 0$ **means** $C = B/A$

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A valid graded structure must be obstruction free:

$$B\mathcal{H}^d[G_b, \mathbb{Z}_2] \equiv \{n_d | n_d \in \mathcal{H}^d[G_b, \mathbb{Z}_2] \text{ and } (-)^{f_{d+2}} \in \mathcal{B}^{d+2}[G_b, U(1)]\}$$

f_{d+2} is the Steenrod square Sq^2 of n_d, which maps:

$$n_d \in \mathcal{H}^d(G_b, \mathbb{Z}_2) \to f_{d+2} \in \mathcal{H}^{d+2}(G_b, \mathbb{Z}_2)$$

Towards a complete classification

- The (special) group super cohomology class give rise to a complete classification for fermionic SPT phases in 1D, but not in 2D/3D.

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- Recently, we developed a (general) group supercohomology theory in 2D, which in principle gives rise to a complete classification of fermionic SPT phases in 2D.

(Meng Cheng, Zhen Bi, Yi-Zhuang You, and Zheng-Cheng Gu , arXiv:1501.01313, Zheng-Cheng Gu and Meng Cheng, to appear)

$$0 \to \mathscr{H}^3[G_f, U_T(1)] \to \mathscr{H}^3_{\text{general}}[G_f, U_T(1)] \to H^1(G_b, \mathbb{Z}_2) \to 0$$

Implication for quantum anomaly

Each SPT phase uniquely defines a gauge anomaly or gravitational-gauge mixture anomaly on its boundary.

SPT invariants $\mathbf{Z}_0(\text{sym.twist}) = \mathrm{e}^{\mathrm{i}\mathbf{S}_0(\text{sym.twist})} = \mathrm{e}^{\mathrm{i}\mathbf{S}_0(A)}$

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U(1) symmetry:

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[i\frac{2\pi k}{(\frac{d+2}{2})!(2\pi)^{(d+2)/2}}\int A\wedge F\wedge\ldots],$$ **ABJ anomaly**

Conclusion and future directions:

- We propose to use group (super)cohomology theory to classify SPT phases in interacting boson(fermion) systems. (complete in 1D/2D)

- By studying C,P,T protected topological superconductor, we find that a pair of topological Majorana zero modes in a topological defect carry fractionalized C,P,T symmetries, with $T^4=-1$,$(TP)^4=-1$,$(TC)^4=-1$. We propose to use these topological defects to explain the origin of three generations of neutrinos and their mass mixing matrix. (Z C Gu, arXiv:1308.2488, 1403.1869)

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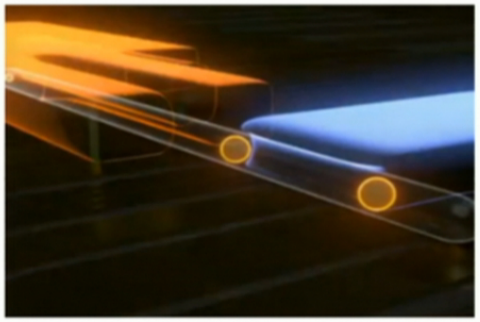
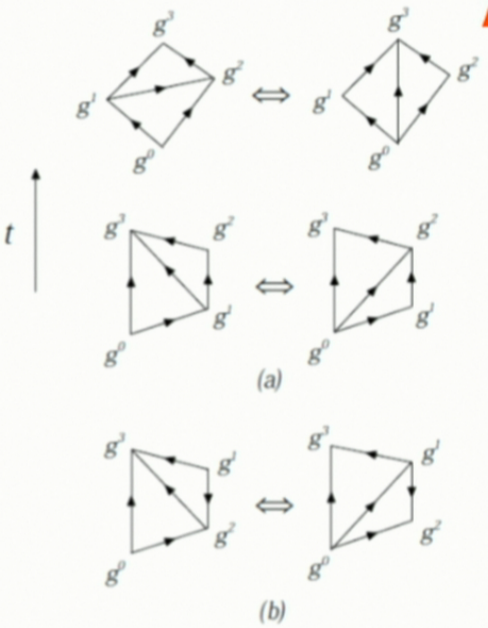
- SPT phases protected by supersymmetry.

- Fermionic TQFT and quantum theory of gravity.

- Realizing SPT phases in experiments.





A topological world,
A new civilization!

Thank you