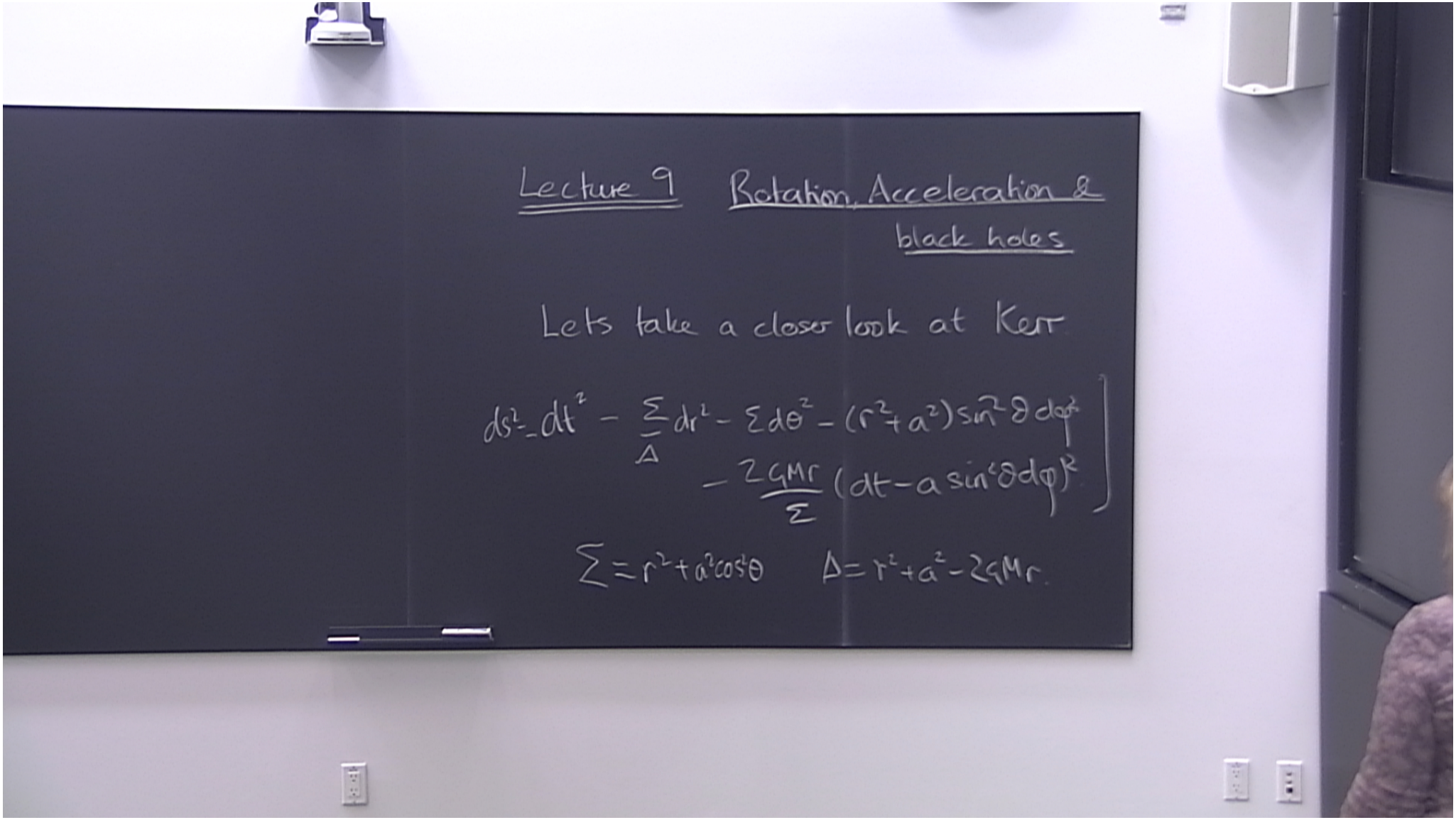
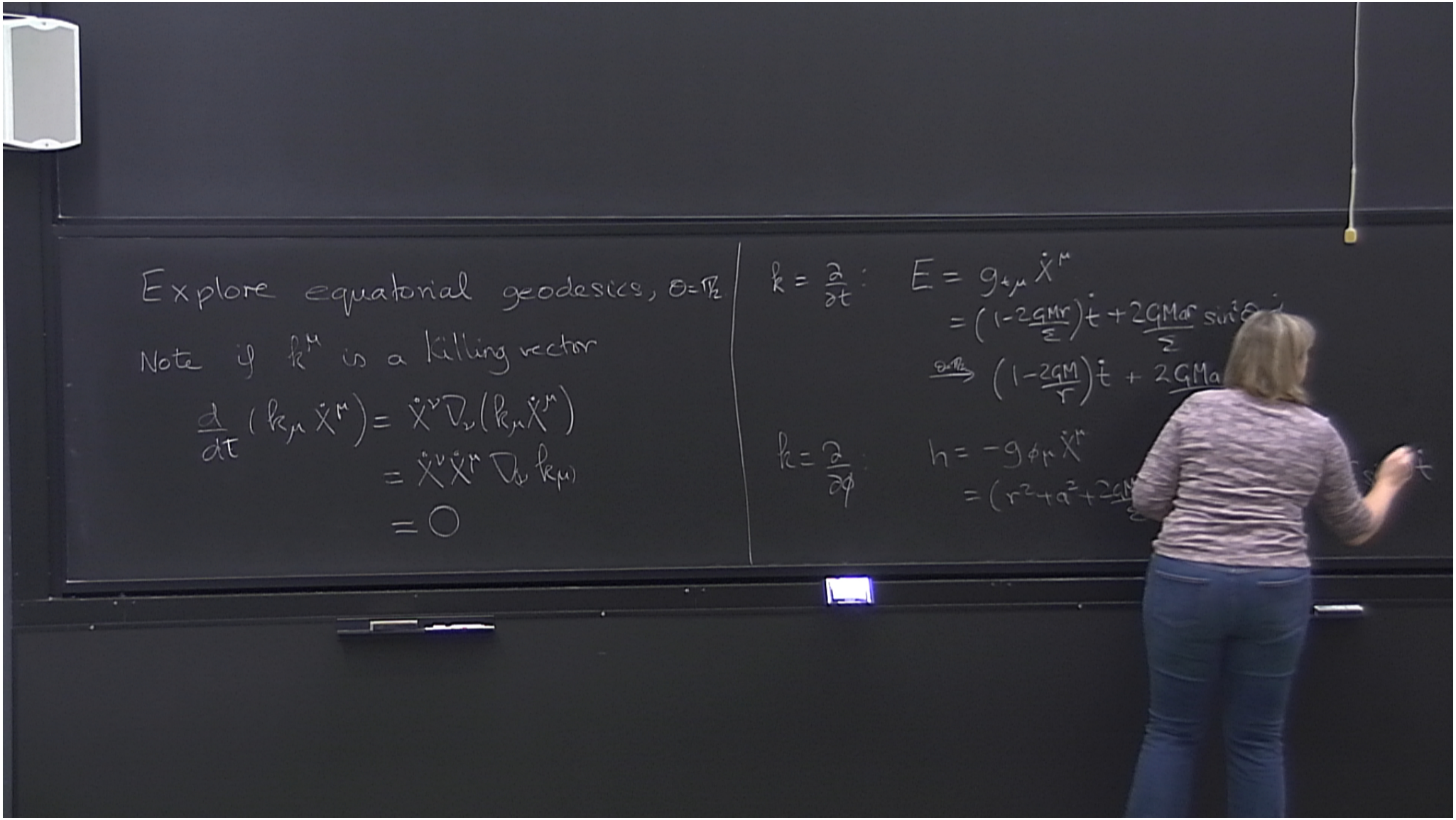
Title: Gravitational Physics-9

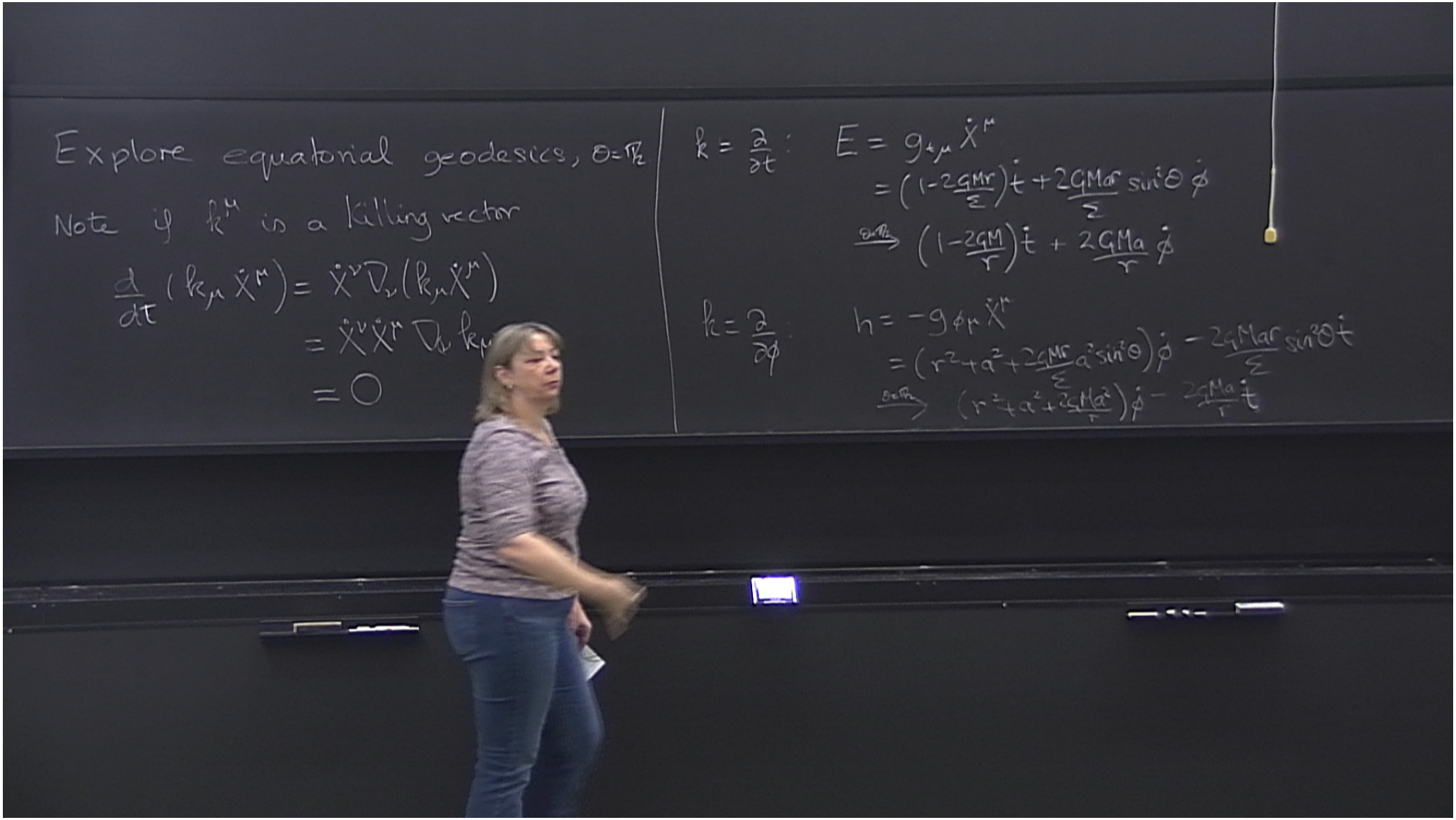
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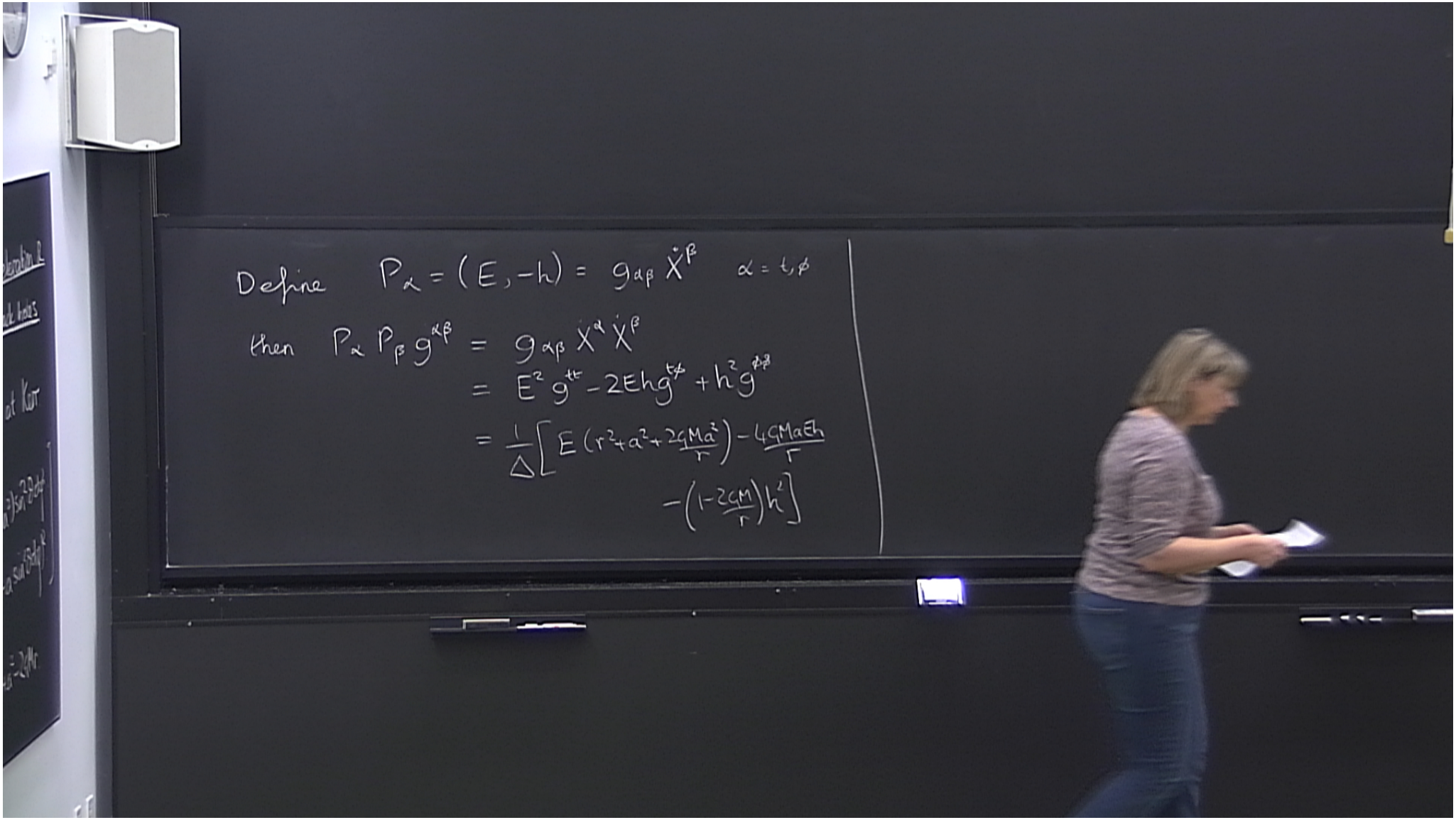
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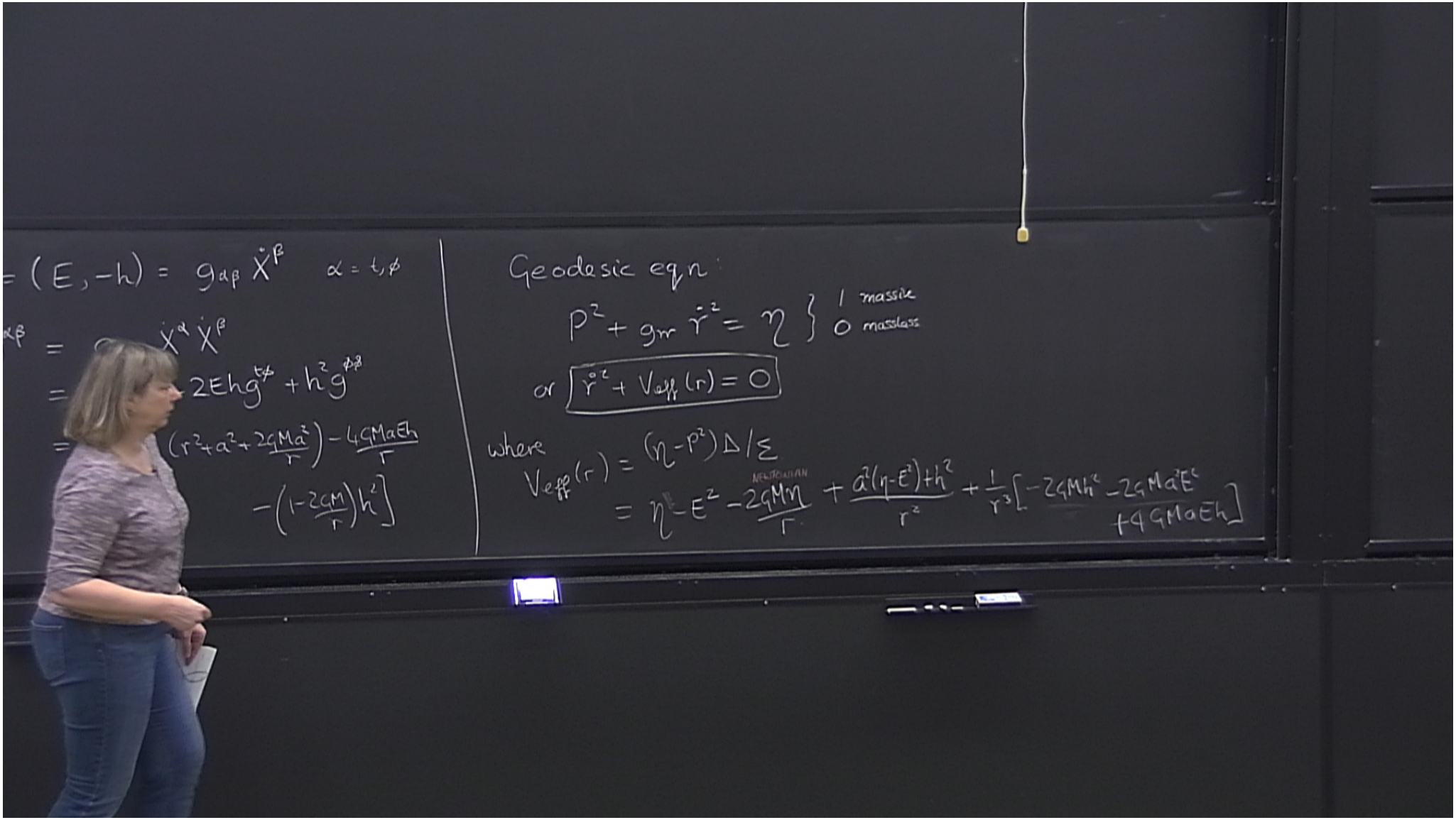
Abstract:


Lecture 9 Rotation, Acceleration & black holes
Lets take a closer look at Kerr
$ds^2 = -dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - (r^2+a^2)\sin^2\theta\, d\phi^2 - \frac{2GMr}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2$
$\Sigma = r^2 + a^2\cos^2\theta \quad \Delta = r^2 + a^2 - 2GMr$


Explore equatorial geodesics, $\theta=\pi/2$
Note if k^μ is a Killing vector
$\frac{d}{dt}(k_\mu \dot{x}^\mu) = \dot{x}^\nu \nabla_\nu (k_\mu \dot{x}^\mu)$
$= \dot{x}^\nu \dot{x}^\mu \nabla_\nu k_\mu$
$= 0$
$k = \frac{\partial}{\partial t}$: $E = g_{t\mu}\dot{x}^\mu$
$= (1-\frac{2GMr}{\Sigma})\dot{t} + \frac{2GMar}{\Sigma}\sin^2\theta\,\dot{\phi}$
$\theta=\pi/2 \Rightarrow (1-\frac{2GM}{r})\dot{t} + 2GMa$
$k = \frac{\partial}{\partial \phi}$: $h = -g_{\phi\mu}\dot{x}^\mu$
$= (r^2+a^2+\frac{2GM}{\Sigma}$


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$k = \frac{\partial}{\partial t}$: $E = g_{t\mu}\dot{x}^\mu$
$= \left(1-\frac{2GMr}{\Sigma}\right)\dot{t} + \frac{2GMar}{\Sigma}\sin^2\theta\,\dot{\phi}$
$\xrightarrow{\theta=\pi/2} \left(1-\frac{2GM}{r}\right)\dot{t} + \frac{2GMa}{r}\dot{\phi}$
$k = \frac{\partial}{\partial \phi}$: $h = -g_{\phi\mu}\dot{x}^\mu$
$= \left(r^2+a^2+\frac{2GMr}{\Sigma}a^2\sin^2\theta\right)\dot{\phi} - \frac{2GMar}{\Sigma}\sin^2\theta\,\dot{t}$
$\xrightarrow{\theta=\pi/2} \left(r^2+a^2+\frac{2GMa^2}{r}\right)\dot{\phi} - \frac{2GMa}{r}\dot{t}$


Define $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{X}^\beta \quad \alpha = t, \phi$
then $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta}\dot{X}^\alpha \dot{X}^\beta$
$= E^2 g^{tt} - 2Ehg^{t\phi} + h^2 g^{\phi\phi}$
$= \frac{1}{\Delta}\left[E\left(r^2 + a^2 + \frac{2GMa^2}{r}\right) - \frac{4GMaEh}{r}\right.$
$\left. - \left(1 - \frac{2GM}{r}\right)h^2\right]$


$= (E, -h) = g_{\alpha\beta}\dot{X}^{\beta}$ $\alpha = t, \phi$
$= \ldots \dot{X}^{\alpha}\dot{X}^{\beta}$
$= \ldots 2Ehg^{t\phi} + h^2 g^{\phi\phi}$
$= \ldots \left(r^2 + a^2 + \frac{2GMa^2}{r}\right) - \frac{4GMaEh}{r}$
$-\left(1 - \frac{2GM}{r}\right)h^2\Big]$
Geodesic eqn
$p^2 + g_{rr}\dot{r}^2 = \eta$ } 1 massive, 0 massless
or $\boxed{\dot{r}^2 + V_{eff}(r) = 0}$
where
$V_{eff}(r) = (\eta - p^2)\Delta/\Sigma$
$= \eta - E^2 - \frac{2GM\eta}{r}$ (NEWTONIAN) $+ \frac{a^2(\eta - E^2) + h^2}{r^2} + \frac{1}{r^3}\left[-2GMh^2 - 2GMa^2E^2 + 4GMaEh\right]$

$= (E, -h) = g_{\alpha\beta}\dot{X}^{\beta} \quad \alpha = t, \phi$

$= g_{\alpha\beta}\dot{X}^{\alpha}\dot{X}^{\beta}$

$= E^2 g^{tt} - 2Eh g^{t\phi} + h^2 g^{\phi\phi}$

$= \frac{1}{\Delta}\left[E\left(r^2 + a^2 + \frac{2GMa^2}{r}\right) - \frac{4GMaEh}{r} - \left(1 - \frac{2GM}{r}\right)h^2\right]$

Geodesic eqn

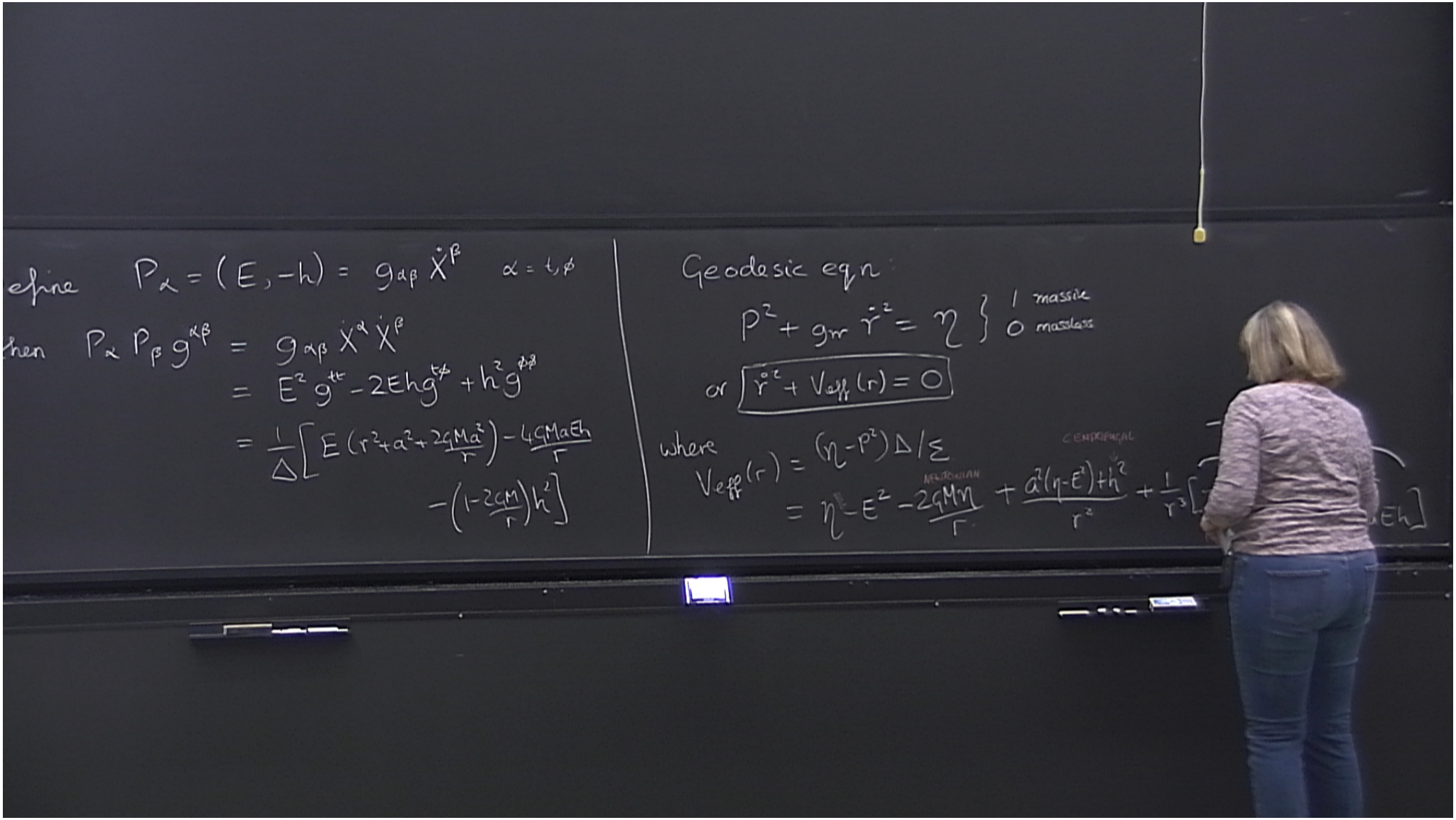
$p^2 + g_{rr}\dot{r}^2 = \eta \Big\} \begin{matrix} 1 \text{ massive} \\ 0 \text{ massless} \end{matrix}$

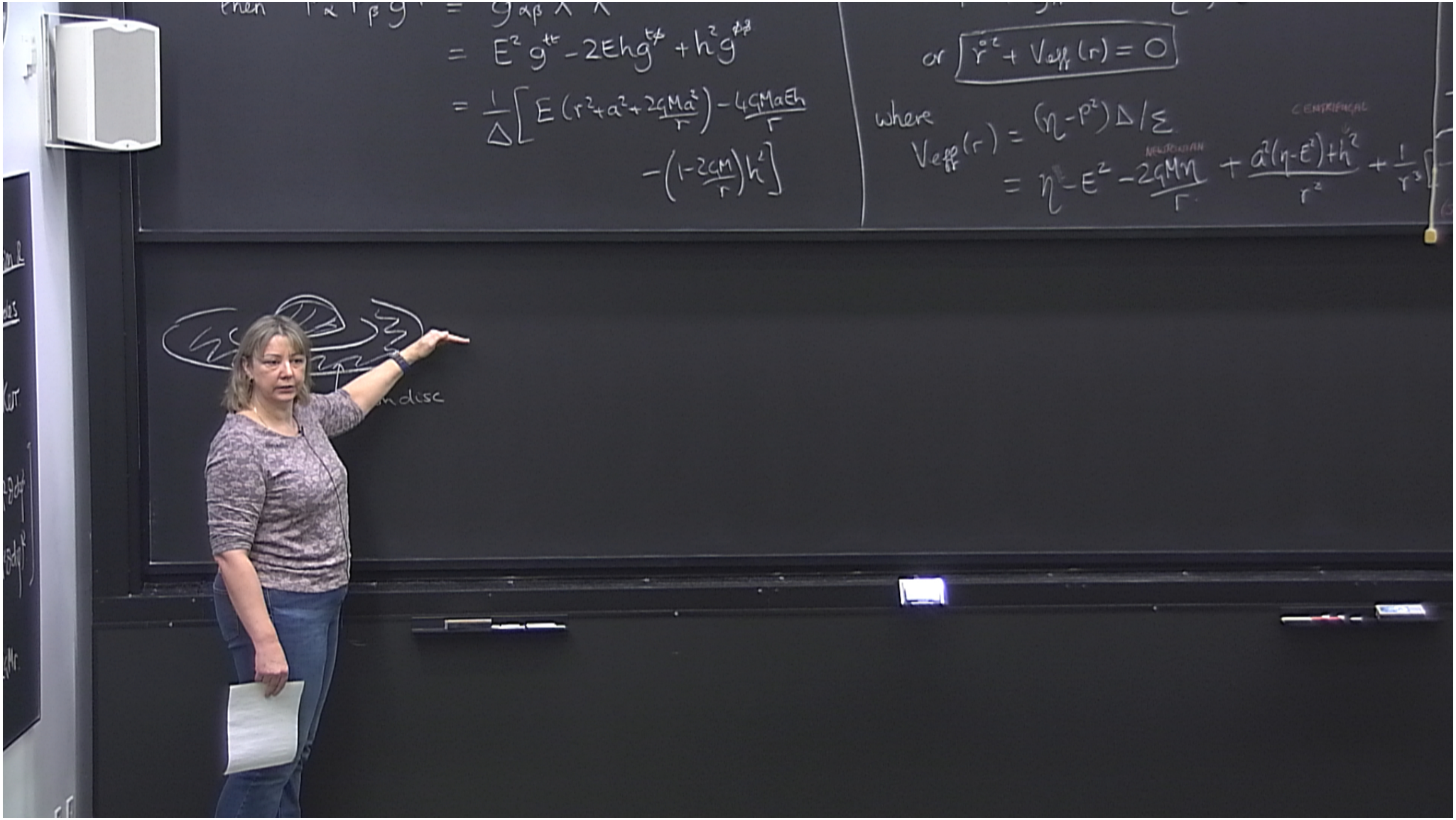
or $\boxed{\dot{r}^2 + V_{eff}(r) = 0}$

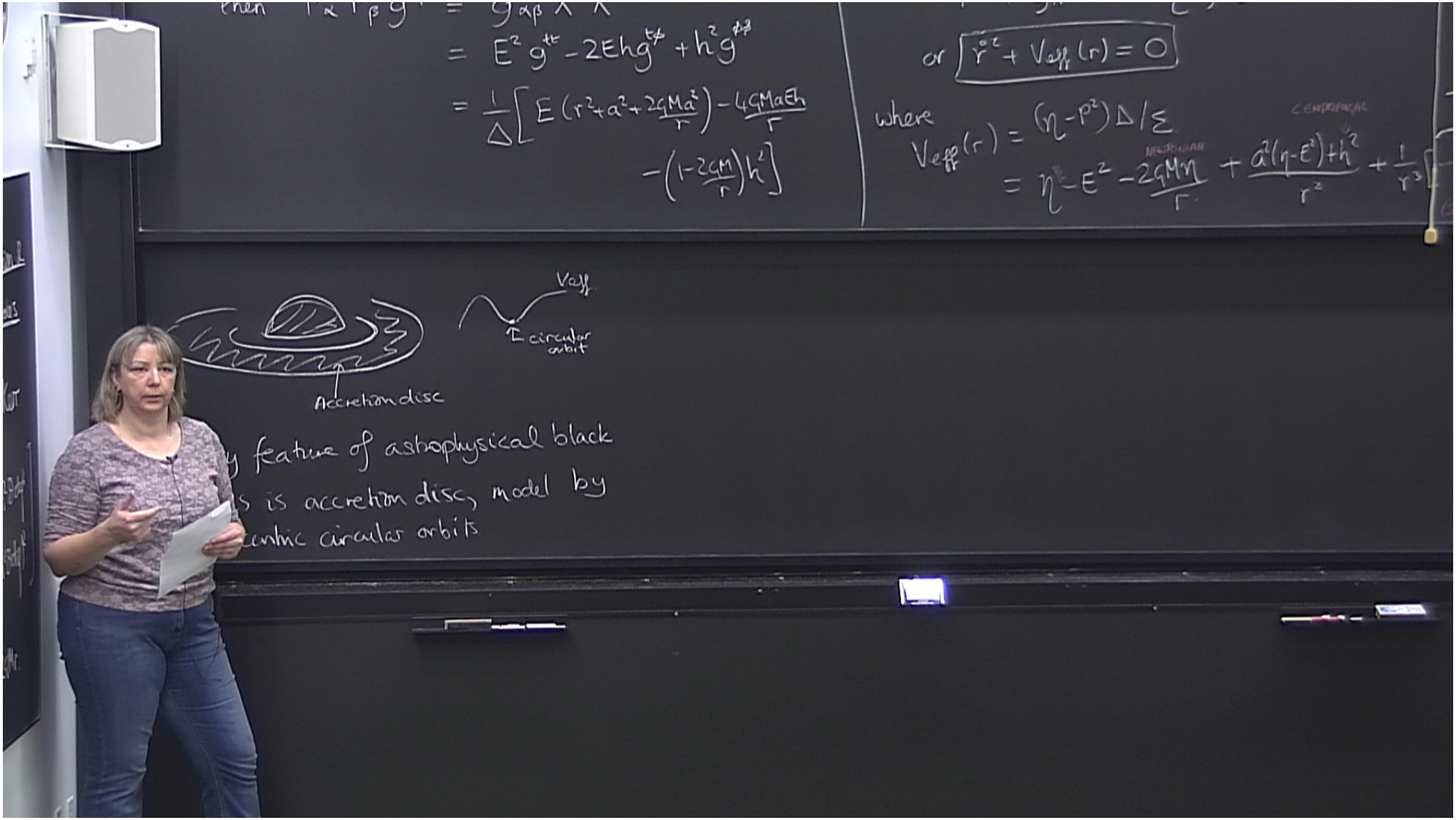
where $V_{eff}(r) = (\eta - p^2)\Delta/\Sigma$

$= \eta - E^2 - \frac{2GM\eta}{r} + \frac{a^2(\eta - E^2) + h^2}{r^2} + \frac{1}{r^3}\left[\overbrace{-2GMh^2 - 2GMa^2E^2 + 4GMaEh}^{-2GM(h-aE)^2}\right]$

NEWTONIAN · CENTRIFUGAL · $(GM)^2$ GR


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$= E^2 g^{tt} - 2Ehg^{t\phi} + h^2 g^{\phi\phi}$
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NEWTONIAN
CENTRIFUGAL
disc


$= E^2 g^{tt} - 2Ehg^{t\phi} + h^2 g^{\phi\phi}$
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NEWTONIAN
CENTRIFUGAL
V_{eff}
circular orbit
Accretion disc
feature of astrophysical black
is accretion disc, model by
centric circular orbits

$= E^2 g^{tt} - 2Ehg^{t\phi} + h^2 g^{\phi\phi}$

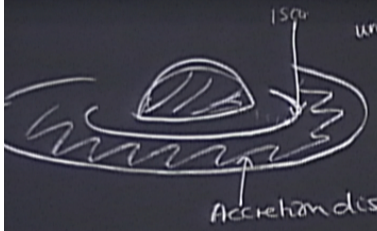
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KERR GR $- 2GM(h-aE)^2$



Key feature of astrophysic… holes is accretion disc… concentric circular orbits…

the innermost boundary of the accretion disc occurs where unstable + stable roots for $V'=0$ meet:

Innermost Stable Circular Orbit

$V = V' = 0$ for circular orbit

$V'' \to 0$ for ISCO $(\eta = 1)$

$3r^4V' + r^5V'' = 2GMr^2 - 6GM(h-aE)^2$

$4r^3V' + r^4V'' = 4GMr - 2a^2(1-E^2) - 2h^2$

$= E^2 g^{tt} - 2Ehg^{t\phi} + h^2 g^{\phi\phi}$

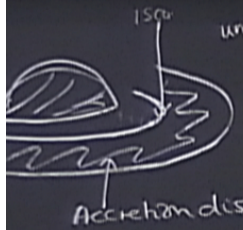
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NEWTONIAN · CENTRIFUGAL · (SCH) GR · KERR GR



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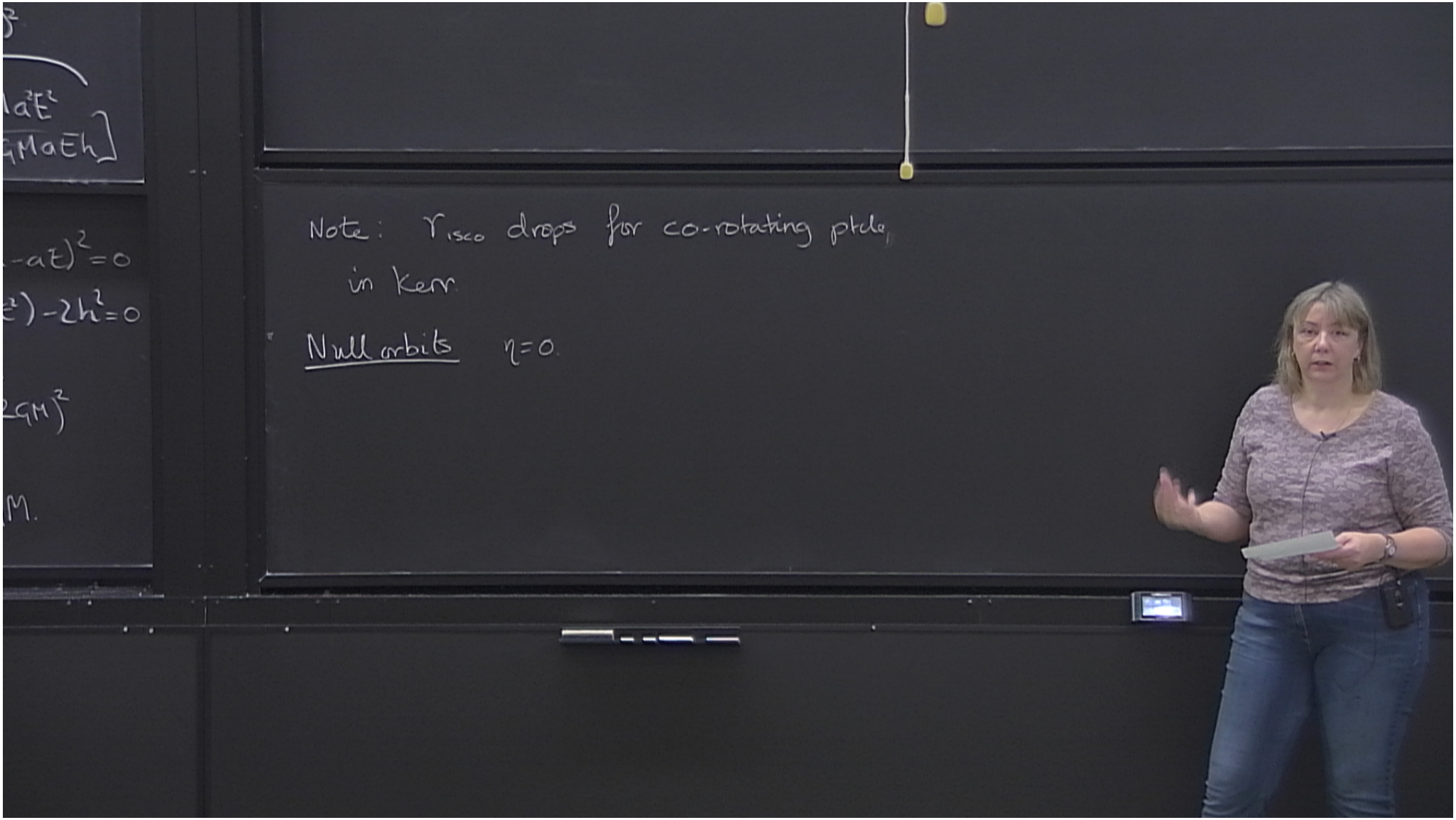
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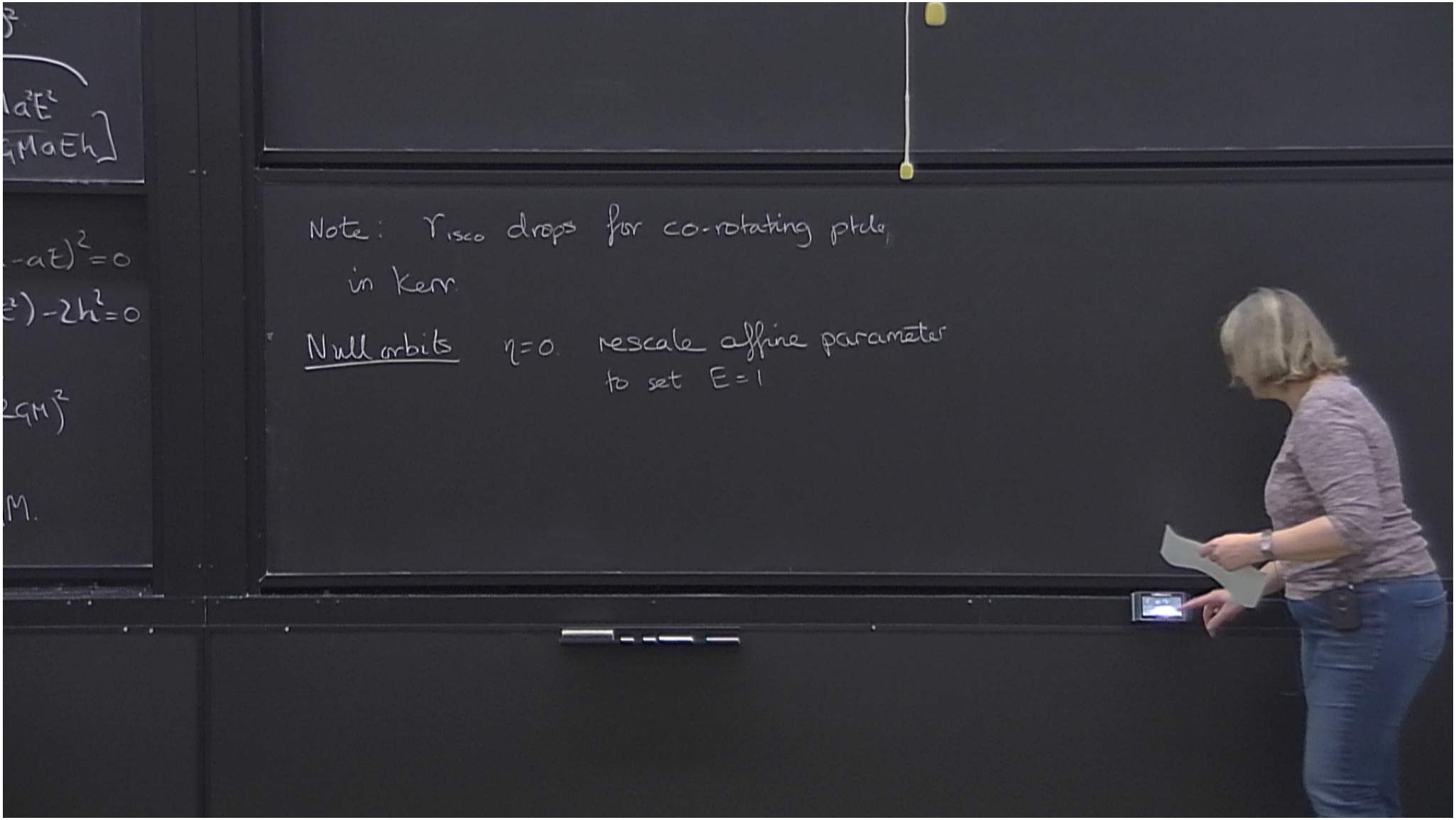
$3r^4V' + r^5V'' = 2GMr^2 - 6GM(h - aE)^2 = 0$

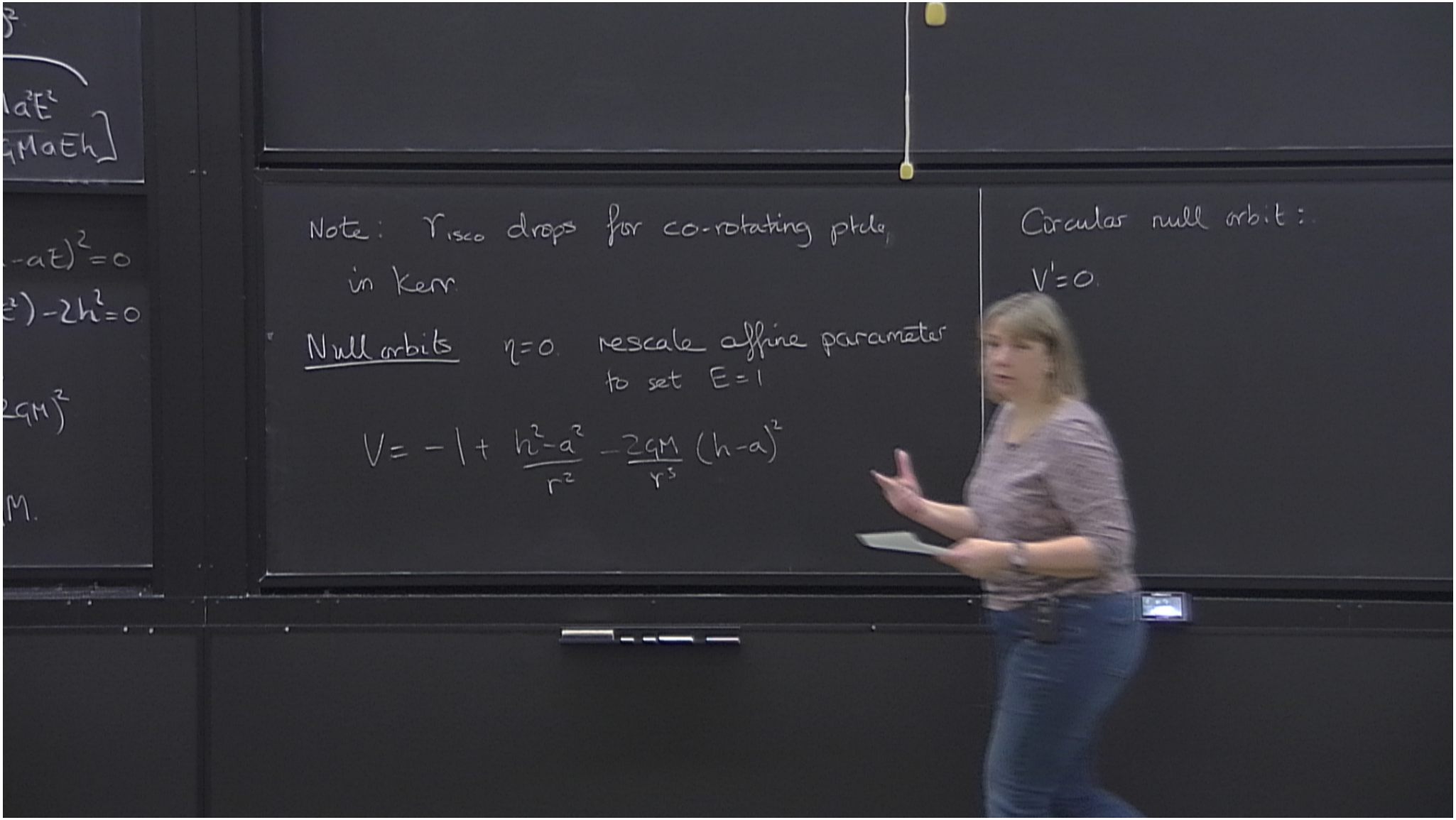
$4r^3V' + r^4V'' = 4GMr - 2a^2(1 - E^2) - 2h^2 = 0$

SCH: $r^2 = 3h^2$, $r = h^2/2GM$ $\Big\}$ $3h^2 = h^4/(2GM)^2$

$\Rightarrow h^2 = 12G^2M^2 \Rightarrow r = 6GM$


Note: r_{isco} drops for co-rotating ptcle in Kerr
Null orbits $\eta = 0$


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$V = -1 + \frac{h^2-a^2}{r^2} - \frac{2GM}{r^3}(h-a)^2$
Circular null orbit:
$V'=0$

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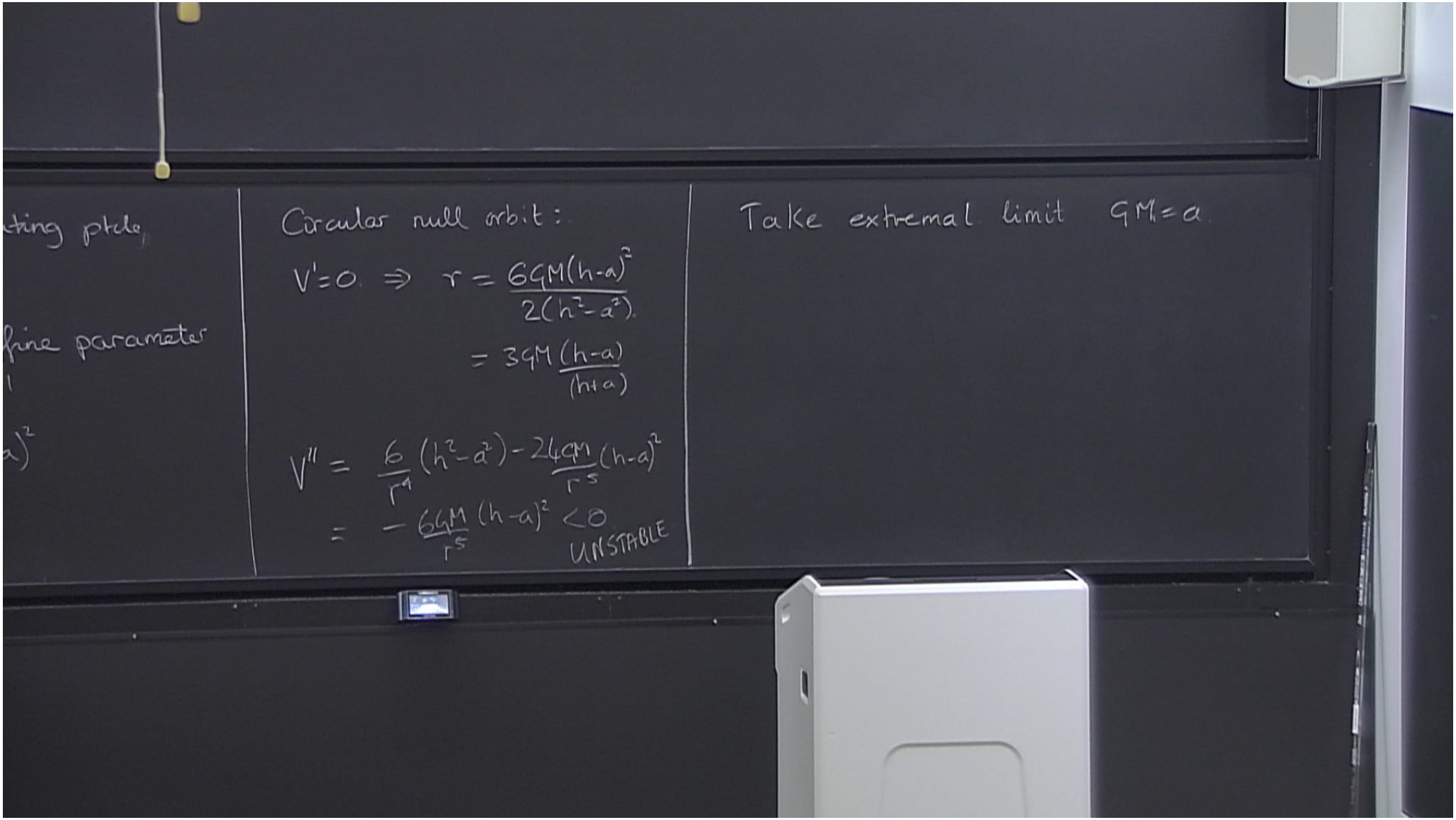
Circular null orbit:

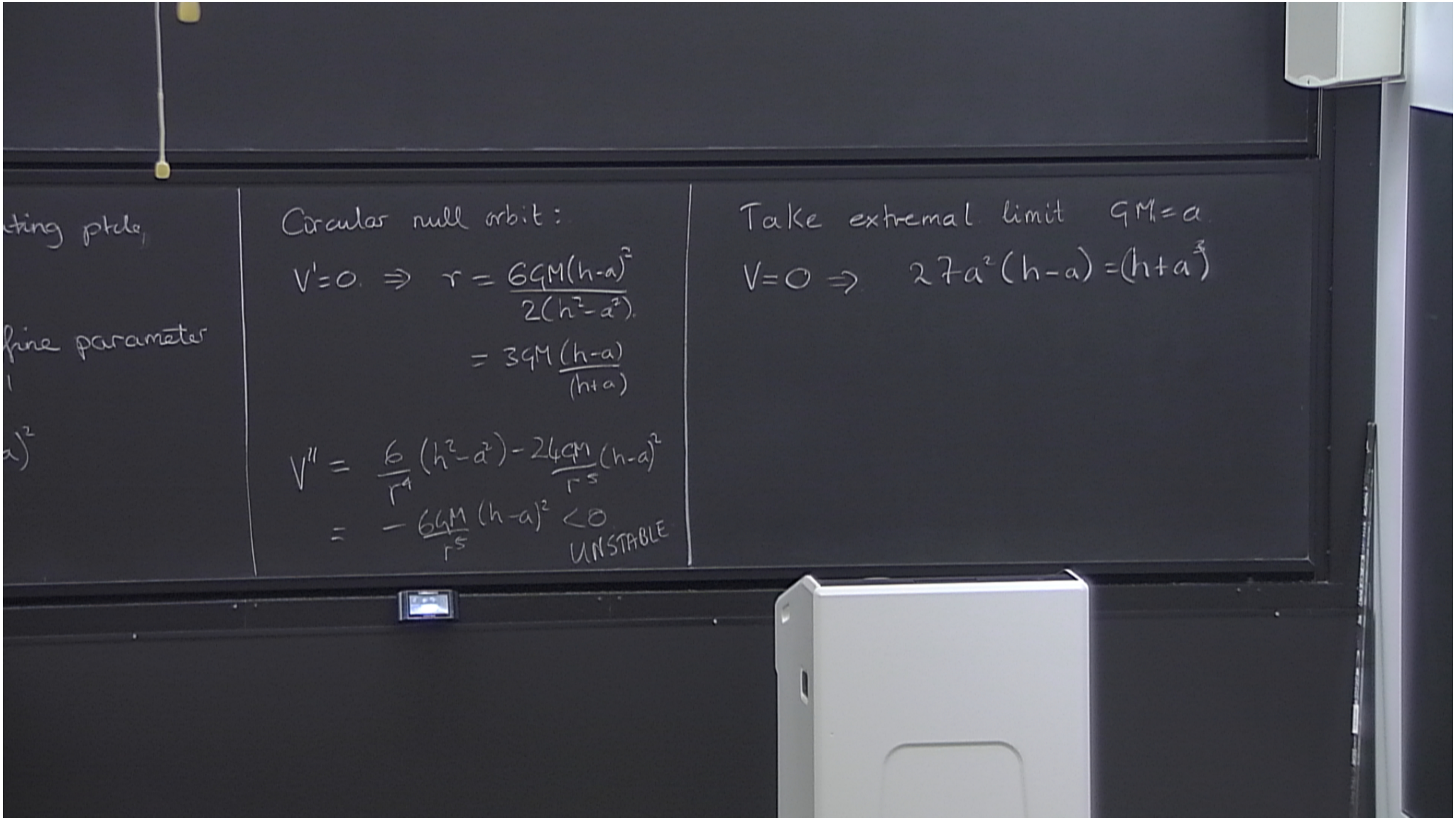
$$V' = 0 \Rightarrow r = \frac{6GM(h-a)^2}{2(h^2-a^2)}$$

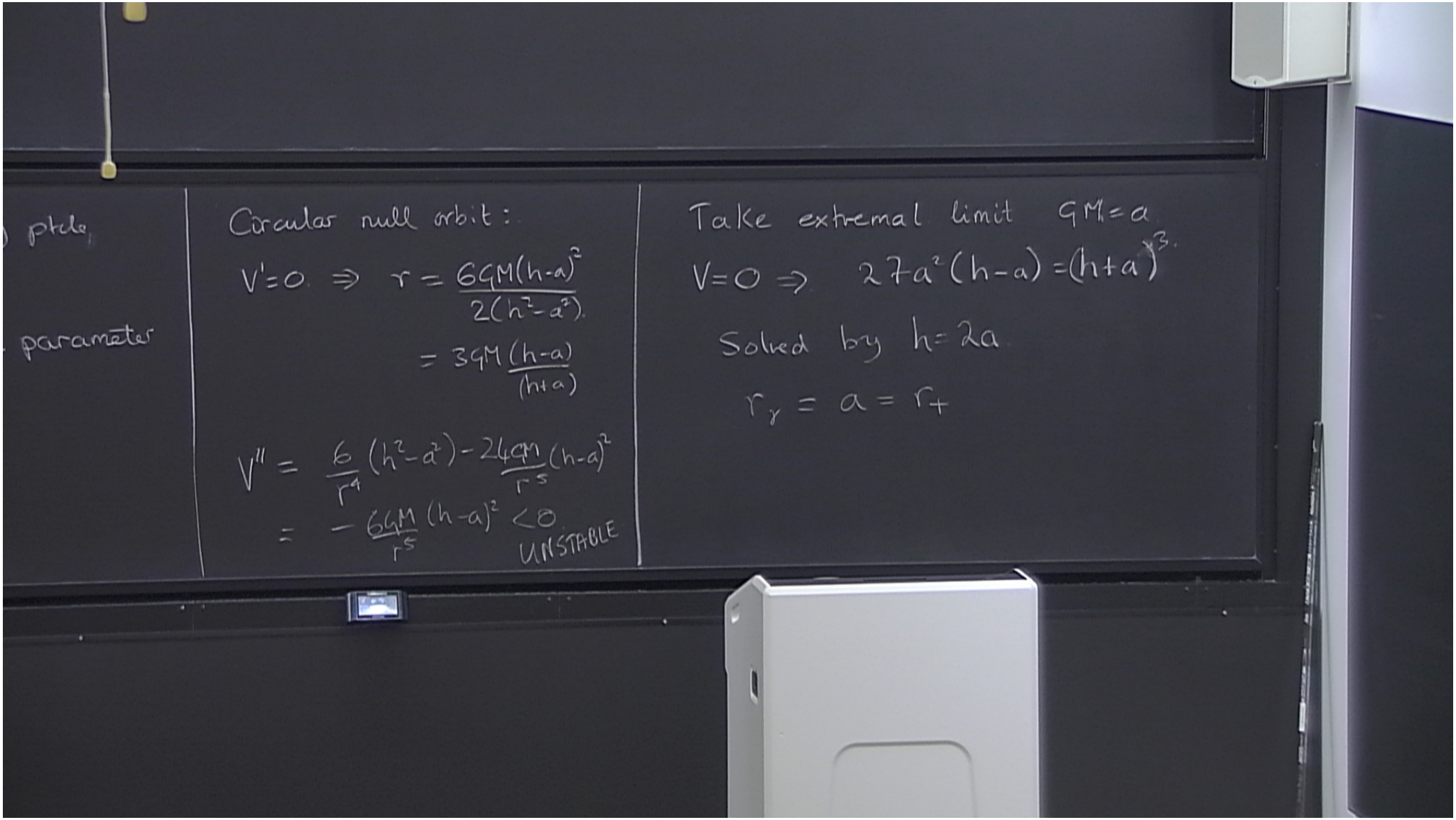
$$= 3GM\frac{(h-a)}{(h+a)}$$

$$V'' = \frac{6}{r^4}(h^2-a^2) - \frac{24GM}{r^5}(h-a)^2$$

$$= -\frac{6GM}{r^5}(h-a)^2 < 0$$


ting ptcle
fine parameter
Circular null orbit:
$V'=0 \Rightarrow r = \frac{6GM(h-a)^2}{2(h^2-a^2)}$
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UNSTABLE
Take extremal limit $GM=a$

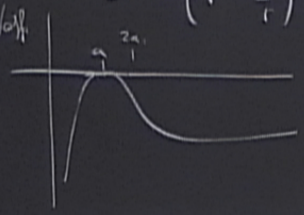

ting ptcle
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$= 3GM\frac{(h-a)}{(h+a)}$
$V'' = \frac{6}{r^4}(h^2-a^2) - 24\frac{GM}{r^5}(h-a)^2$
$= -\frac{6GM}{r^5}(h-a)^2 < 0$
UNSTABLE
Take extremal limit $GM=a$
$V=0 \Rightarrow 27a^2(h-a) = (h+a)^3$


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$= -\frac{6GM}{r^5}(h-a)^2 < 0$
UNSTABLE
Take extremal limit $GM=a$
$V=0 \Rightarrow 27a^2(h-a) = (h+a)^3$.
Solved by $h=2a$
$r_\gamma = a = r_+$

Keep $h = 2a$

$$V = -1 + \frac{3a^2}{r^2} - \frac{2a^3}{r^3}$$

$$= -\left(1 - \frac{a}{r}\right)^2 \left(1 + \frac{2a}{r}\right)$$



null geodesic at $r = a$ escapes to ∞

Acceleration & Rindler space

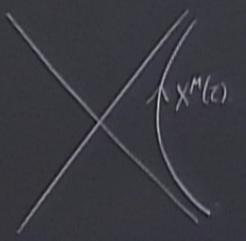
An accelerating observer can be represented by the worldline

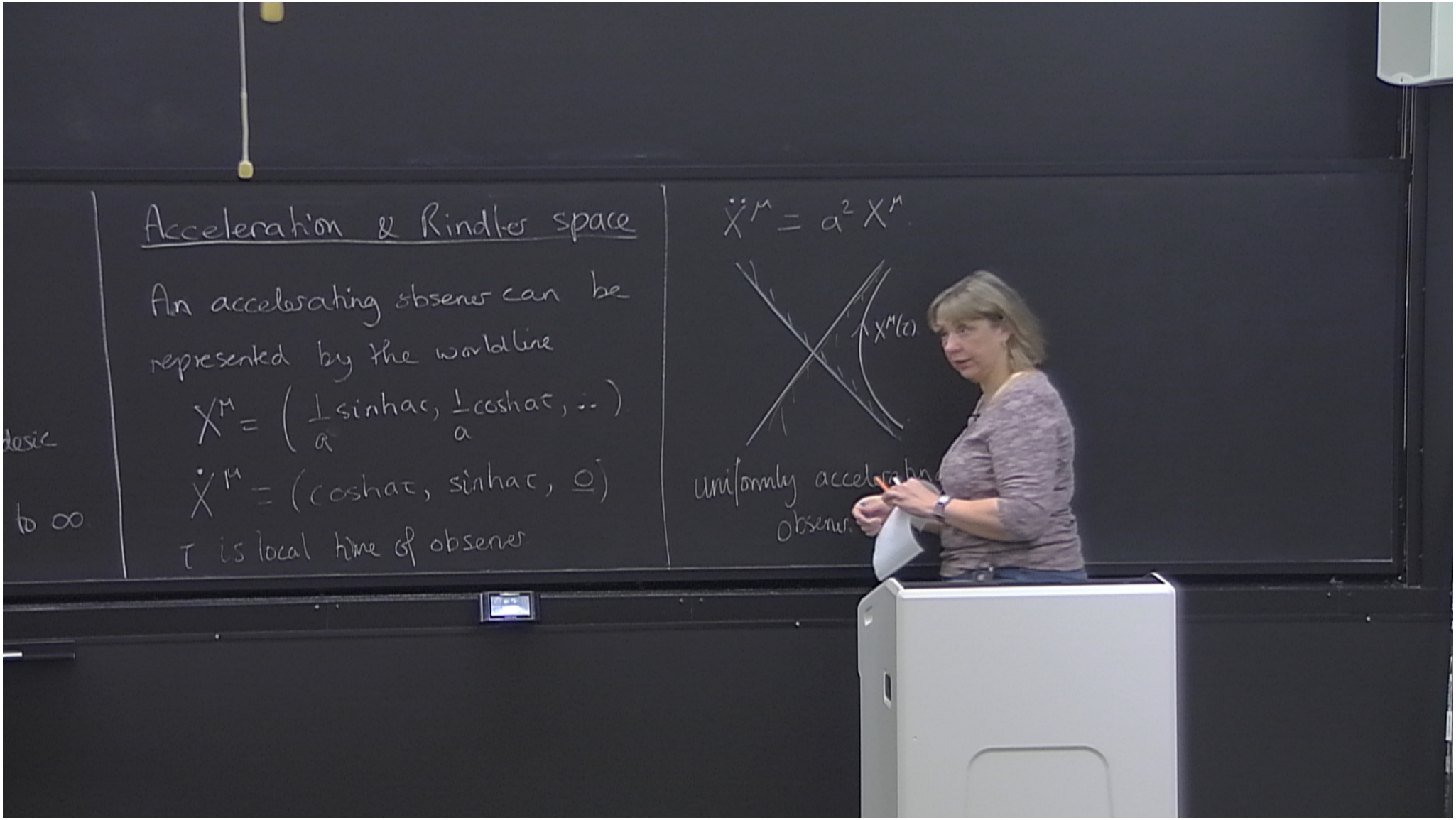
$$X^M = \left(\frac{1}{a}\sinh a\tau, \frac{1}{a}\cosh a\tau, \ldots\right)$$

$$\dot{X}^M = (\cosh a\tau, \sinh a\tau, \underline{0})$$

τ is local time of observer

$$\ddot{X}^M = a^2 X^M$$




Acceleration & Rindler space
An accelerating observer can be represented by the worldline
$X^\mu = \left(\frac{1}{a}\sinh a\tau, \frac{1}{a}\cosh a\tau, ..\right)$
$\dot{X}^\mu = (\cosh a\tau, \sinh a\tau, \underline{0})$
τ is local time of observer
$\ddot{X}^\mu = a^2 X^\mu$
$X^\mu(\tau)$
uniformly
observer

$\dot{X}^\mu = (\cosh a\tau, \sinh a\tau, \underline{0})$

τ is local time of observer

uniformly accelerating observer.

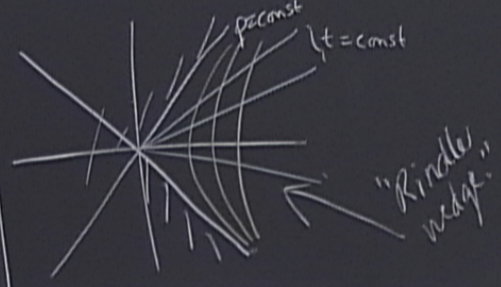
$x = \rho \cosh a\tau$

$\rho = 1/a$ for observer.

This rewriting of MST is called

RINDLER SPACETIME

$x \geq 0 \quad |t| \leq x$

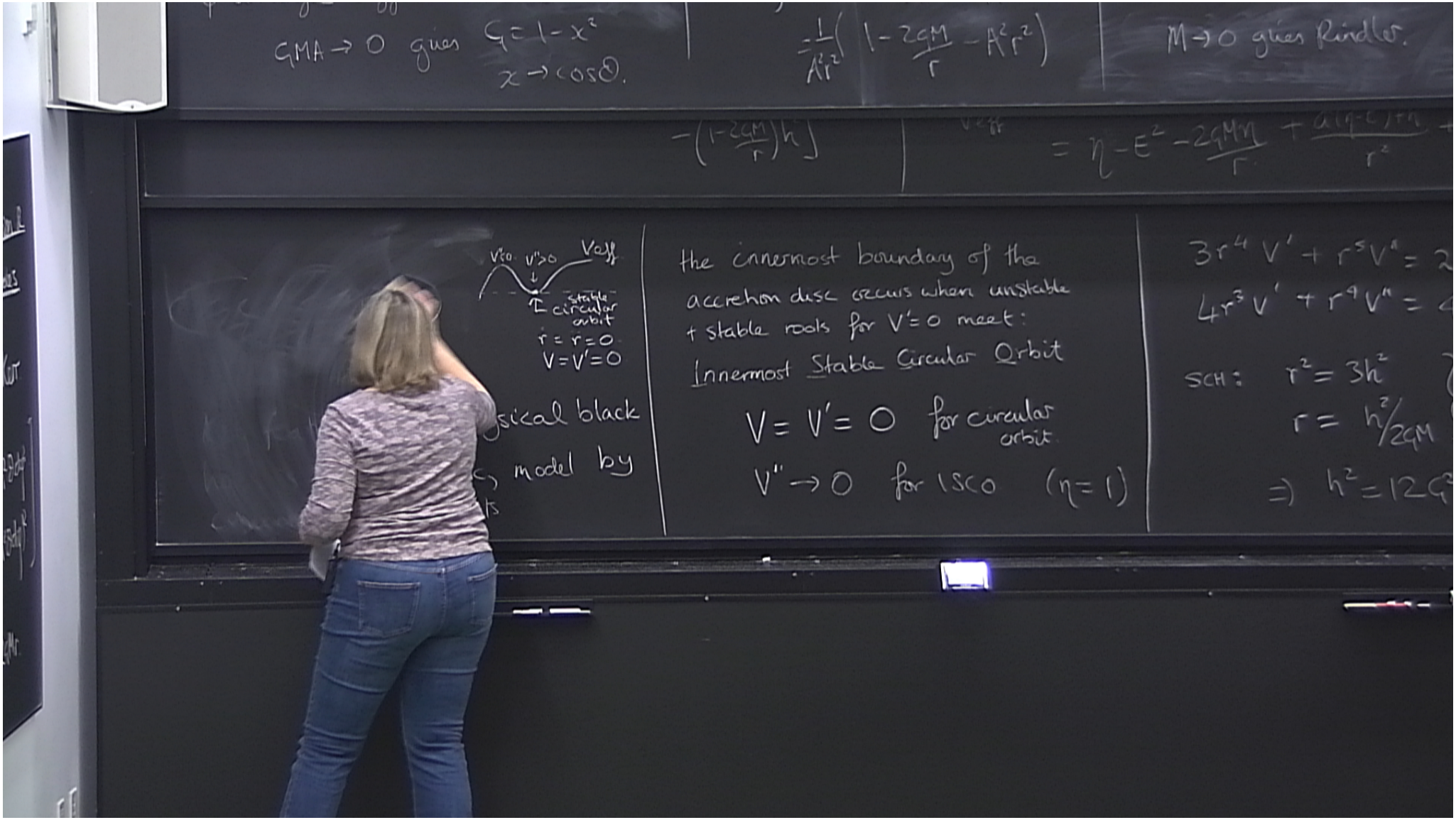


Accelerating black hole C-metric

$$ds^2 = \frac{1}{A^2(x+y)^2}\left[F(y)\,dt^2 - \frac{dy^2}{F(y)} - \frac{dx^2}{G(x)} - G(x)\,d\varphi^2\right]$$

$F(y) = -1 + y^2 - 2qMA\,y^3$

$G(x) = 1 - x^2 - 2qMA\,x^3 \quad (= -F(-x))$


GMA → 0 gives
x → cosθ.
M → 0 gives Rindler.
V'=0
V''>0
Veff
stable circular orbit
r = r = 0
V = V' = 0
the innermost boundary of the accretion disc occurs when unstable + stable roots for V' = 0 meet:
Innermost Stable Circular Orbit
V = V' = 0 for circular orbit
V'' → 0 for ISCO (η = 1)
SCH:
r = h²/2GM

$GMA \to 0$ gives $G = 1 - x^2$, $x \to \cos\theta$.

$= -\frac{1}{A^2 r^2}\left(1 - \frac{2GM}{r} - A^2 r^2\right)$

$M \to 0$ gives Rindler.

$-\left(1 - \frac{2GM}{r}\right)h^2$

$= \eta - E^2 - \frac{2GMn}{r} + \frac{a(q - c/r)}{r^2} + \frac{1}{r^3}\left[-2GMn^2 - 2GM\right.$

Check poles: $x_{c,i}$

$$d\theta_i^2 = \frac{dx^2}{G_i'(x - x_i)} \to \frac{dx^2}{G(x)}$$

$$G(x) \to \frac{G_i'^2}{4}\sin^2\theta_i$$

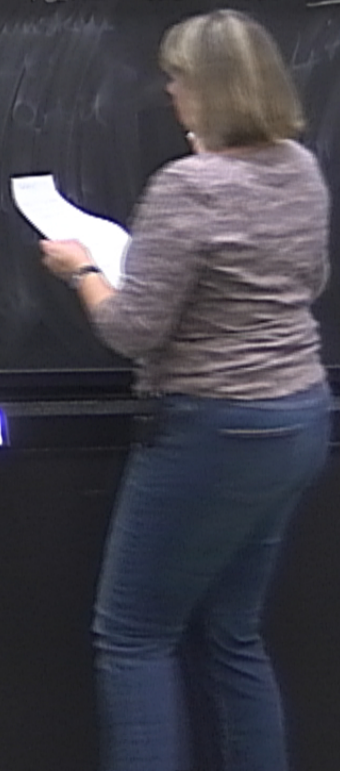
Looks like pole:

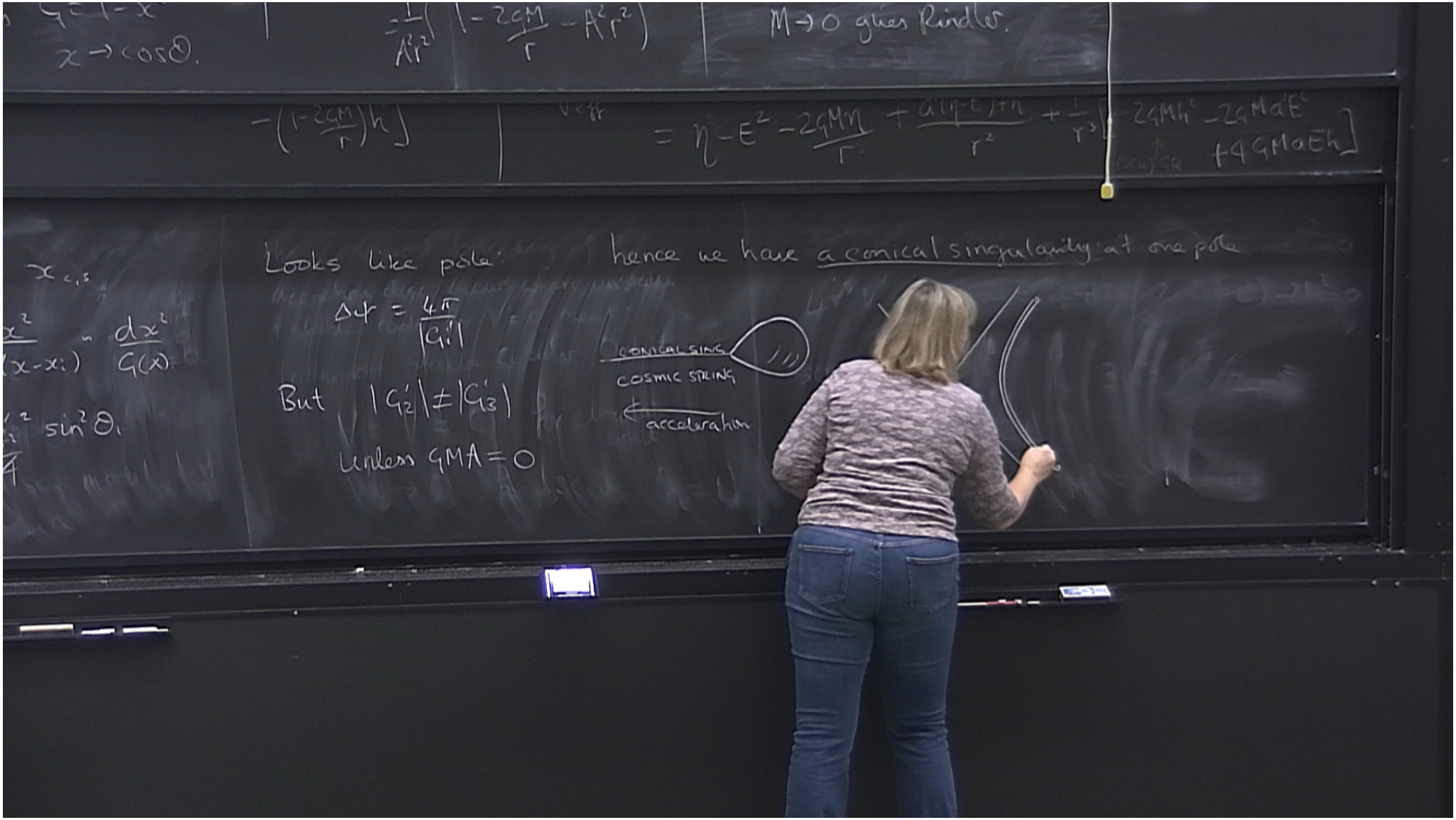
$$\Delta\varphi = \frac{4\pi}{|G_i'|}$$

But $|G_2'| \neq |G_3'|$

Unless $GMA = 0$

hence we have a conical singularity at one pole




M→0 gives Rindler.
Looks like pole
hence we have a conical singularity at one pole
$\Delta\varphi = \frac{4\pi}{|G_1'|}$
But $|G_2'| \neq |G_3'|$
Unless $GMA = 0$
CONICAL SING
COSMIC STRING
acceleration