

Title: Classifications of symmetry protected topological phases in interacting boson/fermion systems

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Abstract: <p>Symmetry protected topological (SPT) states are bulk gapped states with gapless edge excitations. The SPT phases in free fermion systems, like topological insulators, can be classified by K-theory. However, it is not yet known what SPT phases exist in general interacting systems. In this talk, I will first present a systematic way to construct SPT phases in interacting bosonic systems, which allows us to identify many new SPT phases. Just as group theory allows us to construct 230 crystal structures in three dimensions, we find that group cohomology theory allows us to construct many interacting bosonic SPT phases. In my talk, I shall show how topological terms in the path integral description of the system can be constructed from nontrivial group cohomology classes, giving rise to exactly soluble Hamiltonians with explicit ground state wavefunctions. Next, I will discuss the generalization of the classifying scheme to interacting fermionic systems and a new mathematical framework â€“ group supercohomology theory, which predicts a fermionic SPT phase that can neither be realized in free fermionic nor interacting bosonic systems.</p>

<p>Finally, I will briefly mention the deep relationship between SPT phases and chiral anomalies in high energy physics.</p>

# **Classification of Symmetry Protected Topological Phases in Interacting Systems**

**Zhengcheng Gu (P.I.)**

**Collaborators:**

**Prof. Xiao-Gang Wen (PI/MIT)    Prof. M. Levin (U. of Chicago)**  
**Dr. Xie Chen(Caltech)            Dr. Zheng-Xin Liu(Tsinghua U.)**  
**Dr. Meng Chen(Station-Q)        Dr. Peng Ye (Perimeter Institute)**

**PI. Jan. 2015**

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PI, Jun 2015

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# Why do we need a classification?

## Periodic table in chemistry:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18																									
1 <b>H</b> Hydrogen 1.00794	<table border="1"> <tr> <td><b>C</b> Solid</td> <td><b>Hg</b> Liquid</td> <td><b>H</b> Gas</td> <td><b>Rf</b> Unknown</td> </tr> </table>																<b>C</b> Solid	<b>Hg</b> Liquid	<b>H</b> Gas	<b>Rf</b> Unknown	2 <b>He</b> Helium 4.002602																					
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3 <b>Li</b> Lithium 6.941	4 <b>Be</b> Beryllium 9.012182	<table border="1"> <tr> <td><b>Al</b> Alkali metals</td> <td><b>Mg</b> Alkaline earth metals</td> <td><b>La</b> Lanthanoids</td> <td><b>Sc</b> Transition metals</td> <td><b>Zn</b> Poor metals</td> <td><b>B</b> Other nonmetals</td> <td><b>C</b> Other nonmetals</td> <td><b>N</b> Other nonmetals</td> <td><b>O</b> Other nonmetals</td> <td><b>F</b> Other nonmetals</td> <td><b>Ne</b> Noble gases</td> </tr> <tr> <td></td> <td></td> <td><b>Ac</b> Actinoids</td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> </tr> </table>										<b>Al</b> Alkali metals	<b>Mg</b> Alkaline earth metals	<b>La</b> Lanthanoids	<b>Sc</b> Transition metals	<b>Zn</b> Poor metals	<b>B</b> Other nonmetals	<b>C</b> Other nonmetals	<b>N</b> Other nonmetals	<b>O</b> Other nonmetals	<b>F</b> Other nonmetals	<b>Ne</b> Noble gases			<b>Ac</b> Actinoids									10 <b>Ne</b> Neon 20.1797	11 <b>Na</b> Sodium 22.98976928	12 <b>Mg</b> Magnesium 24.304	13 <b>B</b> Boron 10.811	14 <b>Si</b> Silicon 28.0855	15 <b>P</b> Phosphorus 30.973761998	16 <b>S</b> Sulfur 32.06	17 <b>Cl</b> Chlorine 35.453	18 <b>Ar</b> Argon 39.948
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19 <b>K</b> Potassium 39.0983	20 <b>Ca</b> Calcium 40.078	21 <b>Sc</b> Scandium 44.955912	22 <b>Ti</b> Titanium 47.867	23 <b>V</b> Vanadium 50.9415	24 <b>Cr</b> Chromium 51.9961	25 <b>Mn</b> Manganese 54.938044	26 <b>Fe</b> Iron 55.845	27 <b>Co</b> Cobalt 58.933195	28 <b>Ni</b> Nickel 58.6934	29 <b>Cu</b> Copper 63.546	30 <b>Zn</b> Zinc 65.38	31 <b>Ga</b> Gallium 69.723	32 <b>Ge</b> Germanium 72.64	33 <b>As</b> Arsenic 74.9216	34 <b>Se</b> Selenium 78.96	35 <b>Br</b> Bromine 79.904	36 <b>Kr</b> Krypton 83.796																									
37 <b>Rb</b> Rubidium 85.4678	38 <b>Sr</b> Strontium 87.62	39 <b>Y</b> Yttrium 88.90584	40 <b>Zr</b> Zirconium 91.224	41 <b>Nb</b> Niobium 92.90638	42 <b>Mo</b> Molybdenum 95.94	43 <b>Tc</b> Technetium (97.9072)	44 <b>Ru</b> Ruthenium 101.07	45 <b>Rh</b> Rhodium 102.90550	46 <b>Pd</b> Palladium 106.42	47 <b>Ag</b> Silver 107.8682	48 <b>Cd</b> Cadmium 112.411	49 <b>In</b> Indium 114.818	50 <b>Sn</b> Tin 118.710	51 <b>Sb</b> Antimony 121.757	52 <b>Te</b> Tellurium 127.6	53 <b>I</b> Iodine 126.905	54 <b>Xe</b> Xenon 131.29																									
55 <b>Cs</b> Cesium 132.90545196	56 <b>Ba</b> Barium 137.327	57-71 <b>Lanthanoids</b>	72 <b>Hf</b> Hafnium 178.49	73 <b>Ta</b> Tantalum 180.94788	74 <b>W</b> Tungsten 183.84	75 <b>Re</b> Rhenium 186.207	76 <b>Os</b> Osmium 190.23	77 <b>Ir</b> Iridium 192.222	78 <b>Pt</b> Platinum 195.084	79 <b>Au</b> Gold 196.966569	80 <b>Hg</b> Mercury 200.59	81 <b>Tl</b> Thallium 204.3833	82 <b>Pb</b> Lead 207.2	83 <b>Bi</b> Bismuth 208.9804	84 <b>Po</b> Polonium (209)	85 <b>At</b> Astatine (210)	86 <b>Rn</b> Radon 222.01753																									
87 <b>Fr</b> Francium (223)	88 <b>Ra</b> Radium (226)	89-103 <b>Actinoids</b>	104 <b>Rf</b> Rutherfordium (261)	105 <b>Db</b> Dubnium (262)	106 <b>Sg</b> Seaborgium (263)	107 <b>Bh</b> Bohrium (264)	108 <b>Hs</b> Hassium (277)	109 <b>Mt</b> Meitnerium (268)	110 <b>Ds</b> Darmstadtium (271)	111 <b>Rg</b> Roentgenium (272)	112 <b>Uub</b> Ununbium (285)	113 <b>Uut</b> Ununtrium (284)	114 <b>Uuq</b> Ununquadium (289)	115 <b>Uup</b> Ununpentium (288)	116 <b>Uuh</b> Ununhexium (289)	117 <b>Uus</b> Ununseptium (286)	118 <b>Uuo</b> Ununoctium (284)																									

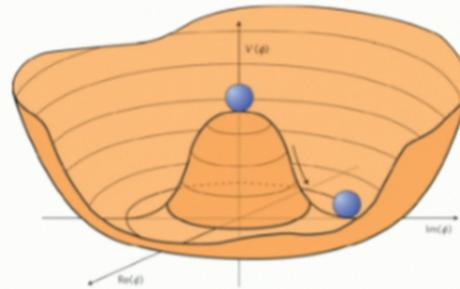
For elements with no stable isotopes, the mass number of the isotope with the longest half-life is in parentheses.

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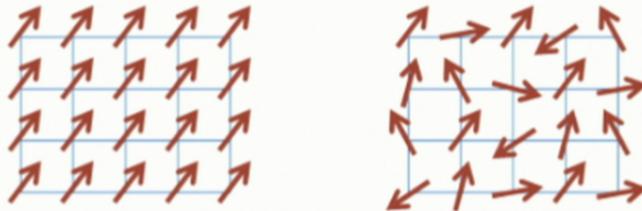
57 <b>La</b> Lanthanum 138.9047	58 <b>Ce</b> Cerium 140.12	59 <b>Pr</b> Praseodymium 140.90766	60 <b>Nd</b> Neodymium 144.242	61 <b>Pm</b> Promethium (145)	62 <b>Sm</b> Samarium 150.36	63 <b>Eu</b> Europium 151.964	64 <b>Gd</b> Gadolinium 157.25	65 <b>Tb</b> Terbium 158.92532	66 <b>Dy</b> Dysprosium 162.500	67 <b>Ho</b> Holmium 164.93032	68 <b>Er</b> Erbium 167.255	69 <b>Tm</b> Thulium 168.93402	70 <b>Yb</b> Ytterbium 173.054	71 <b>Lu</b> Lutetium 174.967
89 <b>Ac</b> Actinium (227)	90 <b>Th</b> Thorium 232.0377	91 <b>Pa</b> Protactinium 231.03688	92 <b>U</b> Uranium 238.02891	93 <b>Np</b> Neptunium (237)	94 <b>Pu</b> Plutonium (244)	95 <b>Am</b> Americium (243)	96 <b>Cm</b> Curium (247)	97 <b>Bk</b> Berkelium (247)	98 <b>Cf</b> Californium (251)	99 <b>Es</b> Einsteinium (252)	100 <b>Fm</b> Fermium (257)	101 <b>Md</b> Mendelevium (258)	102 <b>No</b> Nobelium (259)	103 <b>Lr</b> Lawrencium (260)

# The Landau paradigm of phases and phase transitions -- Symmetry Breaking

Symmetry breaking theory:



- Magnetic orders in spin systems



- Superconductivity



The underlying mathematical framework is group theory

# Topological phases of quantum matter: beyond Landau's paradigm

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Gapped quantum phases without symmetry breaking and long range correlation, but can not be adiabatically connected to a trivial disorder phase without phase transition.

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Two basic classes of topological phases:

### Intrinsic topological phases (long-range-entanglement)

- adiabatical paths with no symmetry

### Symmetry protected topological (SPT) phases

- adiabatical paths with symmetry

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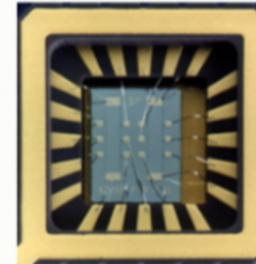
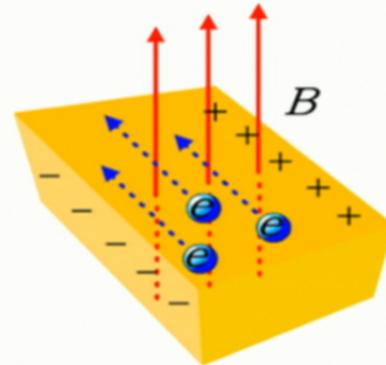
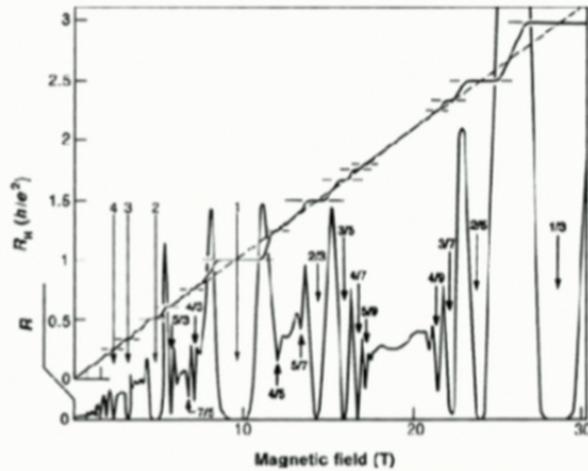
symmetry breaking Hamiltonians

SPT phases  The trivial disorder phase

(Z C Gu, X G Wen 2009)

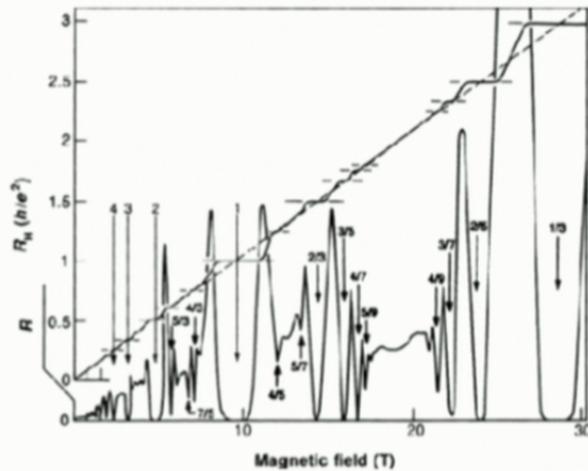
# Examples of intrinsic topological phases in interacting systems (no symmetry)

**Fractional Quantum Hall Effect (FQHE)** D C Tsui, *et al* 1982



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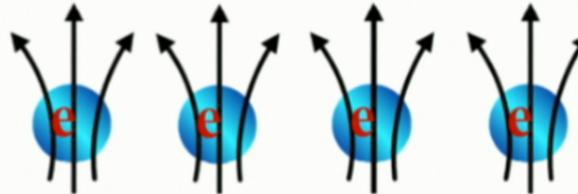
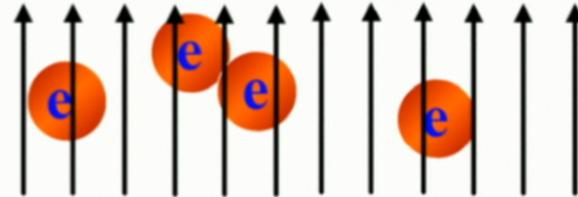
# Topological terms for intrinsic topological phases (no symmetry)

**FQHE**  $\Psi_3 = \prod_{i < j} (z_i - z_j)^3 e^{-\frac{1}{4} \sum_i |z_i|^2}$

$$\mathcal{L}_{\text{eff}} = \frac{2m + 1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$



R B Laughlin 1983  
 E Witten, 1989  
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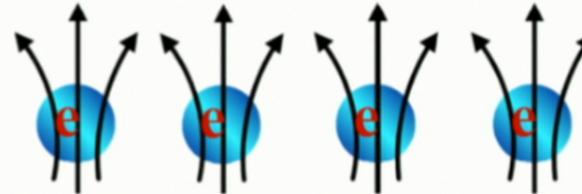
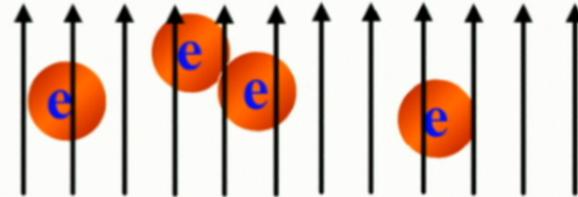
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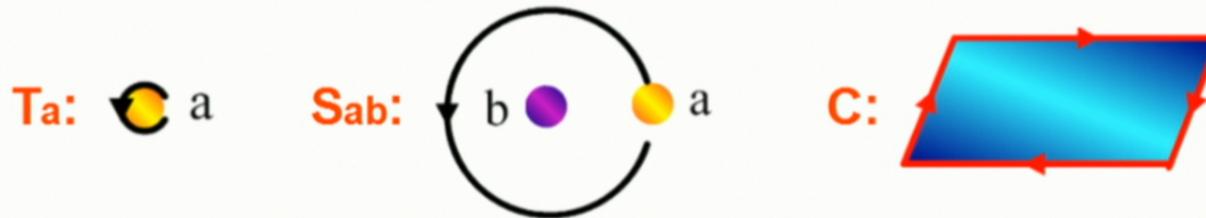
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R B Laughlin 1983  
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 S C Zhang, *et al* 1989  
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**Braiding T, S matrices and chiral central charge as the universal data of topological order (no symmetry).**



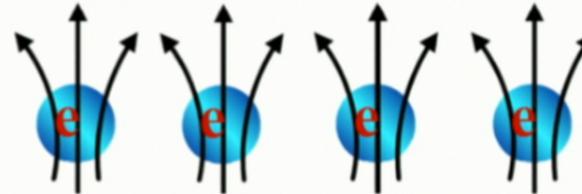
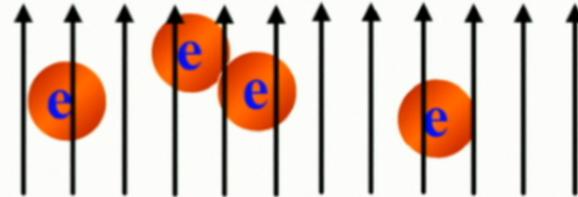
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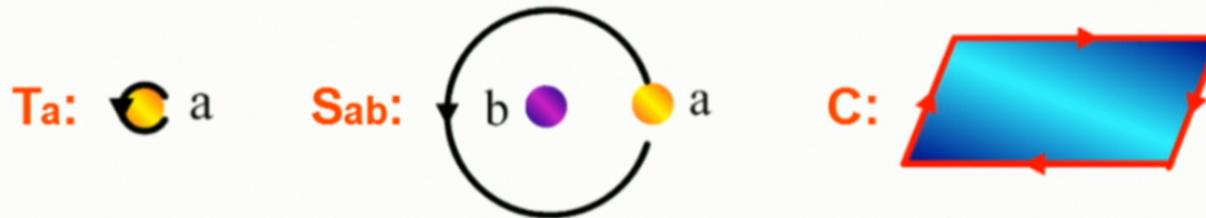
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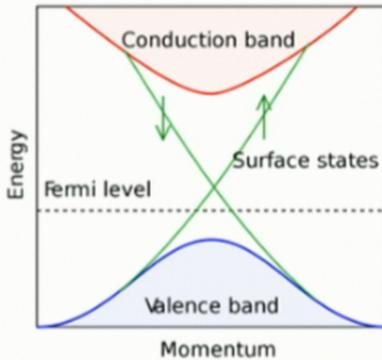
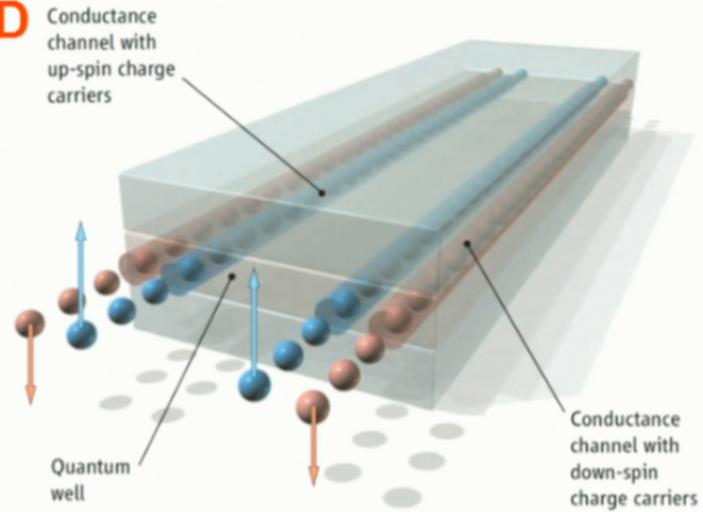
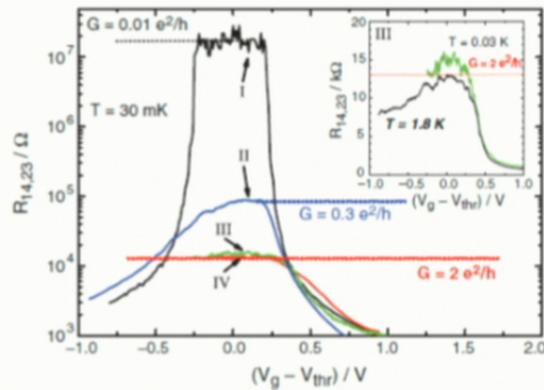
**Braiding T, S matrices and chiral central charge as the universal data of topological order (no symmetry).**



**The mathematical framework for topological phases is known as unitary modular tensor category theory.**

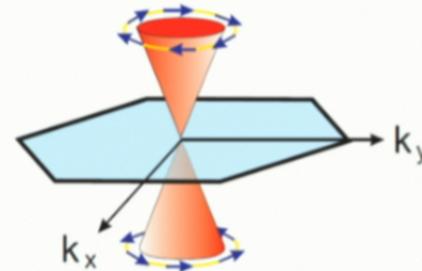
# Examples of symmetry protected topological (SPT) phases in free fermion systems

## Topological insulator in 2D/3D



C L Kane, *et al*, 2005  
 B A Bernevig, *et al* 2006  
 W Molenkamp's group 2007  
 M Zahid Hasan, *et al*, 2008

from Wikipedia



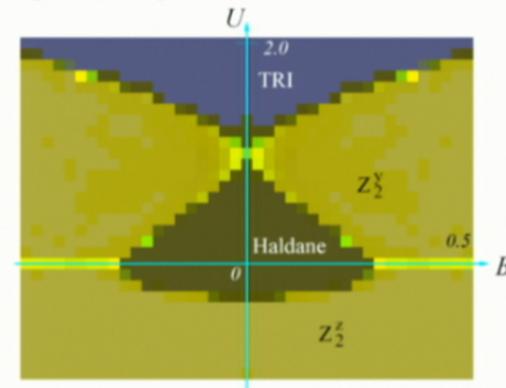
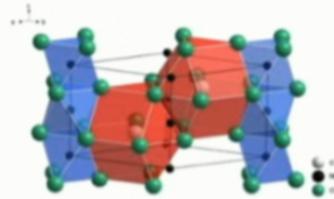
# Examples of SPT phases in interacting models

Spin one Haldane chain realizes 1D topological order

$$H = \sum_i (S_i \cdot S_{i+1} + U(S_i^z)^2 + BS_i^x)$$

$U \sim 1 (B=0)$

$\text{CsNiCl}_3 (U \sim B \sim 0)$



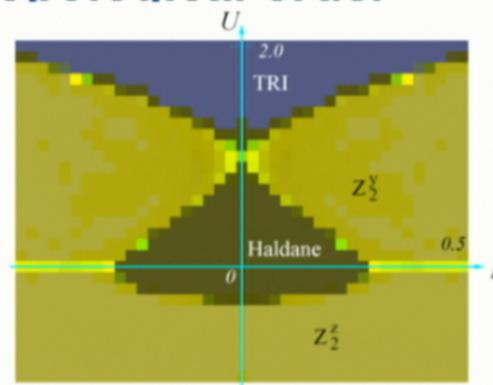
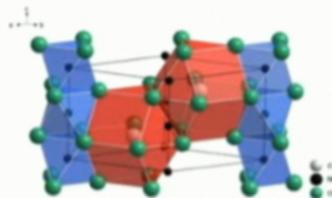
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**Haldane phase requires symmetry!**

- Haldane phase can be protected by many kinds of symmetries: time reversal, spin rotation, etc...

Z C Gu and X G Wen, 2009, F Pollmann, *et al*, 2010

**Fixed point wavefunction: spin-(1/2,1/2) dimer model**



# Projective representation and its (second) group cohomology classification

## Projective representation on left/right dimer ends

$$u_L(g_1)u_L(g_2) = \omega(g_1, g_2)u_L(g_1, g_2); \quad u_R(g_1)u_R(g_2) = \omega(g_1, g_2)^{-1}u_R(g_1, g_2)$$

- Associativity and consistency -- the 2-cocycle group:

$$\mathcal{Z}^2[G, U(1)] = \{\omega \in U(1) | \omega(g_2, g_3)\omega(g_1, g_2g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3)\}$$

- Equivalent projective representation-- 2-coboundary group:

$$u_{L(R)}(g) \sim \beta_{L(R)}(g)u_{L(R)}(g); \quad \beta_{L(R)}(g) \in U(1)$$

$$\mathcal{B}^2[G, U(1)] = \{\omega \in U(1) | \omega(g_1, g_2) = \beta(g_1)\beta(g_2)/\beta(g_1g_2); \beta \in U(1)\}$$

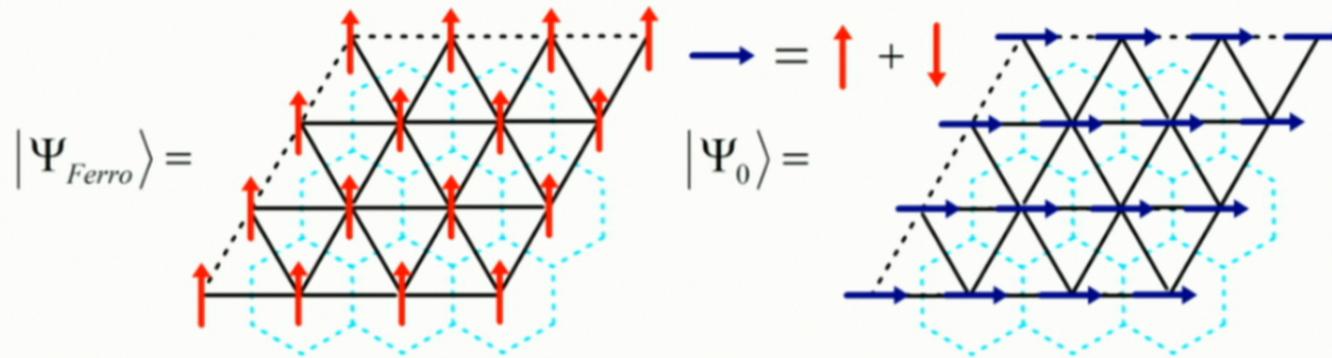
**Projective representation is classified by the quotient group -- the second group cohomology class**

$$\mathcal{H}^2[G, U(1)] = \mathcal{Z}^2[G, U(1)]/\mathcal{B}^2[G, U(1)]$$

**How to classify SPT phases in interacting spin/bosonic systems in higher dimensions with arbitrary (internal) symmetry group  $G$ ?**

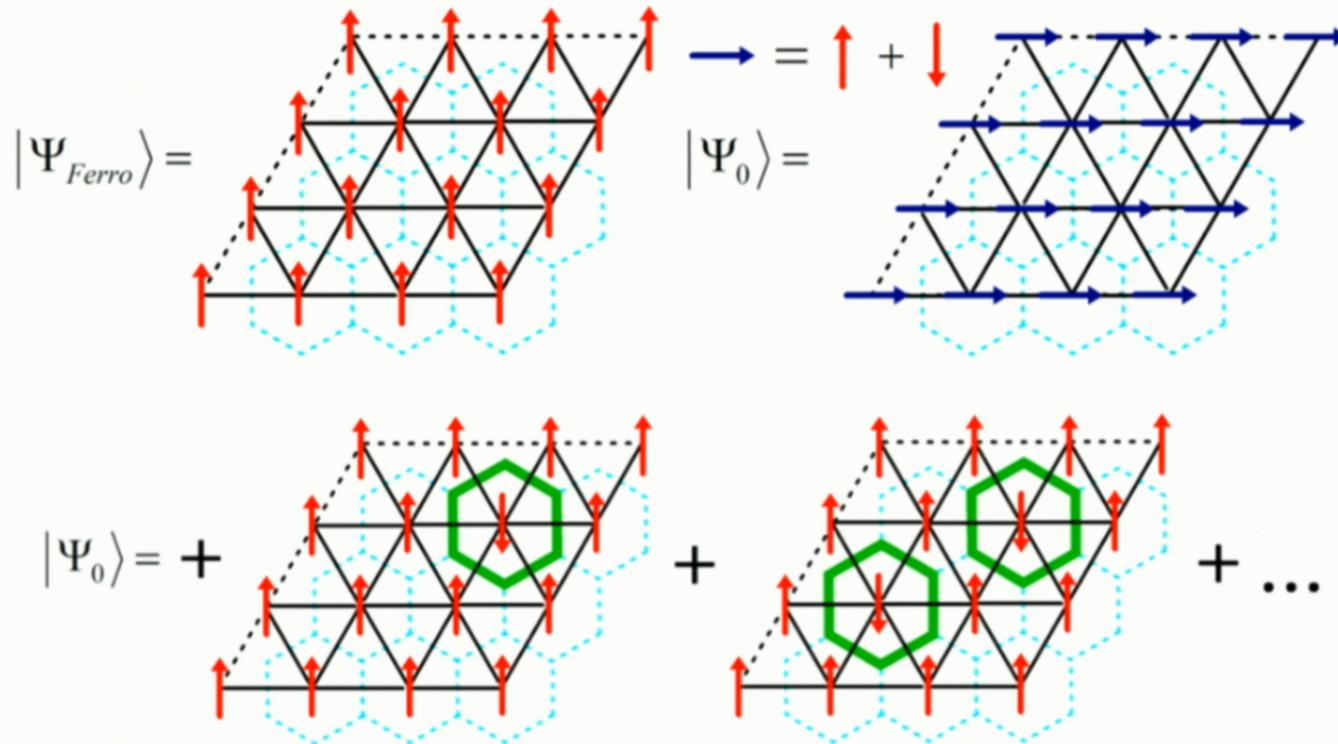
# A revisit of transverse Ising model:

$$H = - \sum_{\langle pq \rangle} \sigma_p^z \sigma_q^z - t \sum_p \sigma_p^x$$



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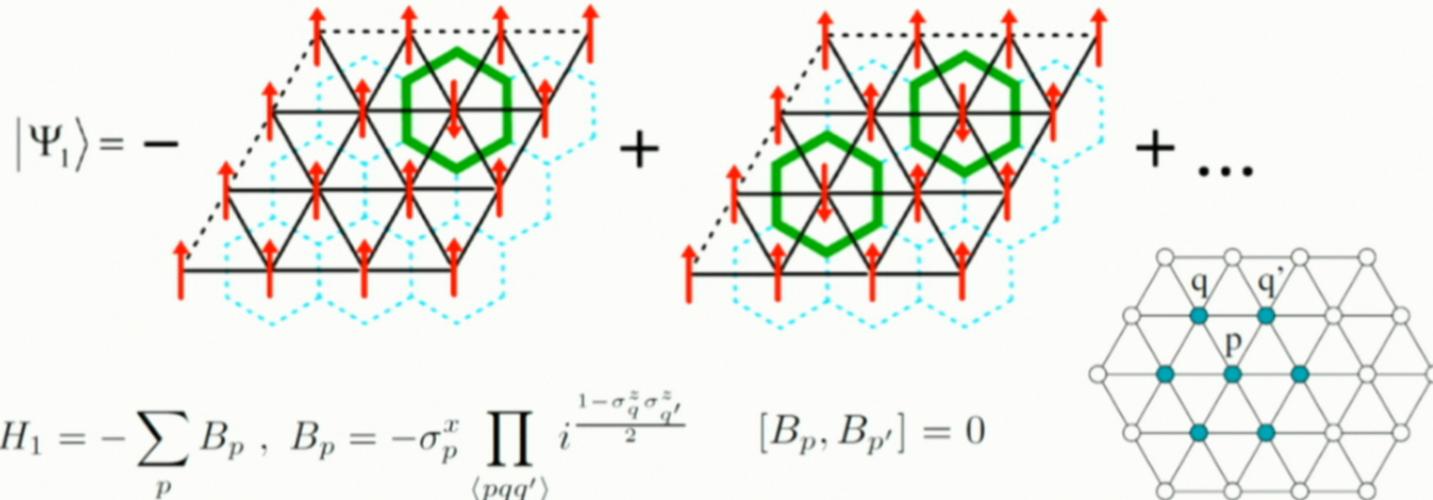
# An example of Ising SPT phase in 2D

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How many different paramagnetic phases?

Two!

(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

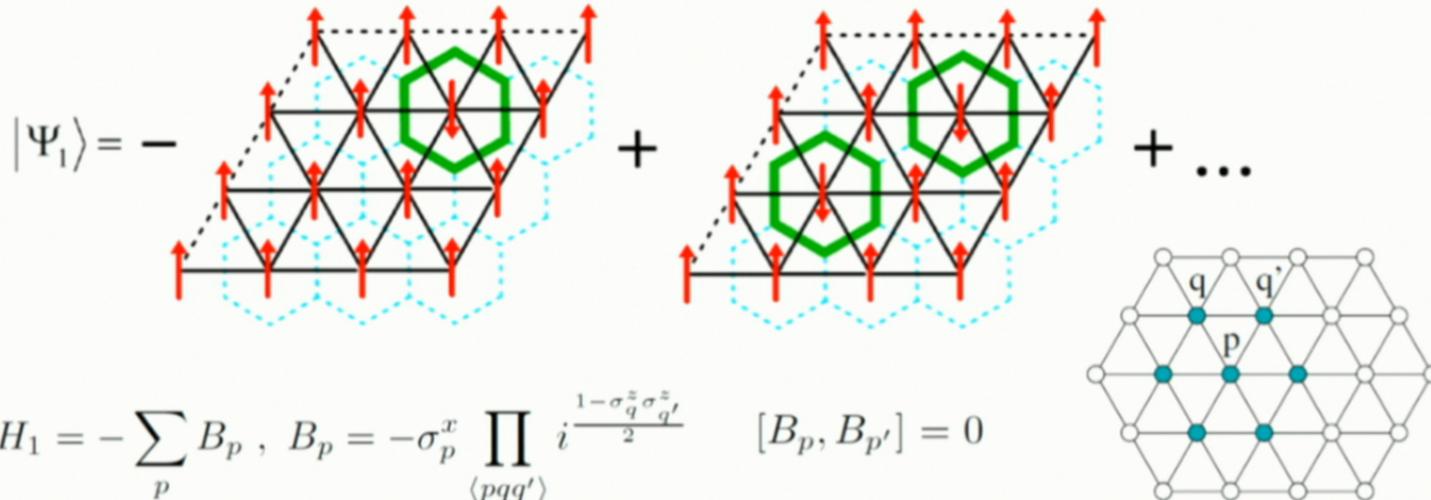


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Domain deformation rule

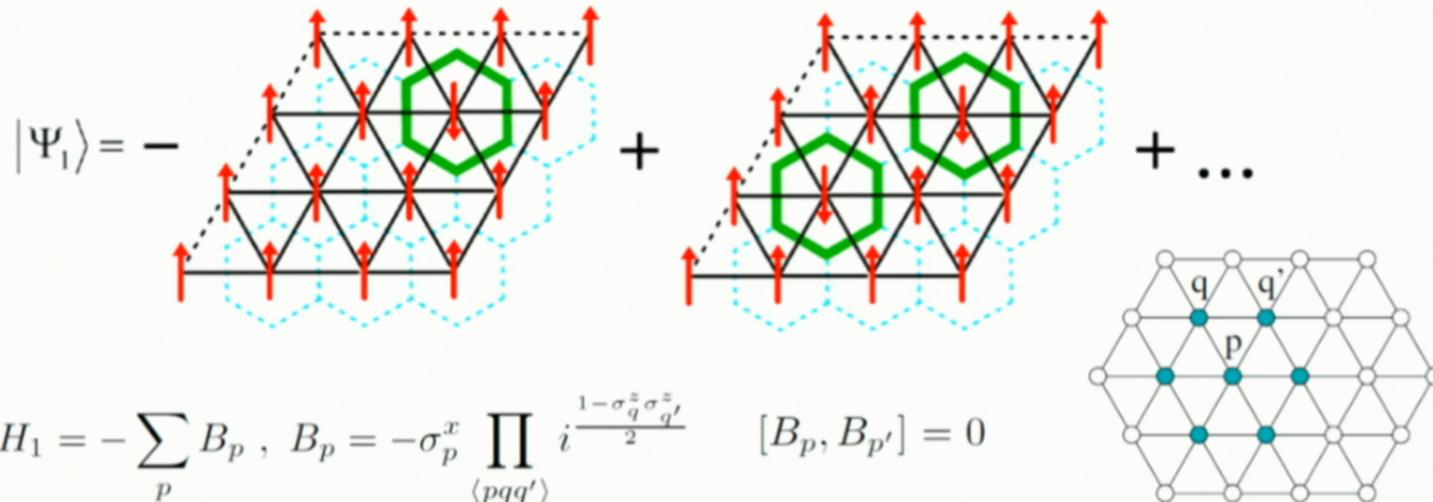


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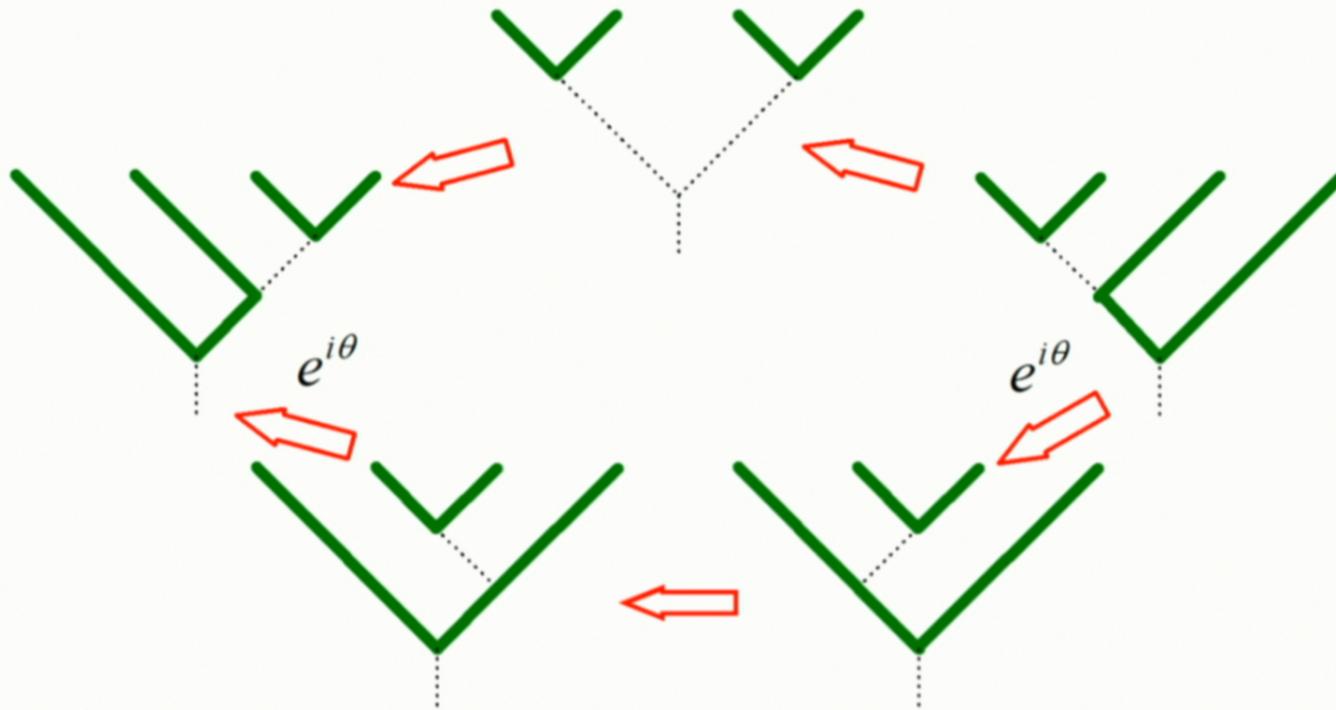


Domain deformation rule

But why not?



# Topologically consistent condition for fixed point wavefunction



$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

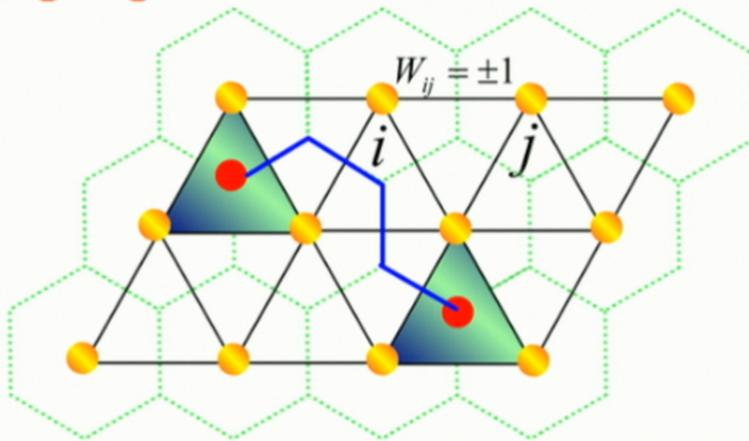
# **Bulk response and the nature of gapless edge**

Assume that Ising spins carry  $Z_2$  gauge charge and can couple to background  $Z_2$  gauge field

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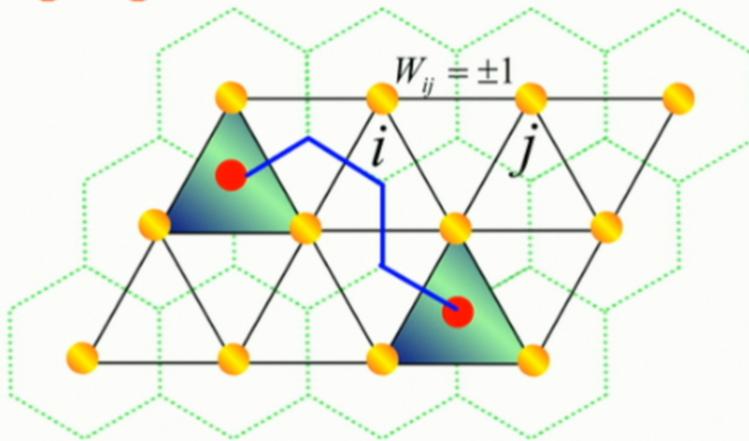
**$Z_2$  gauge flux carries semion statistics!**



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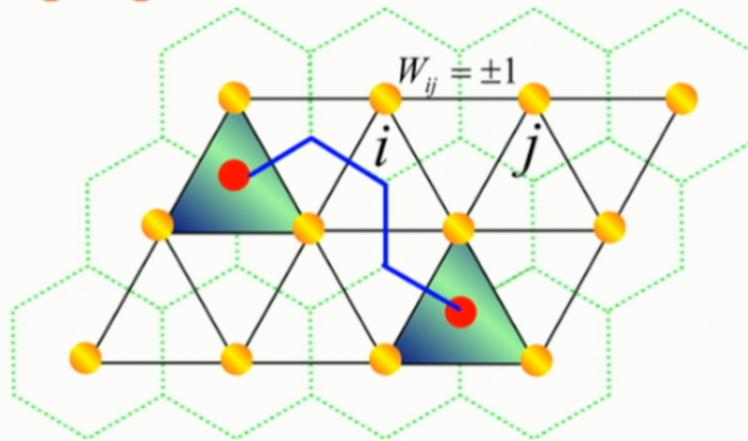
$$\widetilde{W}_\beta |0\rangle = |0\rangle$$

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# Bulk response and the nature of gapless edge

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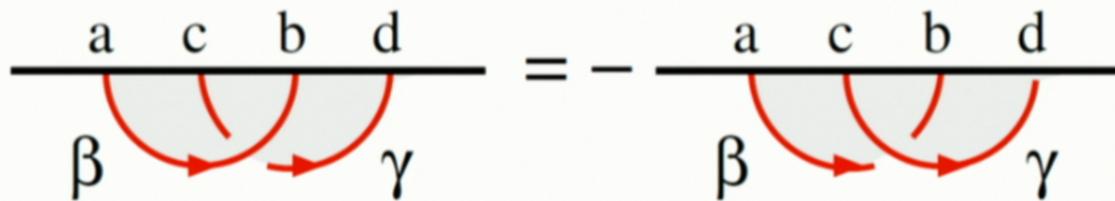
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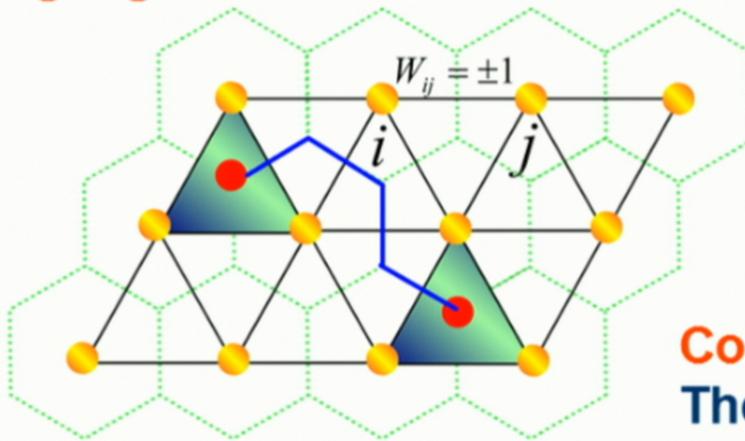
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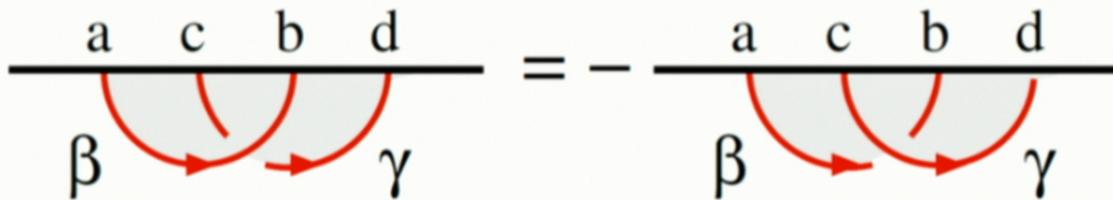
$$\widetilde{W}_\gamma |0\rangle = |0\rangle$$

$$\widetilde{W}_\beta \widetilde{W}_\gamma = -\widetilde{W}_\gamma \widetilde{W}_\beta$$

**Contradiction**

**There is No 1D representation!**

**Non-trivial statistics of flux leads to degenerate edge states!**



## Do we have a systematic way to classify bosonic SPT phases?

- In-equivalent projective representations are classified by **second** group cohomology class, which classifies all 1D bosonic SPT phases. (Xie Chen, Z C Gu, X G Wen PRB 83, 035107,2011)
- In-equivalent flux statistics of  $G$  are classified by **third** group cohomology class, which classifies all 2D bosonic SPT phases. (R. Dijkgraaf and E. Witten, 1990)
- **Conjecture:** Flux line statistics of  $G$  are classified by **fourth** group cohomology class, which classifies all 3D bosonic SPT phases. (C Wang and M Levin, PRL113, 080403 (2014))

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**Topological nonlinear sigma model in discrete space-time to describe/classify SPT phases!**

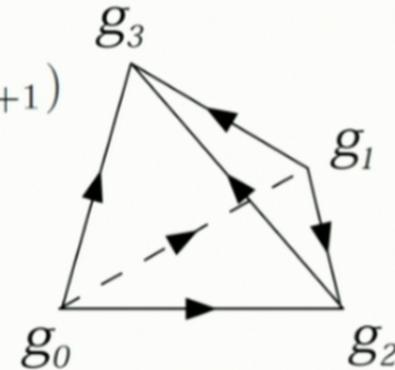
X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen (Science 338, 1604 (2012) )

# (bosonic) SPT phases in any dimensions with any symmetry

$$Z = \frac{1}{|G|^{N_v}} \sum_{\{g_i\}} \prod_{d+1\text{-simplex}} \nu_{d+1}^{s_{01\dots d}}(g_0, g_1, \dots, g_{d+1})$$

- Branched(vertex ordered)  $d+1$ -simplex

$$\nu_{d+1} : G \times G \times \dots \times G \mapsto U(1)$$



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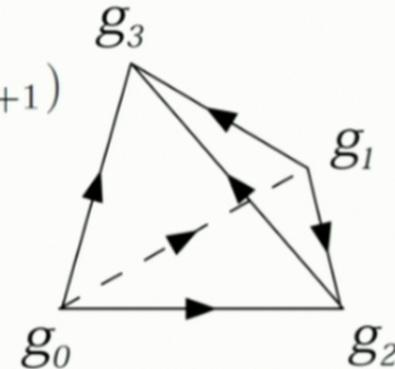
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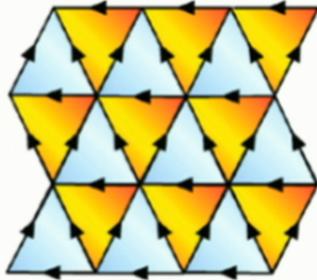
$$\nu_{d+1} : G \times G \times \dots \times G \mapsto U(1)$$

$$\nu_{d+1}(gg_0, gg_1, \dots, gg_{d+1}) = \nu_{d+1}(g_0, g_1, \dots, g_{d+1})$$

co-cycle condition: 
$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{d+2}) = 1$$



## An example of 1+1D case

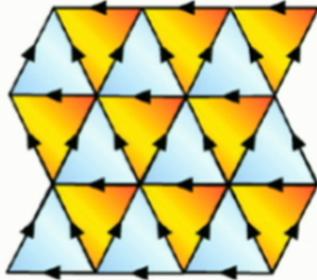


$$Z = |G|^{-N_v} \sum_{\{g_i\}} e^{-S(\{g_i\})}$$

$$e^{-S(\{g_i\})} = \prod_{\{ijk\}} \nu_2^{s_{ijk}}(g_i, g_j, g_k)$$

$$\omega(g_1, g_2) = \nu_2(E, g_1, g_1 g_2)$$

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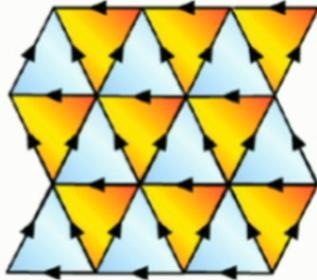


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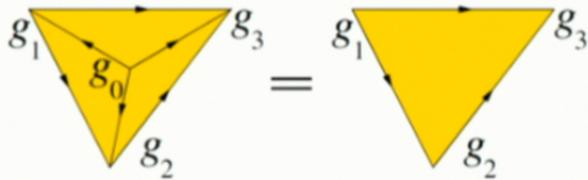
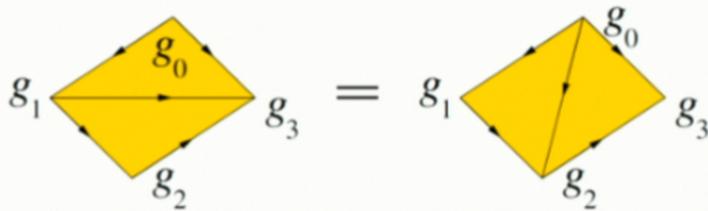


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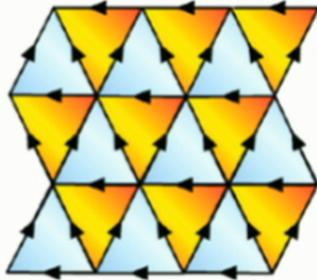
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### Topological invariant

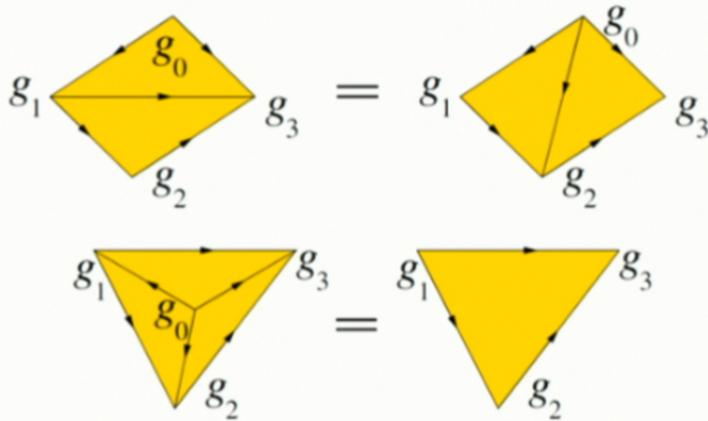


$$\frac{\nu_2(g_1, g_2, g_3) \nu_2(g_0, g_1, g_3)}{\nu_2(g_0, g_2, g_3) \nu_2(g_0, g_1, g_2)} = 1$$

# An example of 1+1D case



## Topological invariant



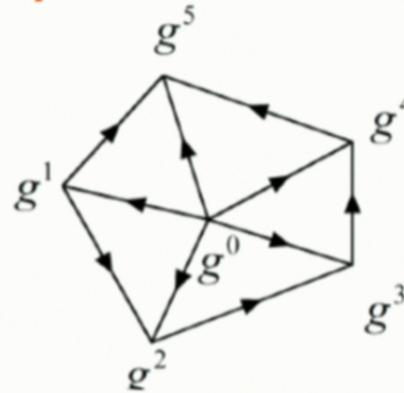
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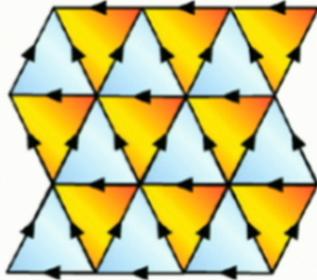
## Fixed point wavefunction



$$g : |\{g_i\}_M\rangle \rightarrow |\{gg_i\}_M\rangle, g \in G$$

$$\Psi(\{g_i\}_M) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$$

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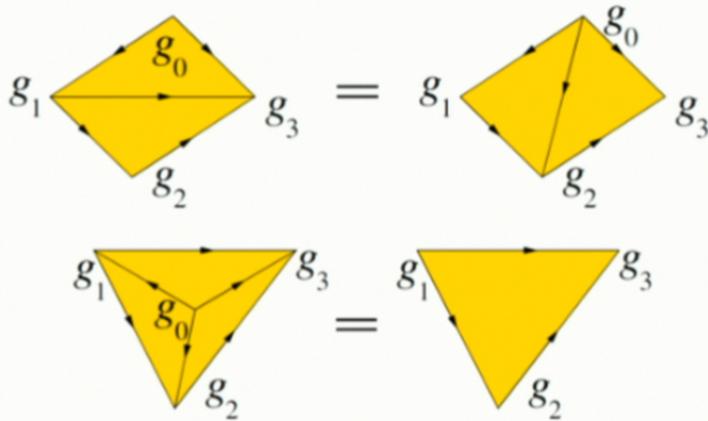


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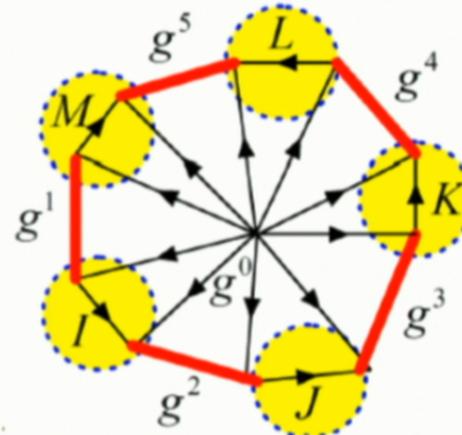
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# Classifications of bosonic SPT phases

Symm. group	$d = 0$	$d = 1$	$d = 2$	$d = 3$
$U(1) \times Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
$Z_n$	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

$Z_2^T$  means time reversal  $T^2 = 1$

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**An almost complete classification, except for time reversal symmetry or (non-local) one-form symmetry in 3D**

(Alexei Kitaev, private communications 2012,

Ashvin Vishwanath, T. Senthil, Phys. Rev. X 3, 011016 (2013)

Anton Kapustin, arXiv:1403.1467, arXiv:1404.6659

Peng Ye, Z C Gu, arXiv:1410.2594, Davide Gaiotto *et al*, arXiv:1412.5148)

### Classifications of bosonic SPT phases

System group	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \times \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$SO(2n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$SO(2n) \times \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times U(1) \times U(1)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$

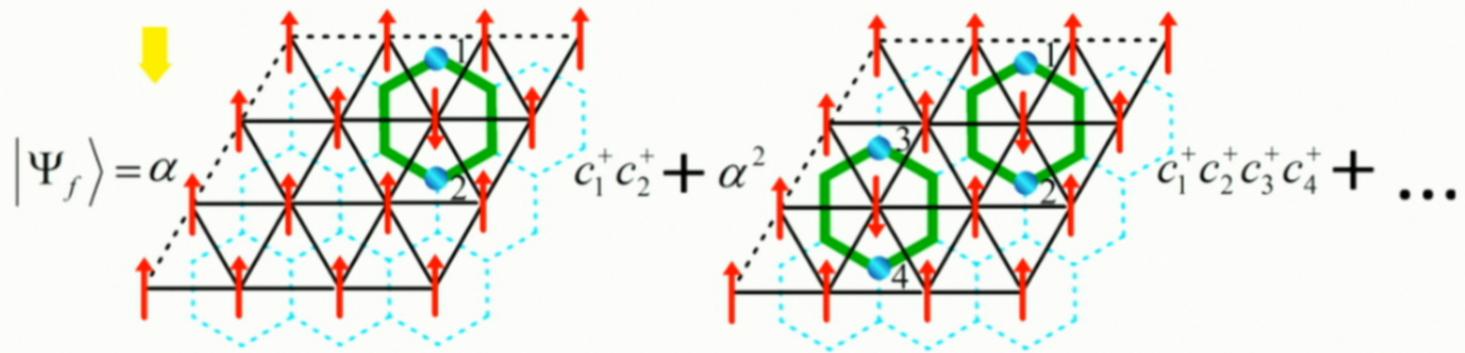
$\mathbb{Z}$  means time reversal  $T^2 = -1$   
 An identical classification is obtained for the general case of  $U(1) \times G$  where  $G$  is a compact Lie group.  
 (Alexei Kitaev, arXiv:1010.4258 [cond-mat.str-el] 2010)  
 Andrew Strominger, J. High Energy Phys. 1008 (2010) 099  
 Andrew Kitaev, arXiv:1003.4354 [cond-mat.str-el] 2010  
 Peng Ye, Z. C. Fan, arXiv:1410.2008 [cond-mat.str-el] arXiv:1410.2008



# Basic concepts of classifying SPT phases in interacting fermion systems

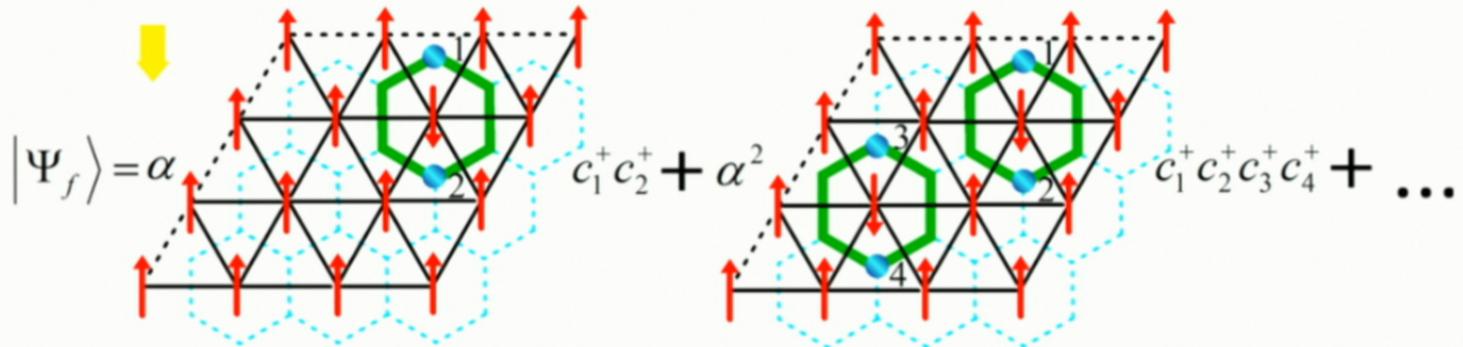
- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry.  
(Xie Chen, Z C Gu, X G Wen, Phys. Rev. B 84, 235128 (2011))

# An example of intrinsic fermionic Ising SPT phase in 2D



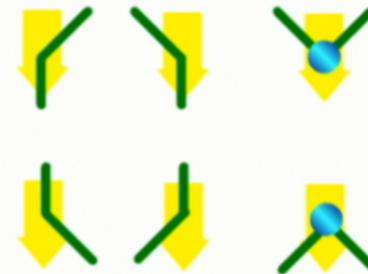
$$\alpha = \pm i$$

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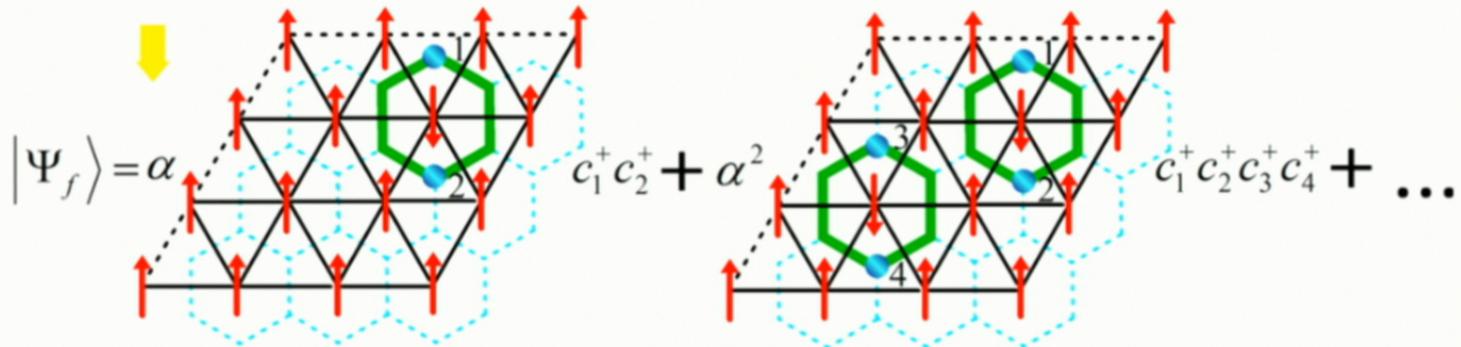


$\alpha = \pm i$

Domain decoration rule:

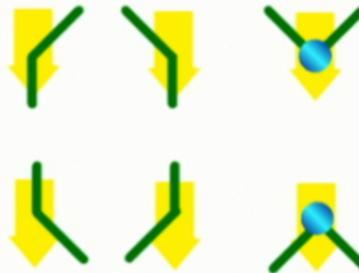


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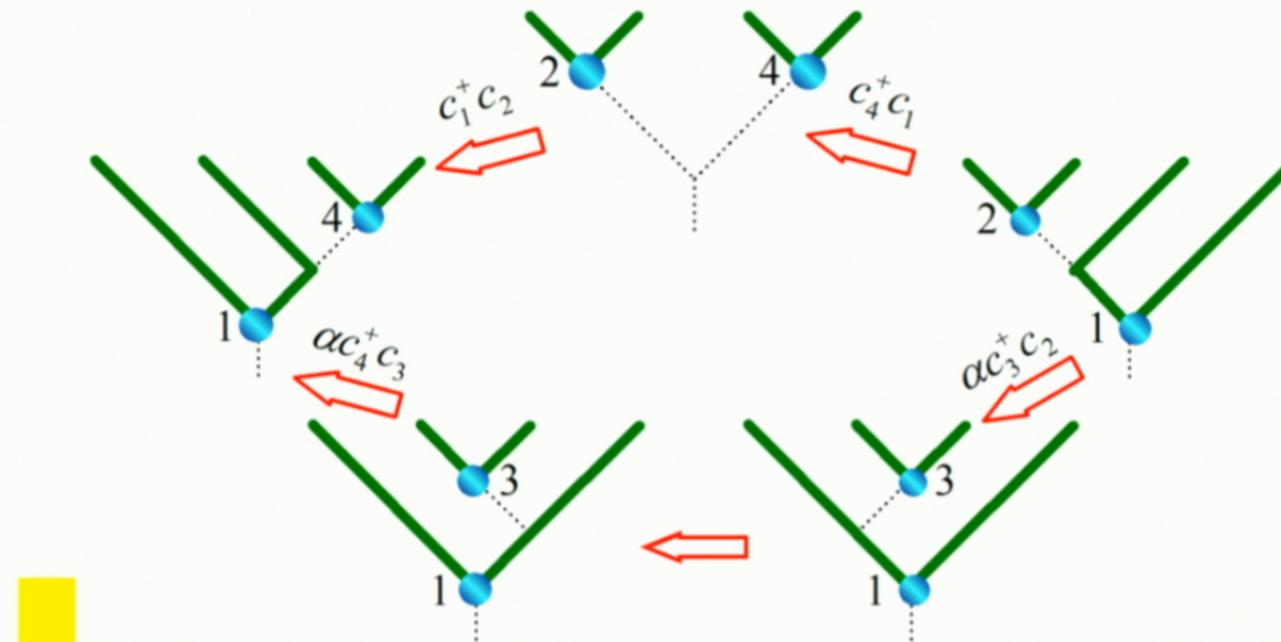
Domain decoration rule:



Domain deformation rule:



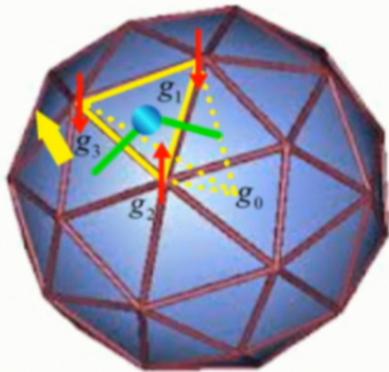
# Topologically consistent condition for fixed point wavefunction



$$\alpha^2 c_4^+ c_3 c_3^+ c_2 = c_1^+ c_2 c_4^+ c_1 \Rightarrow \alpha^2 c_4^+ c_2 = c_2 c_4^+ \Rightarrow \alpha = \pm i$$

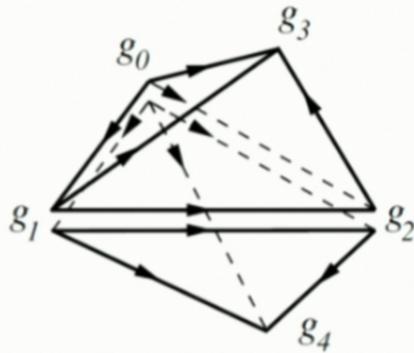
# The concept of Grassmann valued topological nonlinear sigma model

The domain decoration picture for wavefunction implies Grassmann graded amplitude for partition function



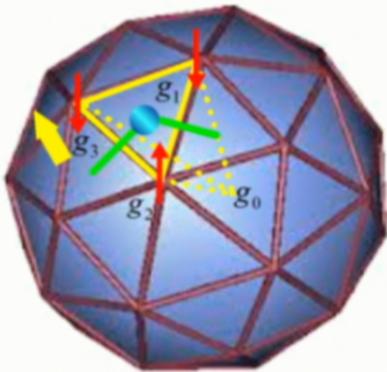
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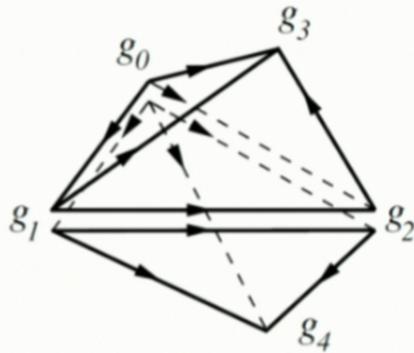
$$\mathcal{V}_3^+(g_0, g_1, g_2, g_3) = \nu_3^+(g_0, g_1, g_2, g_3) \times \theta_{(1,2,3)}^{n_2(g_1, g_2, g_3)} \theta_{(0,1,3)}^{n_2(g_0, g_1, g_3)} \bar{\theta}_{(0,2,3)}^{n_2(g_0, g_2, g_3)} \bar{\theta}_{(0,1,2)}^{n_2(g_0, g_1, g_2)}$$

$$\mathcal{V}_3^-(g_0, g_1, g_2, g_4) = \nu_3^-(g_0, g_1, g_2, g_4) \times \theta_{(0,1,2)}^{n_2(g_0, g_1, g_2)} \theta_{(0,2,4)}^{n_2(g_0, g_2, g_4)} \bar{\theta}_{(0,1,4)}^{n_2(g_0, g_1, g_4)} \bar{\theta}_{(1,2,4)}^{n_2(g_1, g_2, g_4)}$$



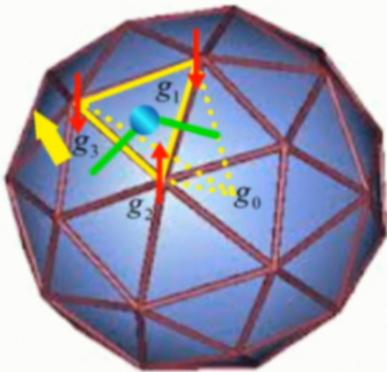
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$$\mathcal{V}_3^+(g_0, g_1, g_2, g_3) = \nu_3^+(g_0, g_1, g_2, g_3) \times \theta_{(1,2,3)}^{n_2(g_1, g_2, g_3)} \theta_{(0,1,3)}^{n_2(g_0, g_1, g_3)} \bar{\theta}_{(0,2,3)}^{n_2(g_0, g_2, g_3)} \bar{\theta}_{(0,1,2)}^{n_2(g_0, g_1, g_2)}$$

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**Total symmetry**

**Z<sub>2</sub> graded structure**

$$G = G_b \otimes Z_2^f$$

$$n_{d-1}(g_i, g_j, \dots, g_k) = 0, 1$$

$$\sum_{i=0}^d n_{d-1}(g_0, \dots, \hat{g}_i, \dots, g_d) = \text{even}$$

## Fermionic topological nonlinear sigma model

$$Z = \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab\dots c]} \mathcal{V}_d^{s(a,b,\dots,c)}$$

# Fermionic topological nonlinear sigma model

$$\begin{aligned}
 Z &= \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab\dots c]} \mathcal{V}_d^{s(a,b,\dots,c)} \\
 &\equiv \int \prod_{(ij\dots k)} d\theta_{(ij\dots k)}^{n_{d-1}(g_i, g_j, \dots, g_k)} d\bar{\theta}_{(ij\dots k)}^{n_{d-1}(g_i, g_j, \dots, g_k)} \times \\
 &\quad \prod_{\{xy\dots z\}} (-)^{m_{d-2}(g_x, g_y, \dots, g_z)} \prod_{[ab\dots c]} \mathcal{V}_d^{s(a,b,\dots,c)}(g_a, g_b, \dots, g_c)
 \end{aligned}$$

$$\begin{aligned}
 &n_{d-1}(g_1, g_2, \dots, g_d) \\
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 \end{aligned}$$

## Super co-cycle condition (consistent domain deformation rules)

Topological invariant conditions enforce  $\nu_{d+1}^\pm$  can be expressed by  $m_{d-1}$  and  $\nu_{d+1}$  that satisfies:

$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{d+2}) = (-)^{f_{d+2}}$$

# Fermionic topological nonlinear sigma model

$$\begin{aligned}
 Z &= \sum_{\{g_i\}} \int_{\text{in}(\Sigma)} \prod_{[ab\dots c]} \mathcal{V}_d^{s(a,b,\dots,c)} \\
 &\equiv \int \prod_{(ij\dots k)} d\theta_{(ij\dots k)}^{n_{d-1}(g_i, g_j, \dots, g_k)} d\bar{\theta}_{(ij\dots k)}^{n_{d-1}(g_i, g_j, \dots, g_k)} \times \\
 &\quad \prod_{\{xy\dots z\}} (-)^{m_{d-2}(g_x, g_y, \dots, g_z)} \prod_{[ab\dots c]} \mathcal{V}_d^{s(a,b,\dots,c)}(g_a, g_b, \dots, g_c)
 \end{aligned}$$

$$\begin{aligned}
 &n_{d-1}(g_1, g_2, \dots, g_d) \\
 &= \sum_{i=1}^d m_{d-2}(g_1, \dots, \hat{g}_i, \dots, g_d) \pmod{2}
 \end{aligned}$$

## Super co-cycle condition (consistent domain deformation rules)

Topological invariant conditions enforce  $\nu_{d+1}^\pm$  can be expressed by  $m_{d-1}$  and  $\nu_{d+1}$  that satisfies:

$$\prod_{i=0}^{d+1} \nu_{d+1}^{(-)^i}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{d+2}) = (-)^{f_{d+2}}$$

## Example in 2+1D:

$$\nu_3^+(g_0, g_1, g_2, g_3) = (-)^{m_1(g_0, g_2)} \nu_3(g_0, g_1, g_2, g_3),$$

$$\nu_3^-(g_0, g_1, g_2, g_3) = (-)^{m_1(g_1, g_3)} / \nu_3(g_0, g_1, g_2, g_3)$$

$$f_4(g_0, g_1, \dots, g_4) = n_2(g_0, g_1, g_2) n_2(g_2, g_3, g_4)$$

The inequivalent solution of  $n_d$  can be classified by  $\mathcal{H}^d[G_b, \mathbb{Z}_2]$ .

## A (special) group super-cohomology theory

Compute group super-cohomology class by using short exact sequence

$d_{sp}$	short exact sequence
0	$0 \rightarrow \mathcal{H}^1[G_b, U_T(1)] \rightarrow \mathcal{H}^1[G_f, U_T(1)] \rightarrow \mathbb{Z}_2 \rightarrow 0$
1	$0 \rightarrow \mathcal{H}^2[G_b, U_T(1)] \rightarrow \mathcal{H}^2[G_f, U_T(1)] \rightarrow \mathcal{H}^1(G_b, \mathbb{Z}_2) \rightarrow 0$
2	$0 \rightarrow \mathcal{H}^3[G_b, U_T(1)] \rightarrow \mathcal{H}^3[G_f, U_T(1)] \rightarrow B\mathcal{H}^2(G_b, \mathbb{Z}_2) \rightarrow 0$
3	$0 \rightarrow \mathcal{H}_{\text{rigid}}^4[G_b, U_T(1)] \rightarrow \mathcal{H}^4[G_f, U_T(1)] \rightarrow B\mathcal{H}^3(G_b, \mathbb{Z}_2) \rightarrow 0$

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  means  $C = B/A$

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2	$0 \rightarrow \mathcal{H}^3[G_b, U_T(1)] \rightarrow \mathcal{H}^3[G_f, U_T(1)] \rightarrow B\mathcal{H}^2(G_b, \mathbb{Z}_2) \rightarrow 0$
3	$0 \rightarrow \mathcal{H}_{\text{rigid}}^4[G_b, U_T(1)] \rightarrow \mathcal{H}^4[G_f, U_T(1)] \rightarrow B\mathcal{H}^3(G_b, \mathbb{Z}_2) \rightarrow 0$

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  means  $C = B/A$

**A valid graded structure must be obstruction free:**

$$B\mathcal{H}^d[G_b, \mathbb{Z}_2] \equiv \{n_d | n_d \in \mathcal{H}^d[G_b, \mathbb{Z}_2] \text{ and } (-)^{f_{d+2}} \in \mathcal{B}^{d+2}[G_b, U(1)]\}$$

$f_{d+2}$  is the Steenrod square  $Sq^2$  of  $n_d$ , which maps:

$$n_d \in \mathcal{H}^d(G_b, \mathbb{Z}_2) \rightarrow f_{d+2} \in \mathcal{H}^{d+2}(G_b, \mathbb{Z}_2)$$

## **Towards a complete classification**

- The (special) group super cohomology class give rise to a complete classification for fermionic SPT phases in 1D, but not in 2D/3D.

# Towards a complete classification

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- Recently, we developed a (general) group supercohomology theory in 2D, which in principle gives rise to a complete classification of fermionic SPT phases in 2D.

(Meng Cheng, Zhen Bi, Yi-Zhuang You, and Zheng-Cheng Gu , arXiv:1501.01313, Zheng-Cheng Gu and Meng Cheng, to appear)

$$0 \rightarrow \mathcal{H}^3[G_f, U_T(1)] \rightarrow \mathcal{H}_{\text{general}}^3[G_f, U_T(1)] \rightarrow H^1(G_b, \mathbb{Z}_2) \rightarrow 0$$

# Implication for quantum anomaly

Each SPT phase uniquely defines a gauge anomaly or gravitational-gauge mixture anomaly on its boundary.

**SPT invariants**  $Z_0(\text{sym.twist}) = e^{iS_0(\text{sym.twist})} = e^{iS_0(A)}$

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**U(1) symmetry:**

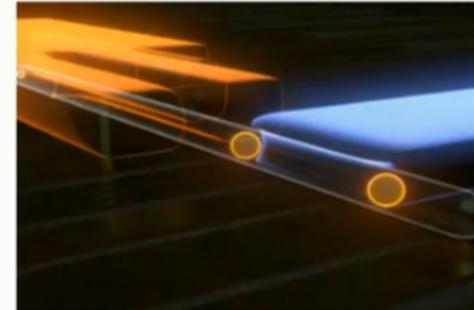
$$Z_0(\text{sym.twist}) = \exp\left[i \frac{2\pi k}{\left(\frac{d+2}{2}\right)! (2\pi)^{(d+2)/2}} \int A \wedge F \wedge \dots\right], \text{ **ABJ anomaly**}$$

## Conclusion and future directions:

- We propose to use group (super)cohomology theory to classify SPT phases in interacting boson(fermion) systems. (complete in 1D/2D)
- By studying C,P,T protected topological superconductor, we find that a pair of topological Majorana zero modes in a topological defect carry fractionalized C,P,T symmetries, with  $T^4=-1, (TP)^4=-1, (TC)^4=-1$ . We propose to use these topological defects to explain the origin of three generations of neutrinos and their mass mixing matrix. (Z C Gu, arXiv:1308.2488, 1403.1869)

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- SPT phases protected by supersymmetry.
- Fermionic TQFT and quantum theory of gravity.
- Realizing SPT phases in experiments.



**A topological world,  
A new civilization!**

Thank you

