Title: How much information can a physical system fundamentally communicate?

Date: Jan 27, 2015 03:30 PM

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Abstract: <p>After a brief motivation of this question, the presentation is divided in two parts. We first introduce the principle of quantum information causality, which states the maximum amount of quantum information that a transmitted quantum system can communicate as a function of its Hilbert space dimension, independently of any quantum physical resources previously shared by the communicating parties. The second part of the talk considers superdense coding within the framework of general probabilistic theories and addresses the question of why in quantum theory, no more than two bits can be communicated by transmission of a single, entangled, qubit. We introduce hyperdense coding in general probabilistic theories: superdense coding in which N</p>

<p>> 2 bits are communicated by transmission of a system that locally</p>

<p>encodes at most one bit, and present protocols with N arbitrarily large. Our hyperdense coding protocols imply superadditive classical</p>

<p>capacities: two entangled systems can encode N > 2 bits, even though each system locally encodes at most one bit. Our protocols violate either a reversibility condition or tomographic locality.</p>

How much information can a physical system fundamentally communicate?

Damián Pitalúa-García

Laboratoire d'Information Quantique, Université libre de Bruxelles

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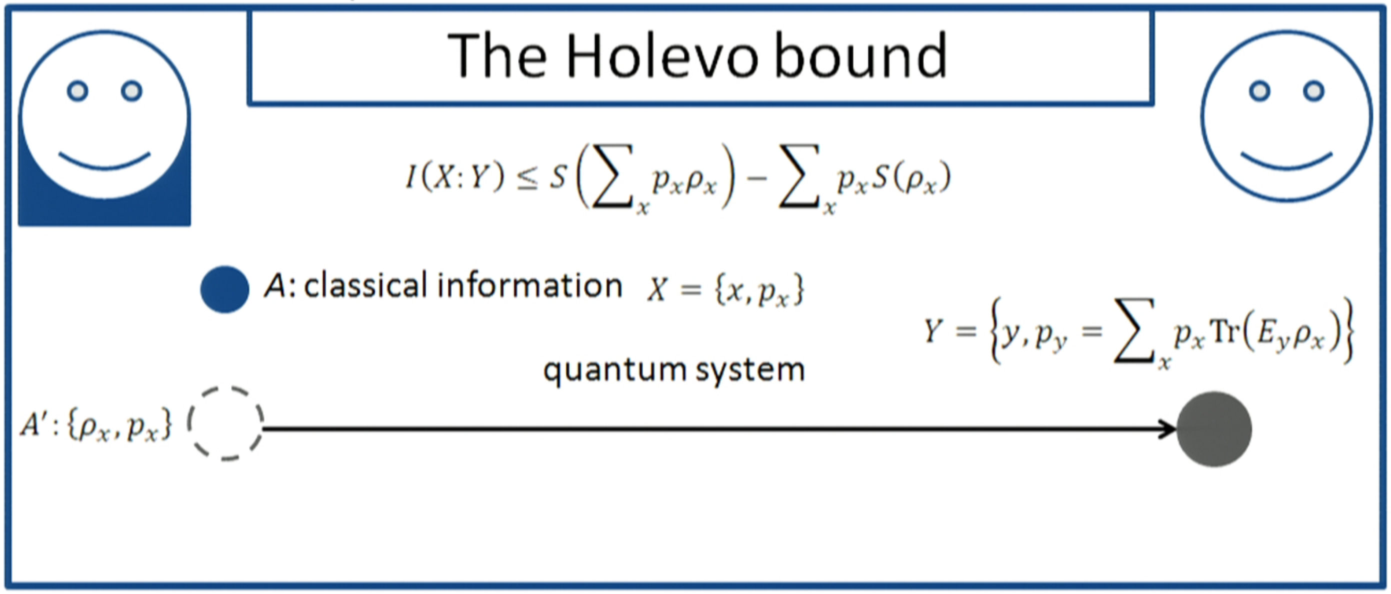
Introduction

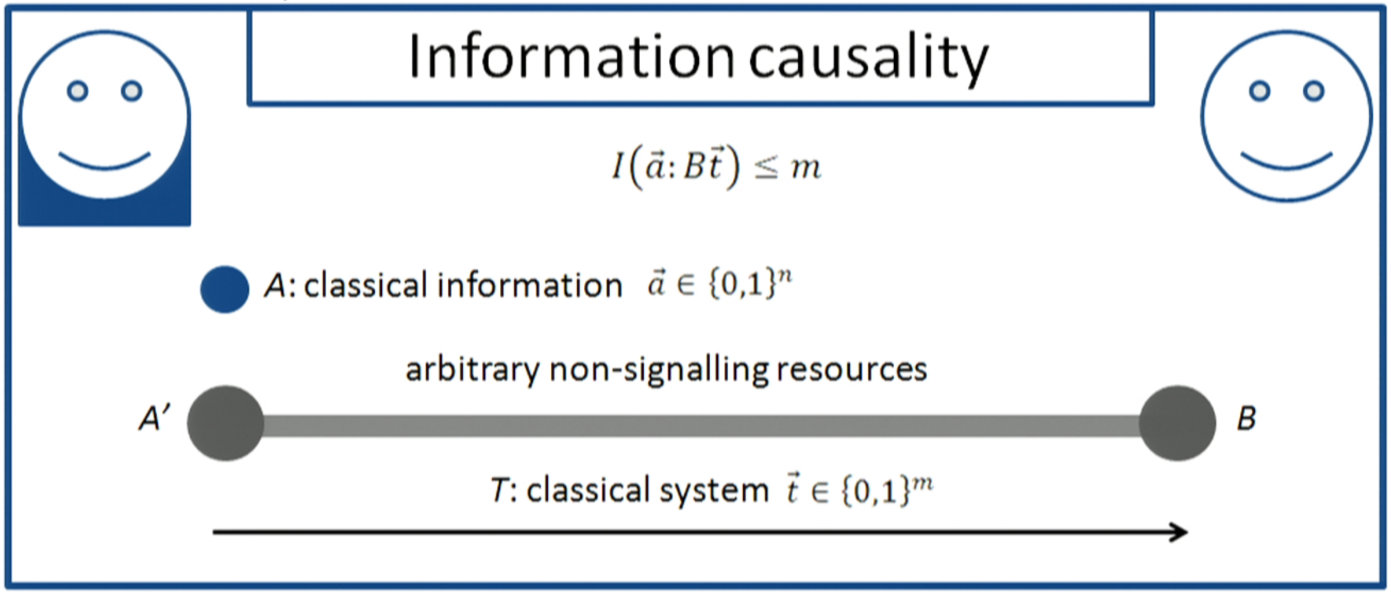
Quantum theory



A. S. Kholevo, Bounds for the quantity of information transmitted by a quantum communication channel, *Problems of Information transmission* **9**, 177 (1973), translated from *Problemy Peredachi Informatsii,* 9(3):3-11,1973.

Introduction

Generalized probabilistic theories



M. Pawloski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, Information causality as a physical principle, *Nature (London)* **461**, 1101 (2009).

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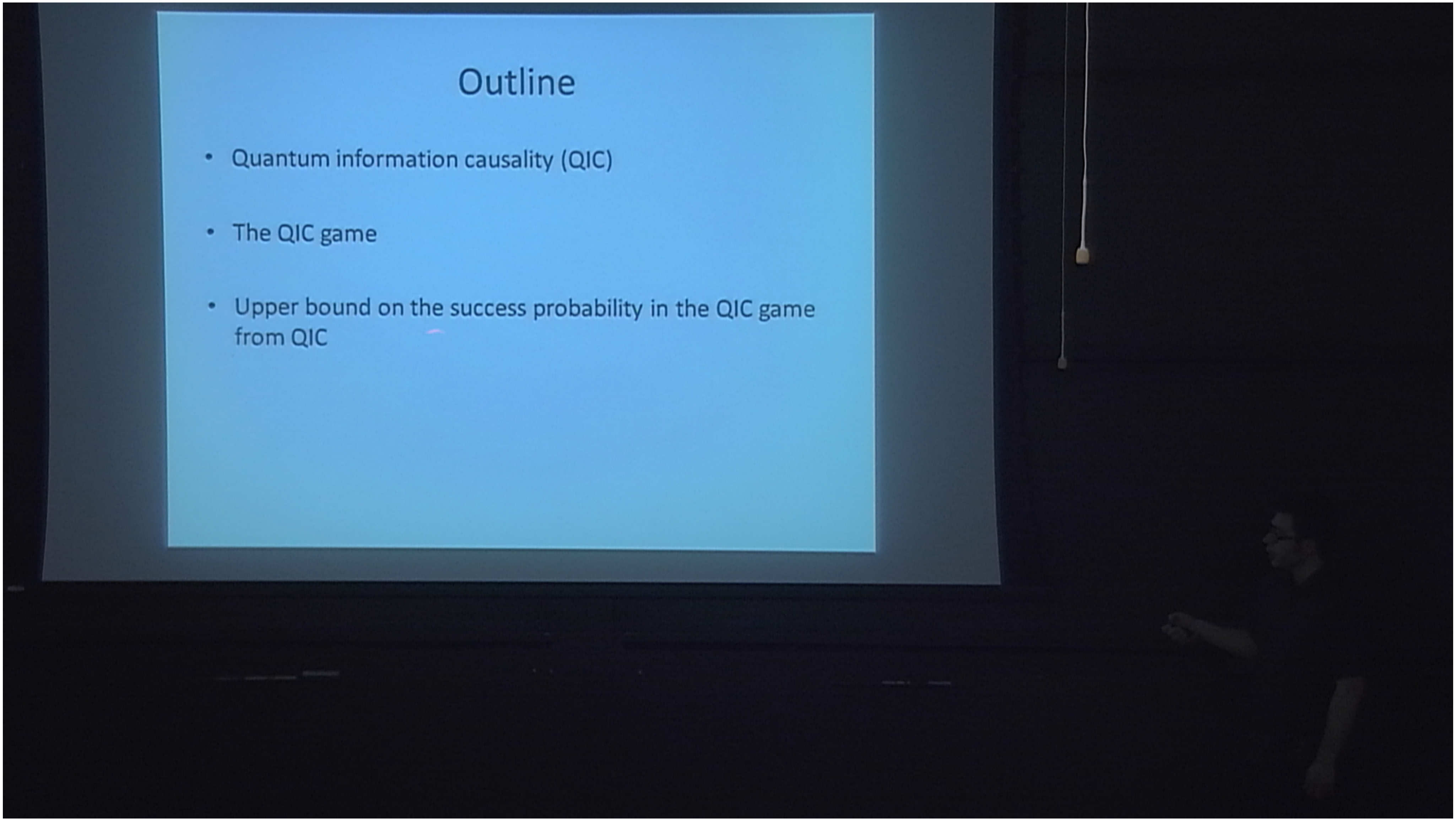
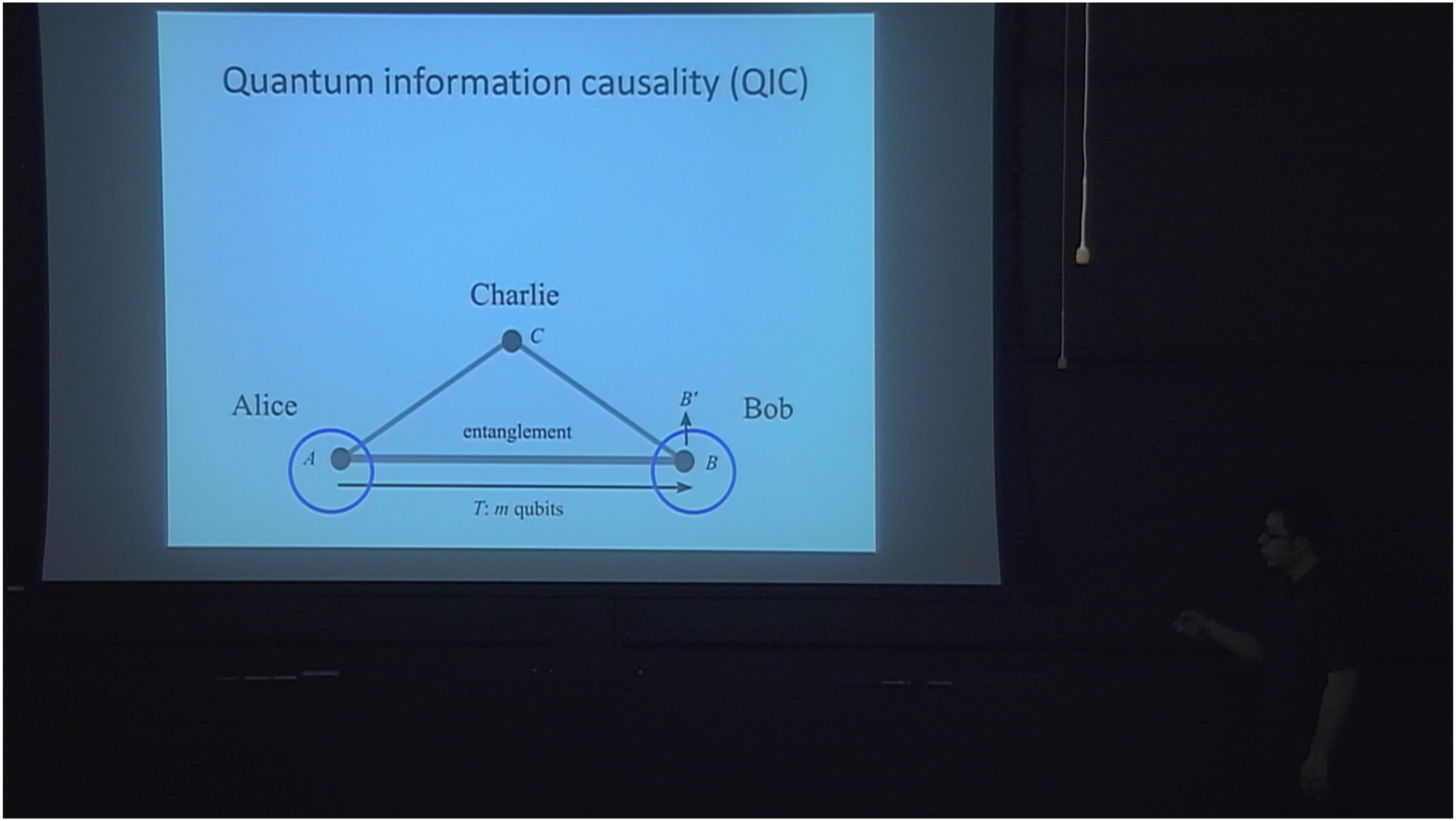
Part I. Quantum information causality

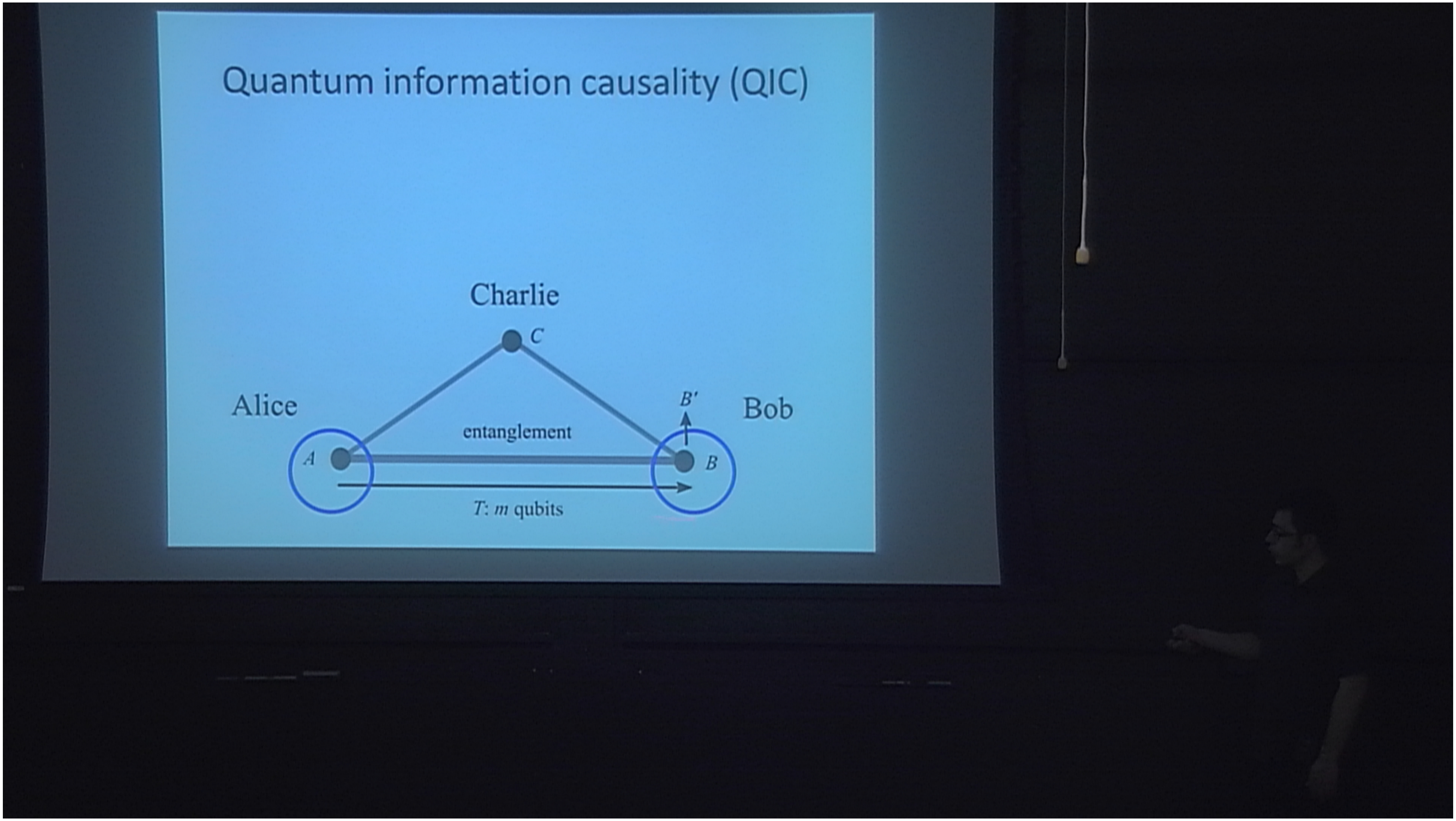
D. Pitalúa-García, Phys. Rev. Lett. 110, 210402 (2013)
Centre for Quantum Information and Foundations, DAMTP, University of Cambridge

Part II. Hyperdense coding and superadditivity in general probabilistic theories

D. Pitalúa-García, S. Massar, and S.Pironio, in preparation
Laboratoire d'Information Quantique, Université libre de Bruxelles


Outline
• Quantum information causality (QIC)
• The QIC game
• Upper bound on the success probability in the QIC game from QIC


Quantum information causality (QIC)
Charlie
C
Alice
B′
Bob
entanglement
A
B
T: m qubits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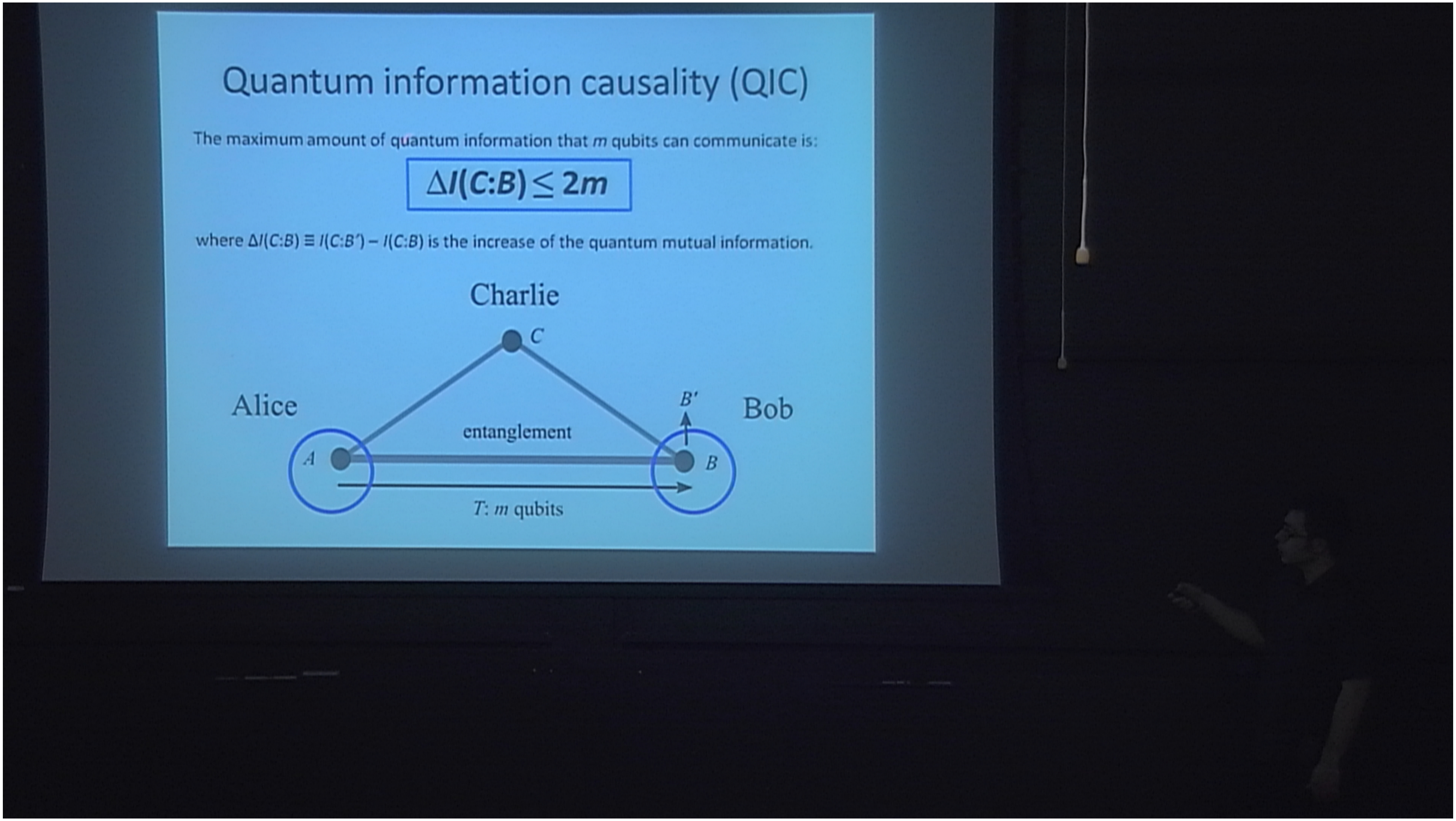
Quantum information causality (QIC)
The maximum amount of quantum information that m qubits can communicate is:
ΔI(C:B) ≤ 2m
where ΔI(C:B) ≡ I(C:B′) − I(C:B) is the increase of the quantum mutual information.
Charlie
C
Alice
A
B′
Bob
B
entanglement
T: m qubits

Proof

To show: $I(C{:}B') - I(C{:}BT) \leq 2m$

Proof

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$$I(C{:}B') \leq I(C{:}BT)$$
$$= S(C) + S(BT) - S(CBT)$$

- Data processing

Proof

To show: $I(C{:}B') - I(C{:}BT) \leq 2m$

$$
\begin{aligned}
I(C{:}B') &\leq I(C{:}BT) \\
&= S(C) + S(BT) - S(CBT) \\
&\leq S(C) + S(B) + S(T) \\
&\qquad - S(CBT) \\
&\leq S(C) + S(B) + S(T) + S(T) \\
&\qquad - S(CB)
\end{aligned}
$$

- Data processing
- Subadditivity
- Triangle inequality

Proof

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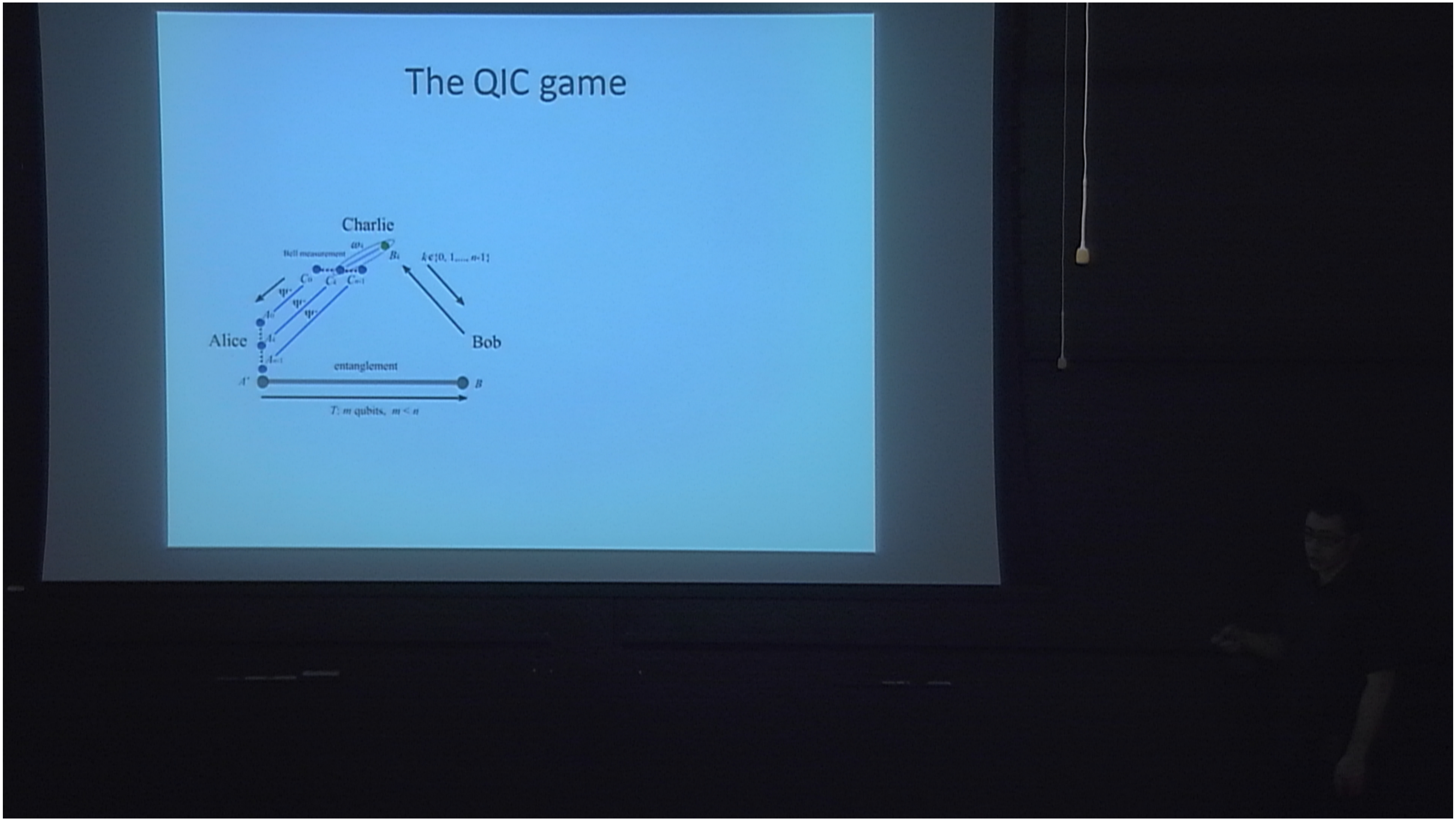
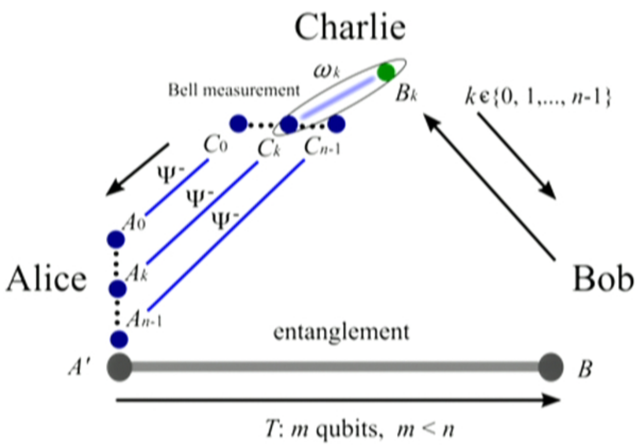
&\quad - S(CBT) \\
&\leq S(C) + S(B) + S(T) + S(T) \\
&\quad - S(CB) \\
&= I(C:B) + 2S(T) \\
&\leq I(C:B) + 2m \quad \blacksquare
\end{aligned}
$$

- Data processing
- Subadditivity
- Triangle inequality


The QIC game
Charlie
Bell measurement
Alice
Bob
entanglement
T: m qubits, m < n

The QIC game

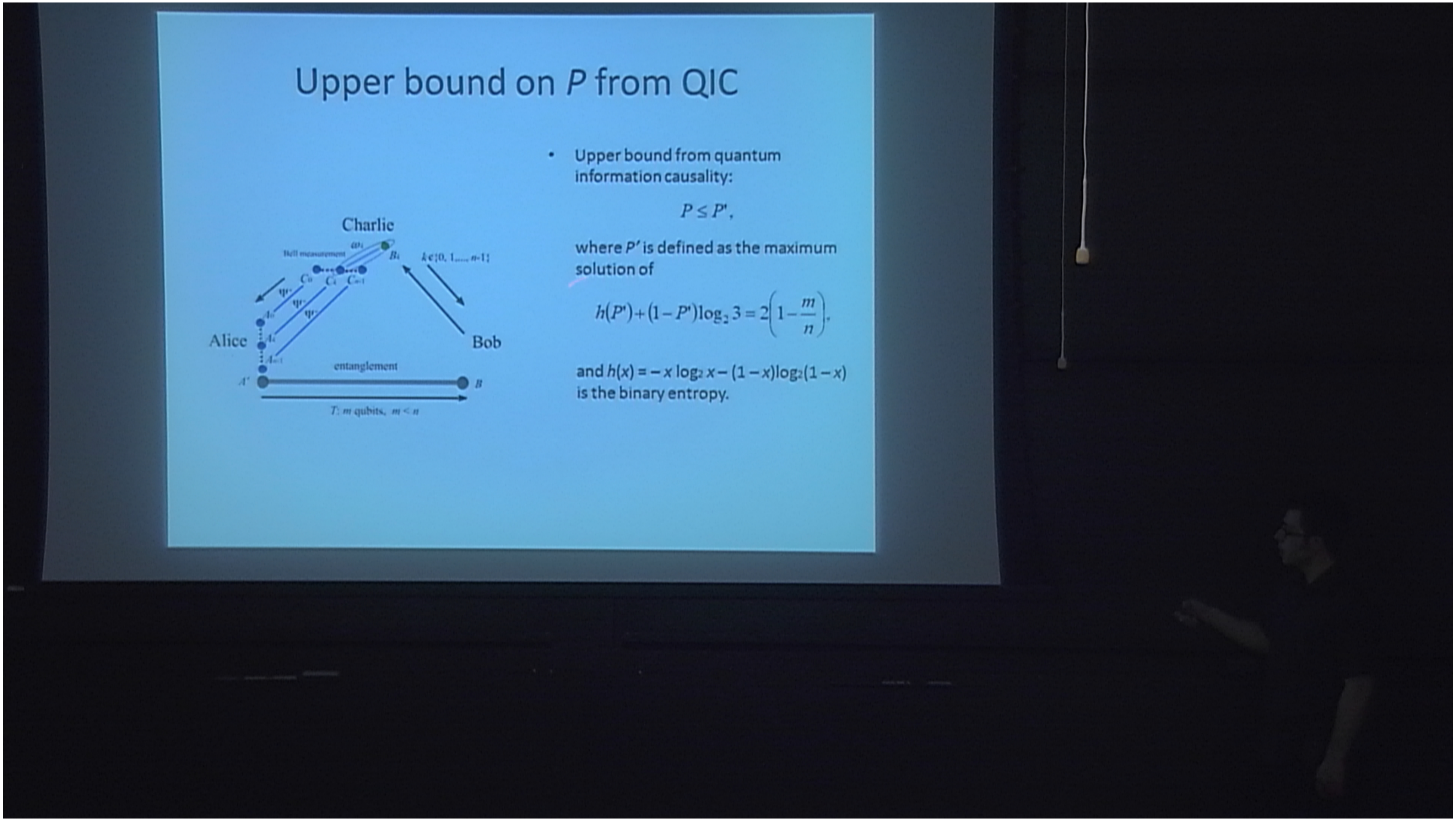


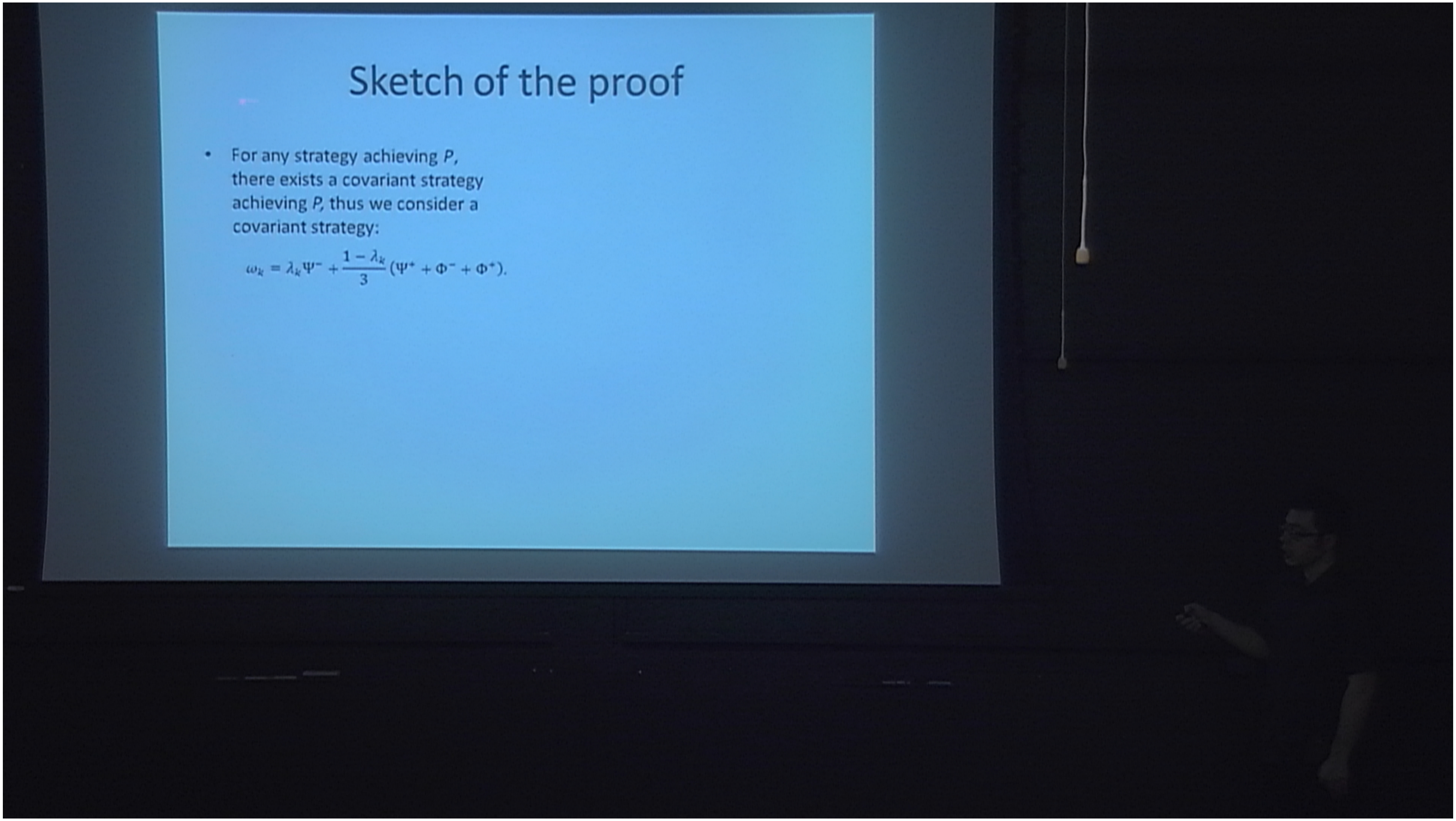
- The success probability is

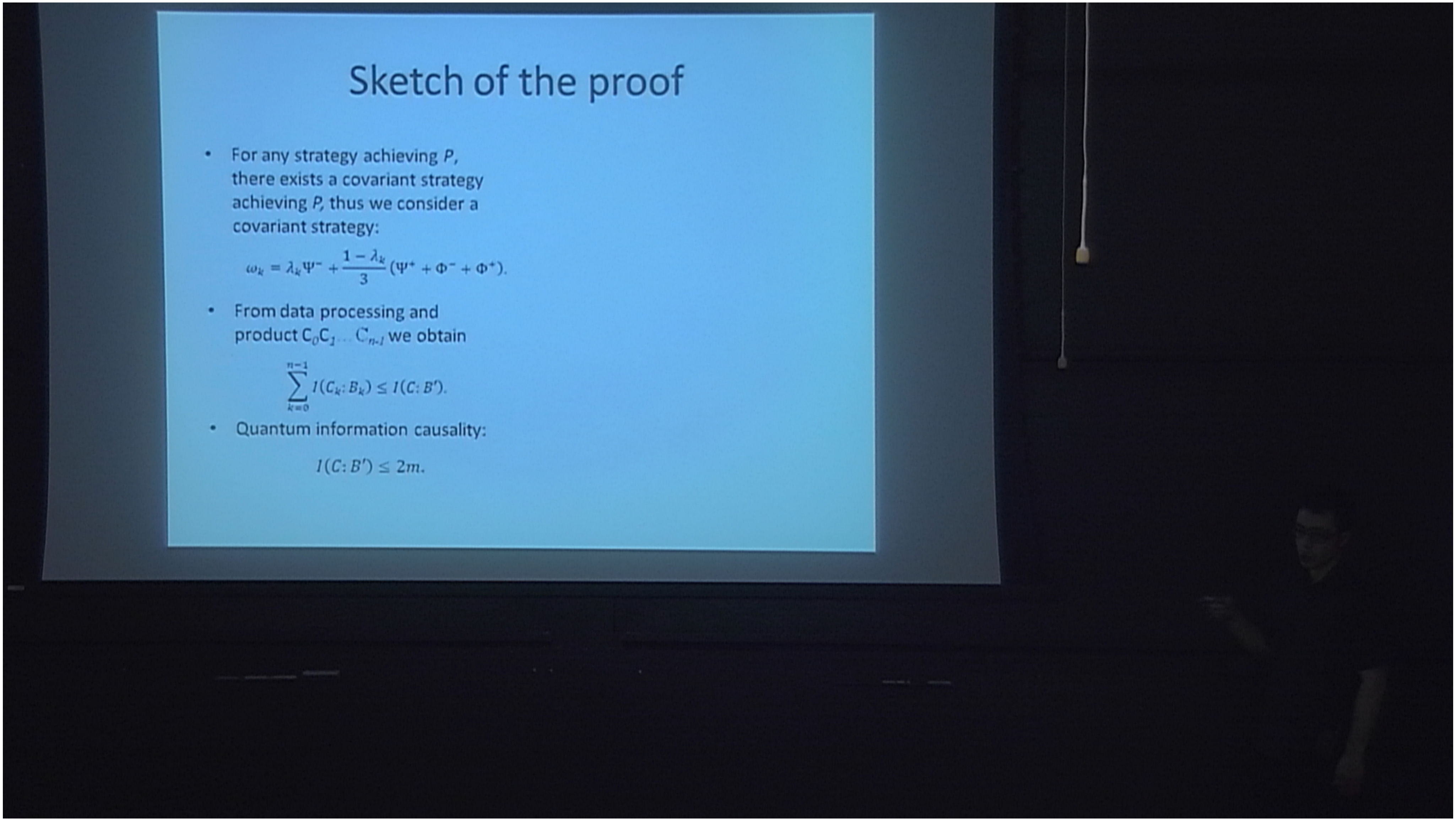
$$P \equiv \frac{1}{n}\sum_{k=0}^{n-1}\langle \Psi^- | \omega_k | \Psi^- \rangle.$$

- *Naive strategy*: Alice sends Bob *m* of the *n* received qubits without applying any operation on these. Its success probability is

$$P_N = \frac{1}{4}\left(1 + 3\frac{m}{n}\right).$$


Upper bound on P from QIC
Charlie
Bell measurement
Alice
Bob
entanglement
T: m qubits, m < n
Upper bound from quantum information causality:
$P \leq P'$,
where P' is defined as the maximum solution of
$h(P') + (1-P')\log_2 3 = 2\left(1 - \frac{m}{n}\right)$,
and $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy.


Sketch of the proof
For any strategy achieving P, there exists a covariant strategy achieving P, thus we consider a covariant strategy:
$\omega_k = \lambda_k \Psi^- + \frac{1-\lambda_k}{3}(\Psi^+ + \Phi^- + \Phi^+).$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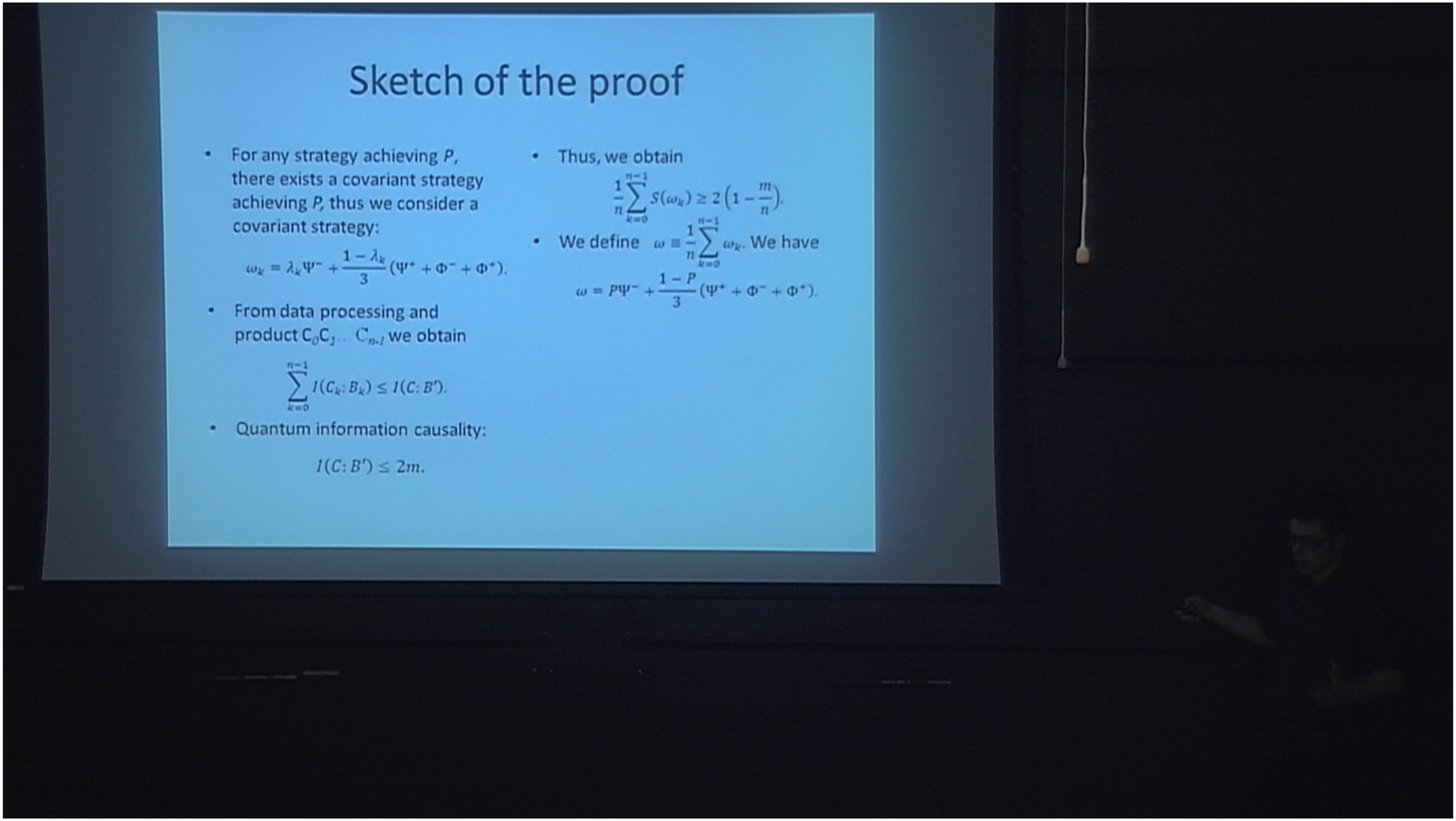
$\omega_k = \lambda_k \Psi^- + \frac{1-\lambda_k}{3}(\Psi^+ + \Phi^- + \Phi^+).$
• From data processing and product $C_0 C_1 \ldots C_{n-1}$ we obtain
$\sum_{k=0}^{n-1} I(C_k : B_k) \leq I(C : B').$
• Quantum information causality:
$I(C : B') \leq 2m.$



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Quantum information causality:
$I(C : B') \le 2m.$
Thus, we obtain
$\frac{1}{n}\sum_{k=0}^{n-1} S(\omega_k) \ge 2\left(1 - \frac{m}{n}\right).$
We define $\omega \equiv \frac{1}{n}\sum_{k=0}^{n-1} \omega_k$. We have
$\omega = P\Psi^- + \frac{1-P}{3}(\Psi^+ + \Phi^- + \Phi^+).$

Sketch of the proof

- For any strategy achieving P, there exists a covariant strategy achieving P, thus we consider a covariant strategy:

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$$\omega = P\Psi^- + \frac{1-P}{3}(\Psi^+ + \Phi^- + \Phi^+).$$

- From concavity we have

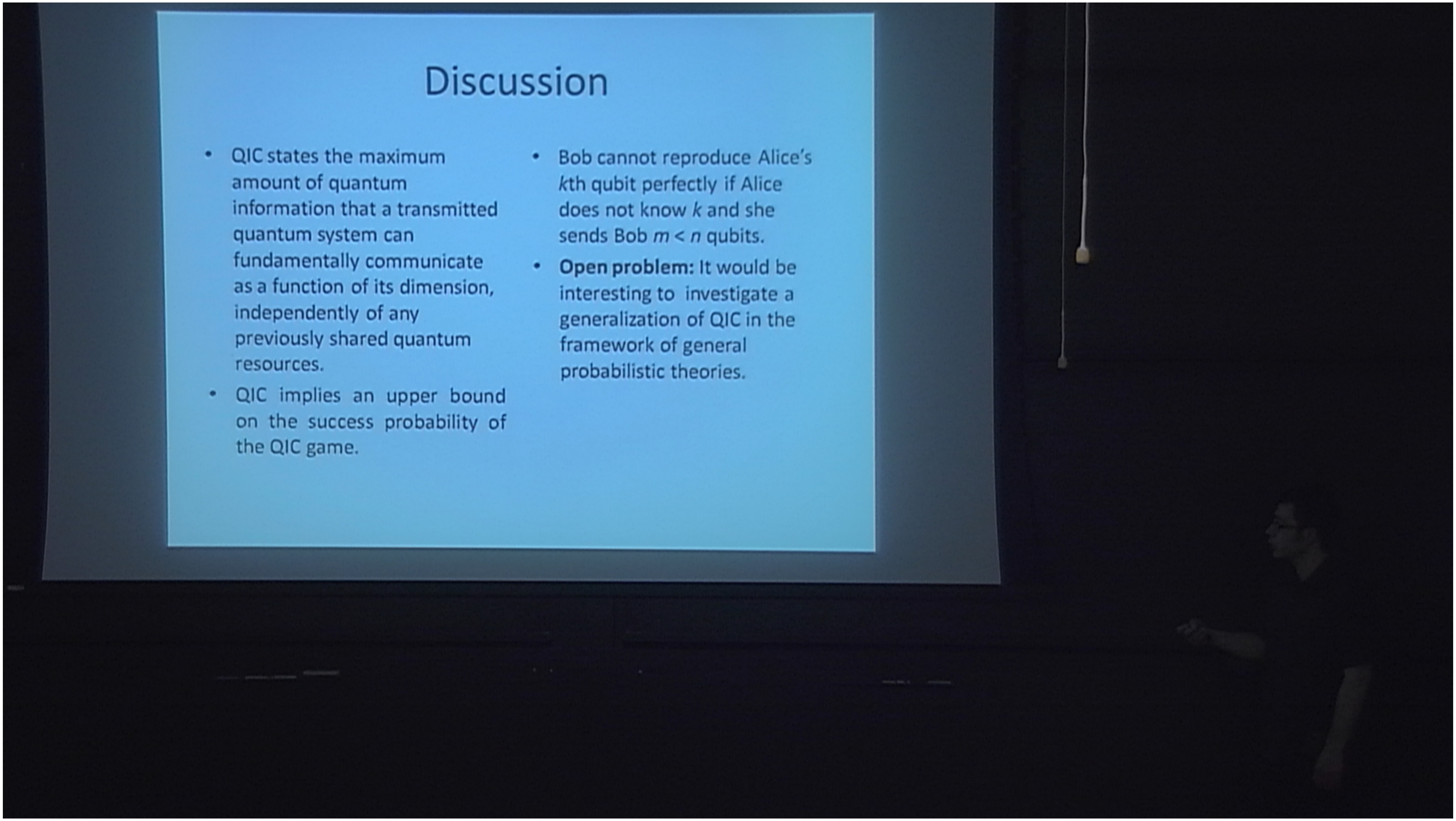
$$S(\omega) \geq \frac{1}{n}\sum_{k=0}^{n-1} S(\omega_k).$$

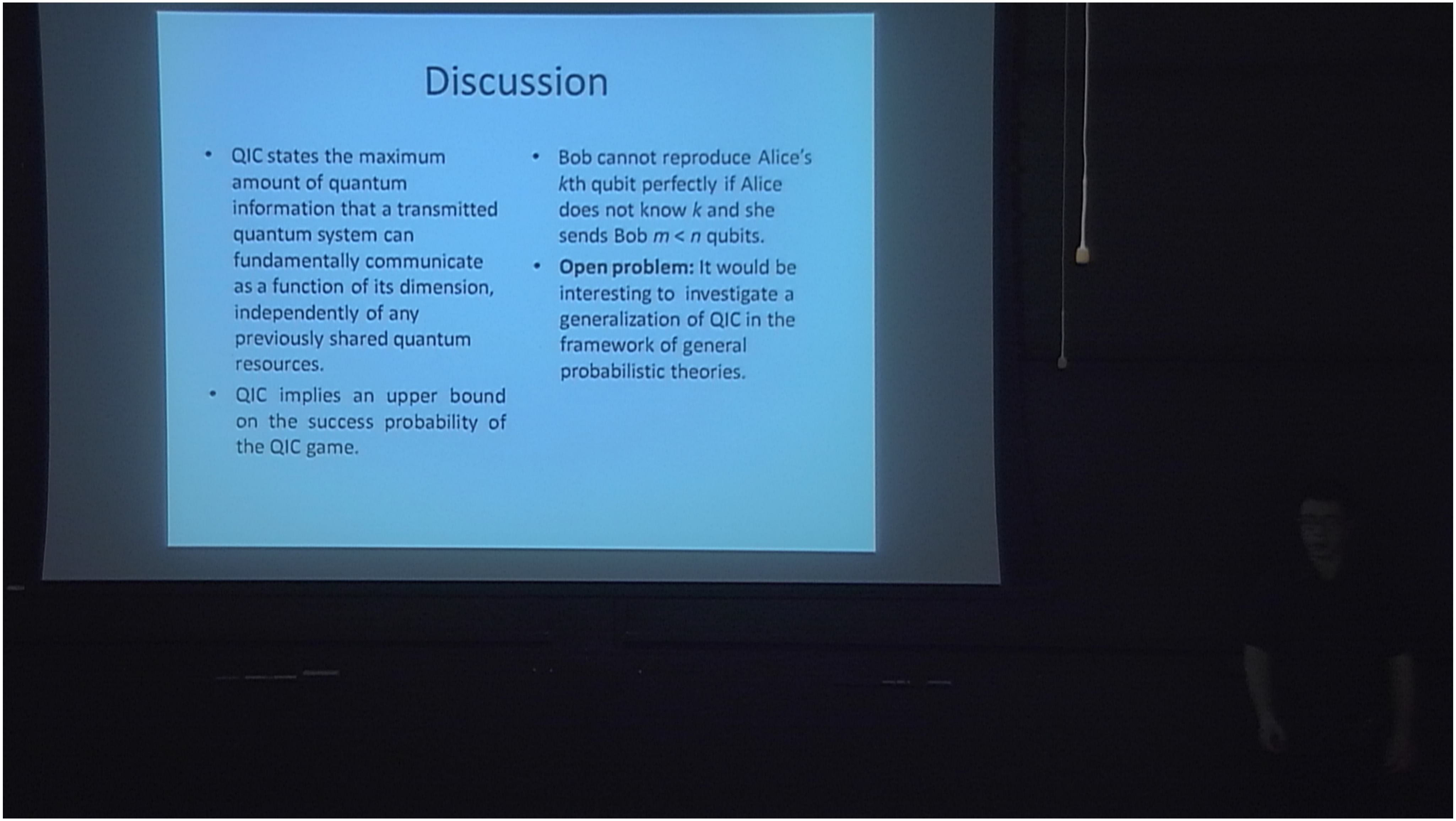
- This implies

$$h(P) + (1-P)\log_2 3 \geq 2\left(1 - \frac{m}{n}\right),$$

and $P \leq P'$, with P' the max sol of

$$h(P) + (1-P)\log_2 3 = 2\left(1 - \frac{m}{n}\right).$$


Discussion
• QIC states the maximum amount of quantum information that a transmitted quantum system can fundamentally communicate as a function of its dimension, independently of any previously shared quantum resources.
• QIC implies an upper bound on the success probability of the QIC game.
• Bob cannot reproduce Alice's kth qubit perfectly if Alice does not know k and she sends Bob m < n qubits.
• Open problem: It would be interesting to investigate a generalization of QIC in the framework of general probabilistic theories.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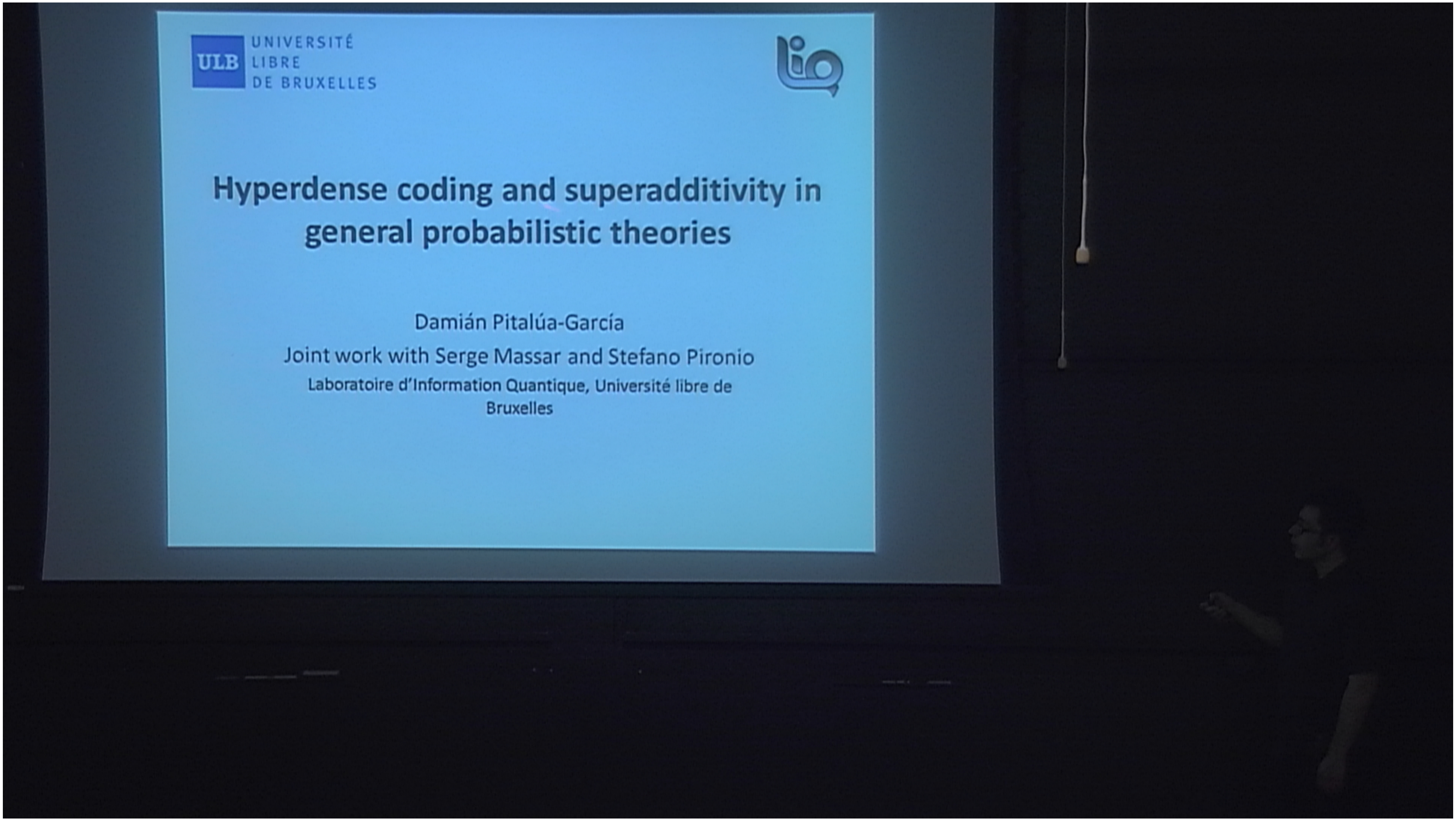
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Hyperdense coding and superadditivity in general probabilistic theories
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Joint work with Serge Massar and Stefano Pironio
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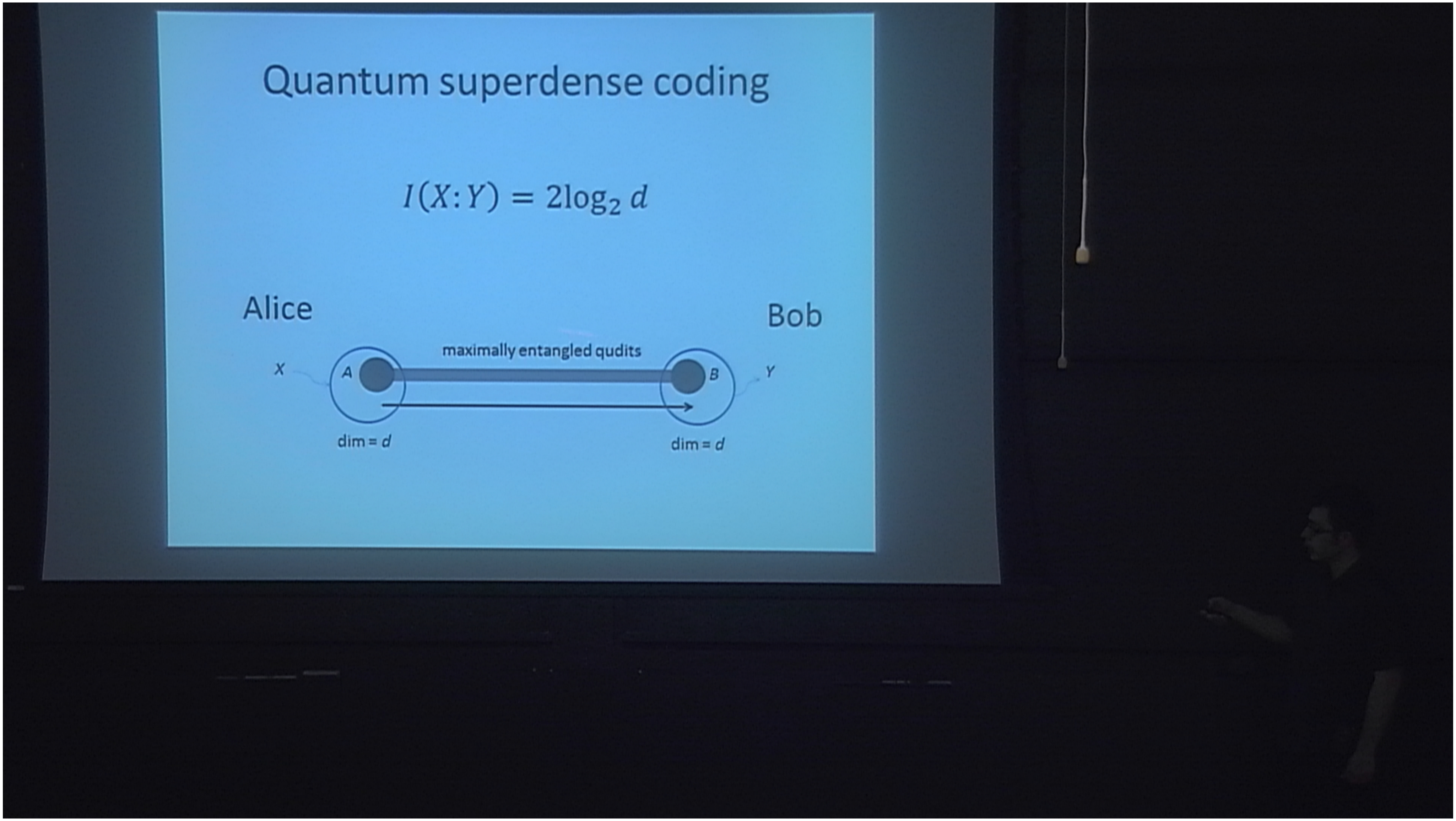
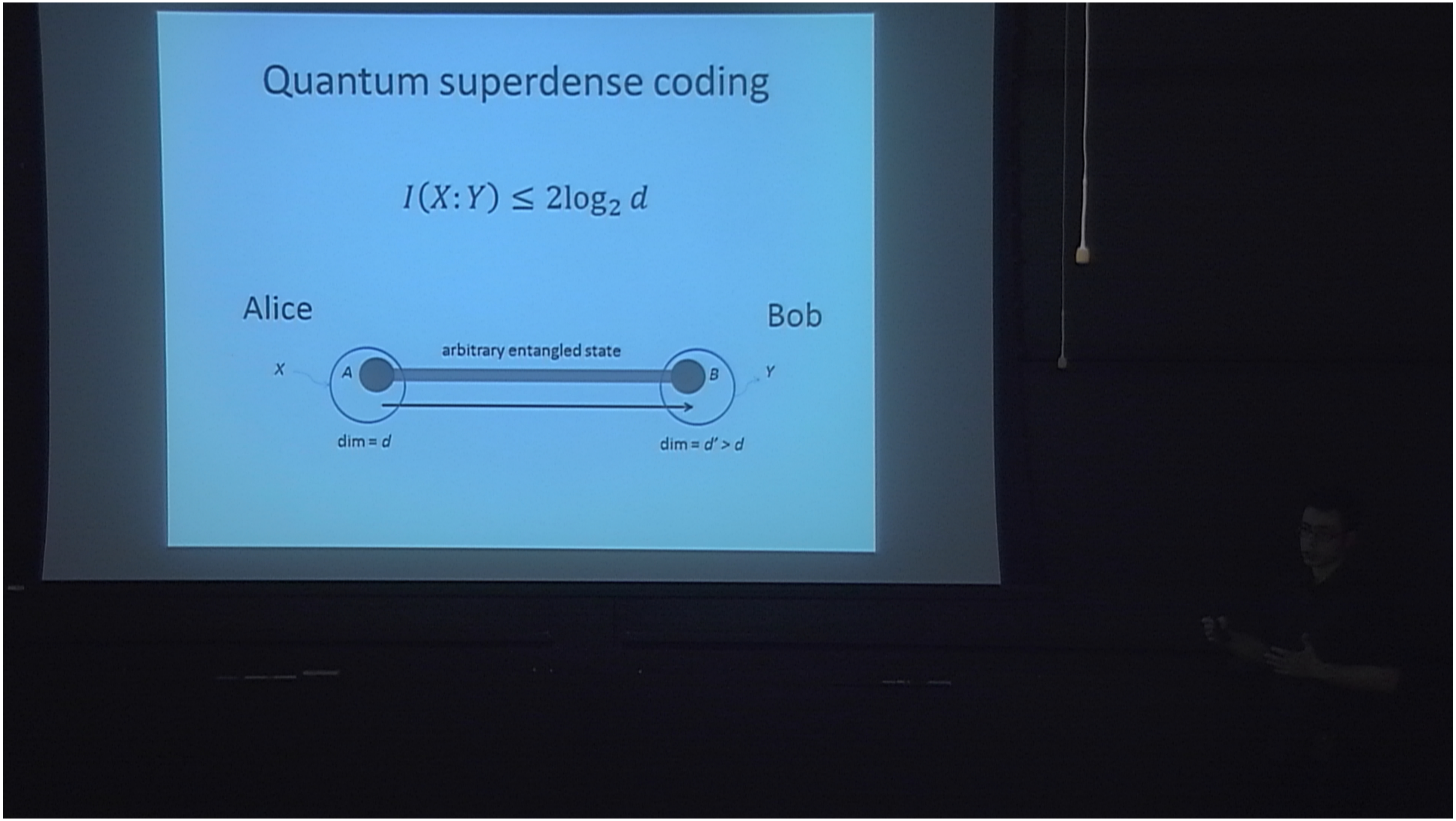
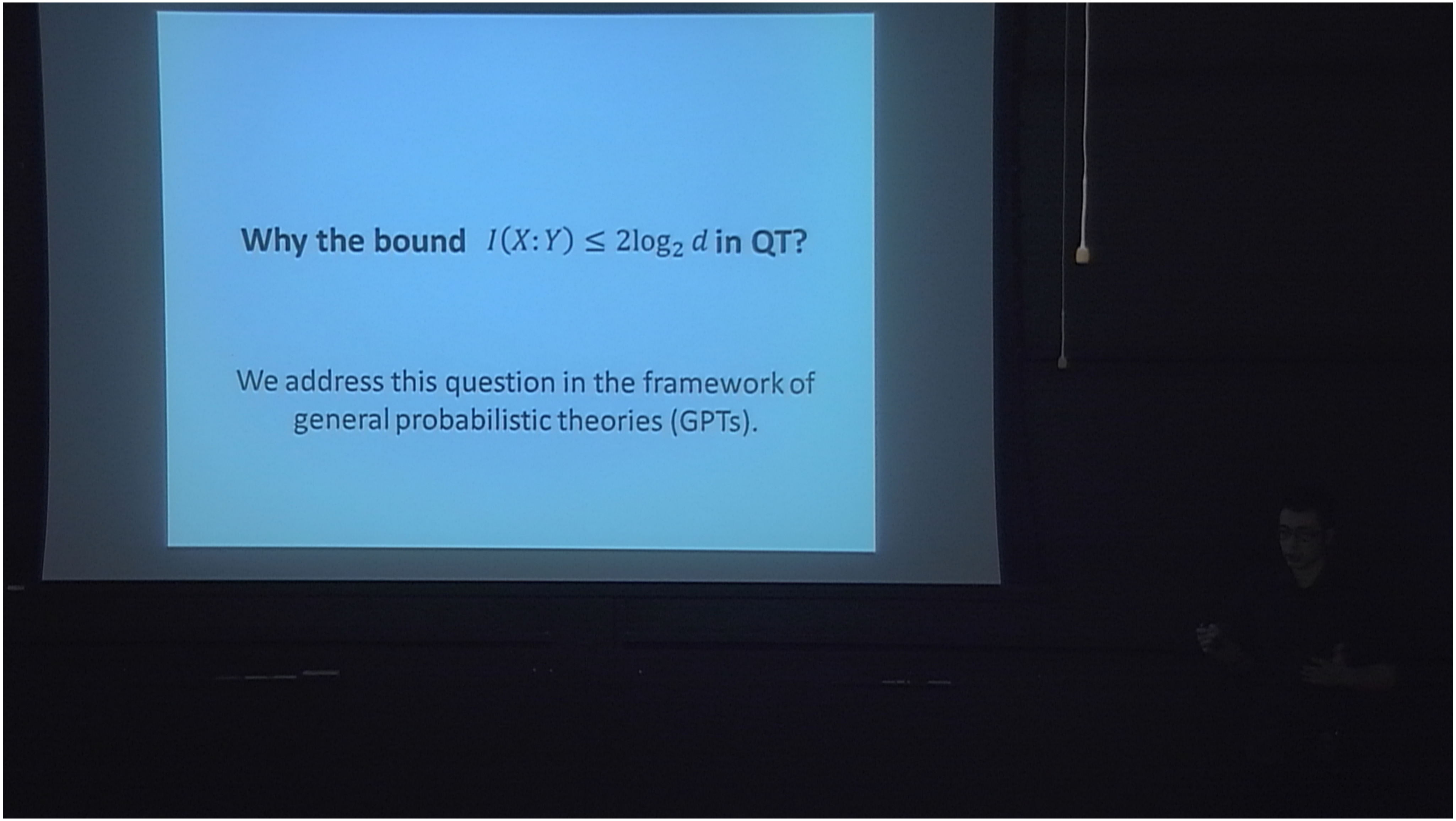

Quantum superdense coding
$I(X:Y) = 2\log_2 d$
Alice
Bob
maximally entangled qudits
X
A
B
Y
dim = d
dim = d


Quantum superdense coding
$I(X:Y) \leq 2\log_2 d$
Alice
Bob
arbitrary entangled state
X
A
B
Y
dim = d
dim = d' > d


Why the bound $I(X:Y) \leq 2\log_2 d$ **in QT?**
We address this question in the framework of
general probabilistic theories (GPTs).

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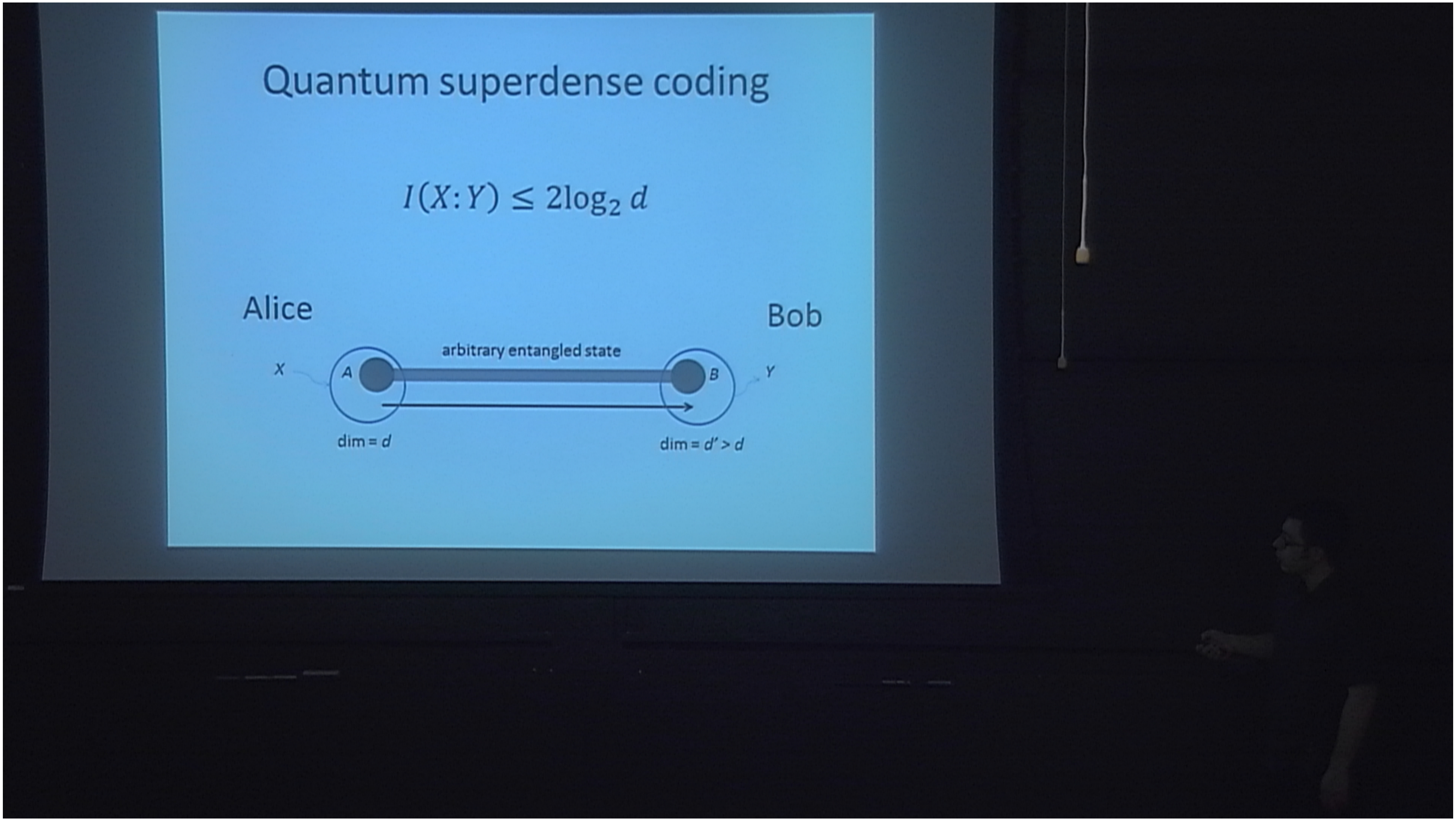
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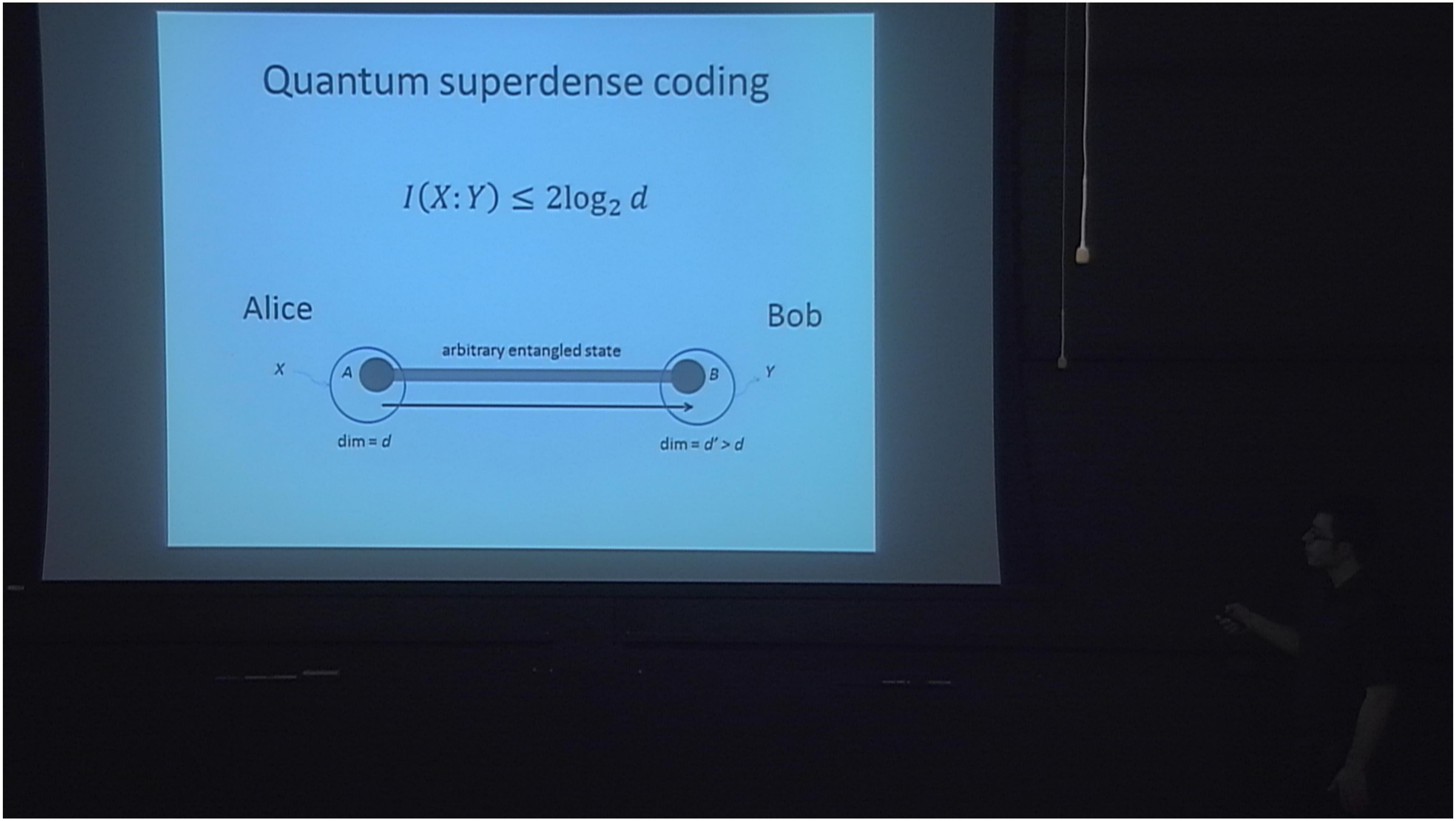
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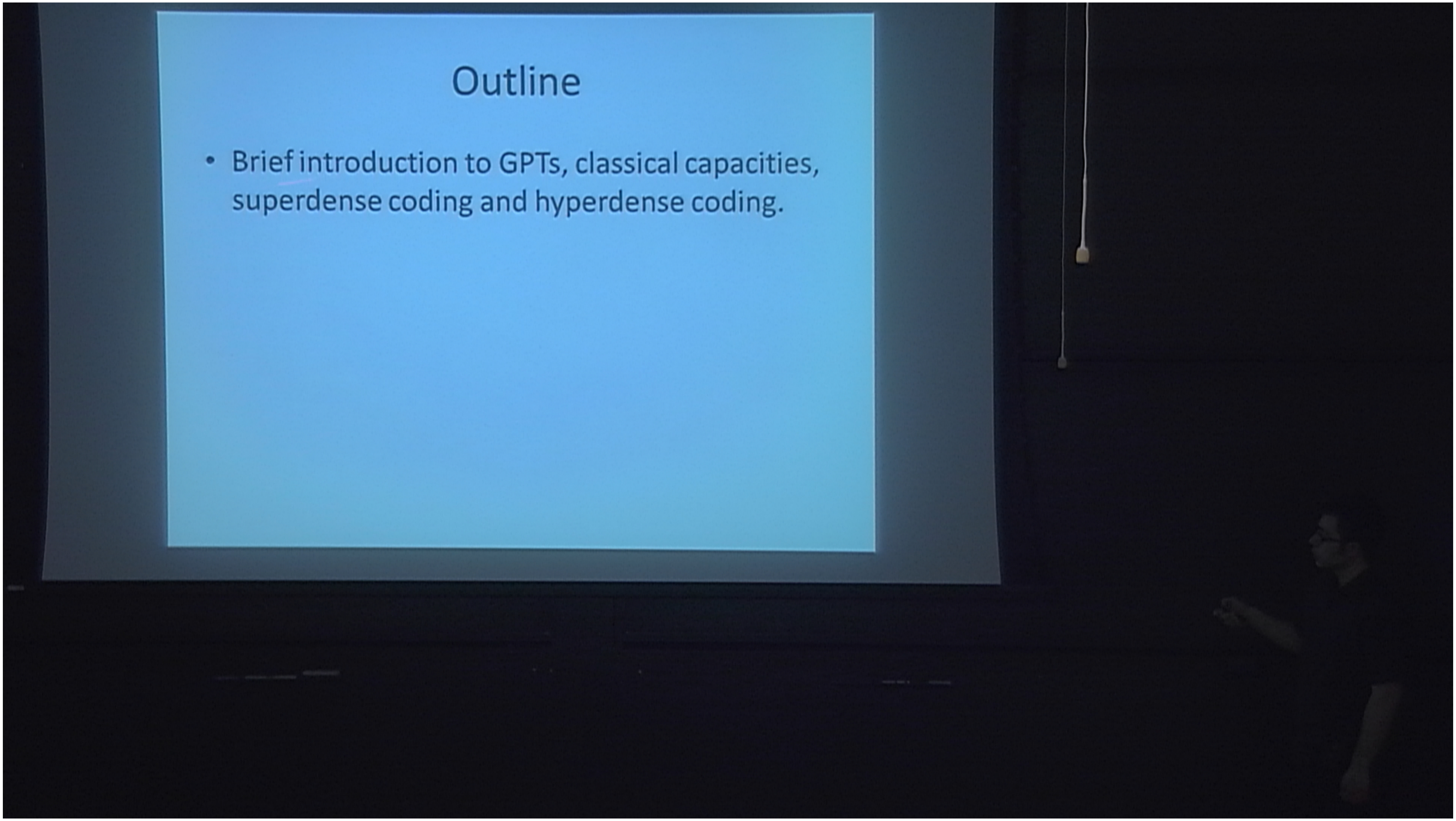
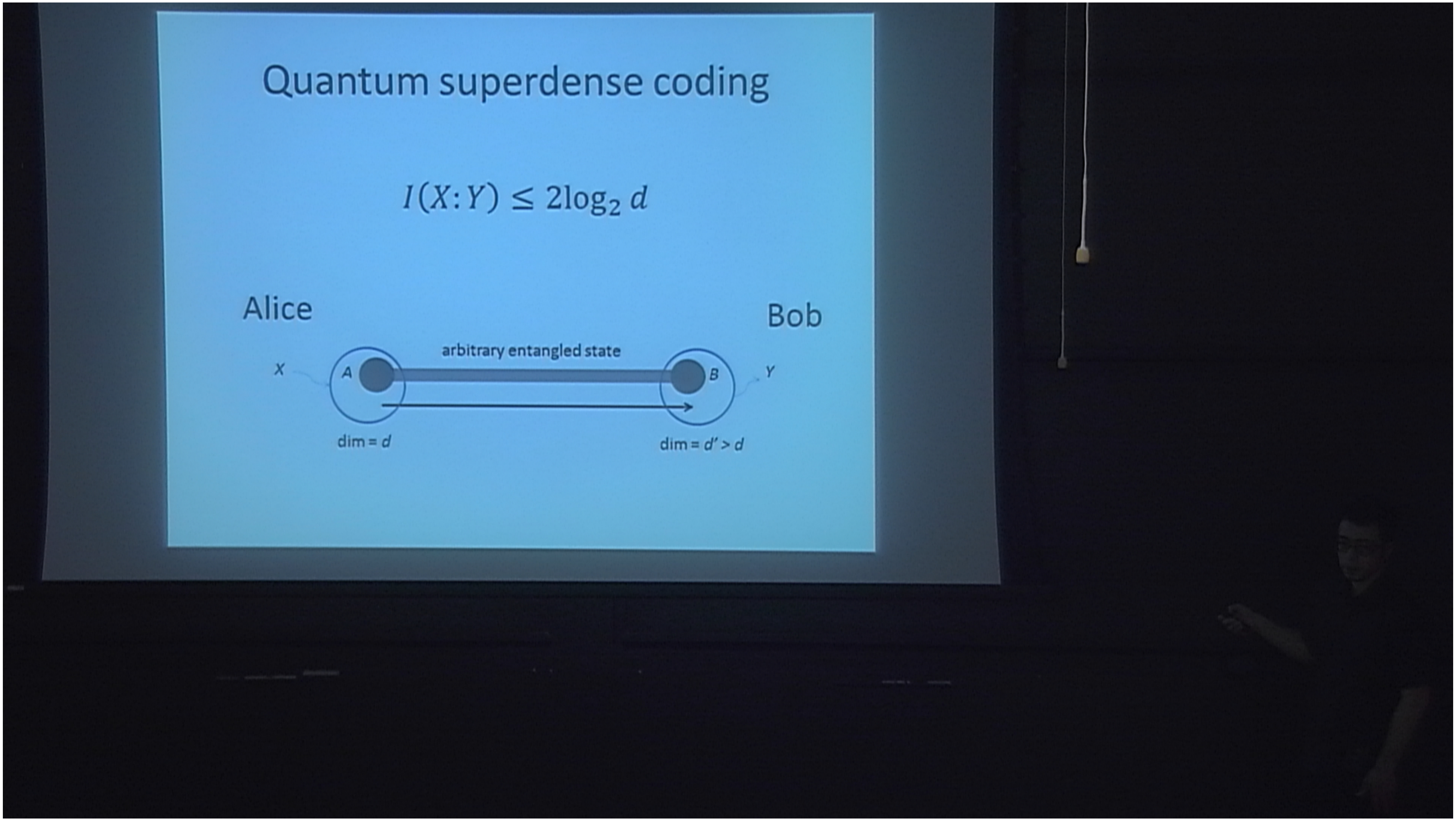

Outline
• Brief introduction to GPTs, classical capacities, superdense coding and hyperdense coding.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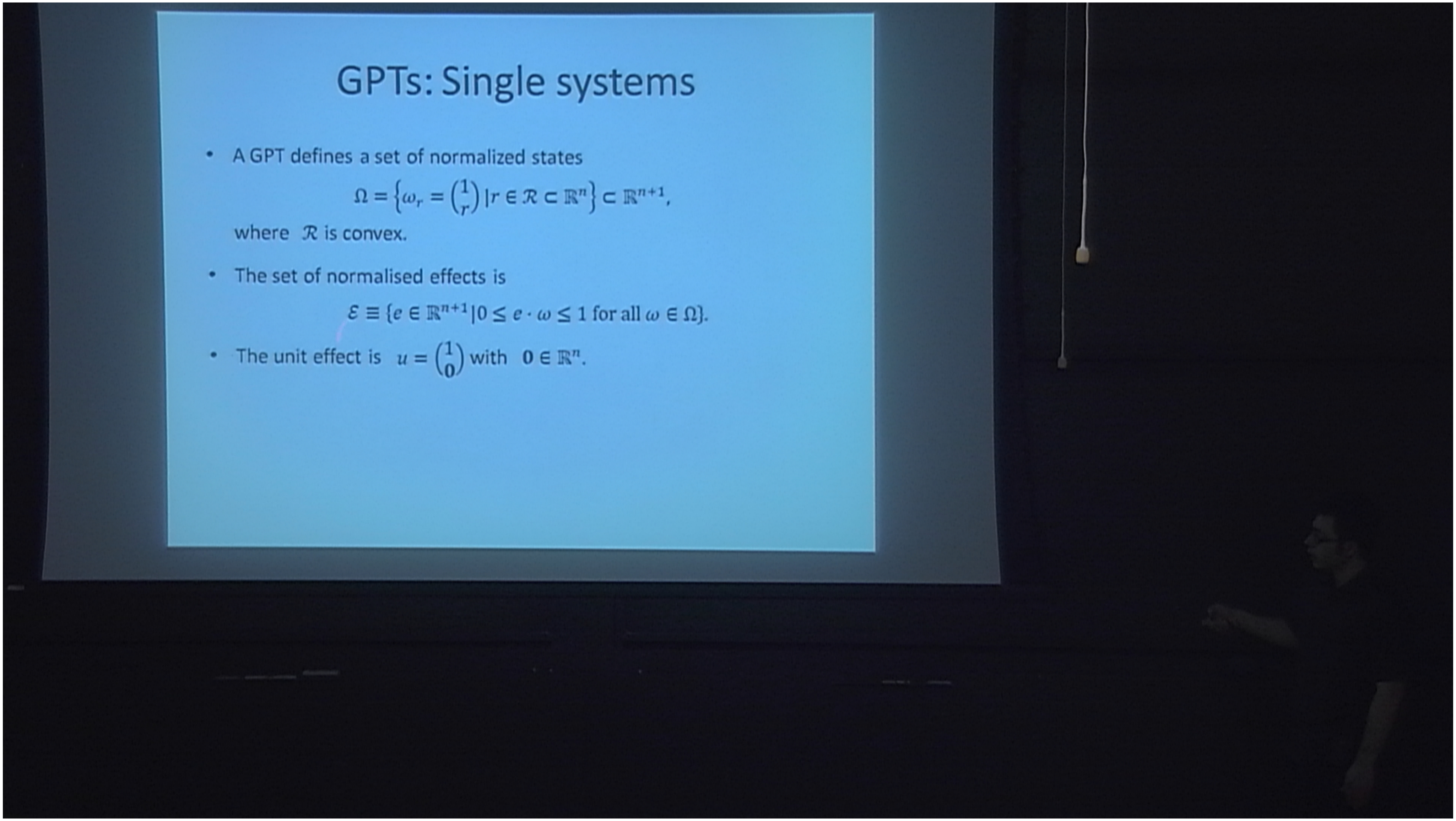
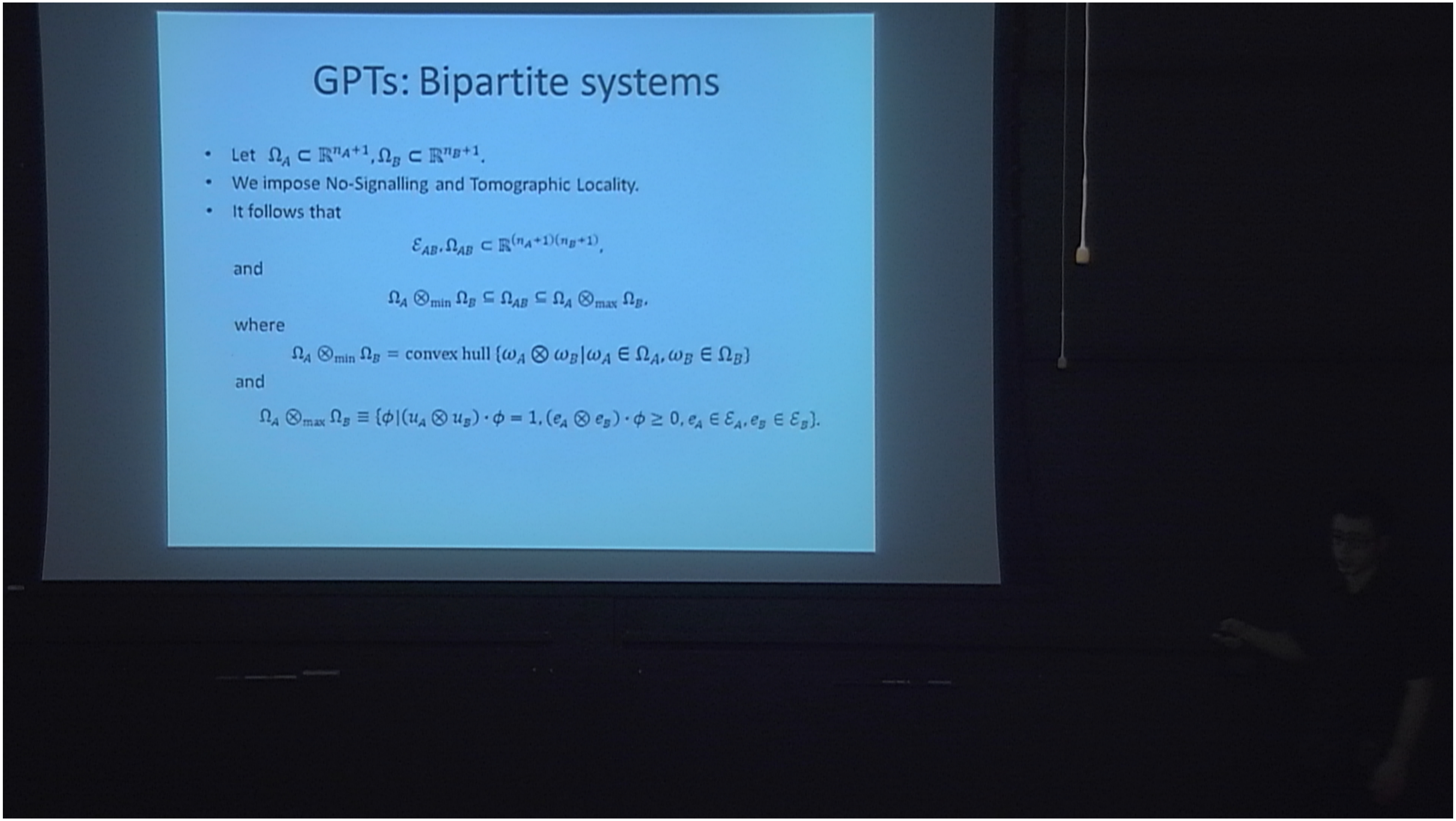
GPTs: Single systems

- A GPT defines a set of normalized states

$$\Omega = \left\{ \omega_r = \begin{pmatrix} 1 \\ r \end{pmatrix} \mid r \in \mathcal{R} \subset \mathbb{R}^n \right\} \subset \mathbb{R}^{n+1},$$

where $\mathcal{R}$ is convex.


GPTs: Single systems
A GPT defines a set of normalized states
$\Omega = \left\{ \omega_r = \binom{1}{r} \mid r \in \mathcal{R} \subset \mathbb{R}^n \right\} \subset \mathbb{R}^{n+1}$,
where $\mathcal{R}$ is convex.
The set of normalised effects is
$\mathcal{E} \equiv \{ e \in \mathbb{R}^{n+1} \mid 0 \leq e \cdot \omega \leq 1 \text{ for all } \omega \in \Omega \}$.
The unit effect is $u = \binom{1}{\mathbf{0}}$ with $\mathbf{0} \in \mathbb{R}^n$.


GPTs: Bipartite systems
• Let $\Omega_A \subset \mathbb{R}^{n_A+1}, \Omega_B \subset \mathbb{R}^{n_B+1}$.
• We impose No-Signalling and Tomographic Locality.
• It follows that
$\mathcal{E}_{AB}, \Omega_{AB} \subset \mathbb{R}^{(n_A+1)(n_B+1)}$,
and
$\Omega_A \otimes_{\min} \Omega_B \subseteq \Omega_{AB} \subseteq \Omega_A \otimes_{\max} \Omega_B$,
where
$\Omega_A \otimes_{\min} \Omega_B = \text{convex hull } \{\omega_A \otimes \omega_B | \omega_A \in \Omega_A, \omega_B \in \Omega_B\}$
and
$\Omega_A \otimes_{\max} \Omega_B \equiv \{\phi | (u_A \otimes u_B) \cdot \phi = 1, (e_A \otimes e_B) \cdot \phi \geq 0, e_A \in \mathcal{E}_A, e_B \in \mathcal{E}_B\}$.

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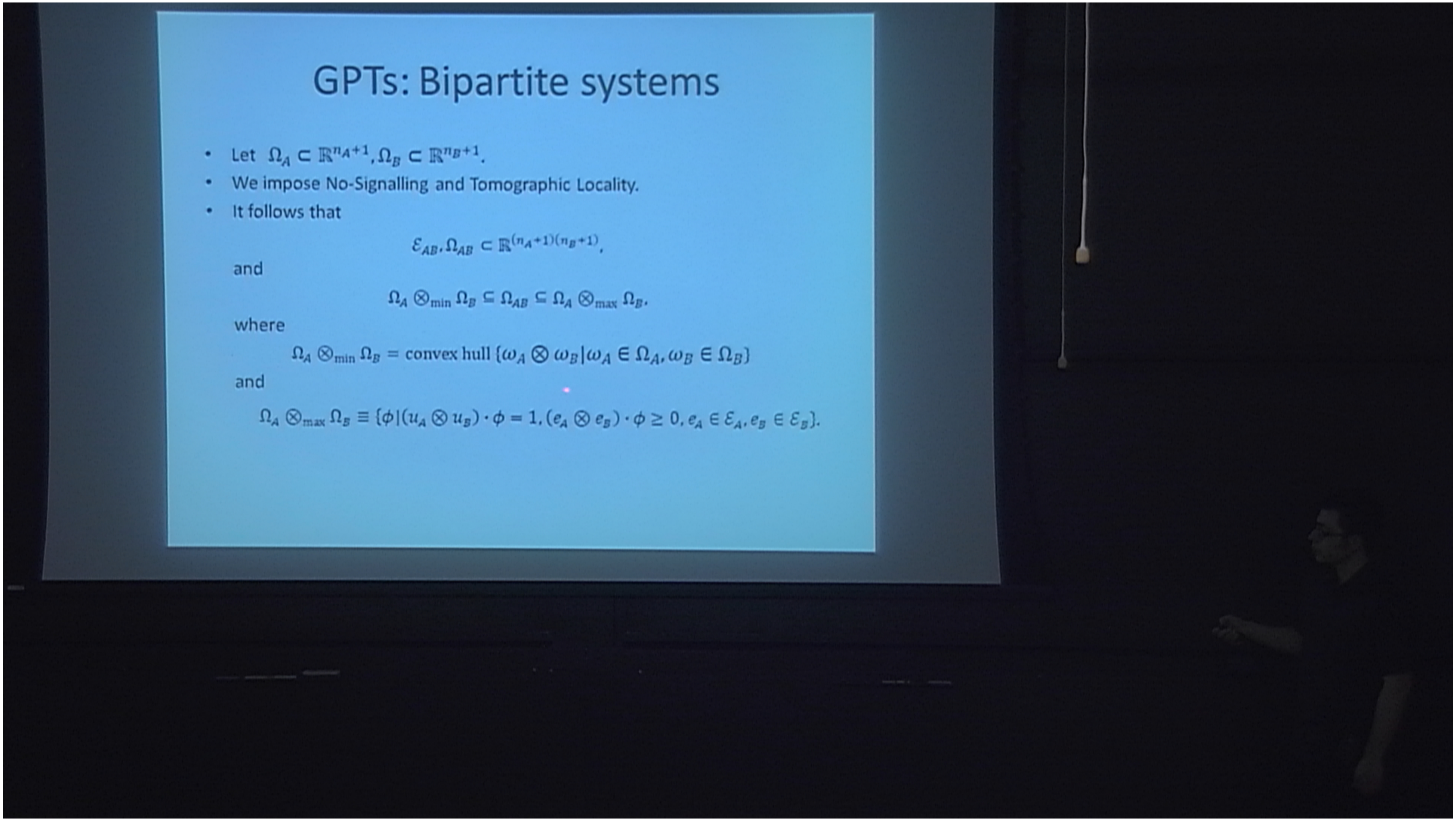
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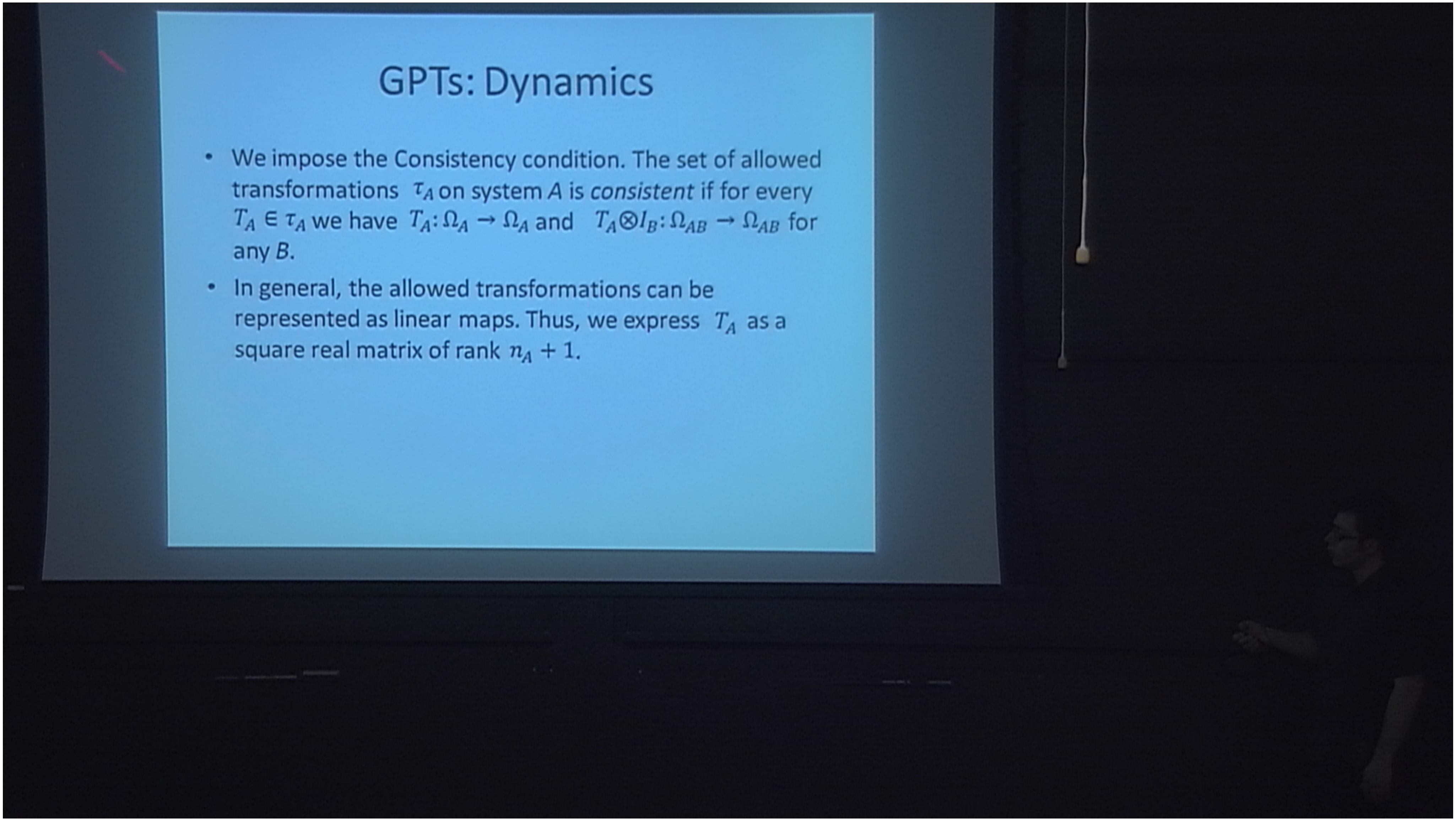
GPTs: Dynamics

- We impose the Consistency condition. The set of allowed transformations τ_A on system A is *consistent* if for every $T_A \in \tau_A$ we have $T_A : \Omega_A \rightarrow \Omega_A$ and $T_A \otimes I_B : \Omega_{AB} \rightarrow \Omega_{AB}$ for any B.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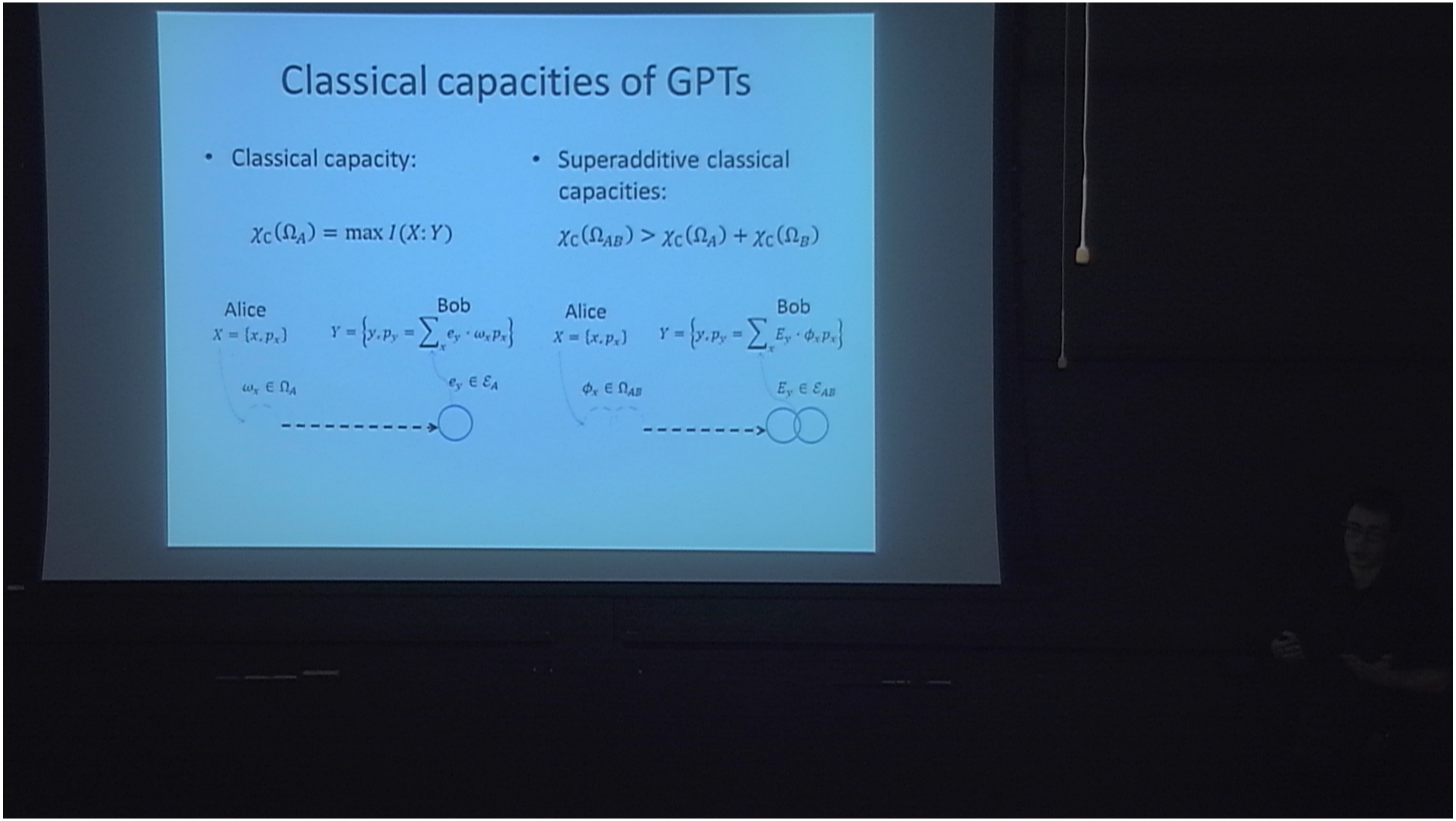
• In general, the allowed transformations can be represented as linear maps. Thus, we express T_A as a square real matrix of rank $n_A + 1$.



Classical capacities of GPTs
• Classical capacity:
$\chi_C(\Omega_A) = \max I(X:Y)$
Alice
$X = \{x, p_x\}$
Bob
$Y = \{y, p_y = \sum_x e_y \cdot \omega_x p_x\}$
$\omega_x \in \Omega_A$
$e_y \in \mathcal{E}_A$
• Superadditive classical capacities:
$\chi_C(\Omega_{AB}) > \chi_C(\Omega_A) + \chi_C(\Omega_B)$
Alice
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Bob
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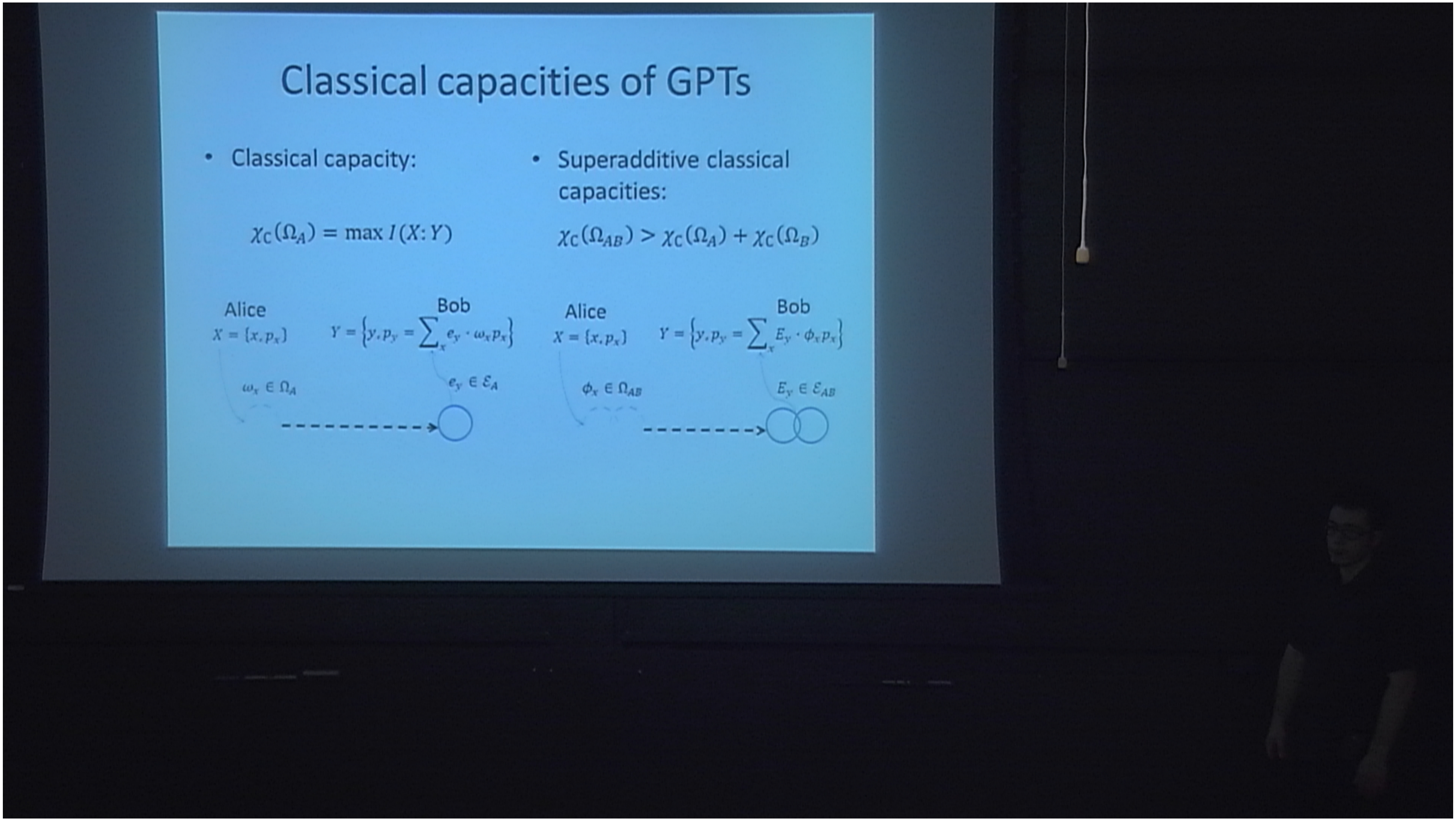
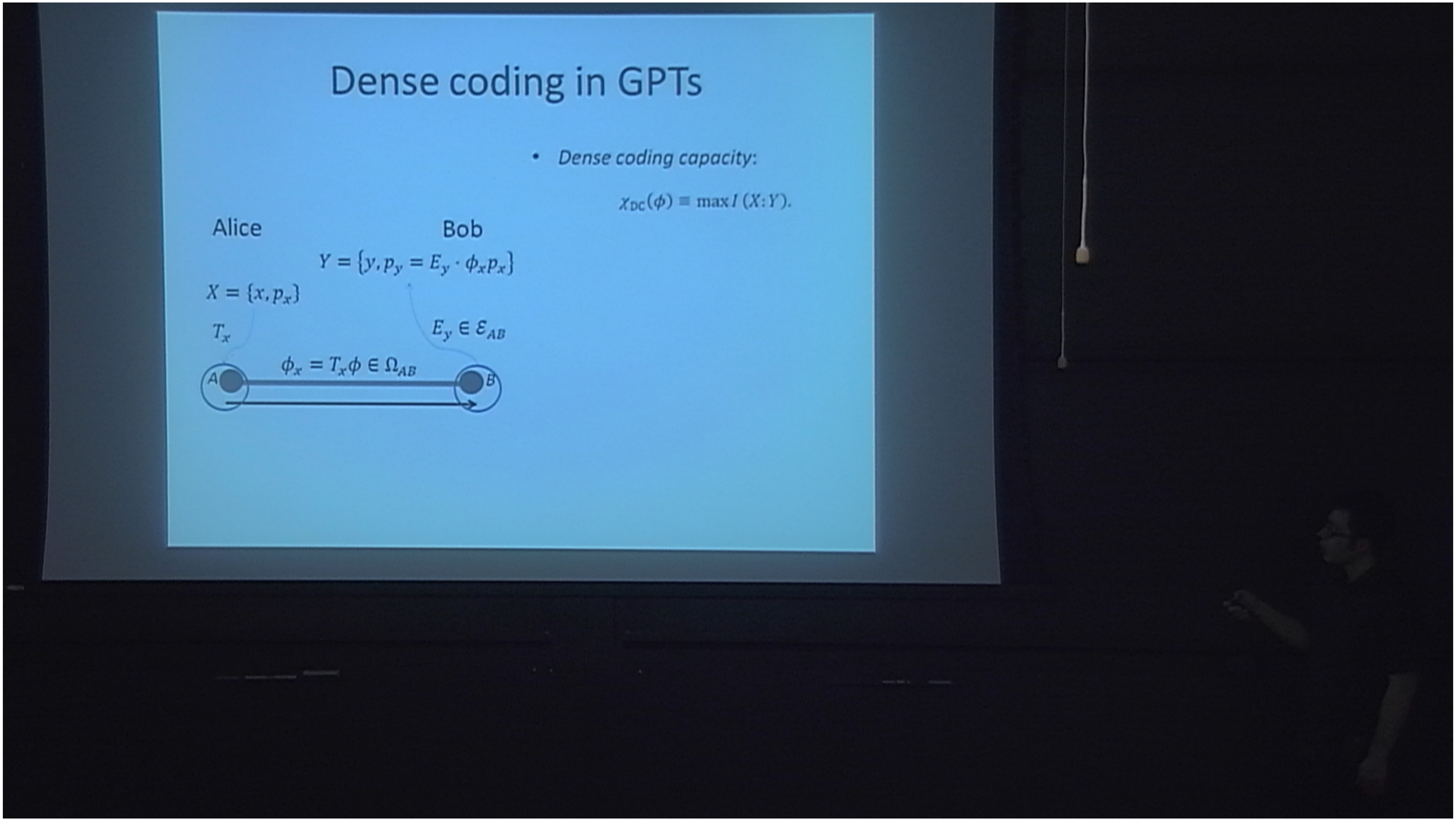
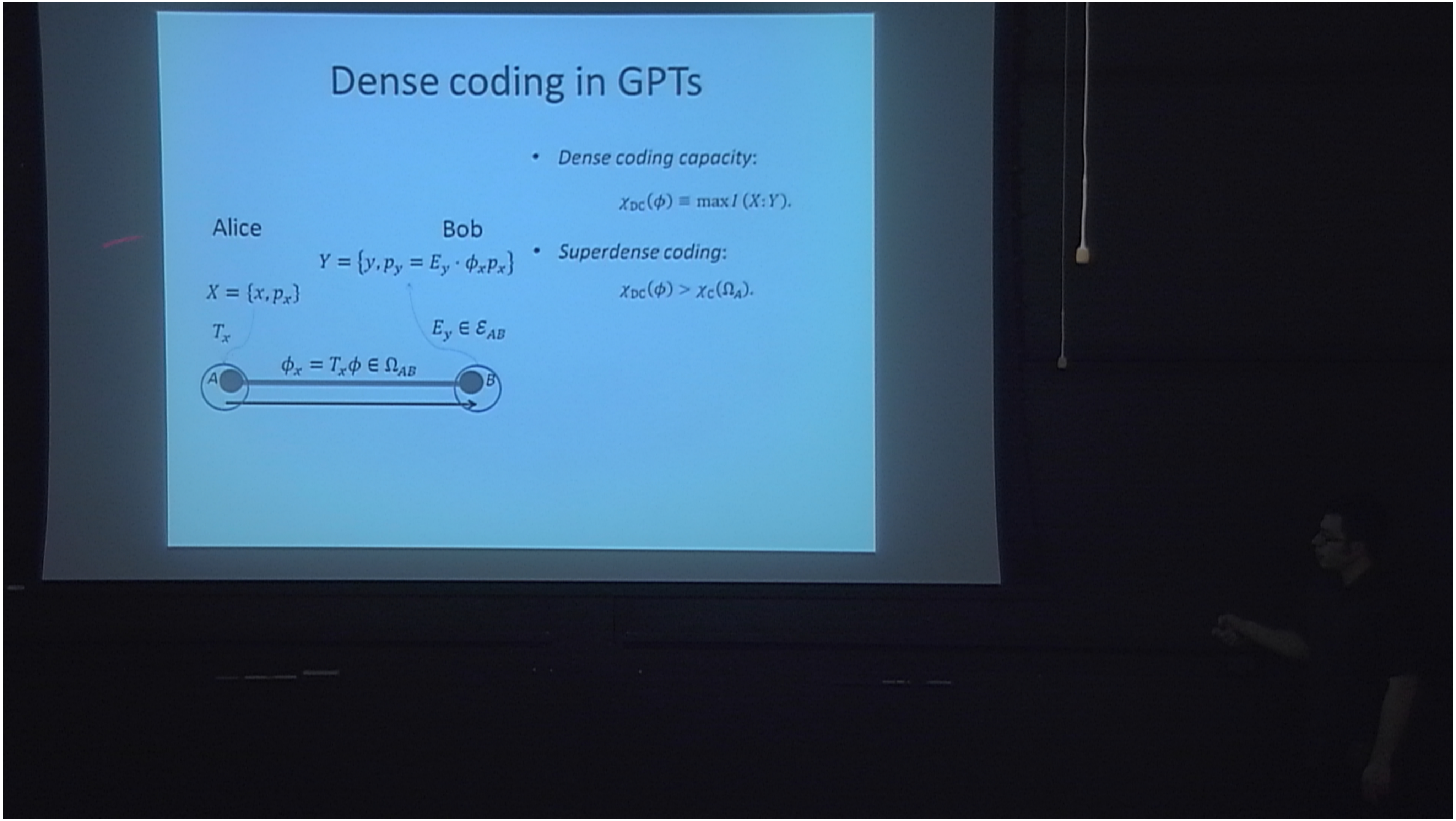
$E_y \in \mathcal{E}_{AB}$


Classical capacities of GPTs
Classical capacity:
$\chi_C(\Omega_A) = \max I(X:Y)$
Alice
$X = \{x, p_x\}$
Bob
$Y = \{y, p_y = \sum_x e_y \cdot \omega_x p_x\}$
$\omega_x \in \Omega_A$
$e_y \in \mathcal{E}_A$
Superadditive classical capacities:
$\chi_C(\Omega_{AB}) > \chi_C(\Omega_A) + \chi_C(\Omega_B)$
Alice
$X = \{x, p_x\}$
Bob
$Y = \{y, p_y = \sum_x E_y \cdot \phi_x p_x\}$
$\phi_x \in \Omega_{AB}$
$E_y \in \mathcal{E}_{AB}$


Dense coding in GPTs
Dense coding capacity:
$\chi_{DC}(\phi) \equiv \max I\,(X{:}Y).$
Alice
Bob
$Y = \{y, p_y = E_y \cdot \phi_x p_x\}$
$X = \{x, p_x\}$
T_x
$E_y \in \mathcal{E}_{AB}$
$\phi_x = T_x\phi \in \Omega_{AB}$
A
B


Dense coding in GPTs
Dense coding capacity:
$\chi_{DC}(\phi) \equiv \max I\,(X{:}Y).$
Superdense coding:
$\chi_{DC}(\phi) > \chi_{C}(\Omega_A).$
Alice
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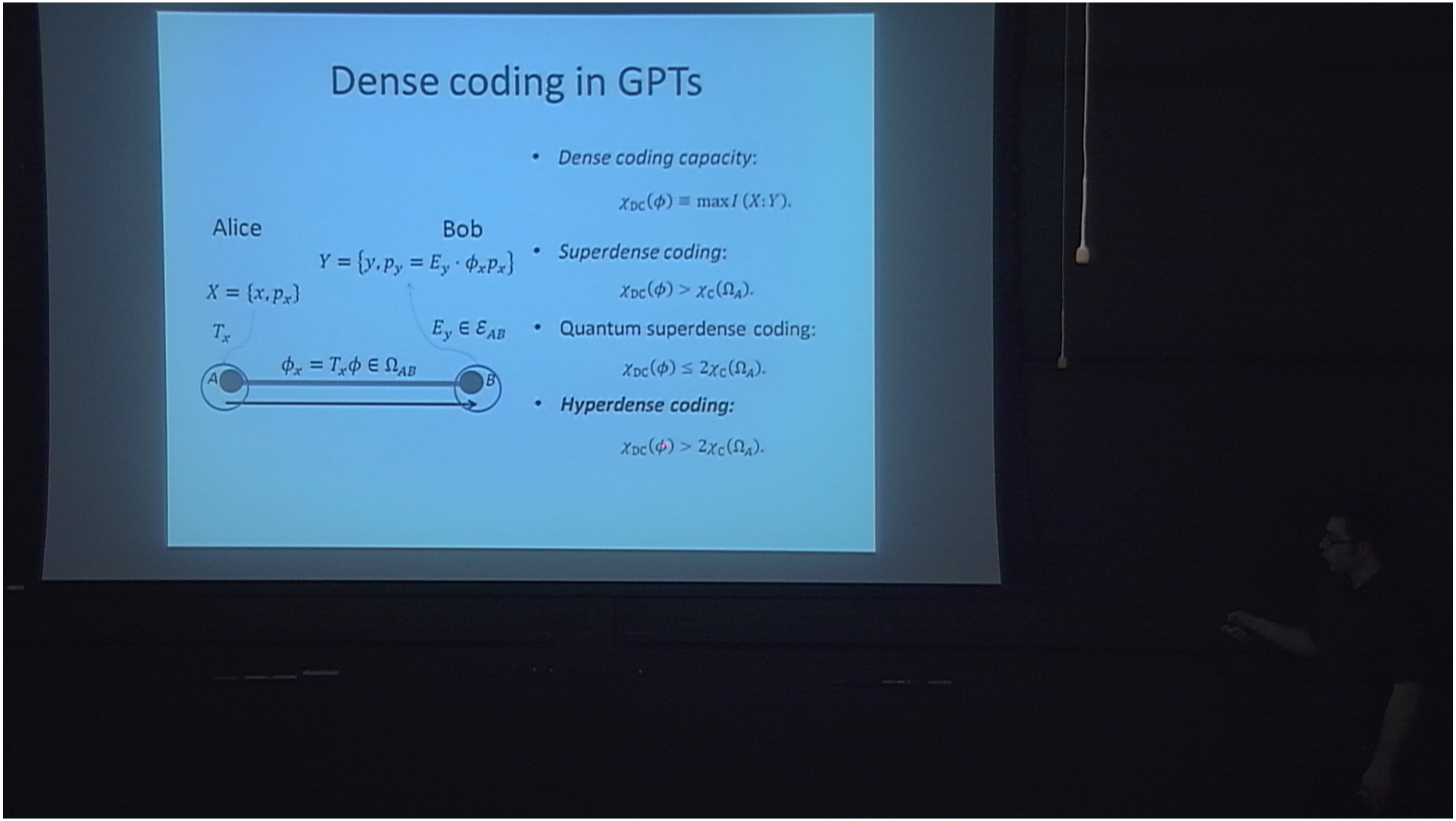
Dense coding in GPTs
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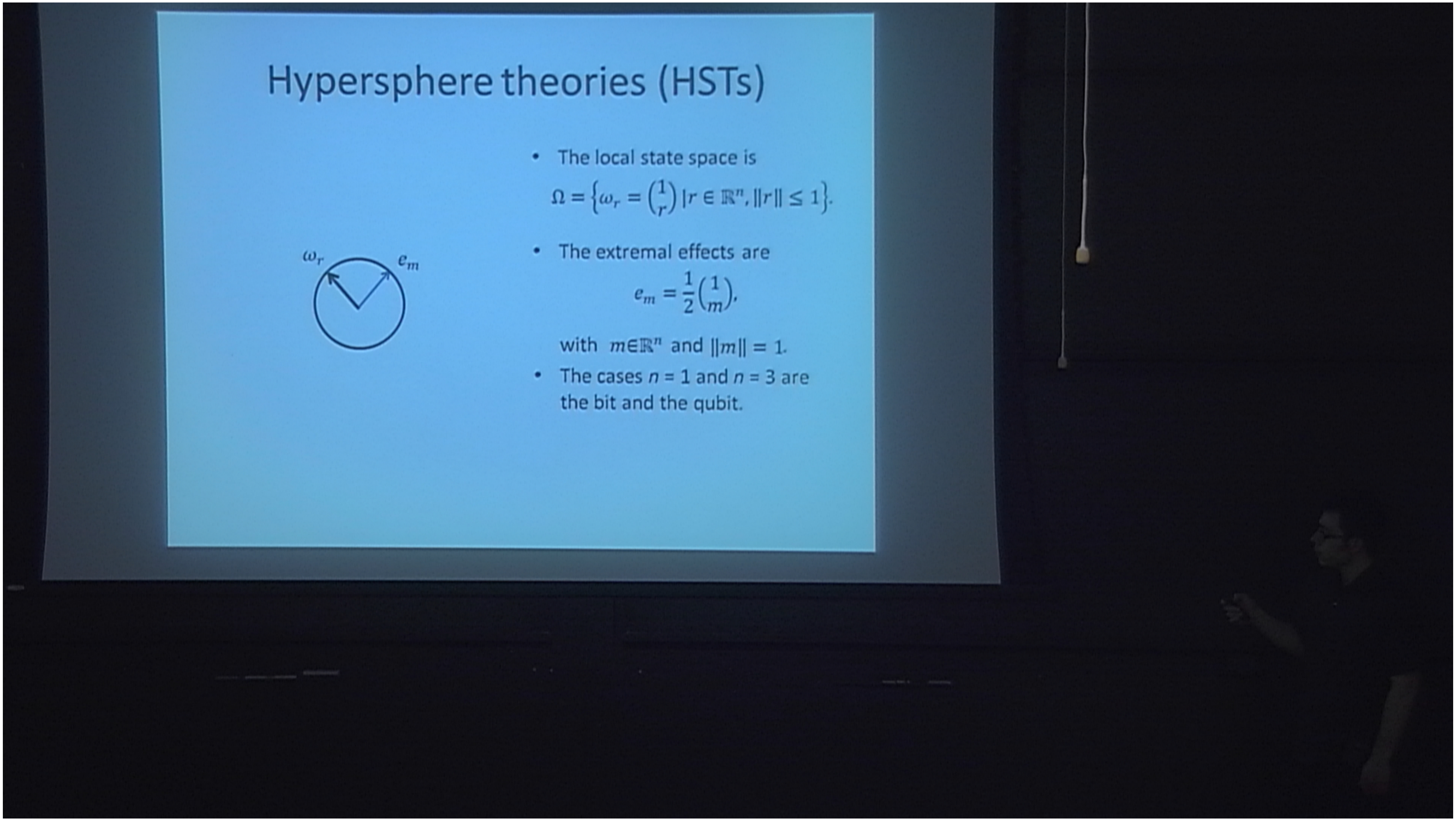
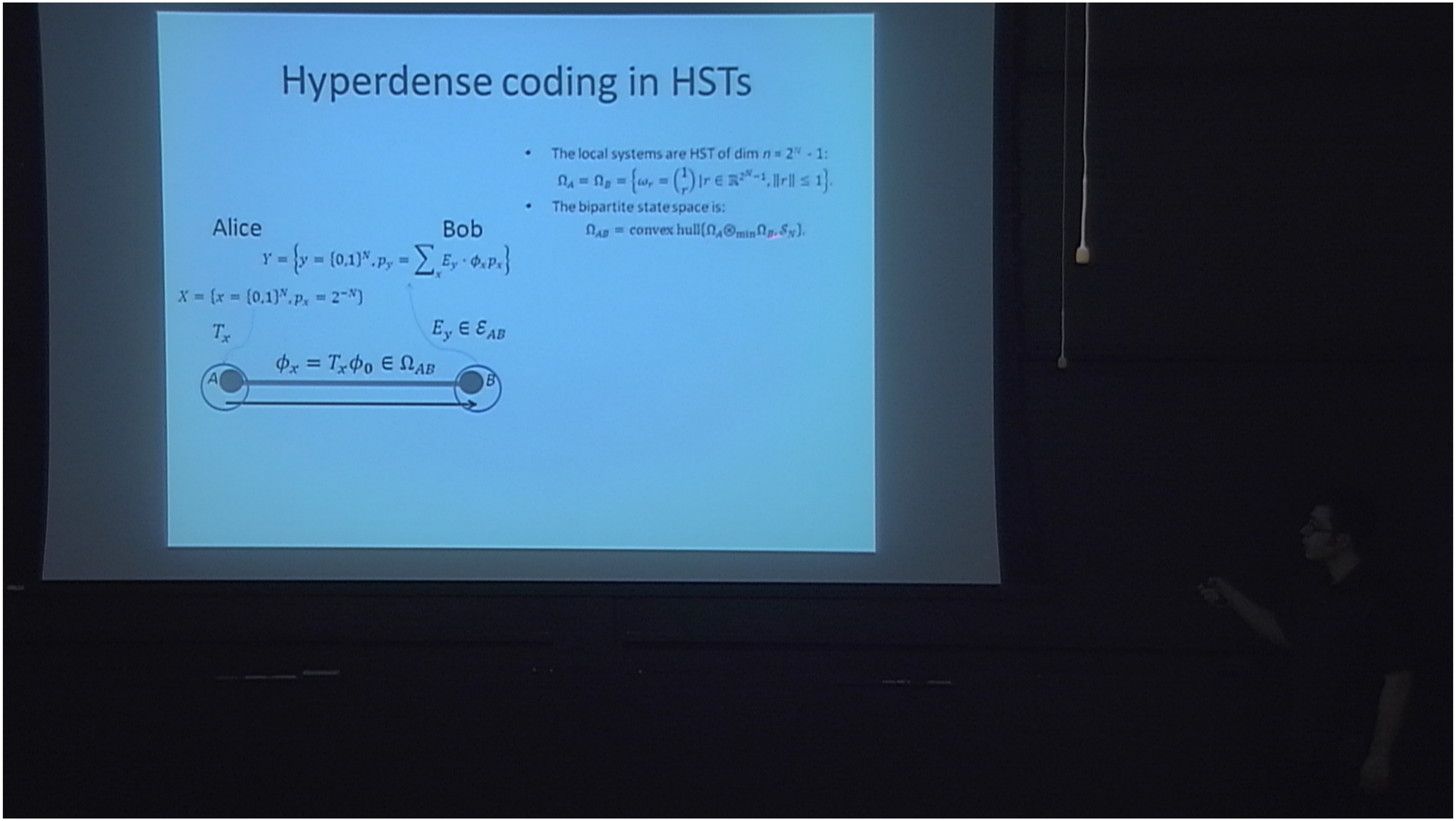
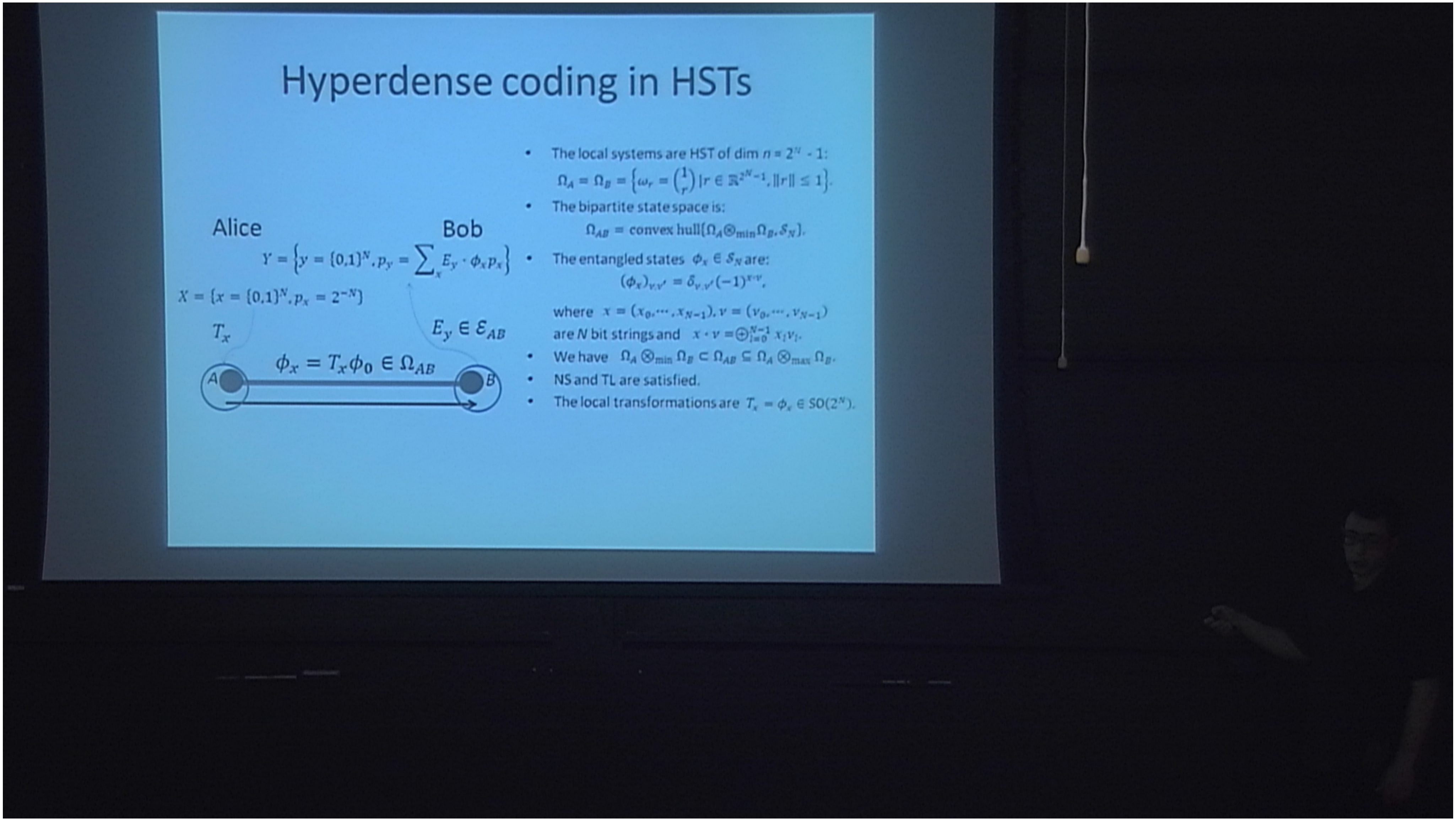
$X = \{x, p_x\}$
T_x
$E_y \in \mathcal{E}_{AB}$
$\phi_x = T_x \phi \in \Omega_{AB}$
A
B
• *Dense coding capacity*:
$\chi_{\mathrm{DC}}(\phi) \equiv \max I\,(X\!:\!Y).$
• *Superdense coding*:
$\chi_{\mathrm{DC}}(\phi) > \chi_{\mathrm{C}}(\Omega_A).$
• Quantum superdense coding:
$\chi_{\mathrm{DC}}(\phi) \leq 2\chi_{\mathrm{C}}(\Omega_A).$
• ***Hyperdense coding:***
$\chi_{\mathrm{DC}}(\phi) > 2\chi_{\mathrm{C}}(\Omega_A).$


Hypersphere theories (HSTs)
ω_r
e_m
• The local state space is
$\Omega = \left\{\omega_r = \binom{1}{r} \,|\, r \in \mathbb{R}^n, \|r\| \leq 1\right\}.$
• The extremal effects are
$e_m = \frac{1}{2}\binom{1}{m},$
with $m \in \mathbb{R}^n$ and $\|m\| = 1.$
• The cases n = 1 and n = 3 are the bit and the qubit.


Hyperdense coding in HSTs
The local systems are HST of dim $n = 2^N - 1$:
$\Omega_A = \Omega_B = \left\{\omega_r = \binom{1}{r} \mid r \in \mathbb{R}^{2^N-1}, \|r\| \leq 1\right\}$.
The bipartite state space is:
$\Omega_{AB} = \text{convex hull}(\Omega_A \otimes_{\min} \Omega_B, \mathcal{S}_N)$.
Alice
Bob
$Y = \left\{y = \{0,1\}^N, p_y = \sum_x E_y \cdot \phi_x p_x\right\}$
$X = \{x = \{0,1\}^N, p_x = 2^{-N}\}$
T_x
$E_y \in \mathcal{E}_{AB}$
$\phi_x = T_x \phi_0 \in \Omega_{AB}$
A
B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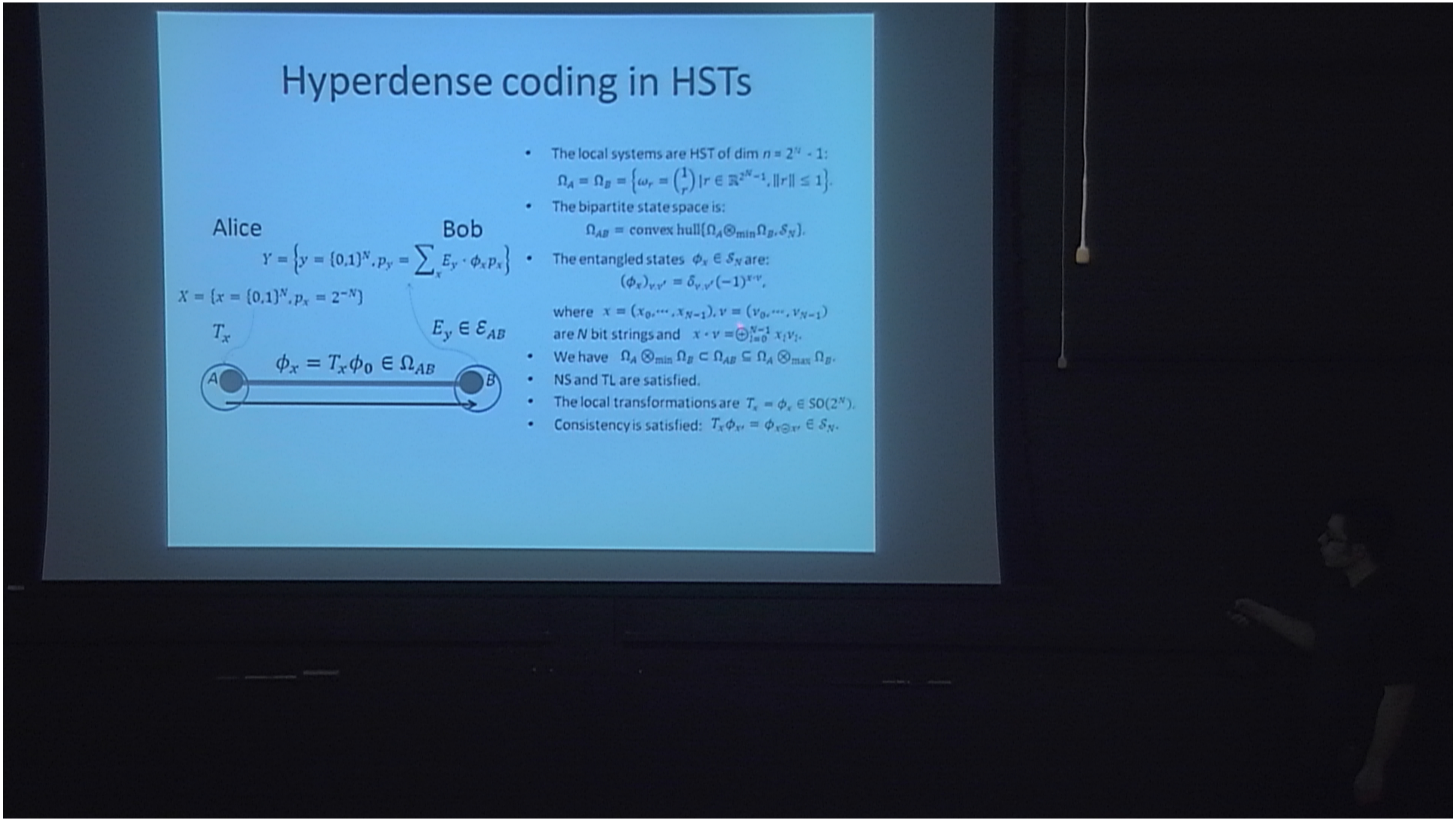
The bipartite state space is:
$\Omega_{AB} = \text{convex hull}\{\Omega_A \otimes_{\min} \Omega_B, \mathcal{S}_N\}$.
The entangled states $\phi_x \in \mathcal{S}_N$ are:
$(\phi_x)_{v,v'} = \delta_{v,v'} (-1)^{x \cdot v}$,
where $x = (x_0, \cdots, x_{N-1}), v = (v_0, \cdots, v_{N-1})$
are N bit strings and $x \cdot v = \bigoplus_{i=0}^{N-1} x_i v_i$.
We have $\Omega_A \otimes_{\min} \Omega_B \subset \Omega_{AB} \subseteq \Omega_A \otimes_{\max} \Omega_B$.
NS and TL are satisfied.
The local transformations are $T_x = \phi_x \in SO(2^N)$.



Hyperdense coding in HSTs
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$X = \{x = \{0,1\}^N, p_x = 2^{-N}\}$
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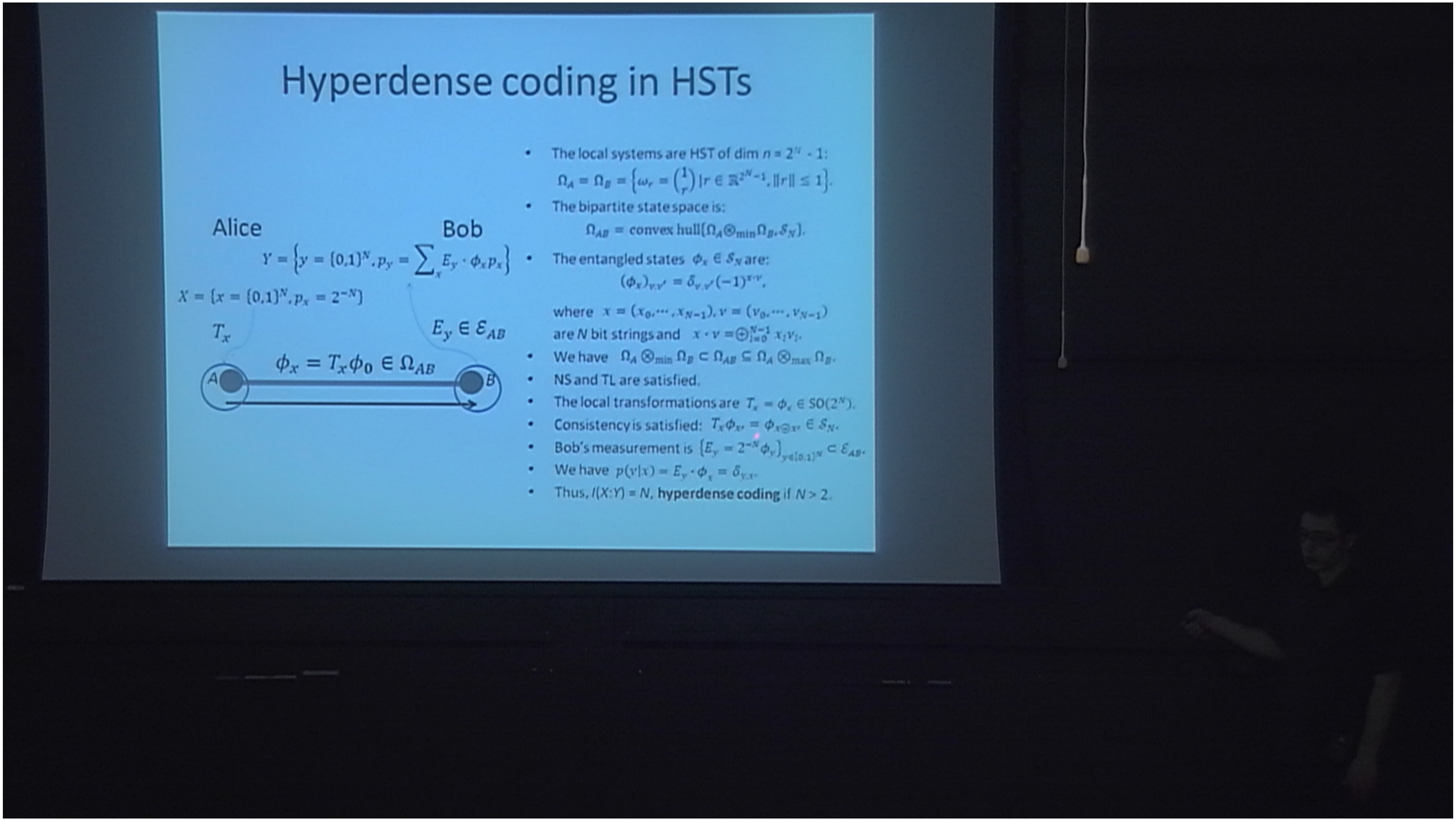
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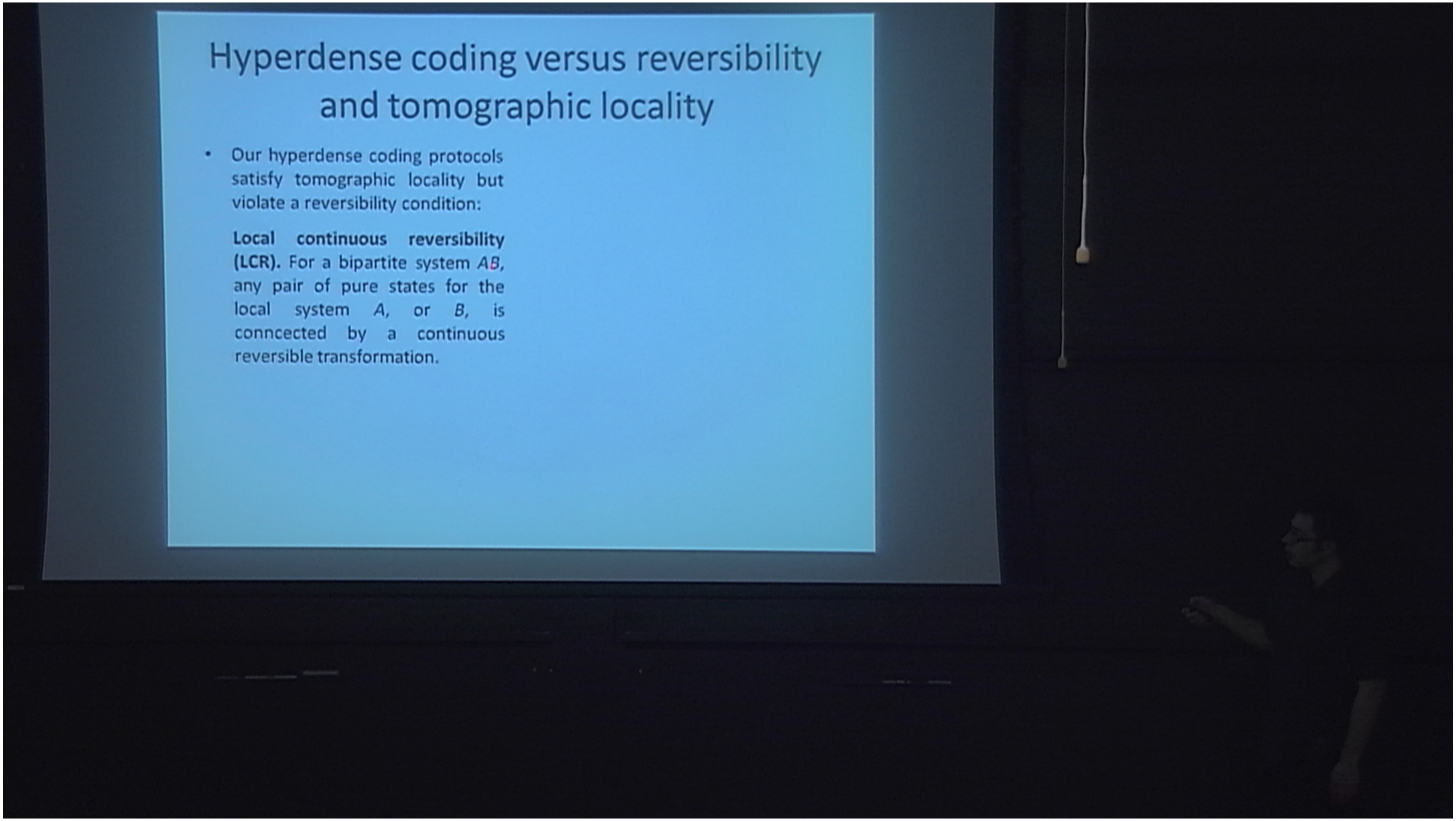
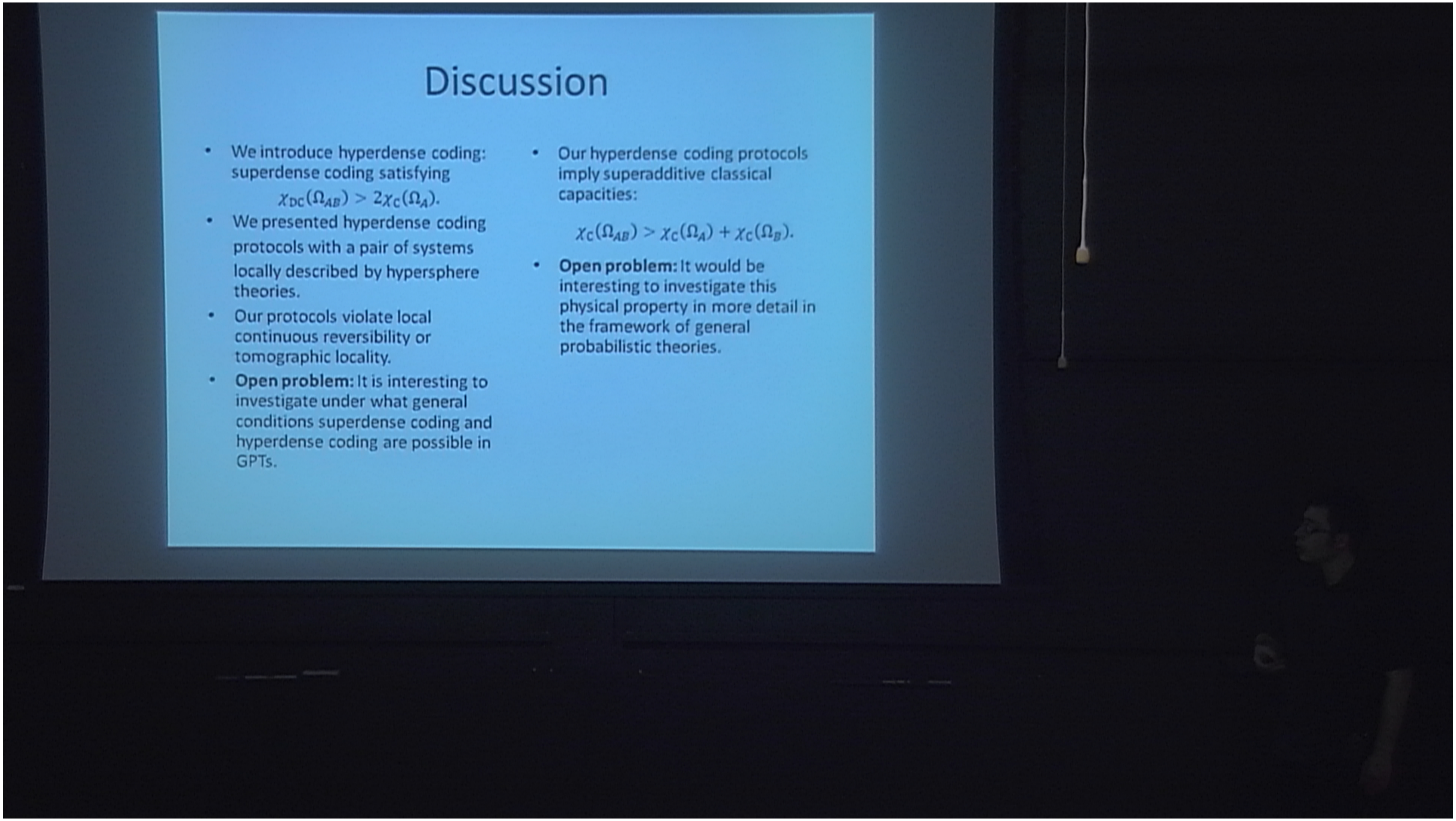
• The local transformations are $T_x = \phi_x \in SO(2^N)$.
• Consistency is satisfied: $T_x \phi_{x'} = \phi_{x \oplus x'} \in \mathcal{S}_N$.
• Bob's measurement is $\{E_y = 2^{-N} \phi_y\}_{y \in \{0,1\}^N} \subset \mathcal{E}_{AB}$.
• We have $p(y|x) = E_y \cdot \phi_x = \delta_{y,x}$.
• Thus, $I(X{:}Y) = N$, **hyperdense coding** if $N > 2$.


Hyperdense coding versus reversibility and tomographic locality
• Our hyperdense coding protocols satisfy tomographic locality but violate a reversibility condition:
Local continuous reversibility (LCR). For a bipartite system AB, any pair of pure states for the local system A, or B, is conncected by a continuous reversible transformation.


Discussion
We introduce hyperdense coding: superdense coding satisfying
$\chi_{\mathrm{DC}}(\Omega_{AB}) > 2\chi_{\mathrm{C}}(\Omega_A)$.
We presented hyperdense coding protocols with a pair of systems locally described by hypersphere theories.


Discussion
We introduce hyperdense coding: superdense coding satisfying
$\chi_{DC}(\Omega_{AB}) > 2\chi_C(\Omega_A)$.
We presented hyperdense coding protocols with a pair of systems locally described by hypersphere theories.
Our protocols violate local continuous reversibility or tomographic locality.
Open problem: It is interesting to investigate under what general conditions superdense coding and hyperdense coding are possible in GPTs.
Our hyperdense coding protocols imply superadditive classical capacities:
$\chi_C(\Omega_{AB}) > \chi_C(\Omega_A) + \chi_C(\Omega_B)$.
Open problem: It would be interesting to investigate this physical property in more detail in the framework of general probabilistic theories.