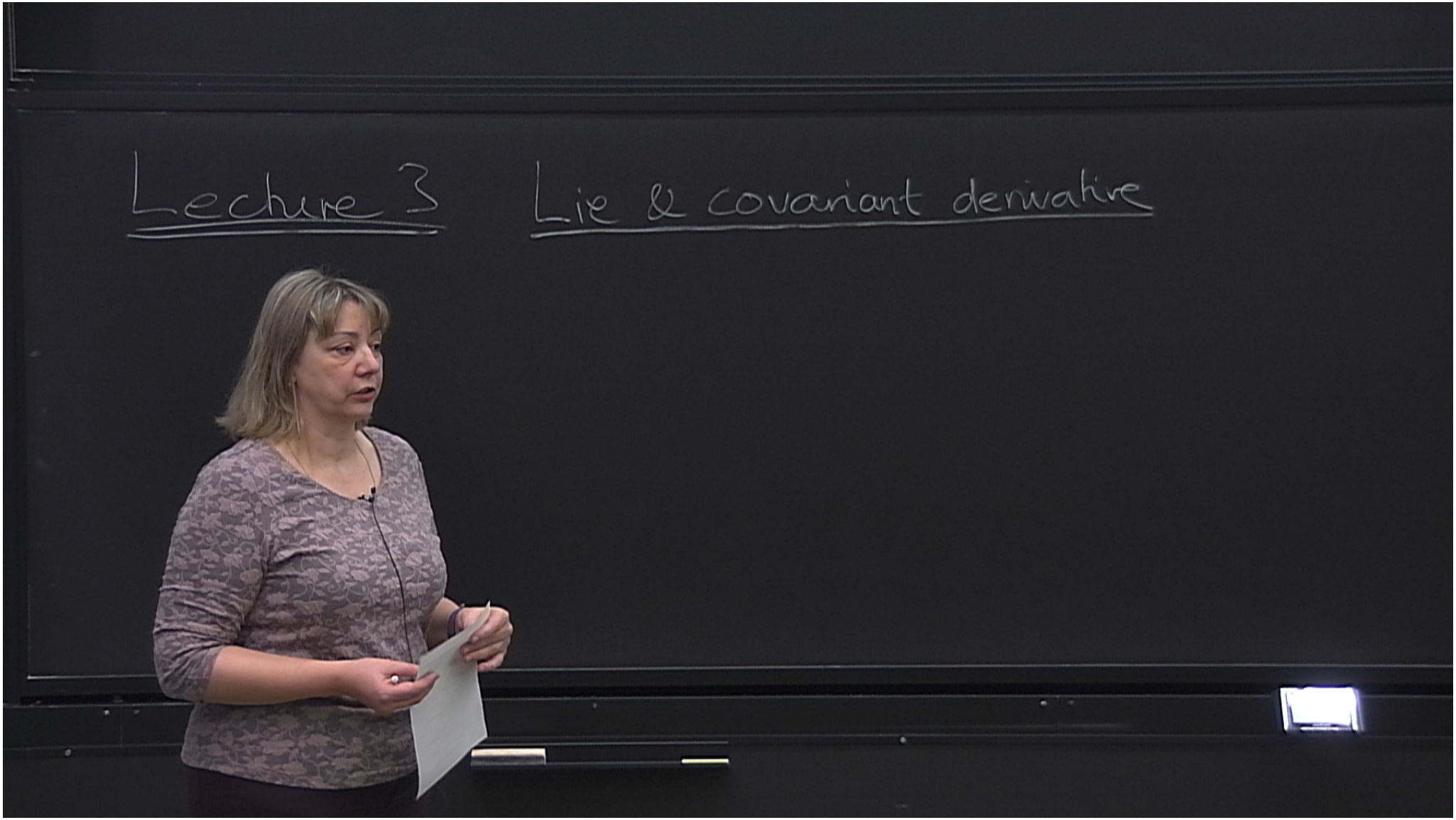
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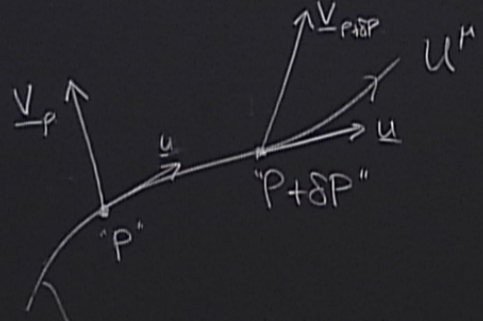
Date: Jan 06, 2015 09:00 AM

URL: http://pirsa.org/15010088

Abstract:


Lecture 3
Lie & covariant derivative

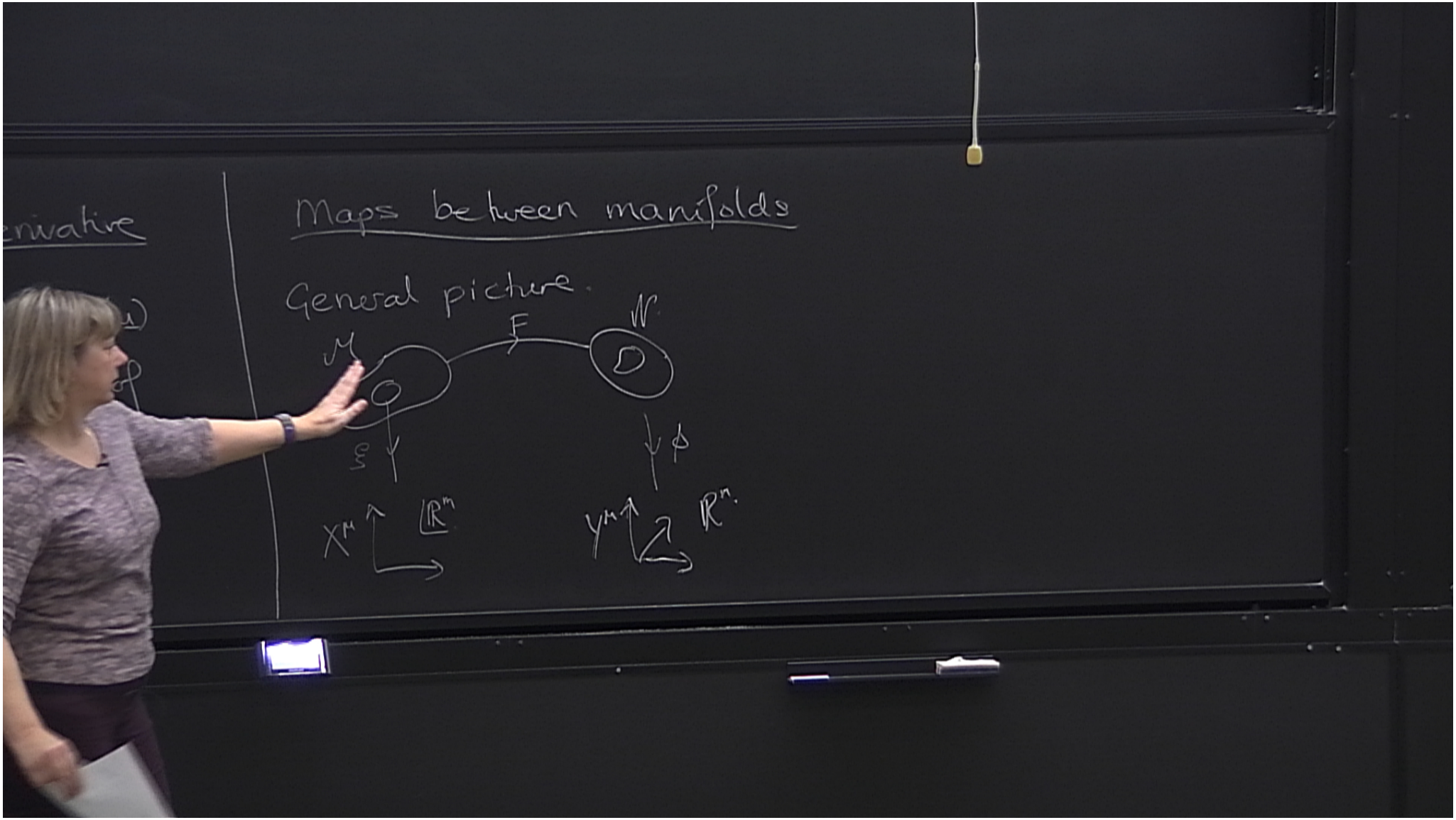
# Lecture 3 Lie & covariant derivative

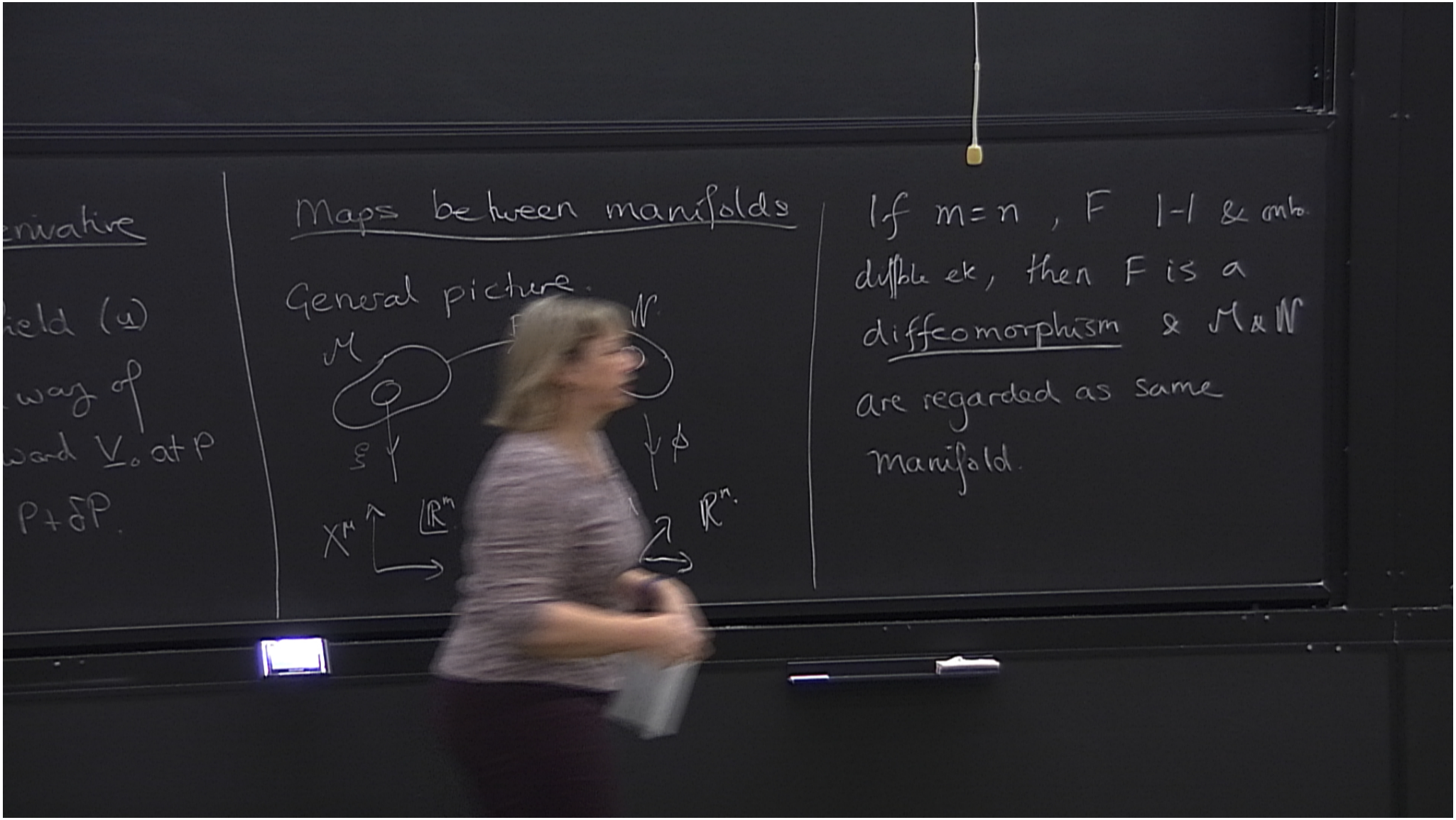


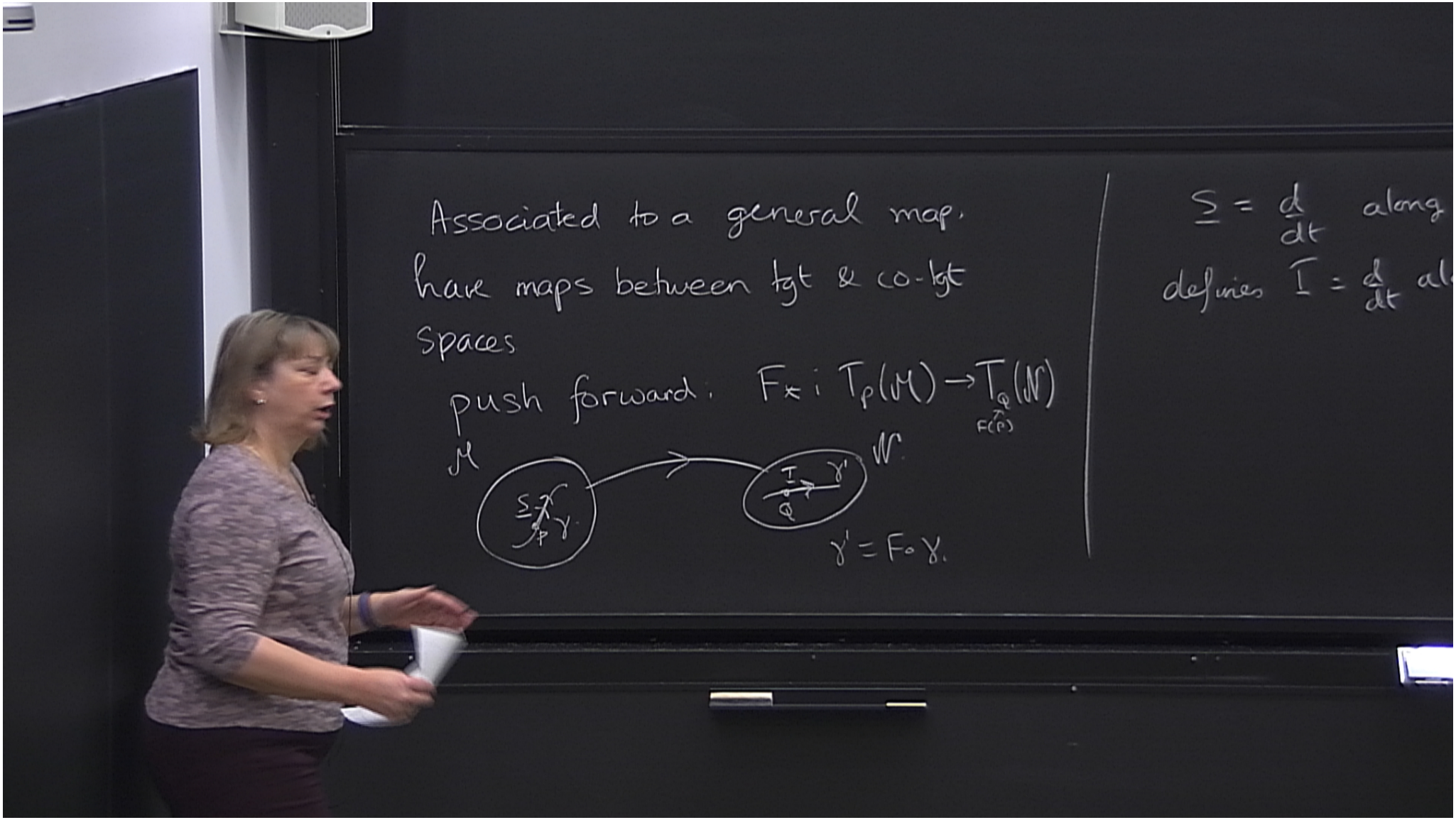

$\gamma$ has tgt $\underline{u}$

$\gamma$ is an integral curve of $\underline{u}$

A vector field ($\underline{u}$) defines a way of pushing forward $\underline{V}_0$ at $P$ to $\underline{V}_*$ at $P+\delta P$.


Maps between manifolds
General picture.
F
M
N
D
$\varphi$
$\psi$
$X^m$
$\mathbb{R}^m$
$Y^n$
$\mathbb{R}^n$.


erivative
ield (v)
way of
ward V at p
P+δP
Maps between manifolds
General picture
M
N
X^μ
ℝ^m
ℝ^n
If m=n, F 1-1 & onto
diffble etc, then F is a
diffeomorphism & M & N
are regarded as same
manifold.


Associated to a general map,
have maps between tgt & co-tgt
spaces
push forward. $F_*: T_p(M) \to T_{F(p)}(N)$
$M$
$N$
$\gamma' = F \circ \gamma$.
$\frac{D}{dt}$ along
defines $\frac{D}{dt}$ al

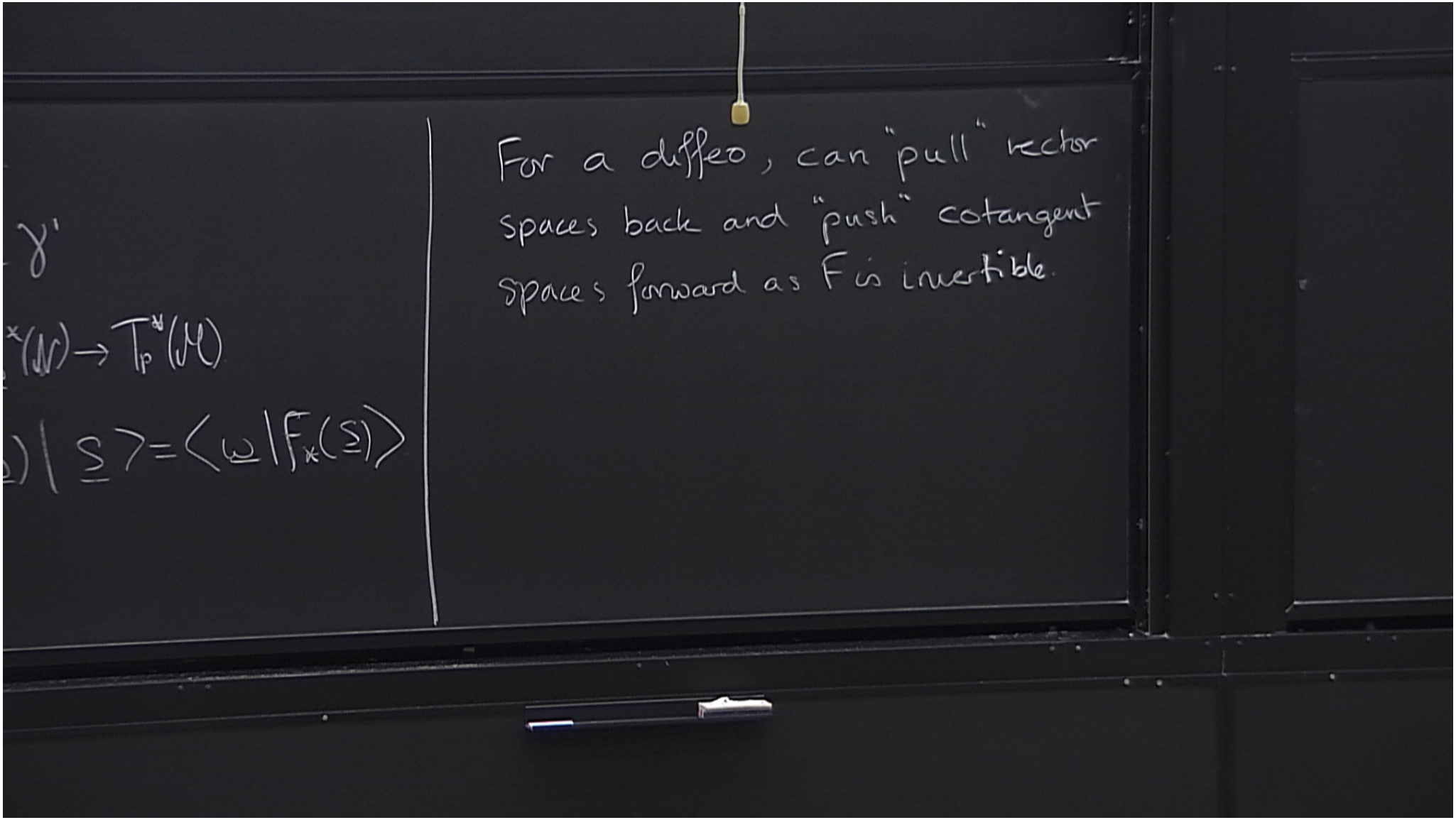
$\underline{S} = \frac{d}{dt}$ along $\gamma$.

defines $\underline{I} = \frac{d}{dt}$ along $\gamma'$

pull back: $F^* : T_Q^*(N) \to T_P^*(M)$

defined via $\langle F^*(\underline{\omega}) \mid \underline{S} \rangle = \langle \underline{\omega} \mid F_*(\underline{S}) \rangle$

$\to T_Q(N)$
$F(P)$

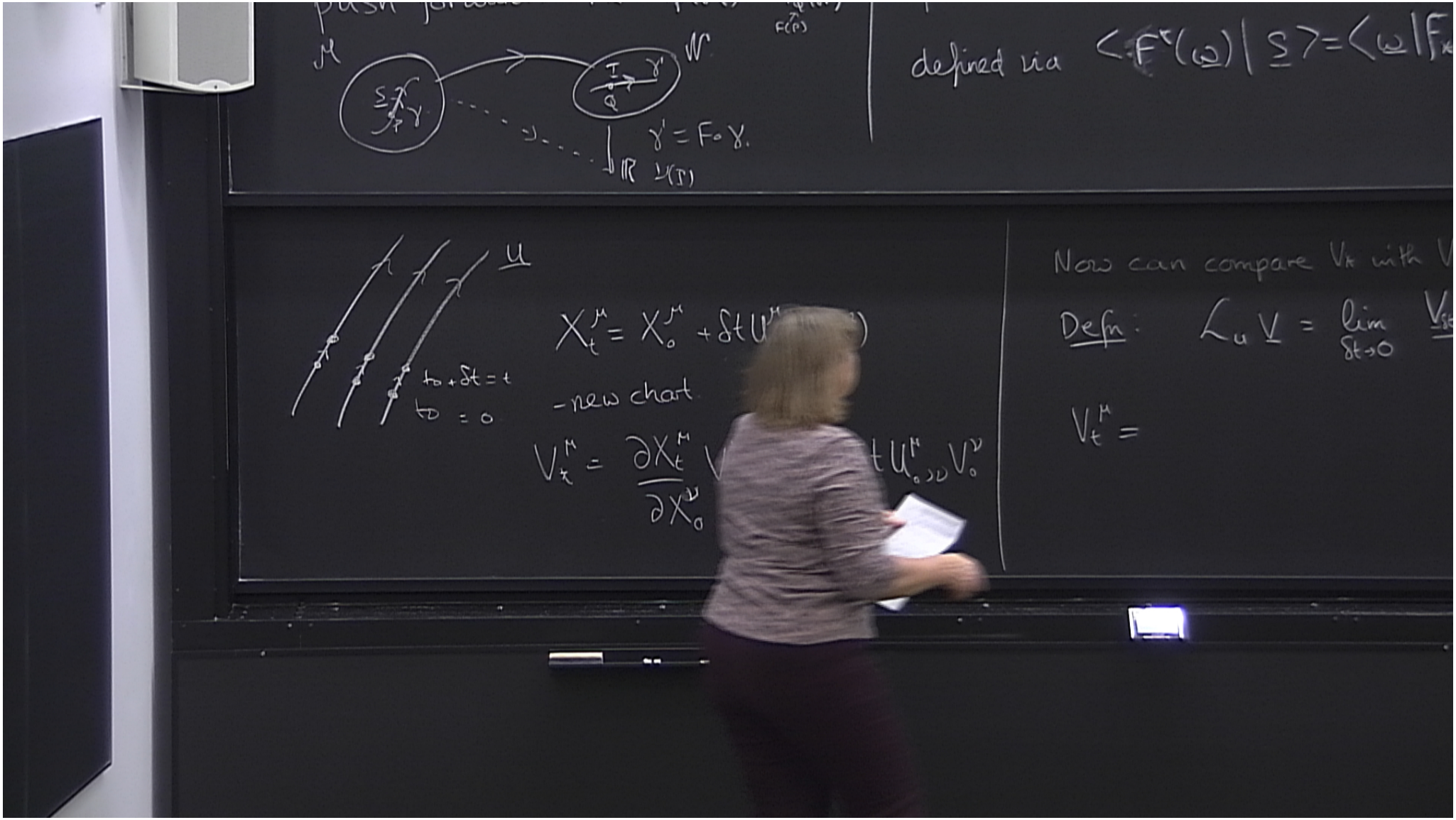

γ'
$T_p^*(N) \to T_p^*(M)$
$\langle \omega | F_*(\xi) \rangle$
For a diffeo, can "pull" vector spaces back and "push" cotangent spaces forward as F is invertible.

For a diffeo, can "pull" vector spaces back and "push" cotangent spaces forward as $F$ is invertible.

//

$|F_*(\Sigma)\rangle$

The vector field $\underline{u}$ defines an infinitesimal co-ord transfm by moving a "$\delta t$" along integral curves of $\underline{u}$.


M
W
$\gamma' = F \circ \gamma$
$\mathbb{R}$
$I(I)$
defined via
$\langle F^*(\omega) | \underline{S} \rangle = \langle \omega | F$
$t_0 + \delta t = t$
$t_0 = 0$
$X^\mu_t = X^\mu_0 + \delta t U$
- new chart
$V^\mu_t = \frac{\partial X^\mu_t}{\partial X^\nu_0}$
Now can compare $V_t$ with
Defn: $\mathcal{L}_u \underline{V} = \lim_{\delta t \to 0}$
$V^\mu_t =$

$F_* : T_p(M) \to T_{F(p)}(N)$

$\gamma' = F \circ \gamma$

$\gamma : I \to M$

pull back: $F^* : T^*_{F(p)}(N) \to T^*_p(M)$

defined via $\langle F^*(\omega) | \underline{S} \rangle = \langle \omega | F_*(\underline{S}) \rangle$

The vector field $\underline{u}$ defines an infinitesimal co-ord transfm by moving a "$\delta t$" along integral curves of $\underline{u}$.

$= X^\mu_0 + \delta t\, U^\mu + o(\delta t^2)$

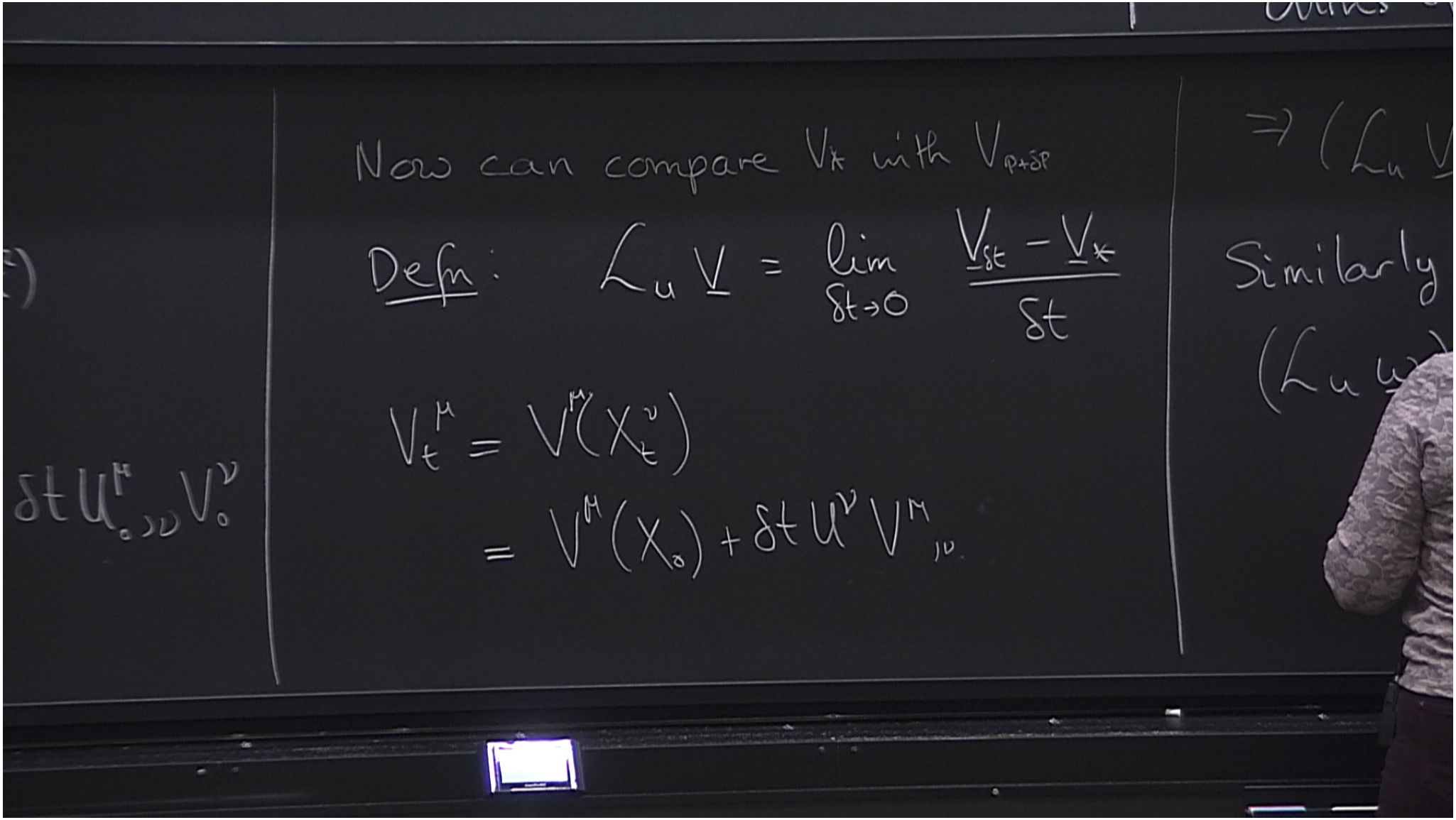
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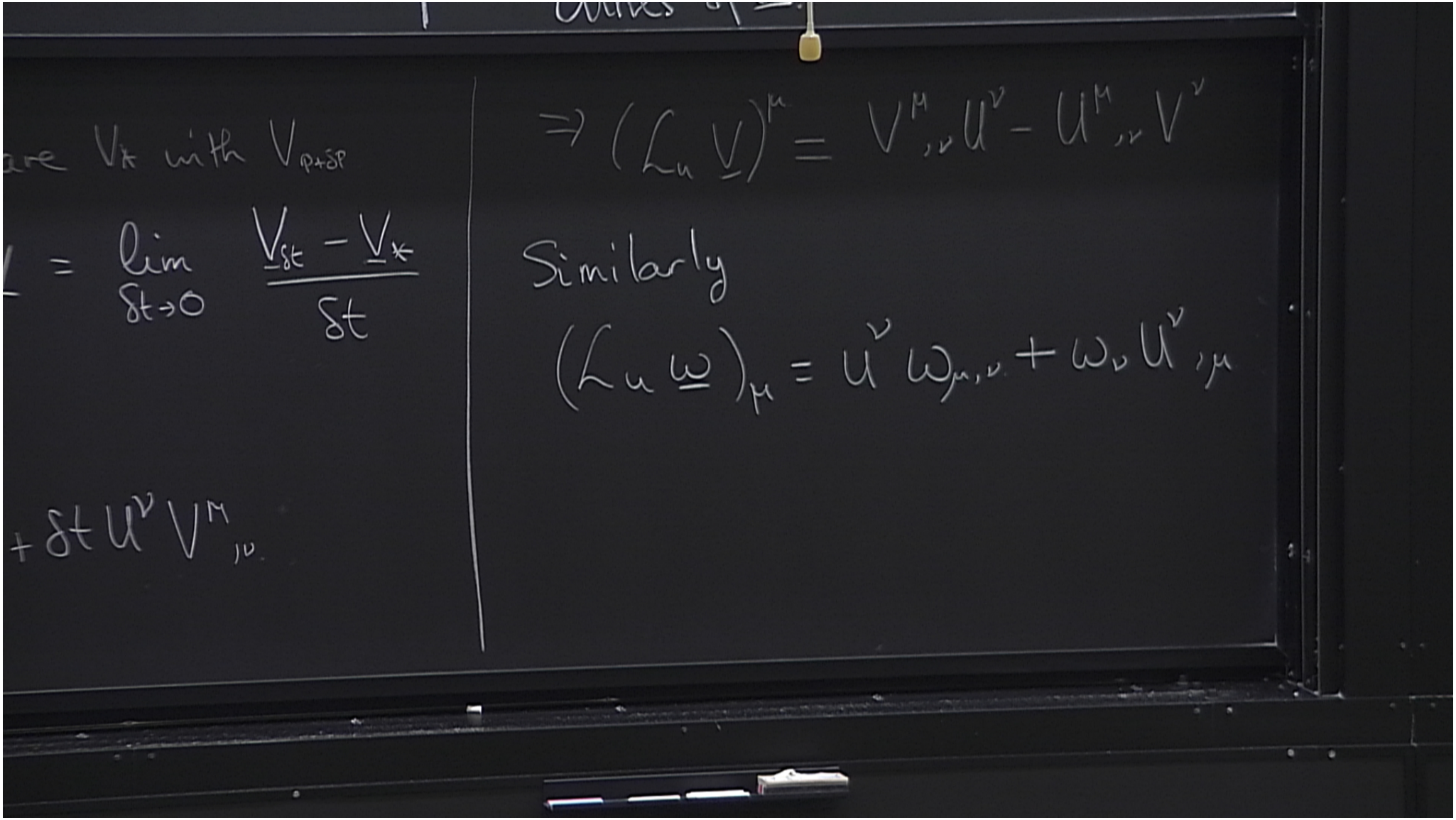
$$\frac{\partial X^\mu_t}{\partial X^\nu_0} V^\nu_0 = V^\mu_0 + \delta t\, U^\mu_{,\nu} V^\nu_0$$

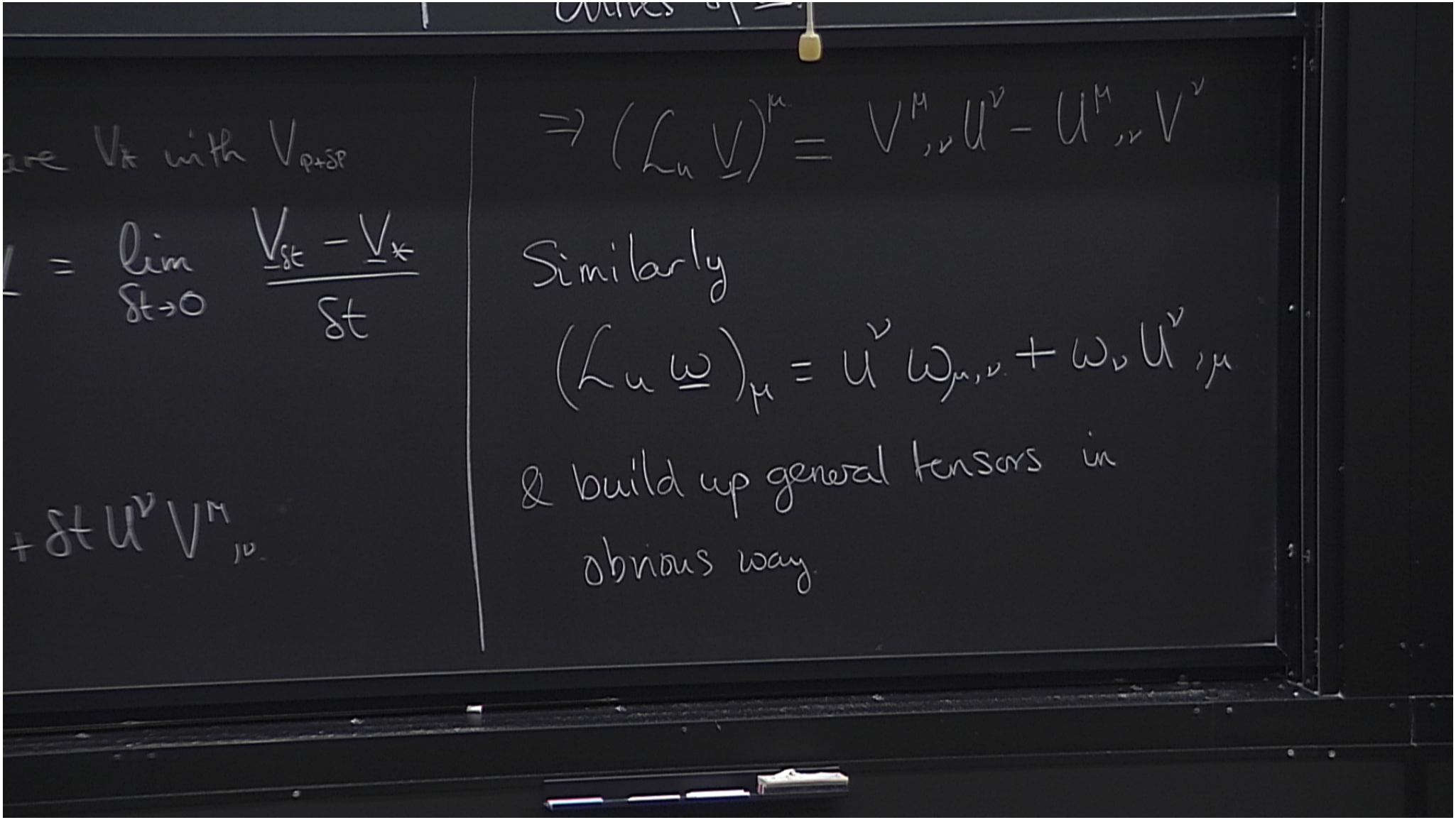
Now can compare $V_t$ with $V_{push}$

**Defn** $\mathcal{L}_u \underline{V} = \lim_{\delta t \to 0} \frac{\underline{V}_{\delta t} - \underline{V}_t}{\delta t}$

$$V^\mu_t = V^\mu(X^\nu_t)$$
$$= V^\mu(X_0) + \delta t\, U^\nu V^\mu_{,\nu}$$


Now can compare $V_*$ with $V_{p+\delta p}$
Defn: $\mathcal{L}_u \underline{V} = \lim_{\delta t \to 0} \frac{\underline{V}_{\delta t} - \underline{V}_*}{\delta t}$
$V_t^\mu = V^\mu(X_t^\nu)$
$= V^\mu(X_0) + \delta t\, U^\nu V^\mu{}_{,\nu}$
$\delta t\, U^\mu_{,\nu} V^\nu_0$
$\Rightarrow (\mathcal{L}_u \underline{V}$
Similarly
$(\mathcal{L}_u \underline{w}$


are $V_*$ with $V_{p+\delta p}$
$= \lim_{\delta t \to 0} \frac{\underline{V}_{\delta t} - \underline{V}_*}{\delta t}$
$+ \delta t\, U^\nu V^\mu{}_{,\nu}$
$\Rightarrow (\mathcal{L}_U \underline{V})^\mu = V^\mu{}_{,\nu} U^\nu - U^\mu{}_{,\nu} V^\nu$
Similarly
$(\mathcal{L}_U \underline{\omega})_\mu = U^\nu \omega_{\mu,\nu} + \omega_\nu U^\nu{}_{,\mu}$


are $V_x$ with $V_{p+\delta p}$
$= \lim_{\delta t \to 0} \frac{\underline{V}_{\delta t} - \underline{V}_*}{\delta t}$
$+ \delta t\, U^\nu V^\mu{}_{,\nu}$
$\Rightarrow (\mathcal{L}_U \underline{V})^\mu = V^\mu{}_{,\nu} U^\nu - U^\mu{}_{,\nu} V^\nu$
Similarly
$(\mathcal{L}_U \underline{\omega})_\mu = U^\nu \omega_{\mu,\nu} + \omega_\nu U^\nu{}_{,\mu}$
& build up general tensors in
obvious way.

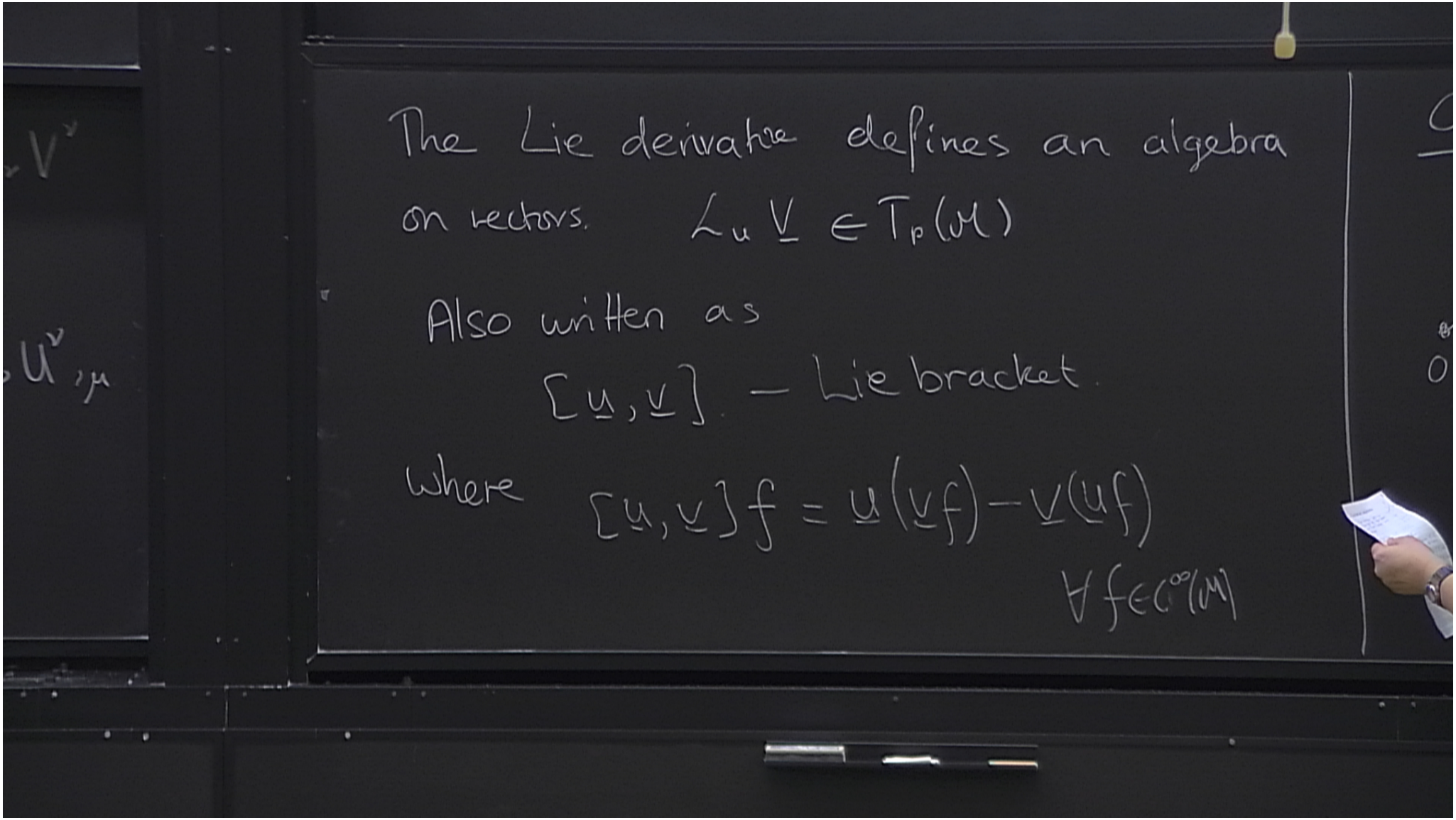
The Lie derivative defines an algebra on vectors. $\mathcal{L}_{\underline{u}} \underline{V} \in T_p(\mathcal{M})$

Also written as

$[\underline{u}, \underline{v}]$ — Lie bracket

where $[\underline{u}, \underline{v}] f = \underline{u}(\underline{v} f) - \underline{v}(\underline{u} f)$

$\forall f \in C^\infty(\mathcal{M})$


The Lie derivative defines an algebra
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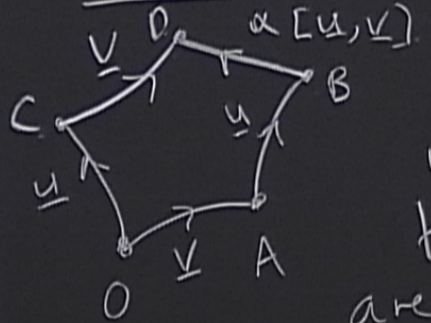
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acket

$)-\underline{v}(\underline{u}f)$

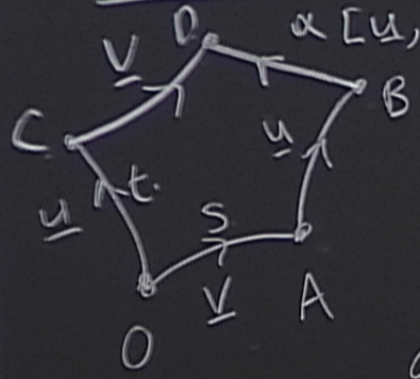
$\forall f \in C^\infty(M)$

## Geometrical Significance




Bracket tells us how far off two vector fields are from co-ord lines.

# Geometrical Significance



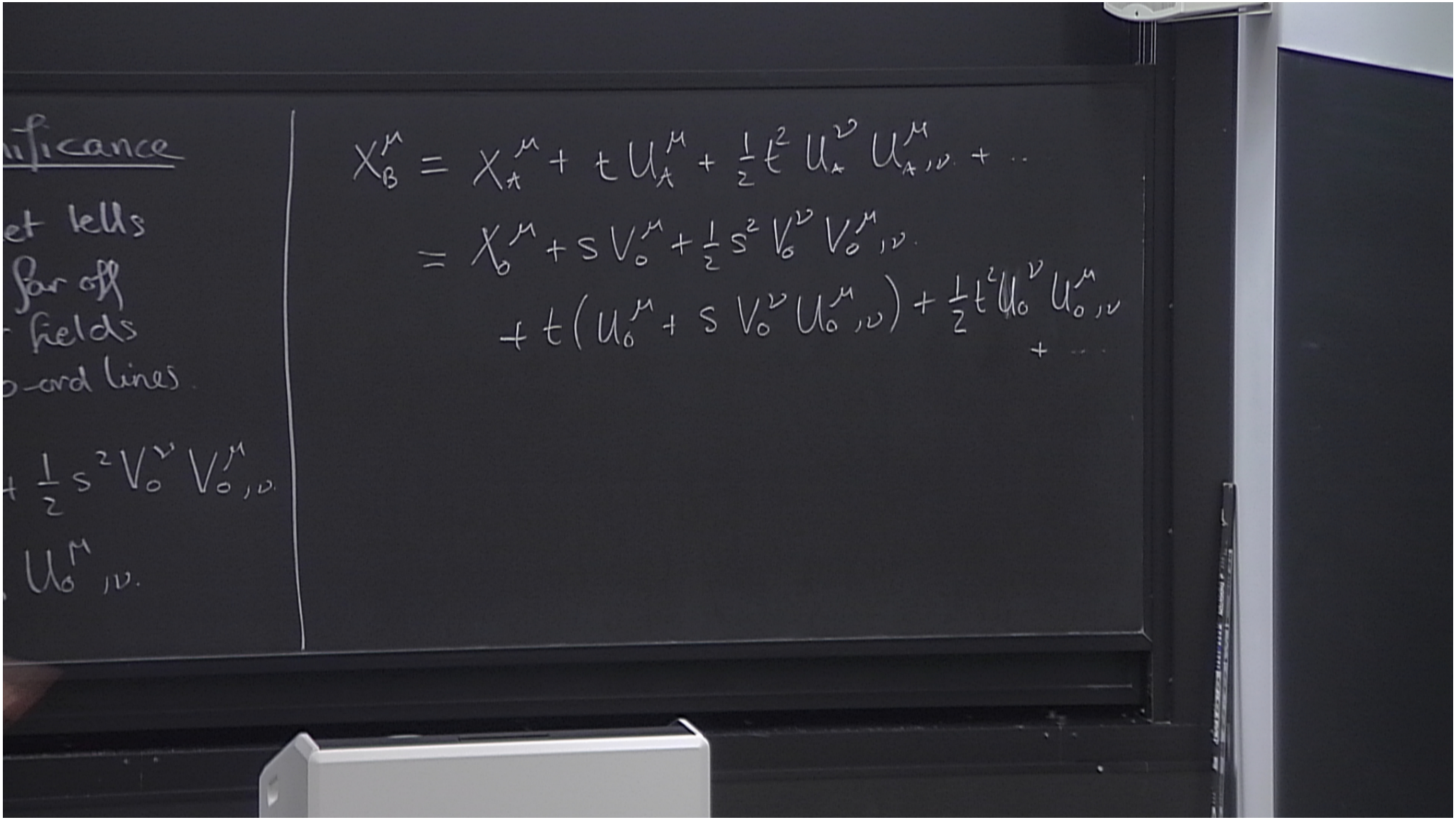

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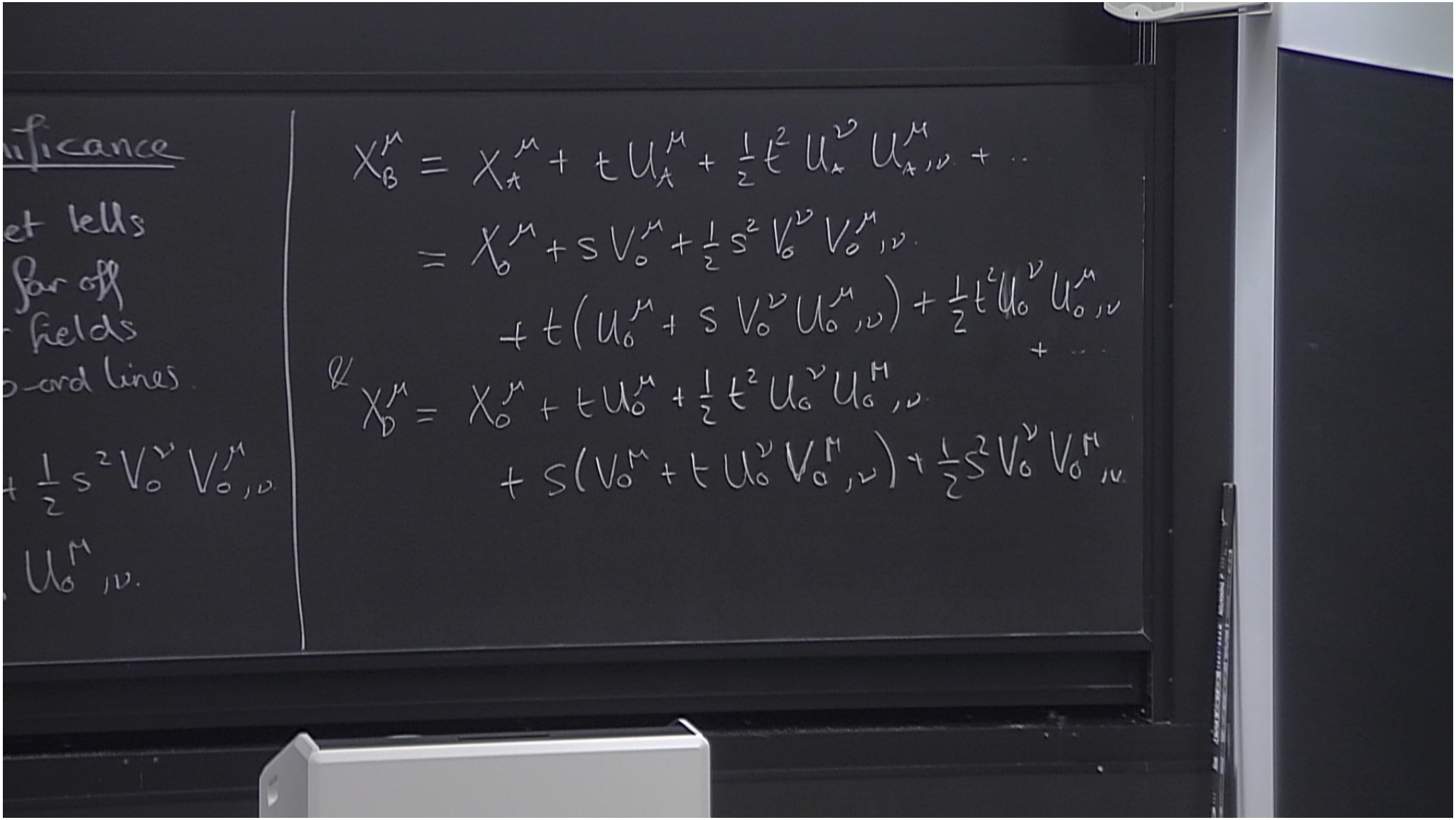
$$X^\mu_A = X^\mu_0 + s V^\mu_0 + \frac{1}{2} s^2 V^\nu_0 V^\mu_{0,\nu}$$

$$U^\mu_A = U^\mu_0 + s V^\nu_0 U^\mu_{0,\nu}$$

$$X^\mu_B = X^\mu_A +$$

$$= X^\mu_0 +$$


nificance
et tells
far off
fields
o-ord lines
$+\frac{1}{2}s^2 V_0^\nu V_{0,\nu}^\mu$
$U_{0,\nu}^\mu$
$X_B^\mu = X_A^\mu + t U_A^\mu + \frac{1}{2}t^2 U_A^\nu U_{A,\nu}^\mu + \ldots$
$= X_b^\mu + s V_0^\mu + \frac{1}{2}s^2 V_0^\nu V_{0,\nu}^\mu$
$+ t(U_0^\mu + s V_0^\nu U_{0,\nu}^\mu) + \frac{1}{2}t^2 U_0^\nu U_{0,\nu}^\mu$
$+ \ldots$


nificance
et tells
far off
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$X_B^\mu = X_A^\mu + t U_A^\mu + \frac{1}{2} t^2 U_A^\nu U_{A,\nu}^\mu + \ldots$
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$+ t(U_o^\mu + s V_o^\nu U_{o,\nu}^\mu) + \frac{1}{2} t^2 U_o^\nu U_{o,\nu}^\mu + \ldots$
& $X_D^\mu = X_o^\mu + t U_o^\mu + \frac{1}{2} t^2 U_o^\nu U_{o,\nu}^\mu$
$+ s(V_o^\mu + t U_o^\nu V_{o,\nu}^\mu) + \frac{1}{2} s^2 V_o^\nu V_{o,\nu}^\mu$

$$X_B^\mu = X_A^\mu + t U_A^\mu + \frac{1}{2} t^2 U_A^\nu U_{A,\nu}^\mu + \dots$$

$$= X_0^\mu + s V_0^\mu + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$$

$$+ t\left(U_0^\mu + s V_0^\nu U_{0,\nu}^\mu\right) + \frac{1}{2} t^2 U_0^\nu U_{0,\nu}^\mu + \dots$$

$$\& \quad X_D^\mu = X_0^\mu + t U_0^\mu + \frac{1}{2} t^2 U_0^\nu U_{0,\nu}^\mu$$

$$+ s\left(V_0^\mu + t U_0^\nu V_{0,\nu}^\mu\right) + \frac{1}{2} s^2 V_0^\nu V_{0,\nu}^\mu$$

$$\Rightarrow X_D^\mu - X_B^\mu = st\left[U^\nu V^\mu{}_{,\nu} - V^\nu U^\mu{}_{,\nu}\right]$$

$$= st\,[U,V]^\mu$$

The other geometrical significance of $\mathcal{L}$ is with symmetries of the metric

Defn. A Killing vector is a vector field along which the metric is Lie invariant

$$\mathcal{L}_k g = 0$$

$k^\mu$ then
of the m

of $\mathcal{L}$

or field

variant

$k^\mu$ then represents a symmetry of the metric

e.g. $S^2$: $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

$k = \dfrac{\partial}{\partial\phi}$ is a k.v. $(k^\phi = 1)$

We

$\mathcal{L}$

<

$$(\mathcal{L}_k g)_{\mu\nu} = k^\sigma g_{\mu\nu,\sigma} + k^\sigma{}_{,\mu}\, g_{\sigma\nu} + k^\sigma{}_{,\nu}\, g_{\mu\sigma}$$

$$= \frac{\partial}{\partial \phi}(g_{\mu\nu}) = 0$$

We also have a relation between

$\mathcal{L} \propto d$.

$$\langle d\omega \,|\, u, v\rangle = u(\langle \omega | v\rangle)$$

$$- v(\langle \omega | u\rangle) - \langle \omega | [u, v]\rangle$$

tor field $\underline{u}$ defines
esimal co-ord transfm
a "$\delta t$" along integral
of $\underline{u}$.

along which the metric is Lie invariant

$$\mathcal{L}_k g = 0$$

$k = \frac{\partial}{\partial \phi}$ is a k v

$(\mathcal{L}_u \underline{v})^\mu = V^\mu{}_{,\nu} U^\nu - U^\mu{}_{,\nu} V^\nu$

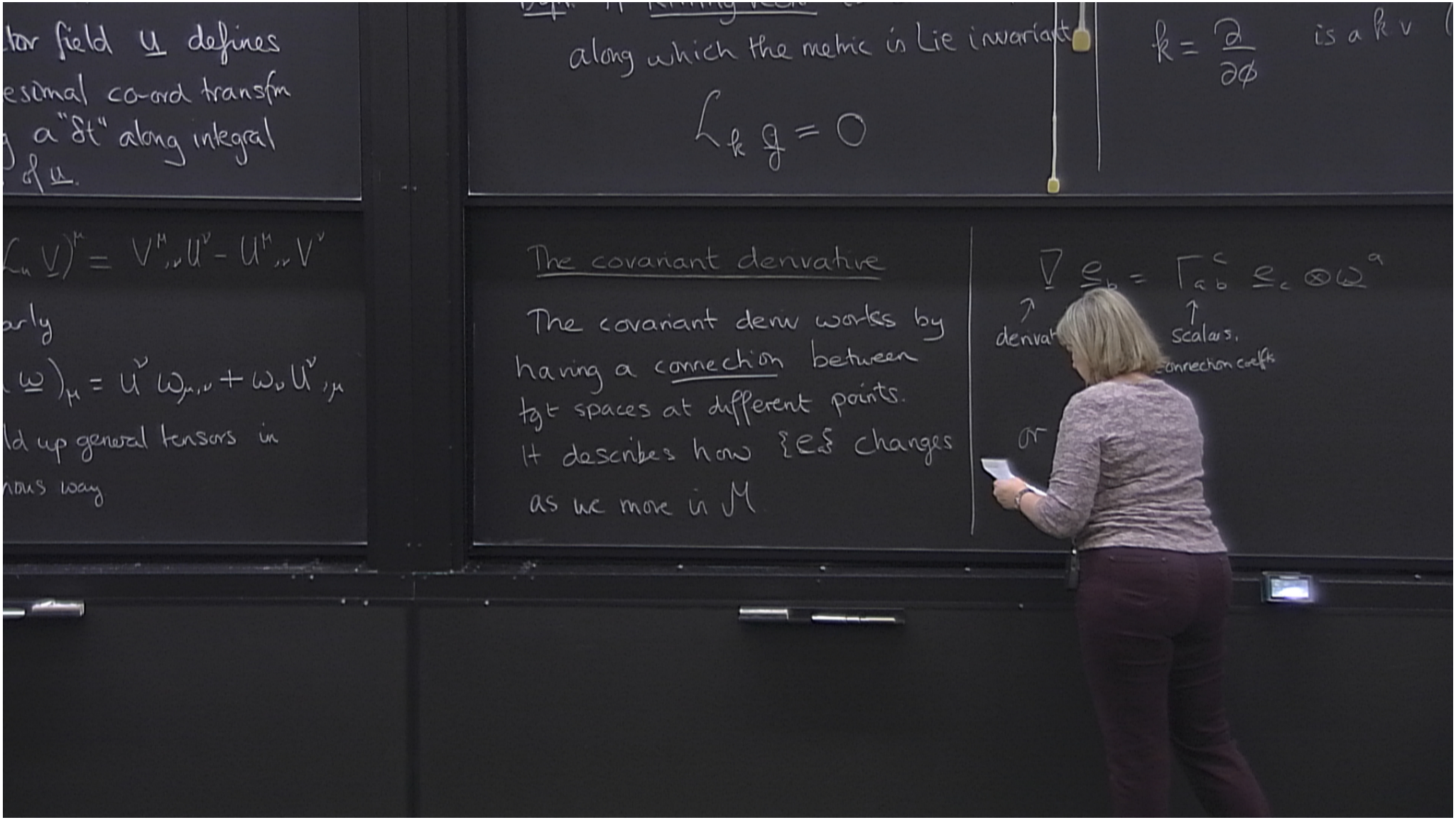
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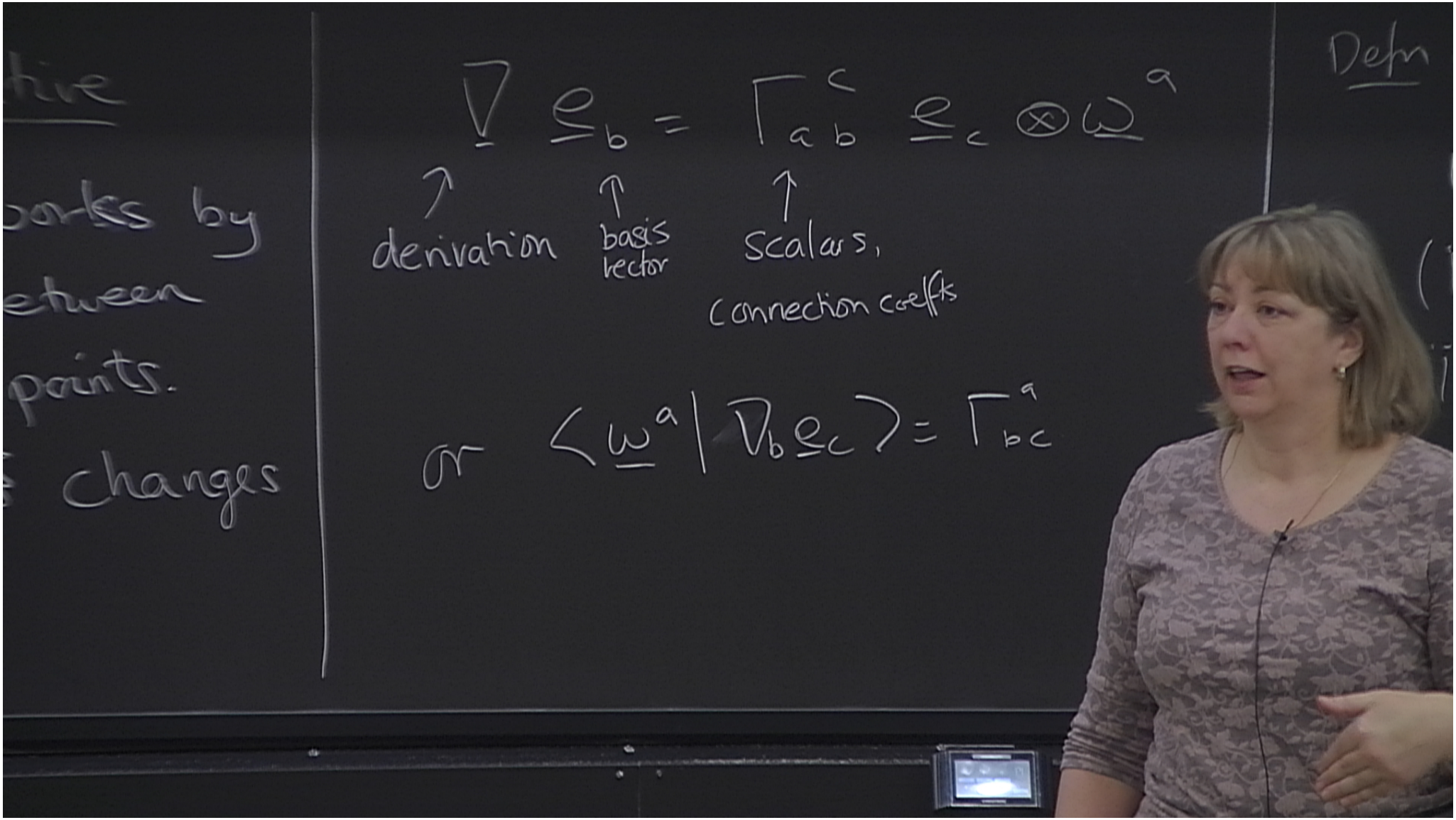
$(\mathcal{L}_u \underline{\omega})_\mu = U^\nu \omega_{\mu,\nu} + \omega_\nu U^\nu{}_{,\mu}$

ld up general tensors in
ious way

## The covariant derivative

The covariant deriv works by having a connection between tgt spaces at different points.


tor field u defines
esimal co-ord transfm
a "δt" along integral
of u.
along which the metric is Lie invariant
$\mathcal{L}_k g = 0$
$k = \frac{\partial}{\partial \phi}$ is a k v
$\mathcal{L}_u v)^\mu = V^\mu{}_{,\nu} U^\nu - U^\mu{}_{,\nu} V^\nu$
arly
$\omega)_\mu = U^\nu \omega_{\mu,\nu} + \omega_\nu U^\nu{}_{,\mu}$
ld up general tensors in
nous way
The covariant derivative
The covariant deriv works by having a connection between tgt spaces at different points.
It describes how {e_a} changes as we move in M
$\nabla e_b = \Gamma^c_{ab} e_c \otimes \omega^a$
derivat
scalars.
connection coeffs
or

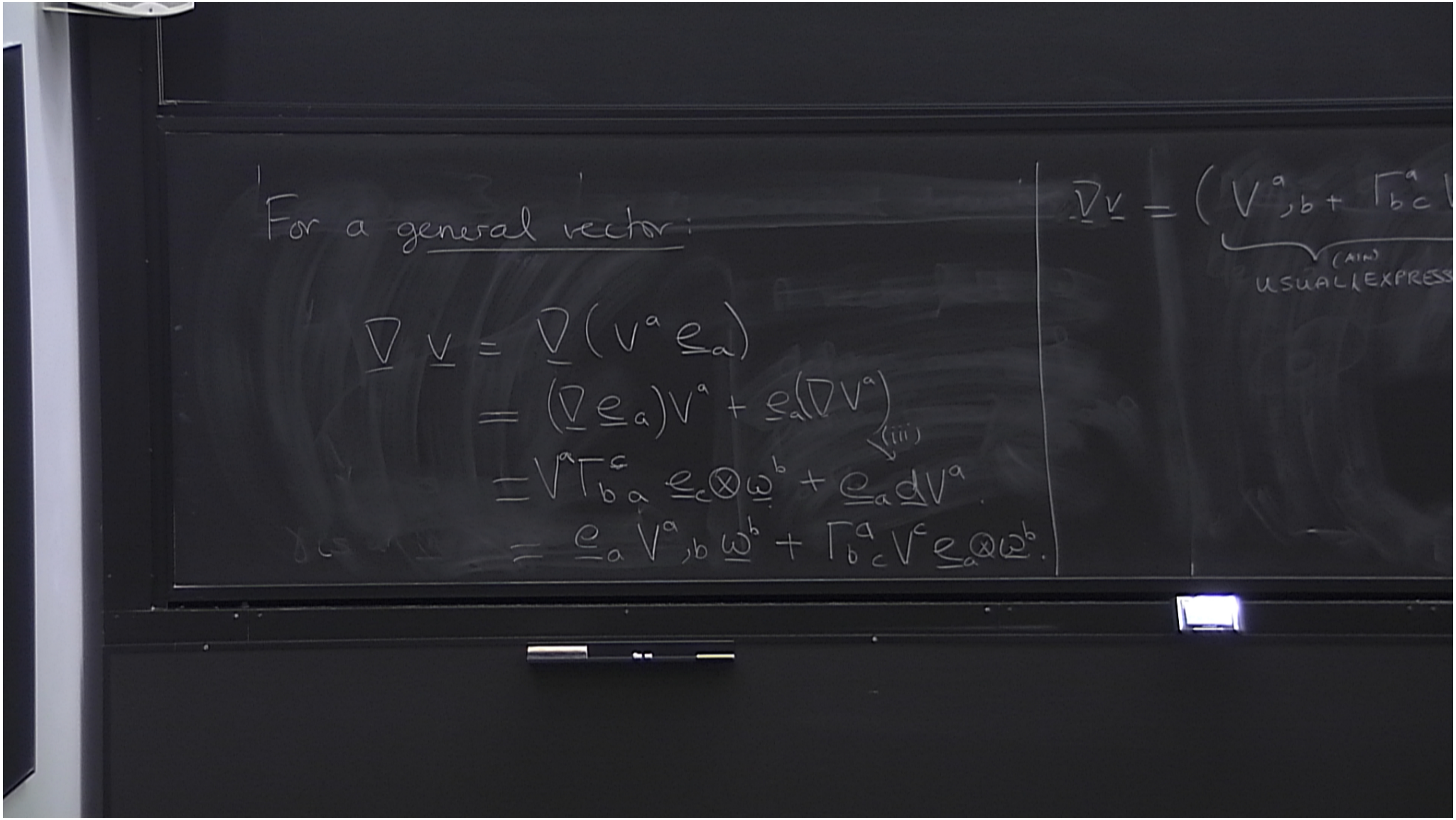

tive
orks by
etween
points.
changes
$\nabla \underline{e}_b = \Gamma^c_{ab} \underline{e}_c \otimes \underline{\omega}^a$
derivation
basis vector
scalars, connection coeffts
or $\langle \underline{\omega}^a | \nabla_b \underline{e}_c \rangle = \Gamma^a_{bc}$
Defn

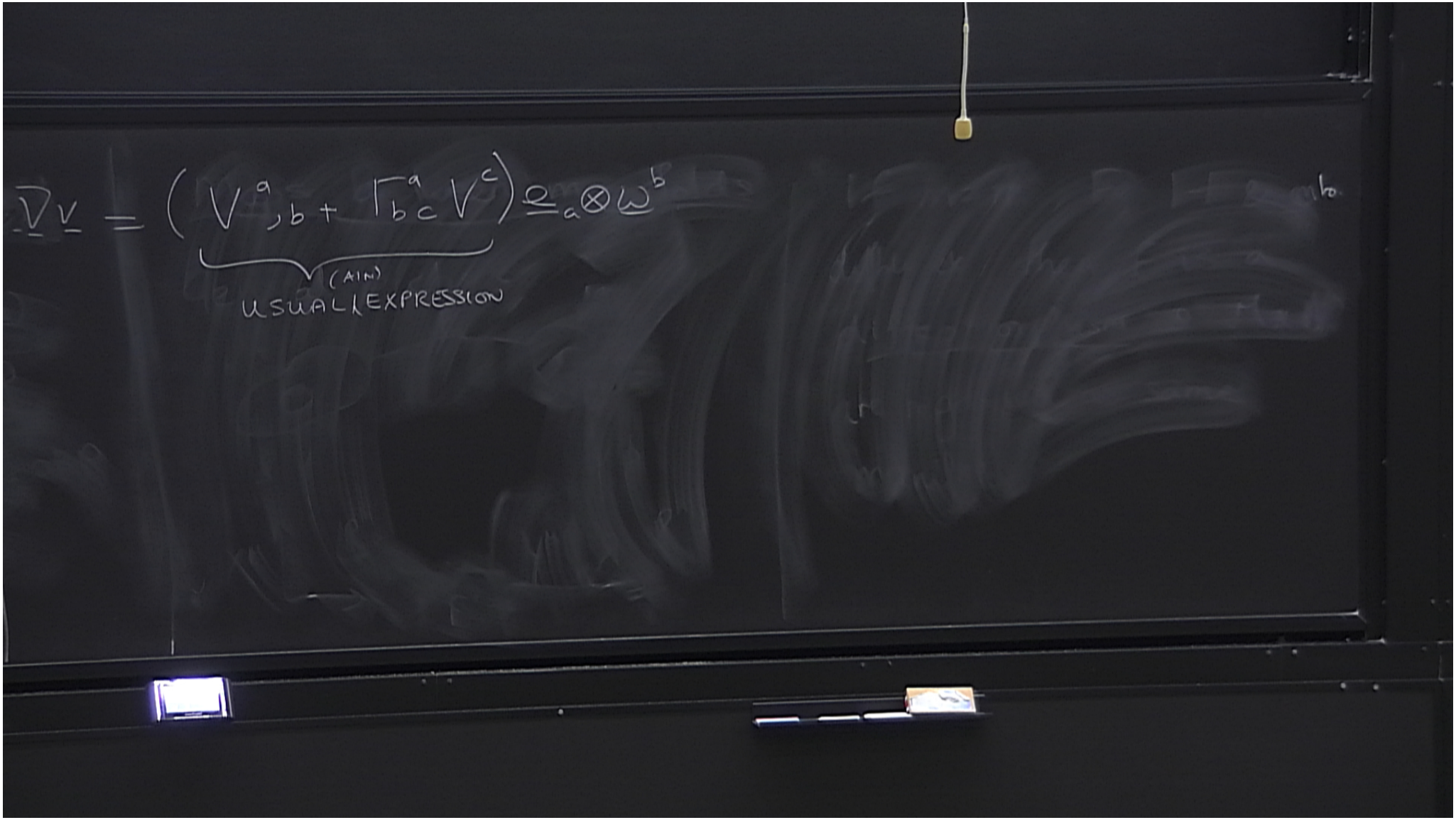
Def: $\nabla$ is a derivation which

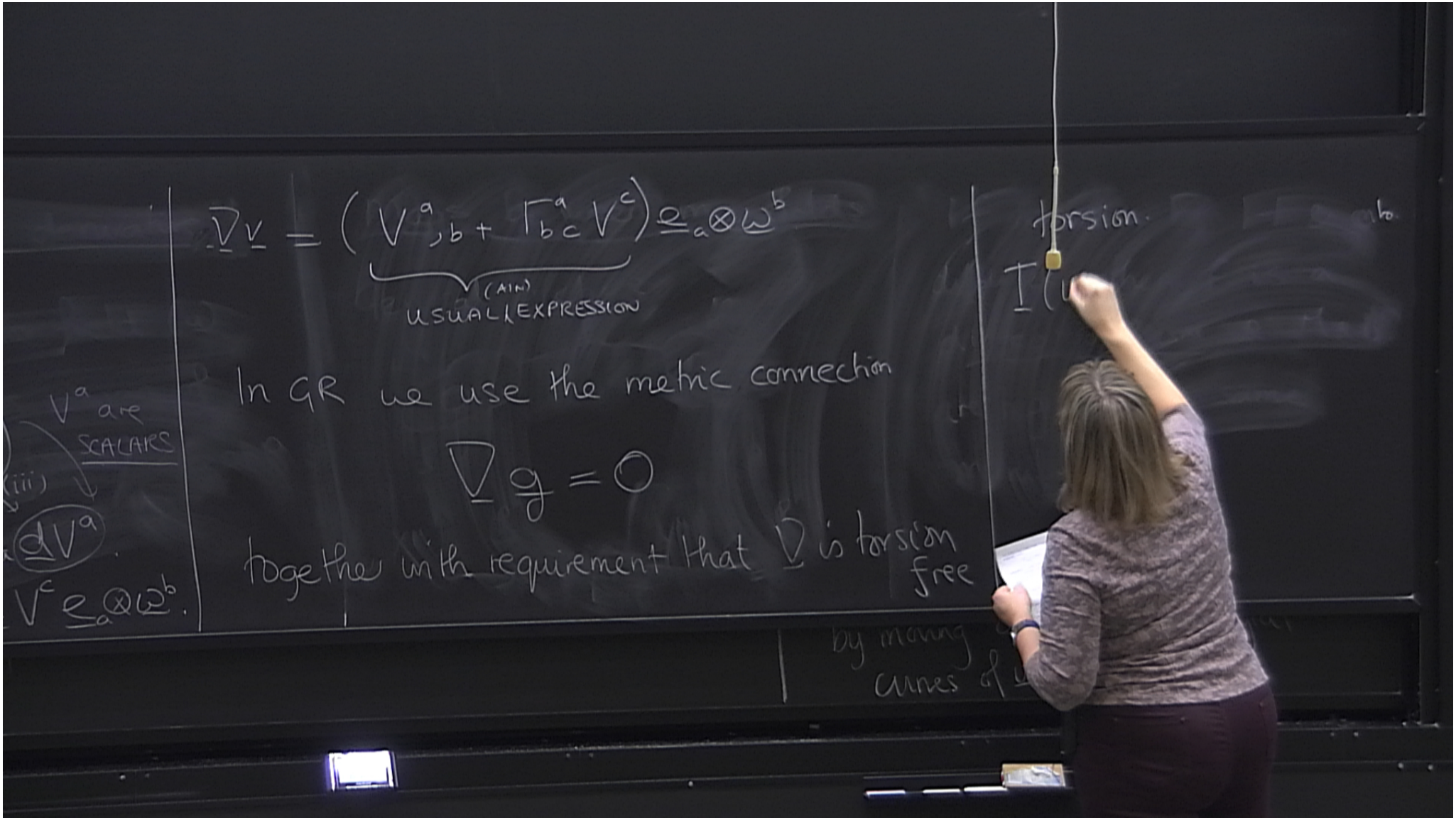
(i) commutes with contractions

(ii) Leibnitzian

(iii) reduces to $d$ on fns.

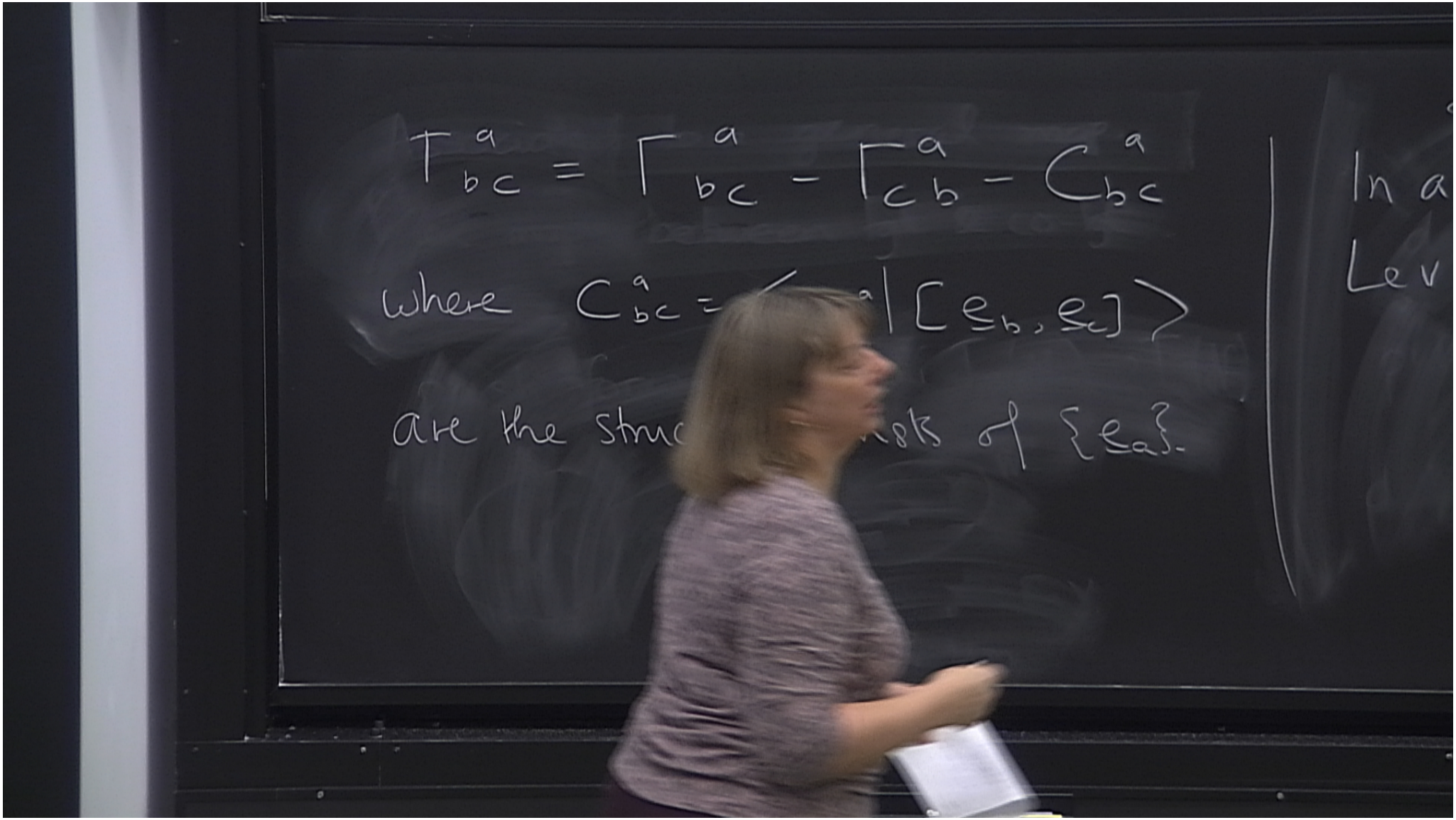

For a general vector:
$\underline{\nabla}\,\underline{v} = \underline{\nabla}(v^a \underline{e}_a)$
$= (\underline{\nabla}\,\underline{e}_a) v^a + \underline{e}_a (\nabla v^a)$
(iii)
$= v^a \Gamma^c_{ba}\, \underline{e}_c \otimes \underline{\omega}^b + \underline{e}_a \underline{d} v^a$
$= \underline{e}_a v^a{}_{,b}\, \underline{\omega}^b + \Gamma^a_{bc} v^c \underline{e}_a \otimes \underline{\omega}^b$.
$\underline{\nabla}\,\underline{v} = ( v^a{}_{,b} + \Gamma^a_{bc}$
(AIN)
USUAL EXPRESS


$\nabla \underline{V} = \left( \underbrace{V^a{}_{,b} + \Gamma^a_{bc} V^c}_{\text{(AIN) USUAL EXPRESSION}} \right) \underline{e}_a \otimes \underline{\omega}^b$


$\underline{\nabla}\underline{v} = (\underbrace{V^a{}_{,b} + \Gamma^a_{bc} V^c}_{\text{USUAL (AIN) EXPRESSION}})\, \underline{e}_a \otimes \underline{\omega}^b$
In GR we use the metric connection
$\underline{\nabla}\, \underline{g} = 0$
together with requirement that $\underline{\nabla}$ is torsion free
$V^a$ are SCALARS
$dV^a$
$V^c \underline{e}_a \otimes \underline{\omega}^b$.
torsion.
by moving
curves of

For a diffeo, can "pull" vector spaces back and "push" cotangent spaces forward as $F$ is invertible.

The vector field $\underline{u}$ defines an infinitesimal co-ord transfm by moving a "$\delta t$" along integral curves of $\underline{u}$.


$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$
where $C^a_{bc} = \langle \; | [e_b, e_c] \rangle$
are the stru
of $\{e_a\}$.
In a
Lev

$\nabla_b - C^a_{bc}$

$[e_b, e_c] >$

ks of $\{e_a\}$.

In a coord basis we get the familiar Levi-Civita connection

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left( g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\mu\nu,\sigma} \right)$$

For
spaces
space

The
an infi
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cur

familiar

$g_{\mu\nu,\bar{0}}$

For a diffeo, can "pull" vector spaces back and "push" cotangent spaces forward as $F$ is invertible.

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