Title: Gravitational Physics-5

Date: Jan 07, 2015 10:15 AM

URL: http://pirsa.org/15010021

Abstract:

Lecture 5 Black Hole magic

Take a closer look at static sph. symm.

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$ (% Eriskin)

where $M(r) = \int 4\pi r^2 \rho_0 dr$.

intuitively mass inside radius r.

$$-8\pi G p_r = \frac{1}{r^2} - \left(1 - \frac{2GM(r)}{r}\right)\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right)$$

we also have consvn of energy momentum.

$\nabla_a T^{ab}$ & assume perfect fluid

+ isotropic ($p_r = p_\theta$)

$\nabla_a T$

$$\nabla_a T^{a\hat{r}} = \partial_{\hat{r}} p_r + \Gamma_{\hat{a}}{}^{\hat{a}}{}_{\hat{r}}\, p_r + \Gamma_{\hat{a}\hat{b}}{}^{\hat{r}}\, T^{\hat{a}\hat{b}}$$

$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right) p + \frac{A'}{AB}\rho - \frac{2\rho}{rB}$$

$$= \frac{1}{B}\left(p' + \frac{A'}{A}(p+\rho)\right) = 0.$$

$$\nabla_a T^{a\hat{r}} = \partial_{\hat{r}} p_r + \Gamma_a{}^{\hat{a}}{}_{\hat{r}}\, p_r + \Gamma_{\hat{a}\hat{b}}{}^{\hat{r}}\, T^{\hat{a}\hat{b}}$$

$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \frac{2}{rB}\right) p + \frac{A'}{AB}\rho - \frac{2\rho}{rB}$$

$$= \frac{1}{B}\left(p' + \frac{A'}{A}(p+\rho)\right) = 0.$$

Tolman-Oppenheimer-Volkov.

$$p_r + \Gamma^{\hat{a}}_{\hat{b}\hat{c}} T^{\hat{a}\hat{b}}$$

$$\left(\frac{2}{rB}\right) p + \frac{A'}{AB}\rho - \frac{2\rho}{rB}$$

$$(p+\rho)) = 0.$$

Volkov.

Model star by "top hat" potential

$$\rho = \begin{cases} \rho_0 & r \leq R \\ 0. & r > R \end{cases}$$

$$\Rightarrow M(r) = \begin{cases} \frac{4\pi}{3} r^3 \rho_0 & r \leq R \\ \frac{4}{3}\pi R^3 \rho_0 = M & r > R \end{cases}$$

Pressure, solve

$$p' = -(p+\rho_0)\,\frac{[GM(r)+4\pi G p r^3]}{r(r-2GM(r))}$$

$$= -(p+\rho_0)\cdot\frac{4\pi G r(3p+\rho_0)}{(3-8\pi G r^2\rho_0)}$$

Subject to $p \to 0$ at R

$$p = \rho_0\left[\frac{\sqrt{1-\frac{2GM}{R}}-\sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}}-3\sqrt{1-\frac{2GM}{R}}}\right]$$

Note $p(0) = \rho_0\,\dfrac{\sqrt{1-\frac{2GM}{R}}-1}{1-3\sqrt{1-\frac{2GM}{R}}}$

$$P = \rho_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$$

Note $p(0) = \rho_0 \dfrac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$

$\to \infty$ when $R = \dfrac{9GM}{4}$.

i.e. a given mass cannot fit inside an arbitrarily small radius

$r > R$

$$p = p_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$$

Note $p(0) = p_0 \dfrac{\sqrt{1-\frac{2GM}{R}} - 1}{1 - 3\sqrt{1-\frac{2GM}{R}}}$

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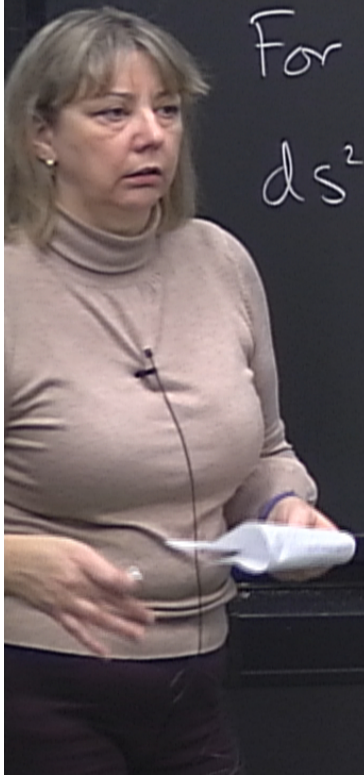
Toy example illustrating the Chandrasekhar Limit

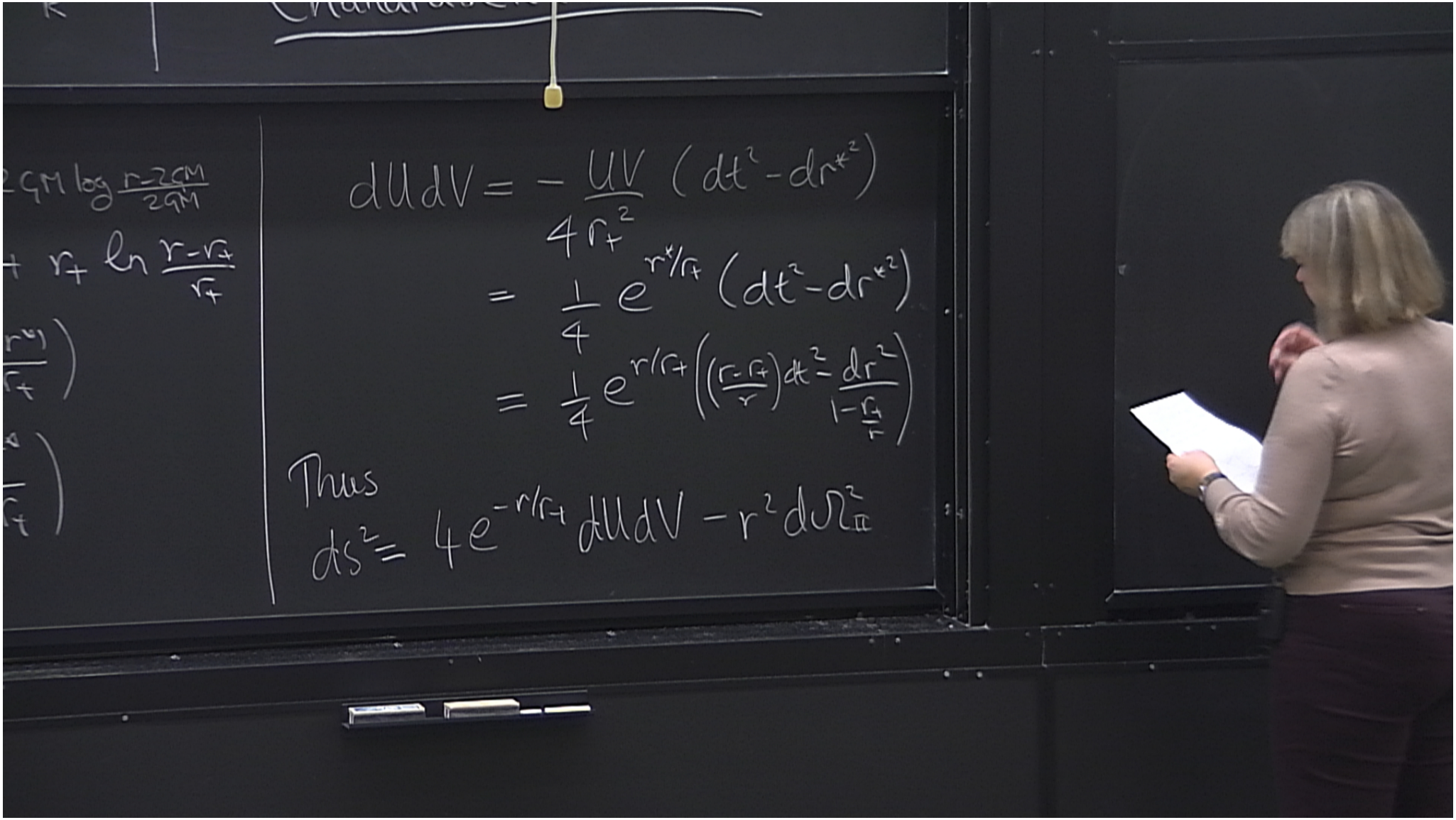
Subject to $p \to 0$ at R.

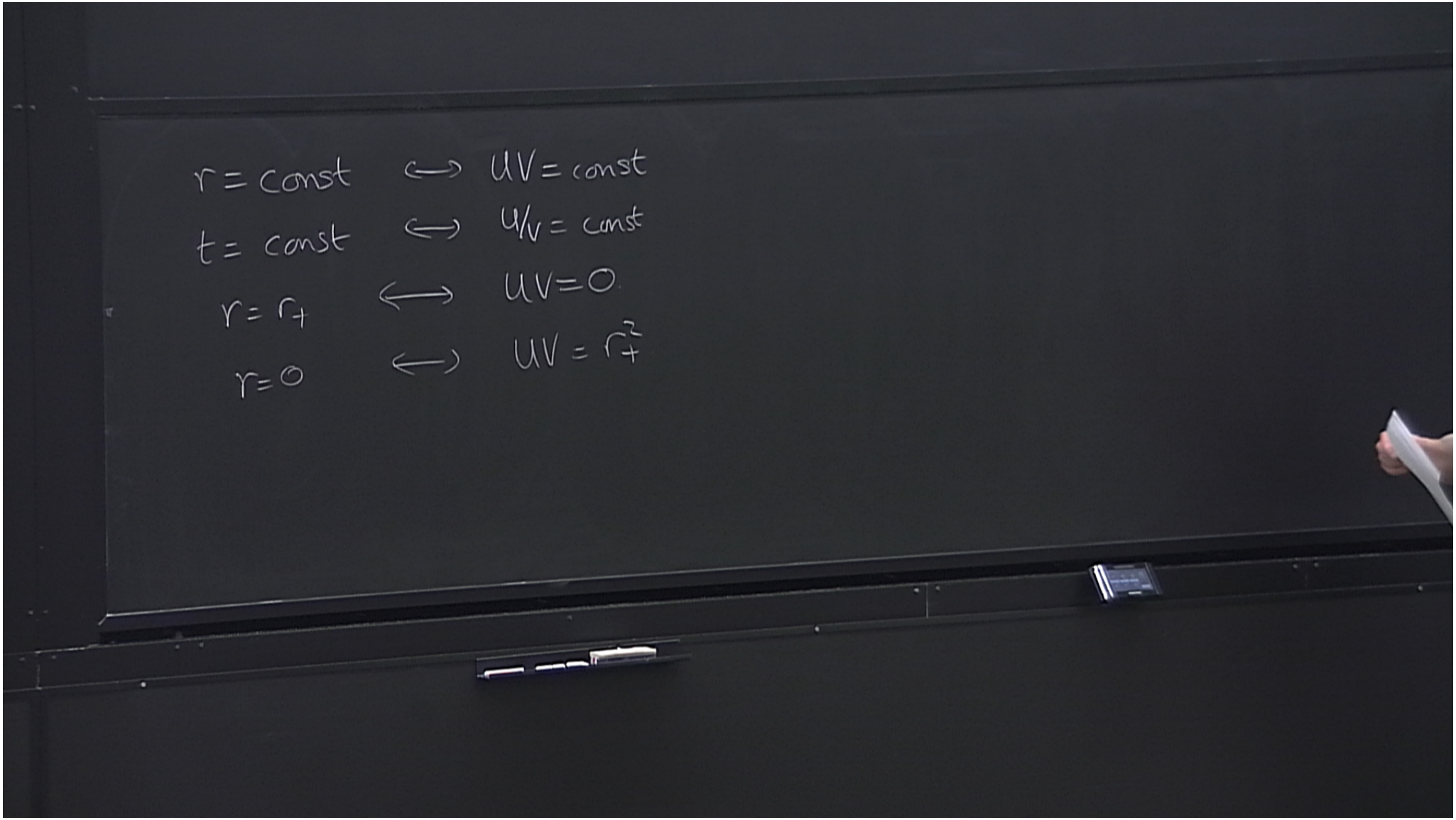
Penrose diagrams

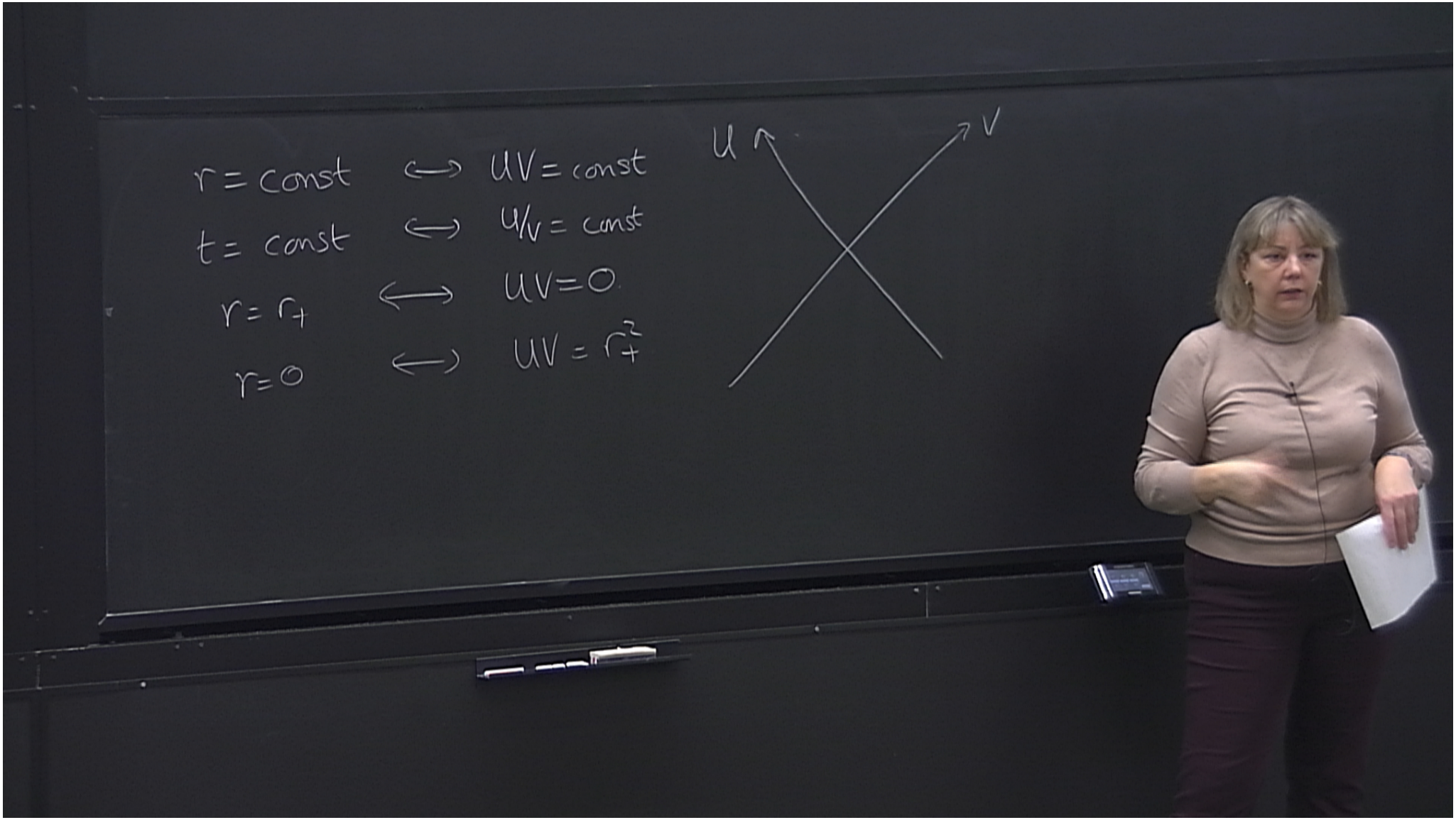
For Schwarzschild:

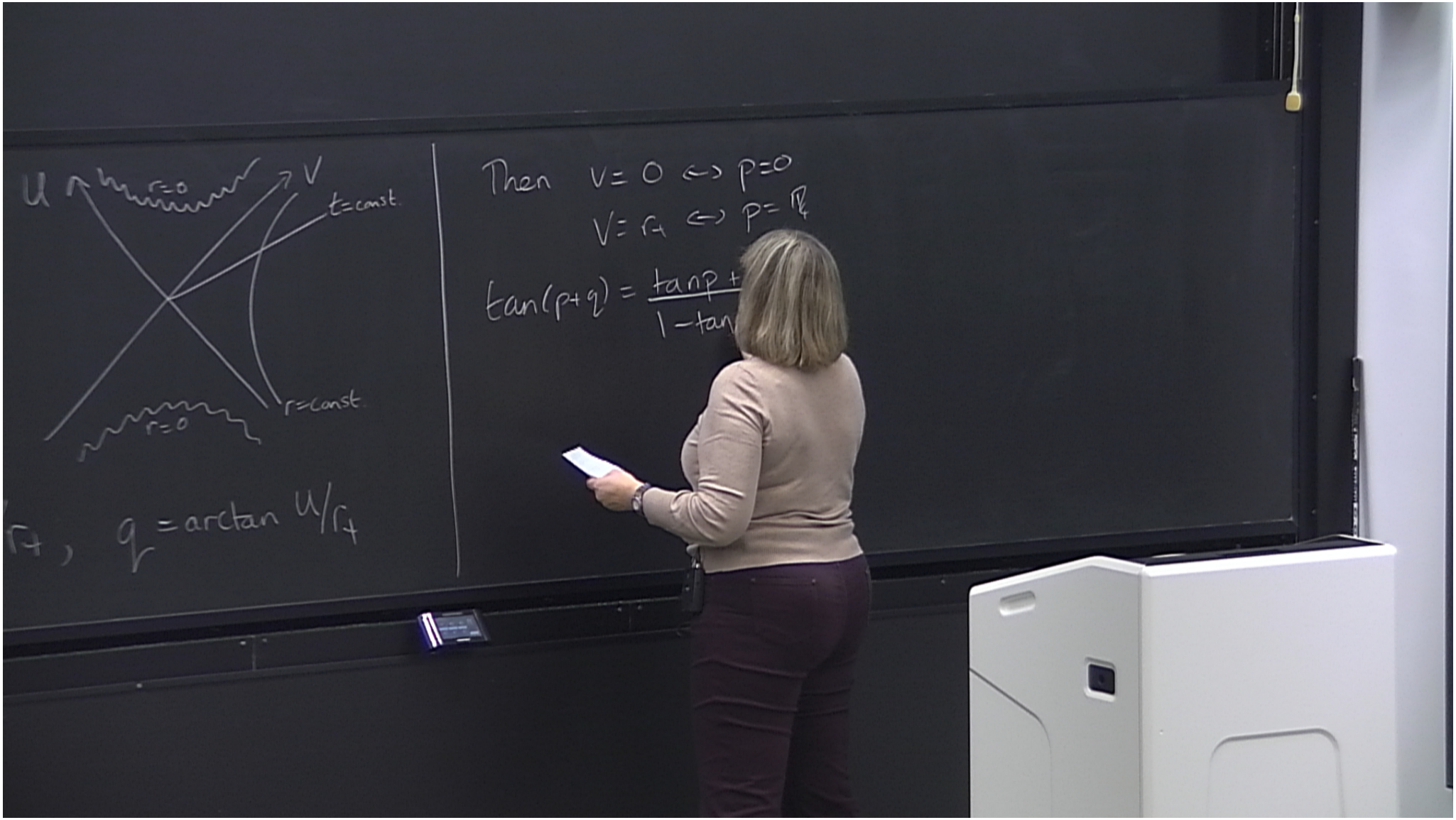
$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

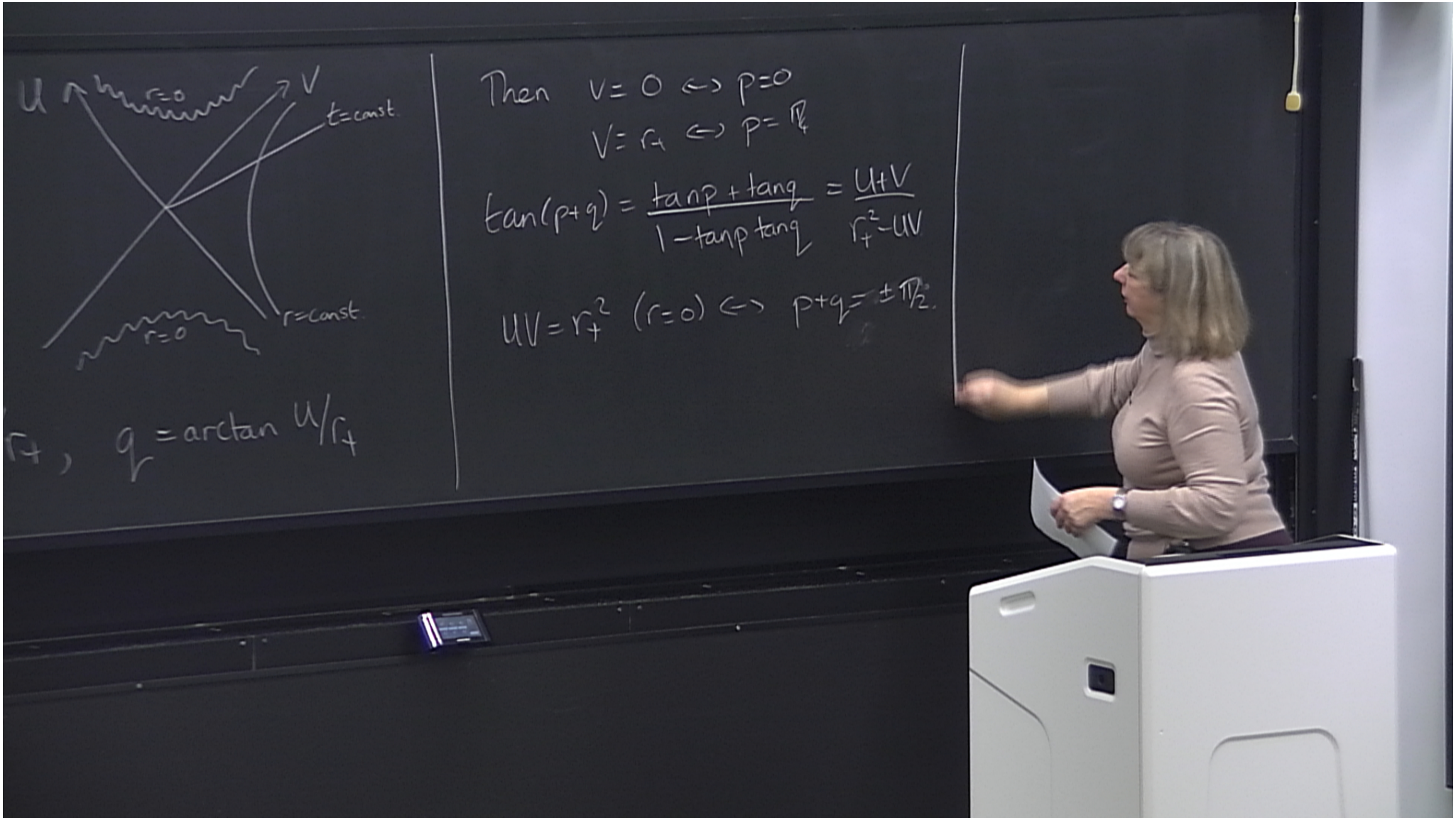


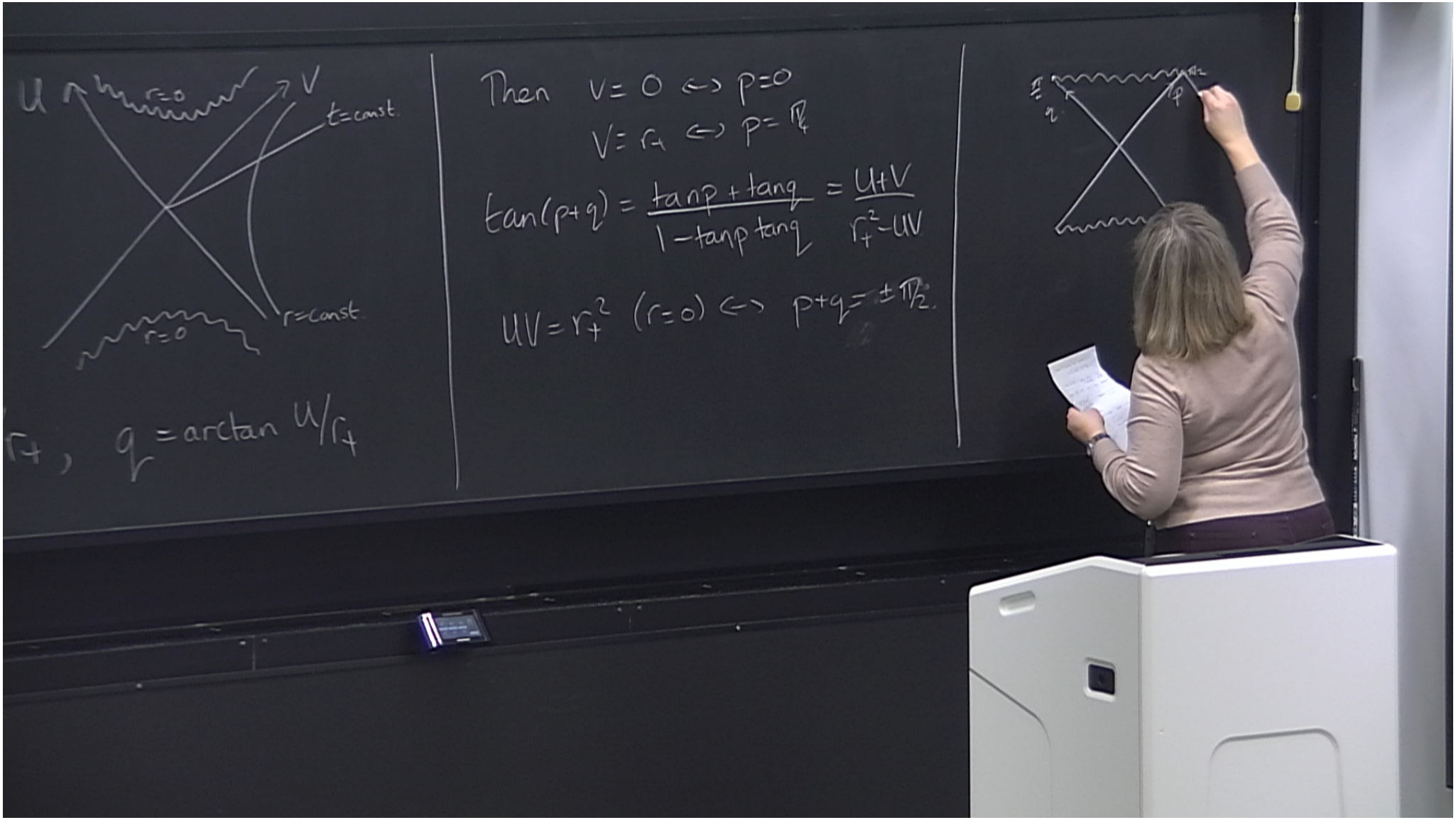

$dUdV = -\frac{UV}{4r_+^2}(dt^2 - dr^{*2})$
$= \frac{1}{4}e^{r^*/r_+}(dt^2 - dr^{*2})$
$= \frac{1}{4}e^{r/r_+}\left(\left(\frac{r-r_+}{r}\right)dt^2 - \frac{dr^2}{1-\frac{r_+}{r}}\right)$
Thus
$ds^2 = 4e^{-r/r_+}dUdV - r^2 d\Omega_{II}^2$

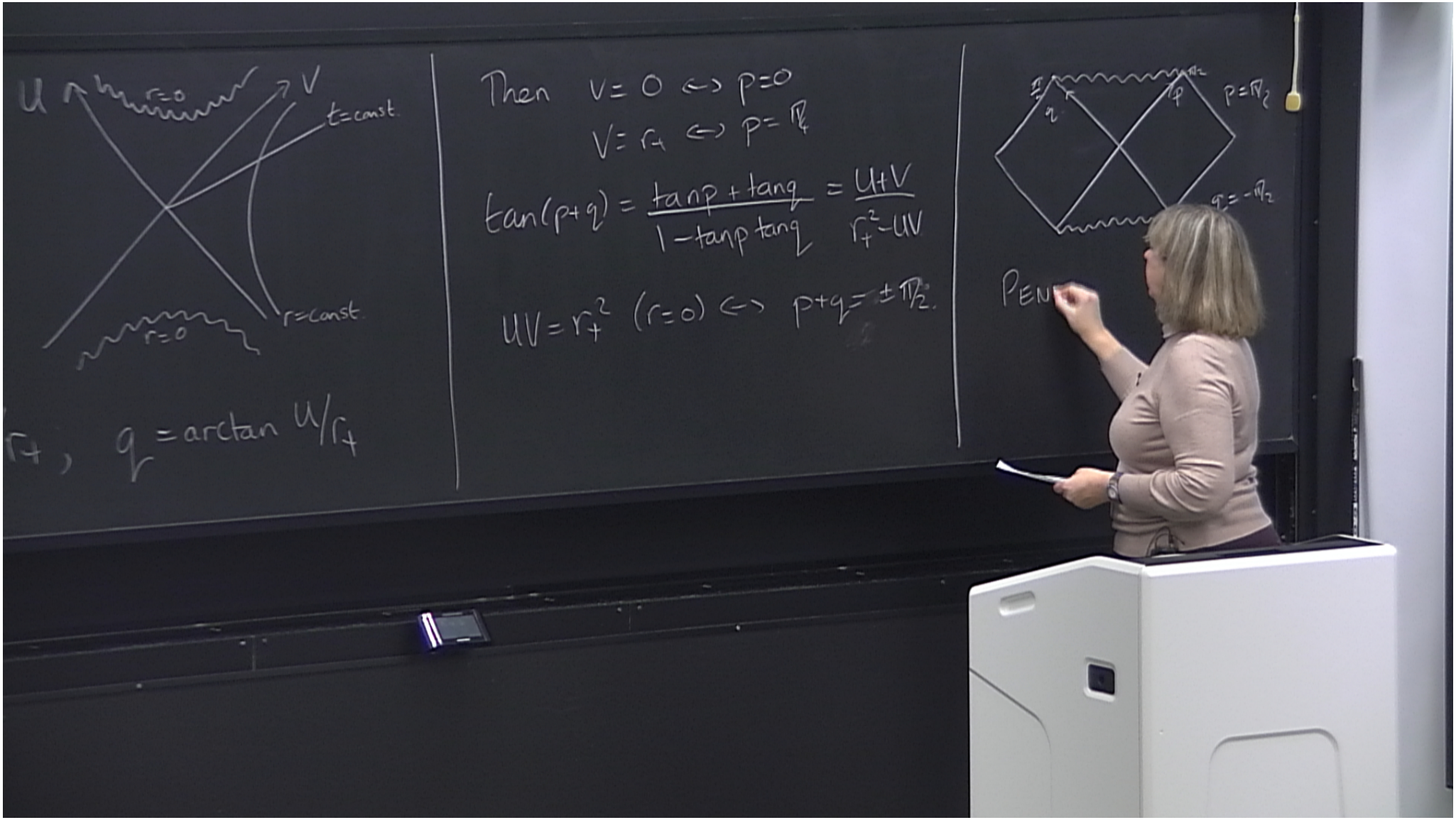

r = const ⟷ UV = const
t = const ⟷ U/V = const
r = r+ ⟷ UV = 0
r = 0 ⟷ UV = r+²

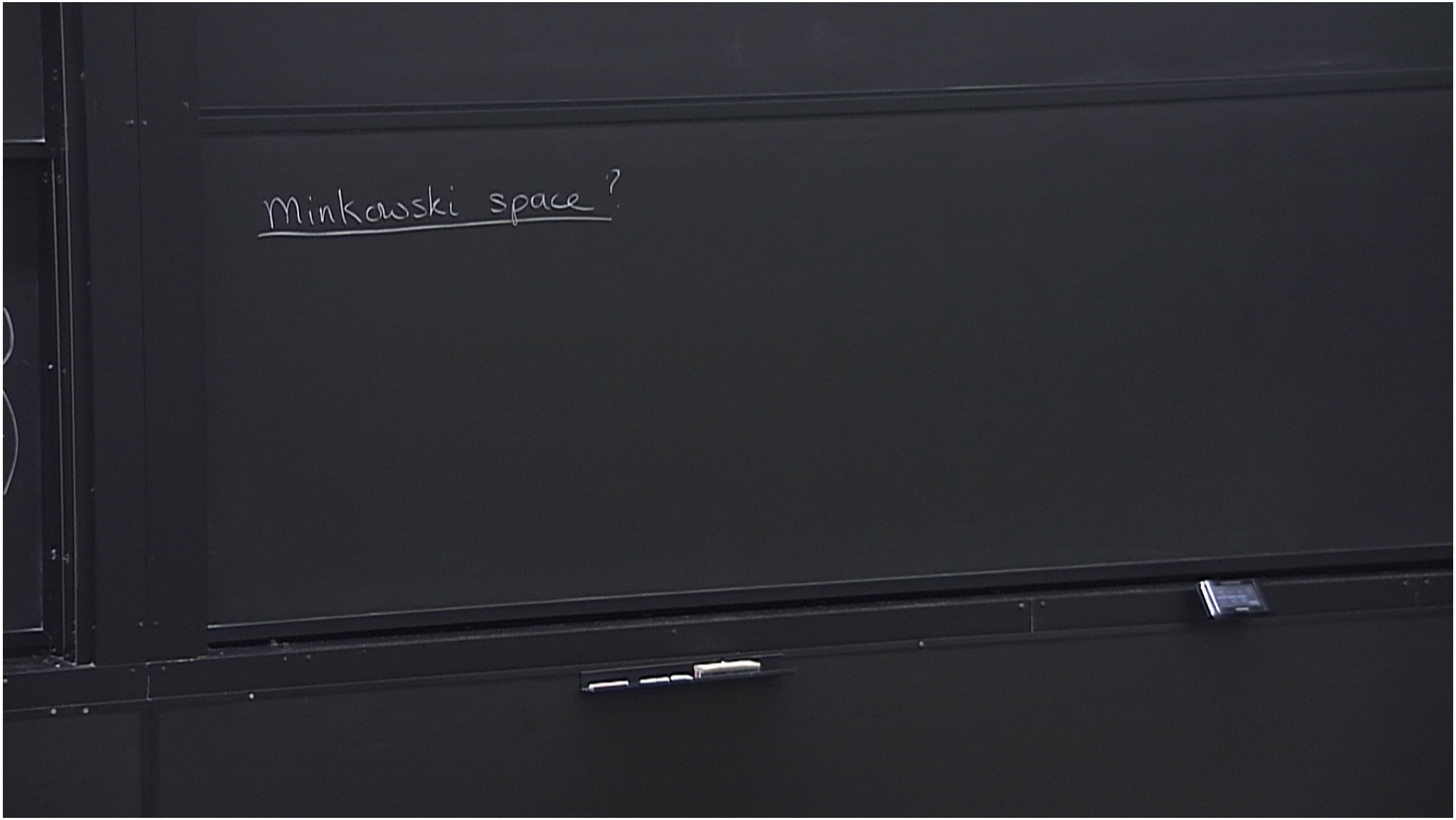

r = const ⟷ UV = const
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U
V

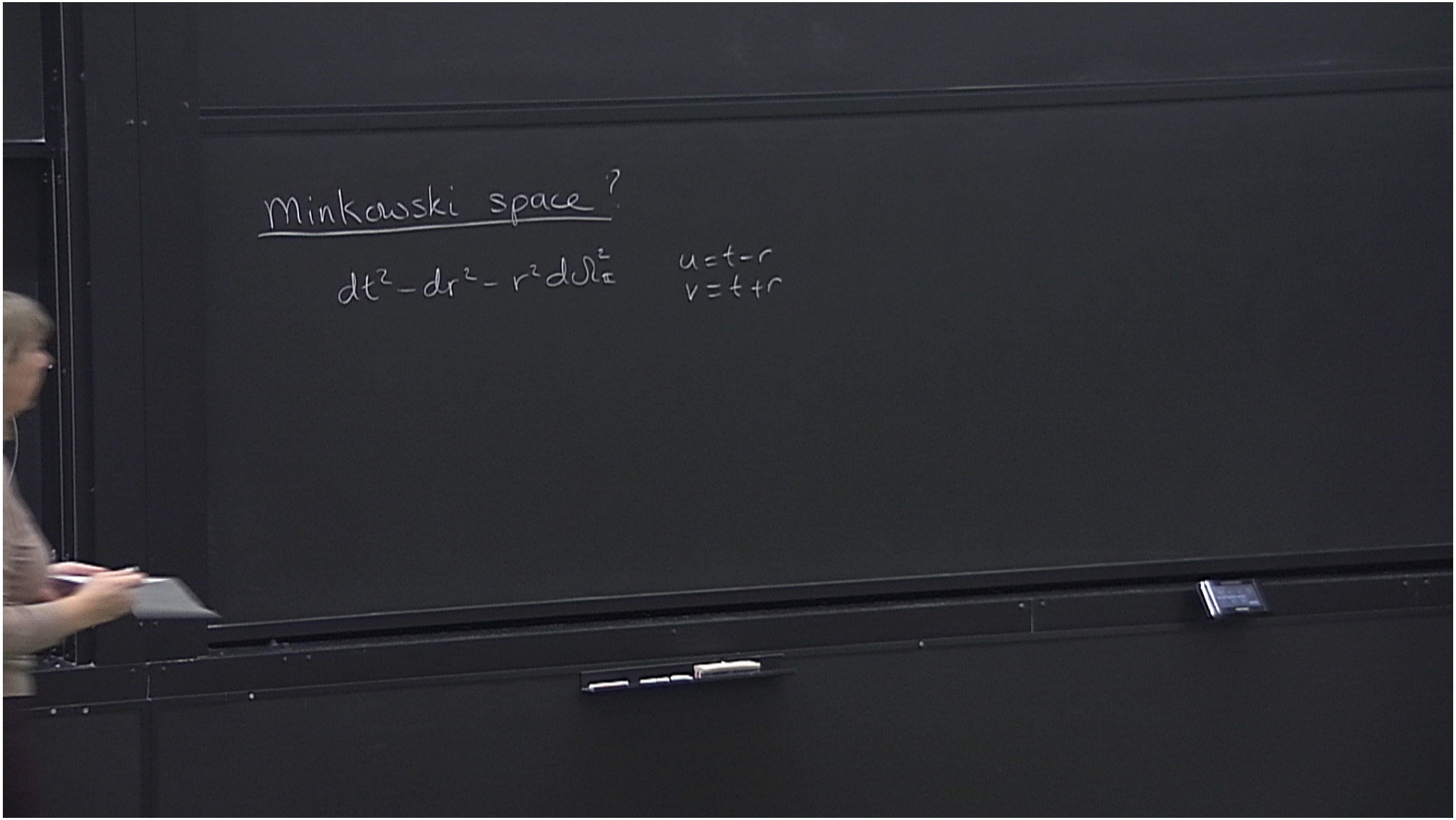

U
V
r=0
t=const.
r=const.
r=0
Then V=0 ⟺ p=0
V=r+ ⟺ p=
tan(p+q) =
1-tan
r+, q=arctan U/r+


U
V
r=0
t=const.
r=const.
r=0
Then V=0 ⟺ p=0
V=r+ ⟺ p=π/4
tan(p+q) = (tanp + tanq)/(1−tanp tanq) = (U+V)/(r+²−UV)
UV=r+² (r=0) ⟺ p+q=±π/2.
r+, q=arctan U/r+


U
V
r=0
t=const.
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Then V=0 ⟺ p=0
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r+, q = arctan U/r+


U
V
r=0
t=const.
r=const.
r=0
r_+, $\varphi = \arctan U/r_+$
Then $V=0 \Leftrightarrow p=0$
$V=r_+ \Leftrightarrow p=\pi/4$
$\tan(p+q) = \frac{\tan p + \tan q}{1-\tan p \tan q} = \frac{U+V}{r_+^2 - UV}$
$UV = r_+^2$ $(r=0) \Leftrightarrow p+q = \pm\pi/2$.
$p=\pi/2$
$q=-\pi/2$
PEN


Minkowski space?

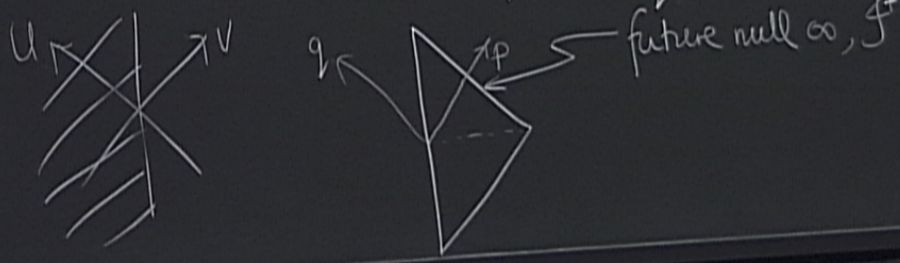

Minkowski space?
$dt^2 - dr^2 - r^2 d\Omega_2^2$
$u = t - r$
$v = t + r$

Minkowski space?

$dt^2 - dr^2 - r^2 d\Omega_{\mathbb{E}}^2$ $\quad u = t - r$, $v = t + r$

$r > 0 \Rightarrow v - u > 0$

$\tan p > \tan q$

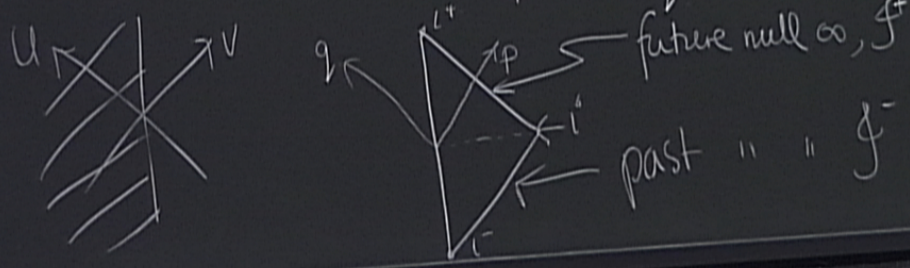


Minkowski space?

$dt^2 - dr^2 - r^2 d\Omega^2_{II}$ $\quad u = t - r$, $v = t + r$

$r > 0 \Rightarrow v - u > 0$ "sch"

$\tan p > \tan q$



i^+ future timelike ∞

i^- past " "

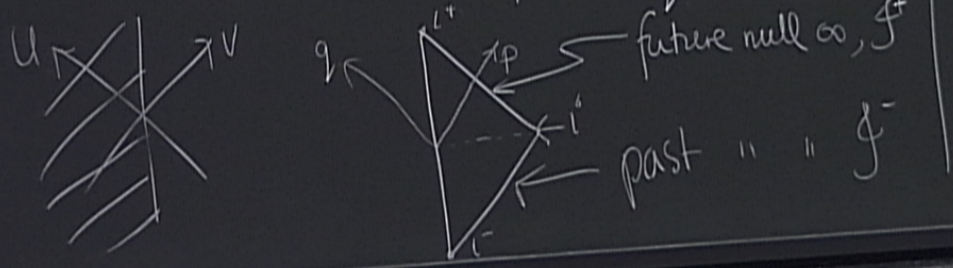
i^0 spacelike ∞

Minkowski space?

$dt^2 - dr^2 - r^2 d\Omega_{II}^2$ $\quad u = t - r$, $v = t + r$

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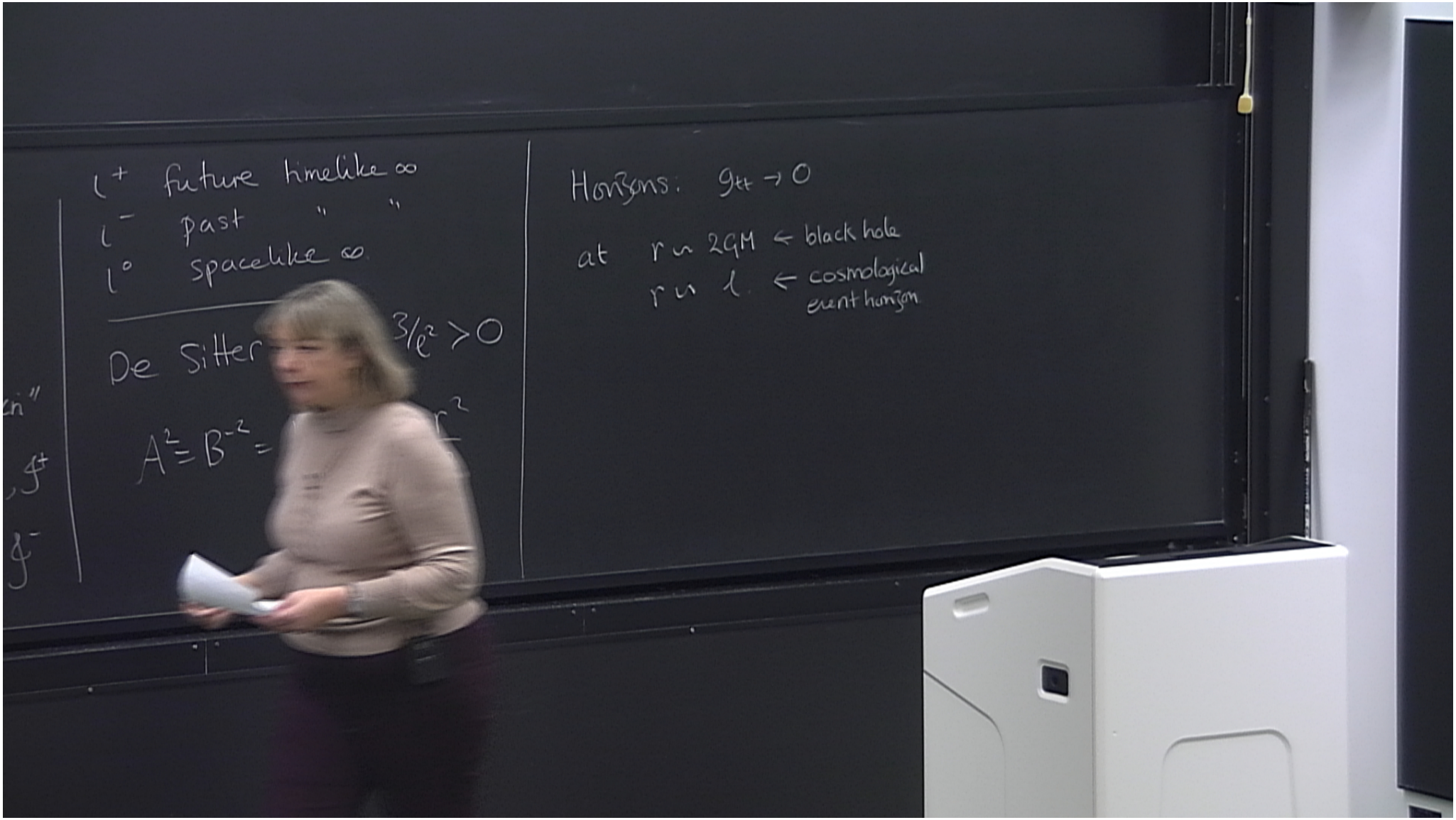
$\tan p > \tan q$

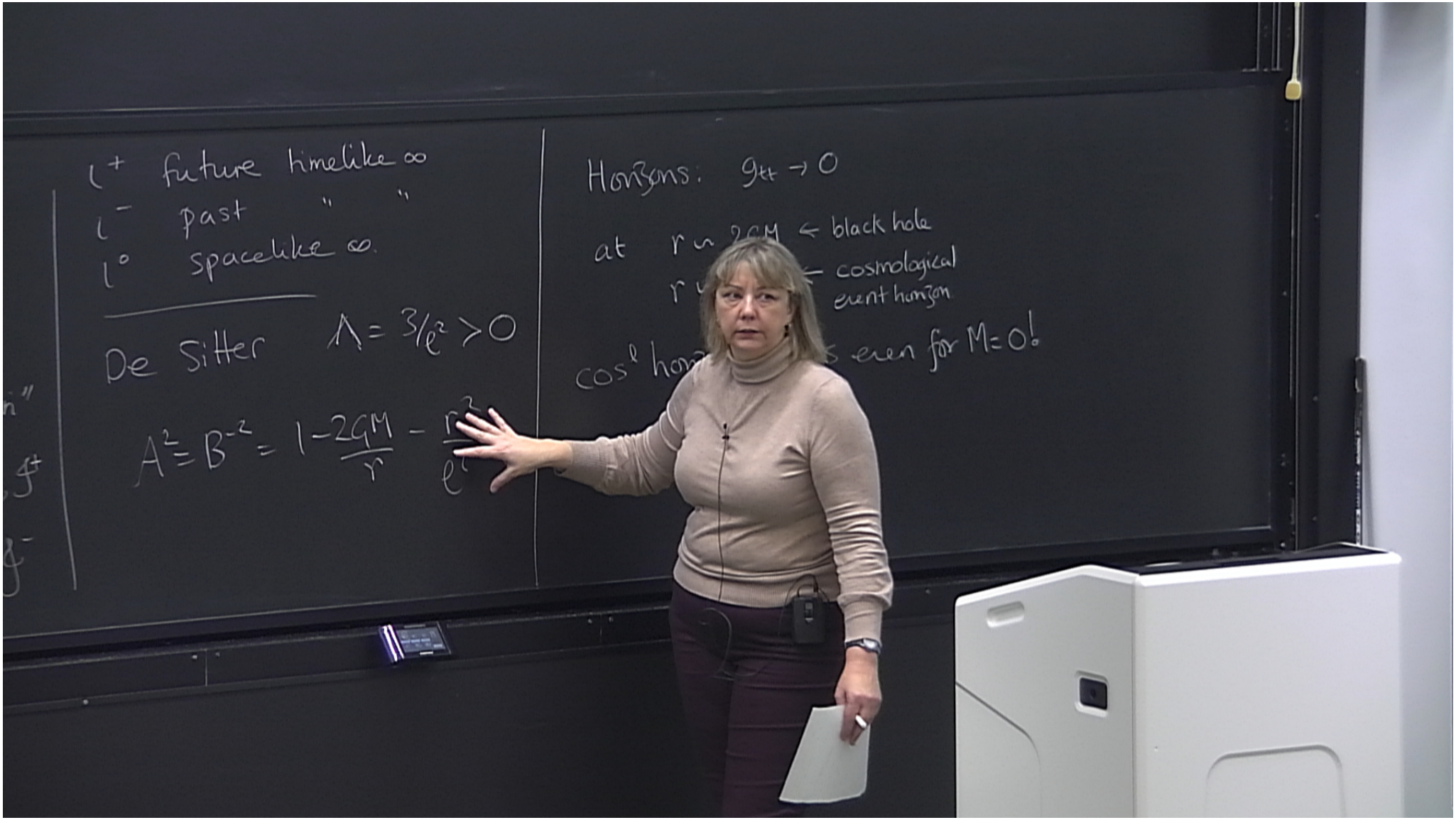


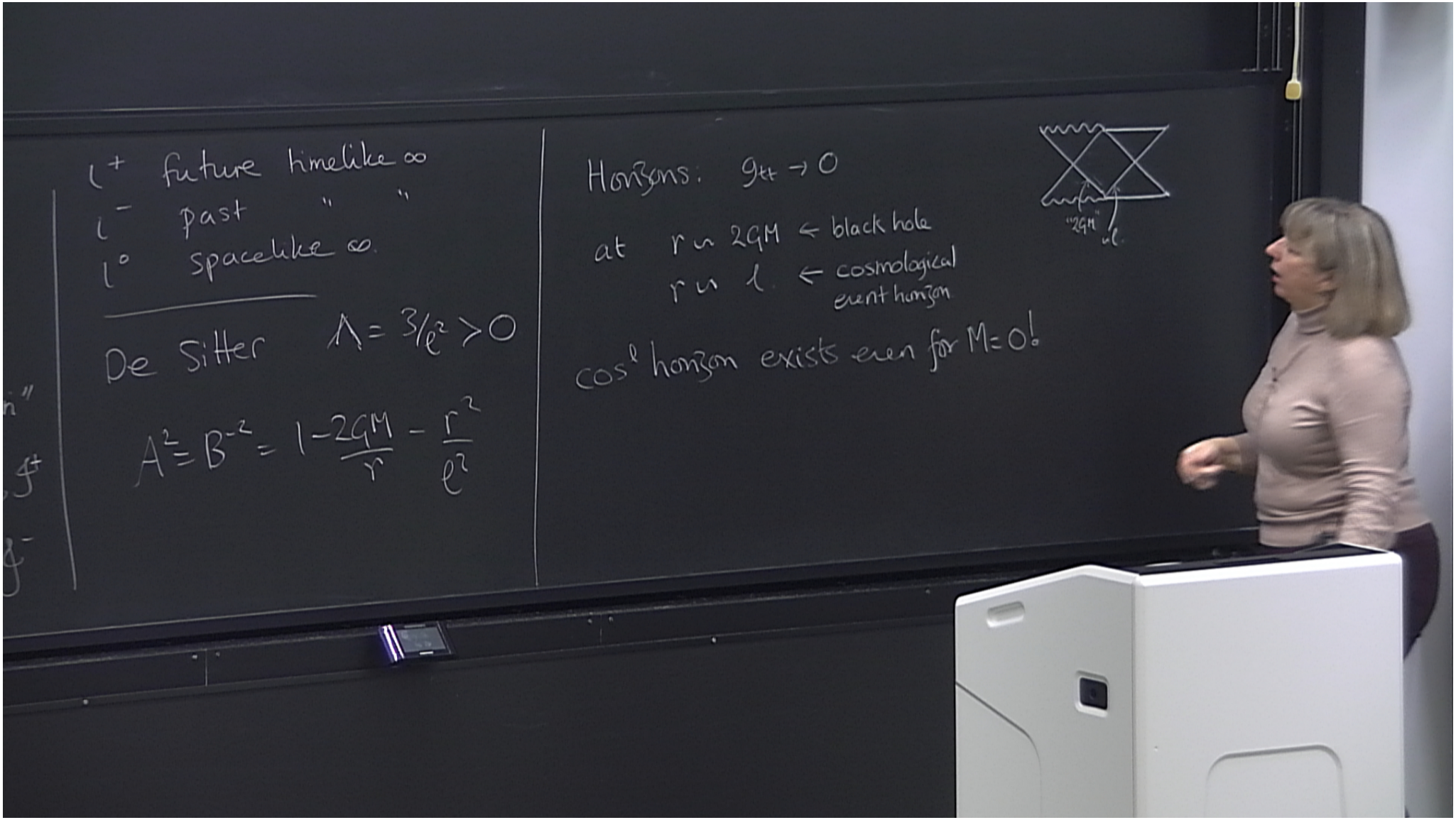
i^+ future timelike ∞

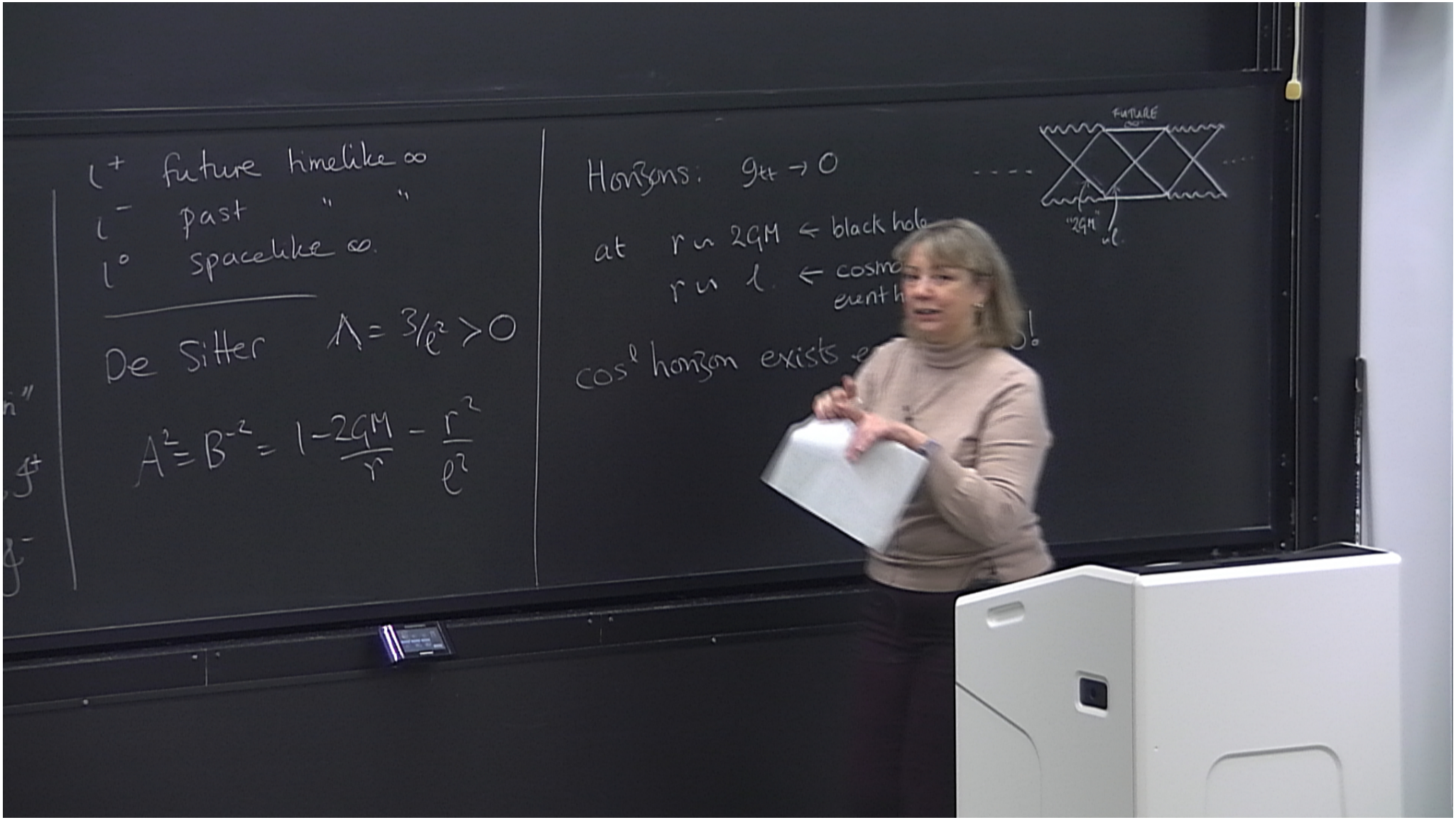
i^- past " "

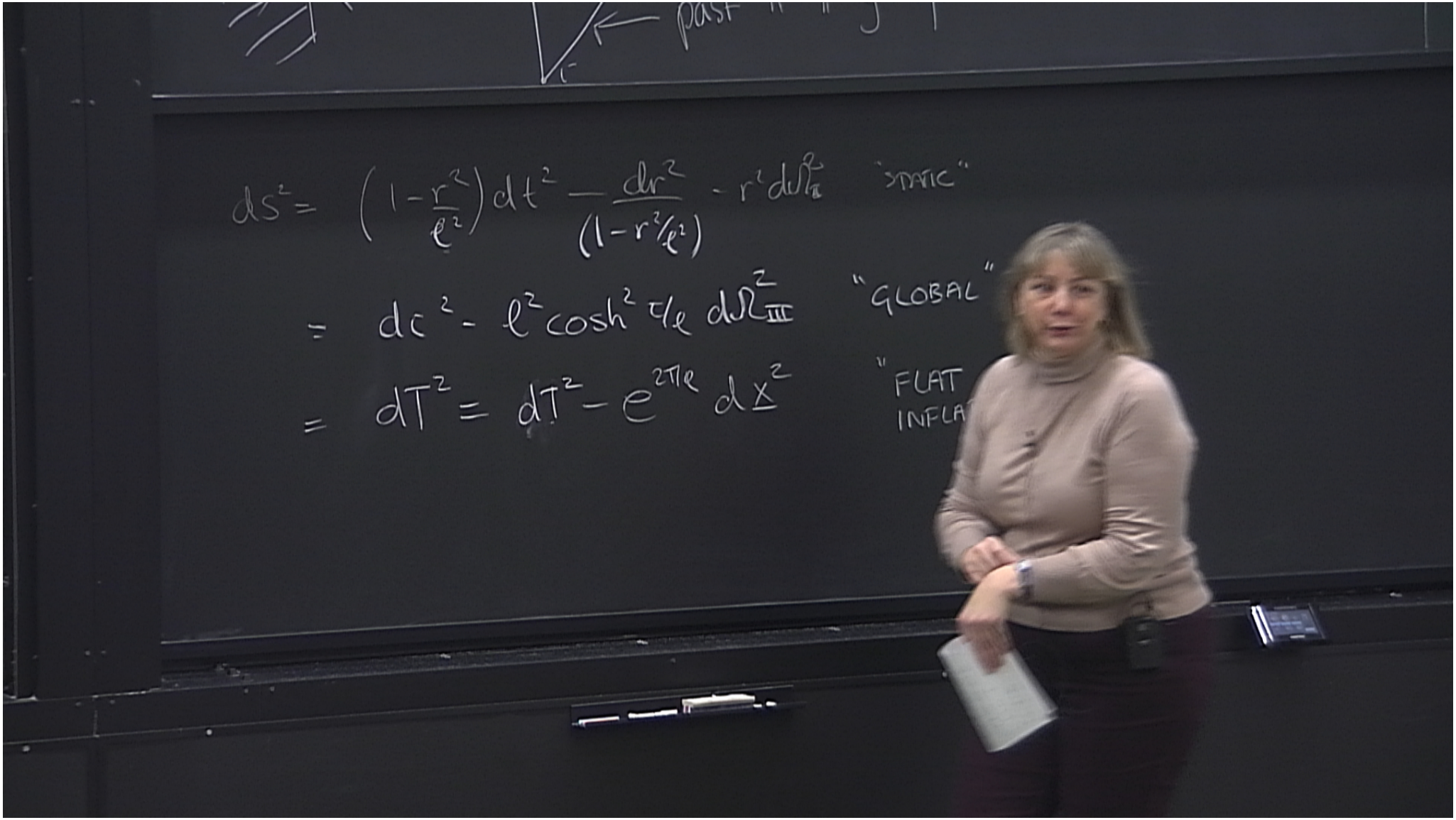
i^0 spacelike ∞

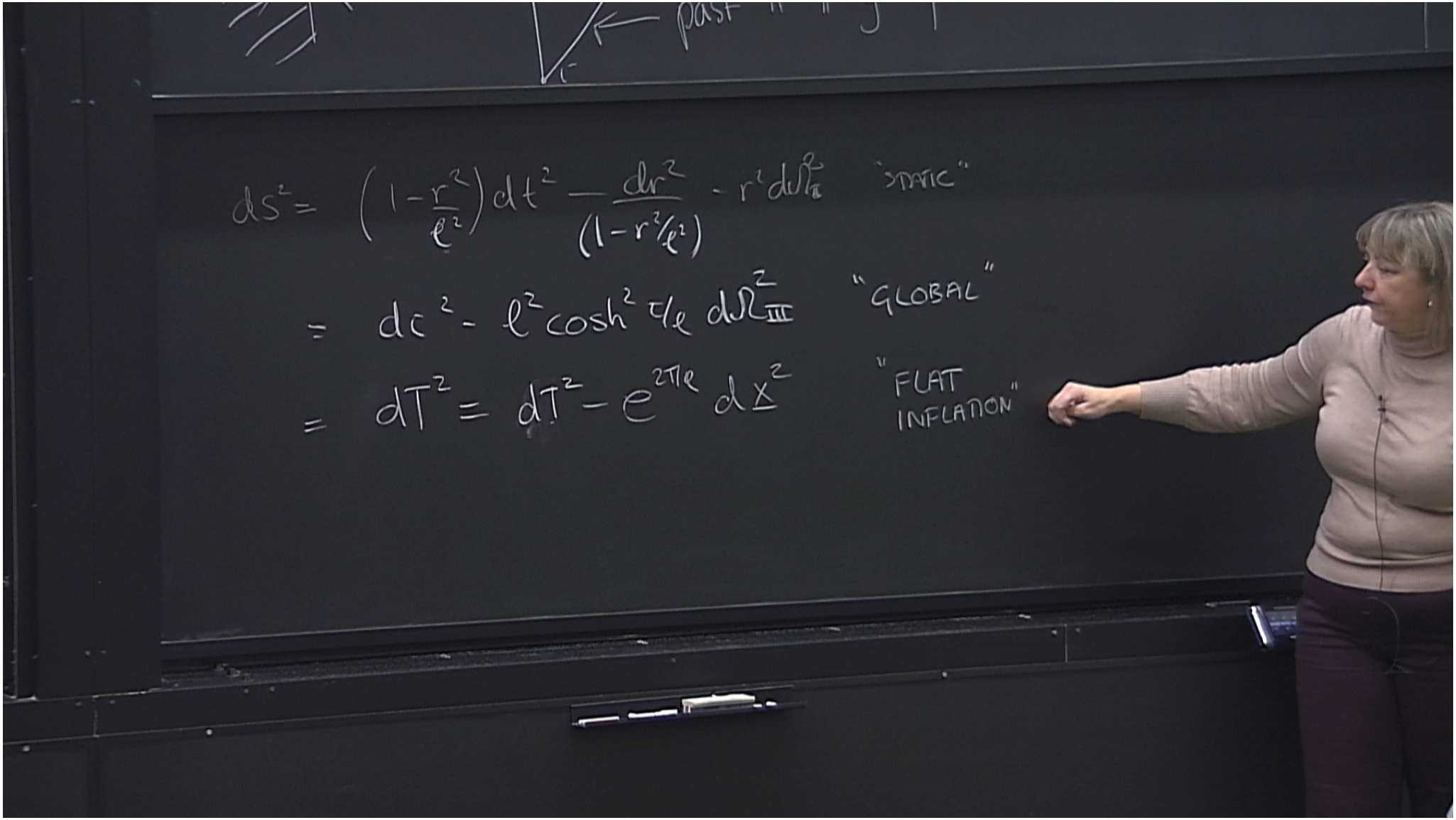

i^+ future timelike ∞
i^- past " "
i^0 spacelike ∞
De Sitter $3/\ell^2 > 0$
$A^2 = B^{-2} =$
Horizons: $g_{tt} \to 0$
at $r \sim 2GM$ ← black hole
$r \sim \ell$ ← cosmological event horizon


i^+ future timelike ∞
i^- past " "
i^0 spacelike ∞.
De Sitter $\Lambda = 3/\ell^2 > 0$
$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{r^2}{\ell^2}$
Horizons: $g_{tt} \to 0$
at $r \sim 2GM$ ← black hole
← cosmological event horizon
even for M=0!


i^+ future timelike ∞
i^- past " "
i^0 spacelike ∞
De Sitter $\Lambda = 3/\ell^2 > 0$
$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{r^2}{\ell^2}$
Horizons: $g_{tt} \to 0$
at $r \sim 2GM$ ← black hole
$r \sim \ell$ ← cosmological event horizon
cosl horizon exists even for $M = 0$!
"2GM"
$\mathcal{I}$


$\mathcal{I}^+$ future timelike ∞
$\mathcal{I}^-$ past " "
$\mathcal{I}^0$ spacelike ∞.
De Sitter $\Lambda = 3/\ell^2 > 0$
$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{r^2}{\ell^2}$
Horizons: $g_{tt} \to 0$
at $r \sim 2GM$ ← black hole
$r \sim \ell$ ← cosmo
event h
cosl horizon exists
FUTURE
"2GM"
$\mathcal{I}^+$


past
$ds^2 = \left(1 - \frac{r^2}{\ell^2}\right) dt^2 - \frac{dr^2}{(1 - r^2/\ell^2)} - r^2 d\Omega_{II}^2$
"STATIC"
$= dt^2 - \ell^2 \cosh^2 t/\ell \, d\Omega_{III}^2$
"GLOBAL"
$= dT^2 = dT^2 - e^{2T/\ell} dX^2$
"FLAT
INFLA

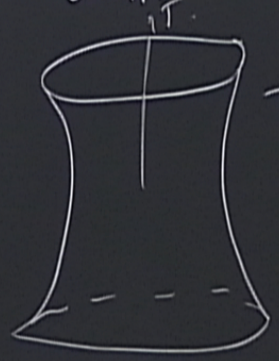

past
$ds^2 = \left(1 - \frac{r^2}{\ell^2}\right) dt^2 - \frac{dr^2}{(1 - r^2/\ell^2)} - r^2 d\Omega_{II}^2$
"STATIC"
$= d\tau^2 - \ell^2 \cosh^2 \tau/\ell \, d\Omega_{III}^2$
"GLOBAL"
$= dT^2 = dT^2 - e^{2T/\ell} d\underline{x}^2$
"FLAT INFLATION"

"STATIC"

"GLOBAL"

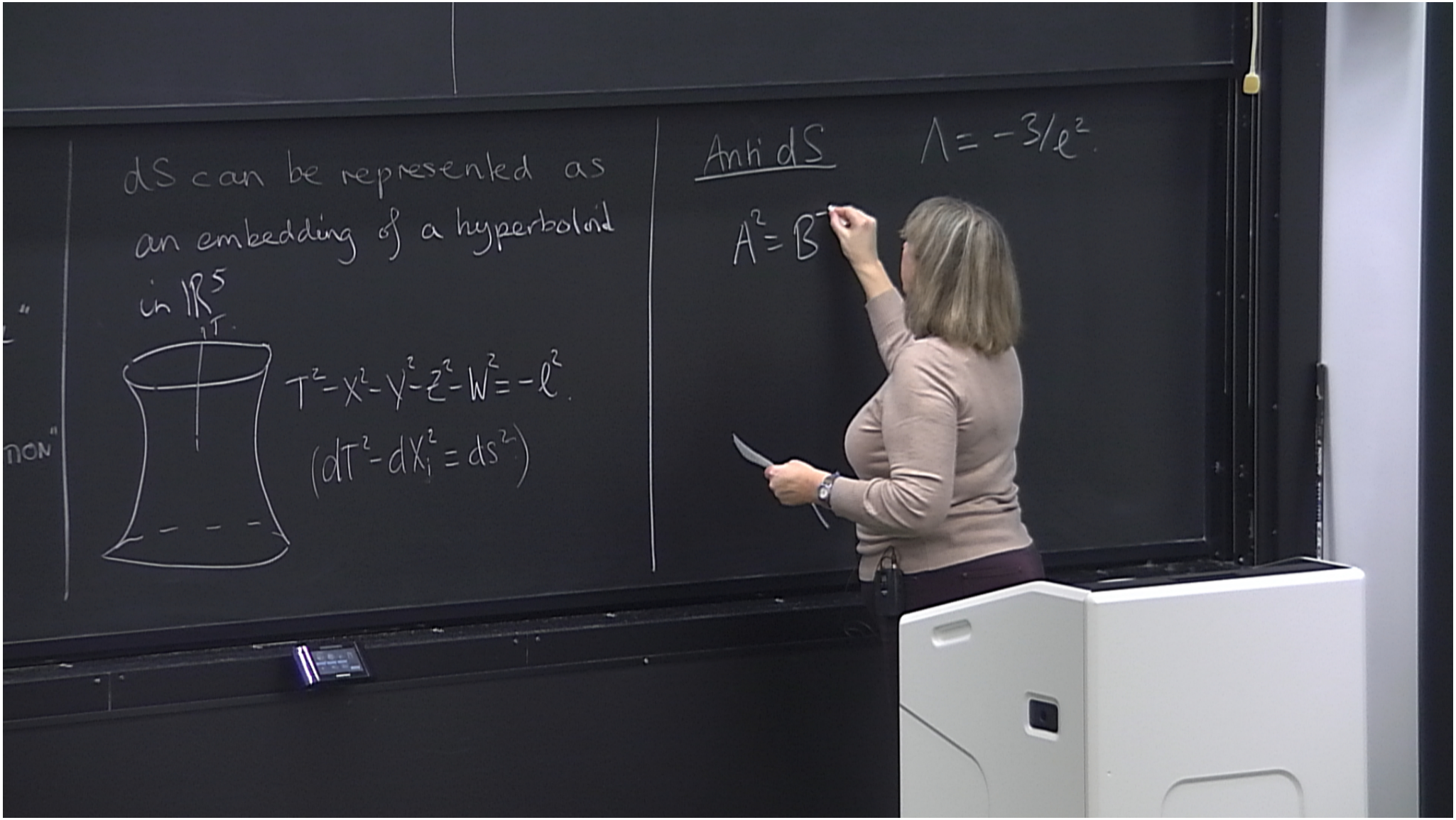
"FLAT INFLATION"

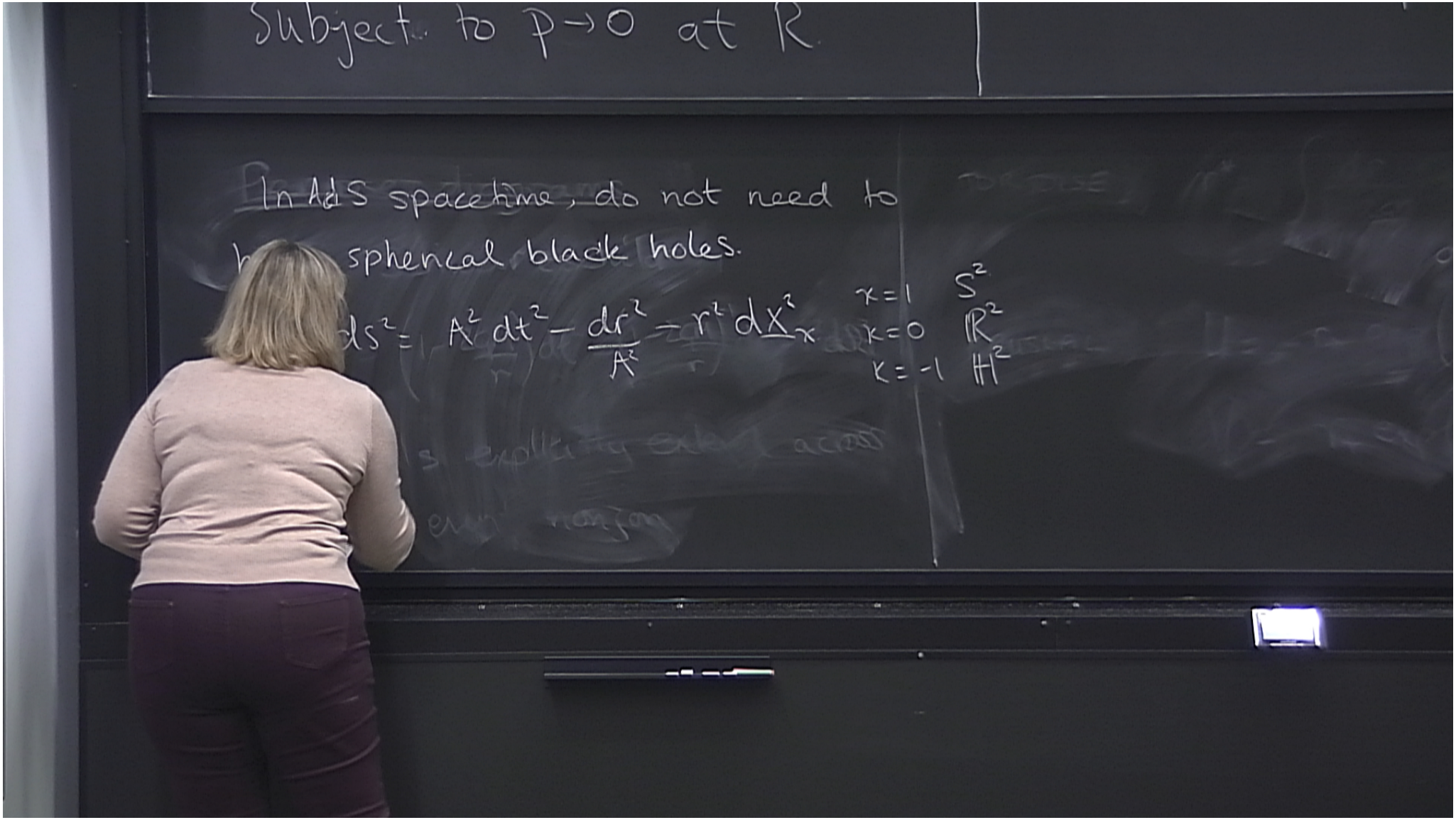
dS can be represented as an embedding of a hyperboloid in $\mathbb{R}^5$

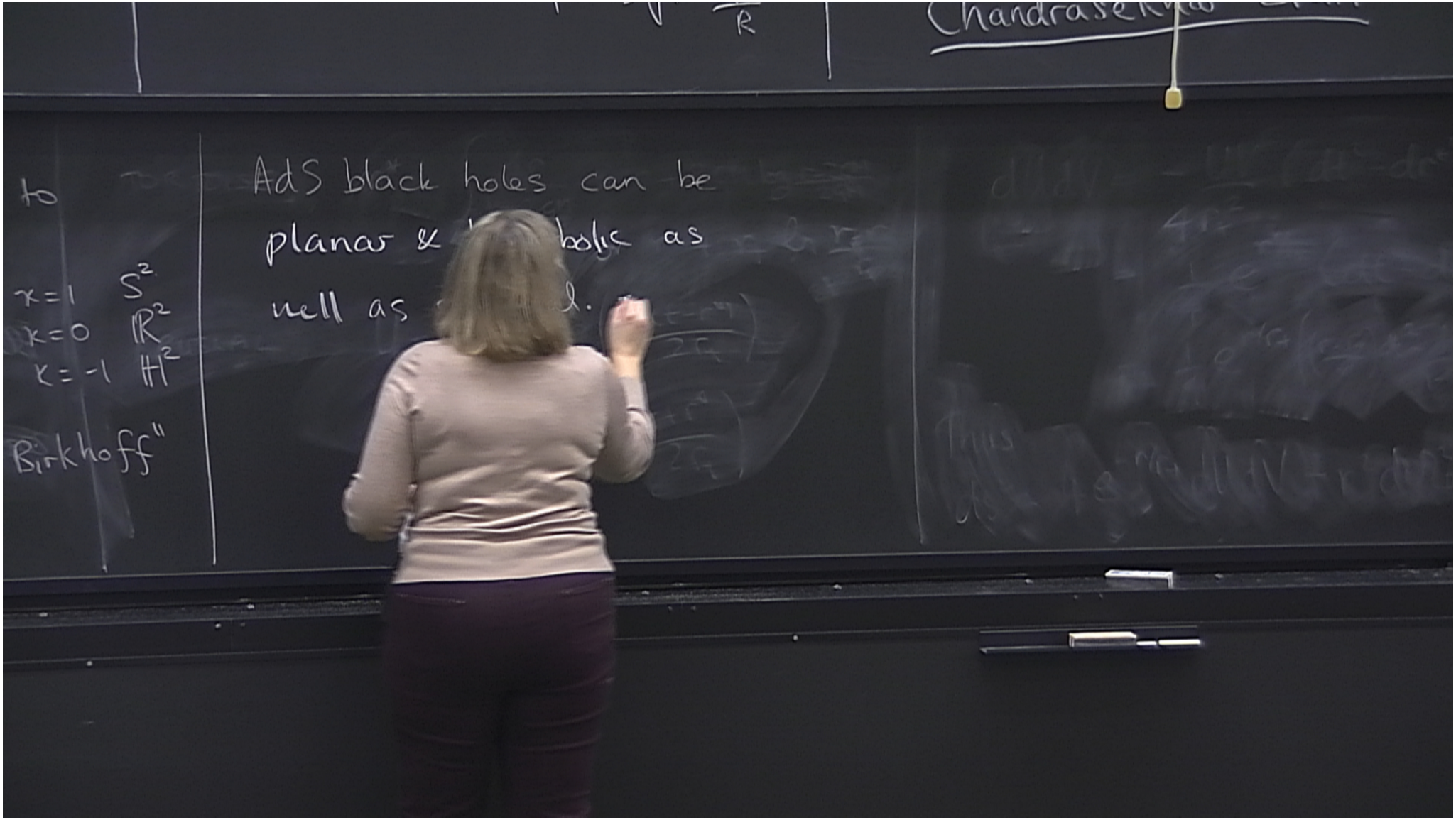


$$T^2 - X^2 - Y^2 - Z^2 - W^2 = -\ell^2$$

$$(dT^2 - dX_i^2 = ds^2)$$


dS can be represented as
an embedding of a hyperboloid
in $\mathbb{R}^5$
T
$T^2 - X^2 - Y^2 - Z^2 - W^2 = -\ell^2$
$(dT^2 - dX_i^2 = ds^2)$
Anti dS
$\Lambda = -3/\ell^2$
$A^2 = B$


Subject to $p \to 0$ at R
In AdS spacetime, do not need to
spherical black holes.
$ds^2 = A^2 dt^2 - \frac{dr^2}{A^2} - r^2 d\underline{x}^2_\kappa$
$\kappa = 1$ S^2
$\kappa = 0$ $\mathbb{R}^2$
$\kappa = -1$ $\mathbb{H}^2$


Chandrasekhar
to
$\kappa=1$ S^2
$\kappa=0$ $\mathbb{R}^2$
$\kappa=-1$ $\mathbb{H}^2$
Birkhoff"
AdS black holes can be
planar &
as
well as

to

$\kappa = 1$ S^2

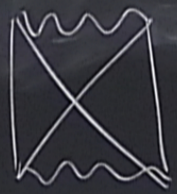
$\kappa = 0$ $\mathbb{R}^2$

$\kappa = -1$ H^2

Birkhoff"

AdS black holes can be planar & hyperbolic as well as spherical. Only

1 horizon $r_+ = (2qM\ell^2)^{1/3}$ $\kappa = 0$

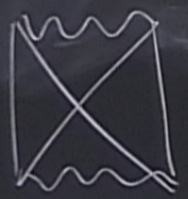


Chandrasekh

$1-3\sqrt{1-\frac{2GM}{R}}$

Chandrasekhar Limit

black holes can be
nar & hyperbolic as
l as spherical. Only
ongon $r_+ = (2GM\ell^2)^{1/3}$ $\kappa = 0$



Euclidean Schwarzschild

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega_{II}^2$$

$1-3\sqrt{1-\frac{2GM}{R}}$

Chandrasekhar Limit

black holes can be

nar & hyperbolic as

l enical. Only

$r_+ = (2GM\ell^2)^{1/3}$ $\kappa = 0$

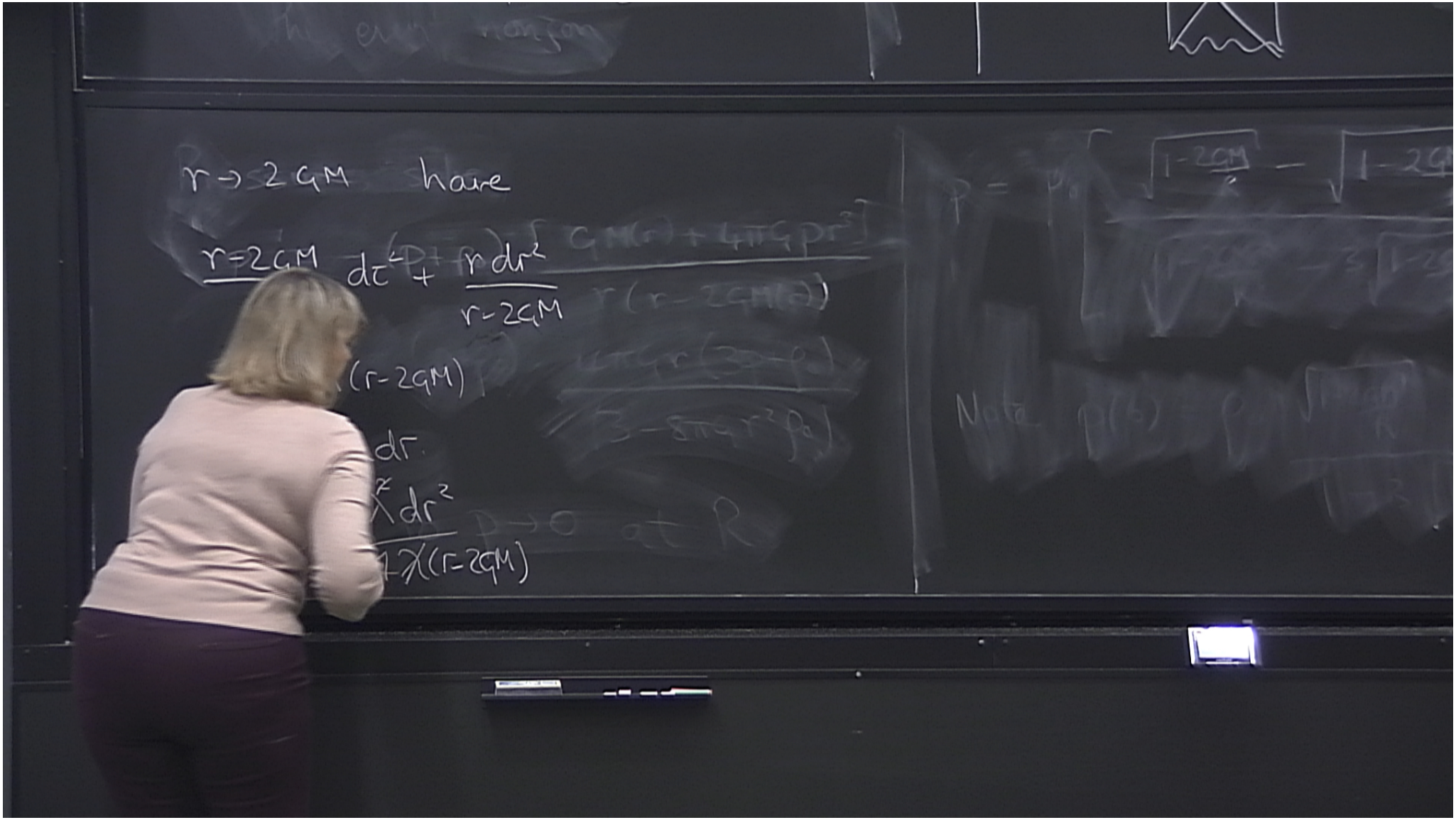


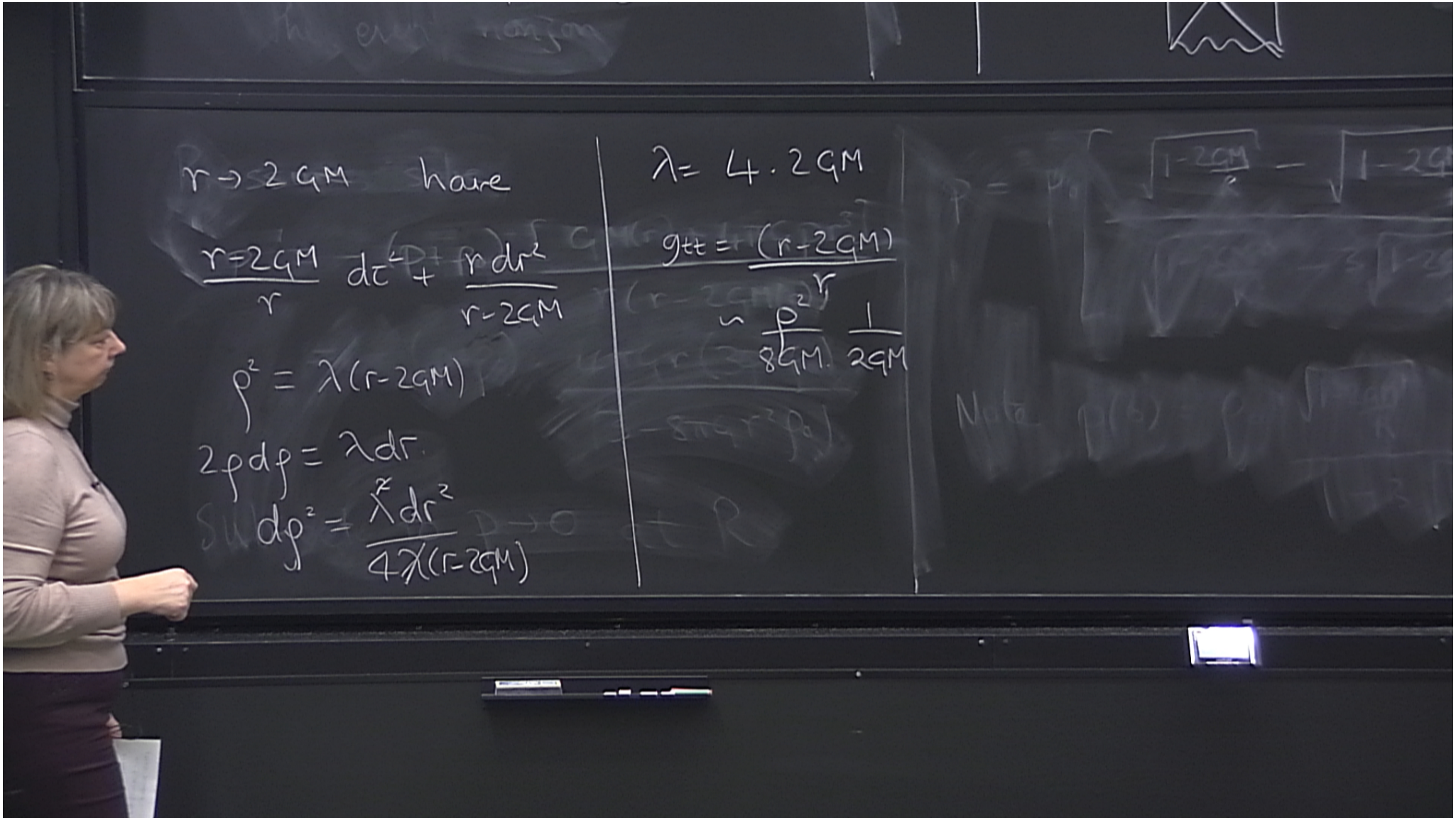
Euclidean Schwarzschild

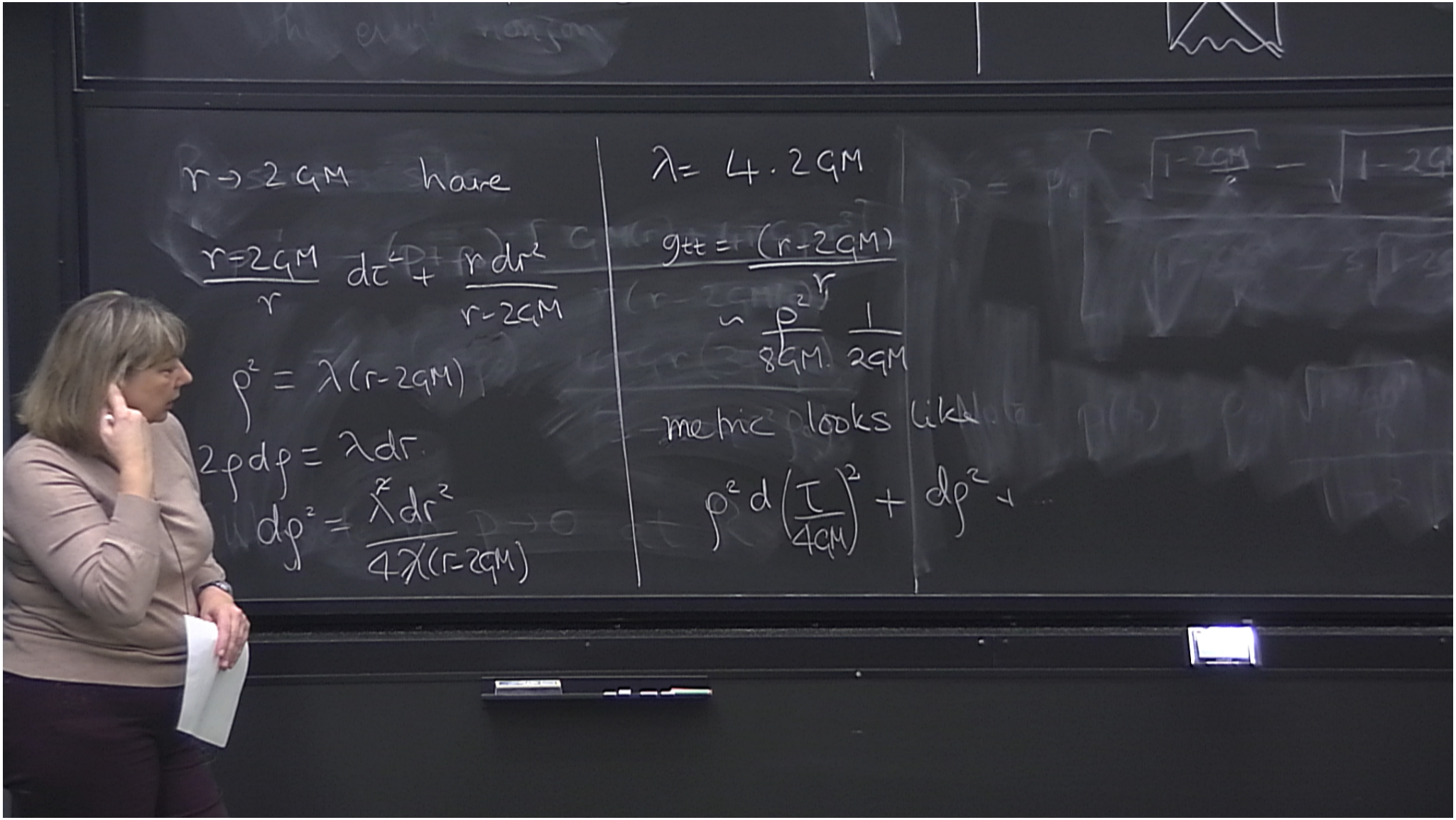
$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)} + r^2 d\Omega_{n-1}^2$$

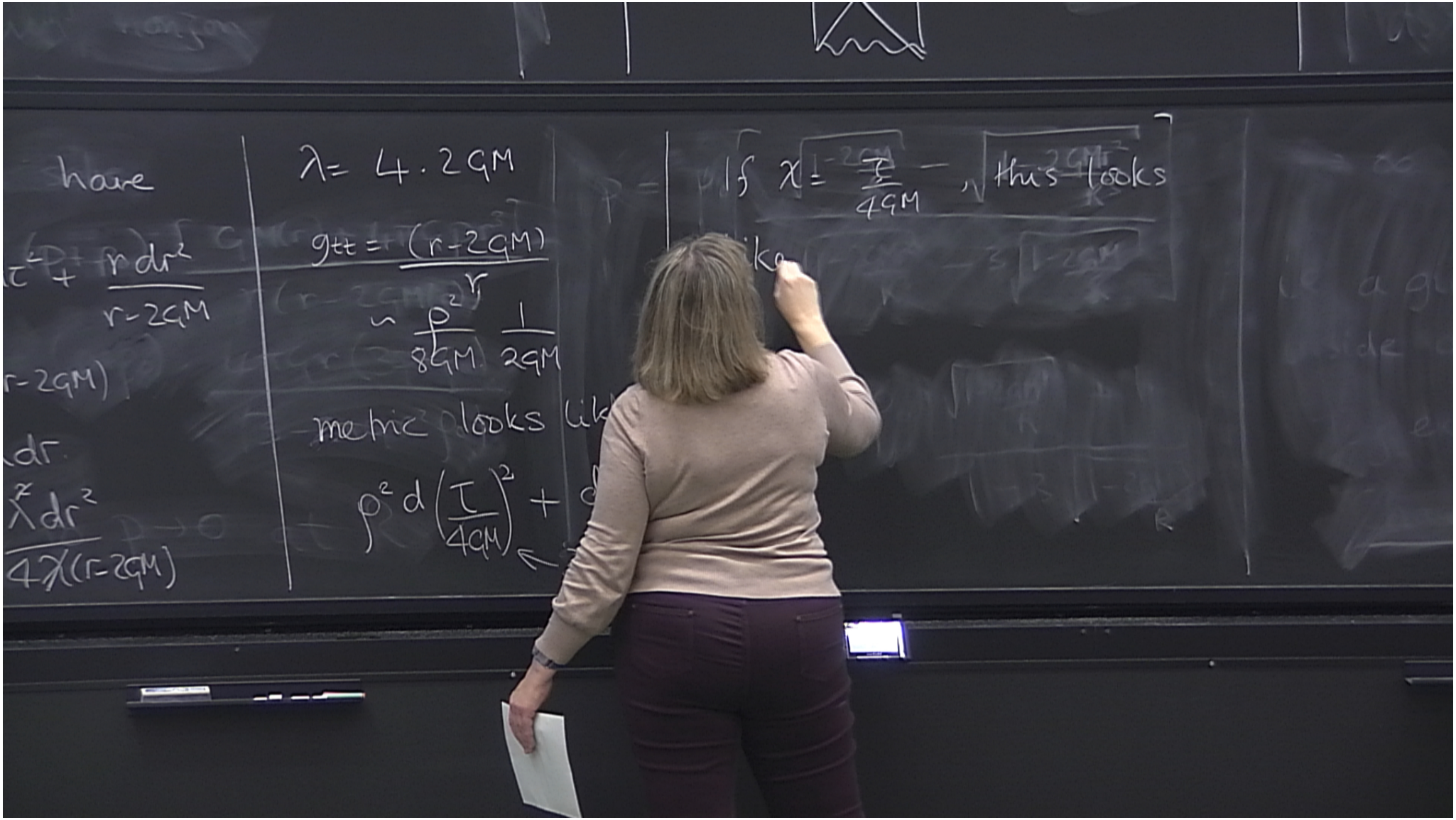
$r \to 2GM$? Boundary to manifold

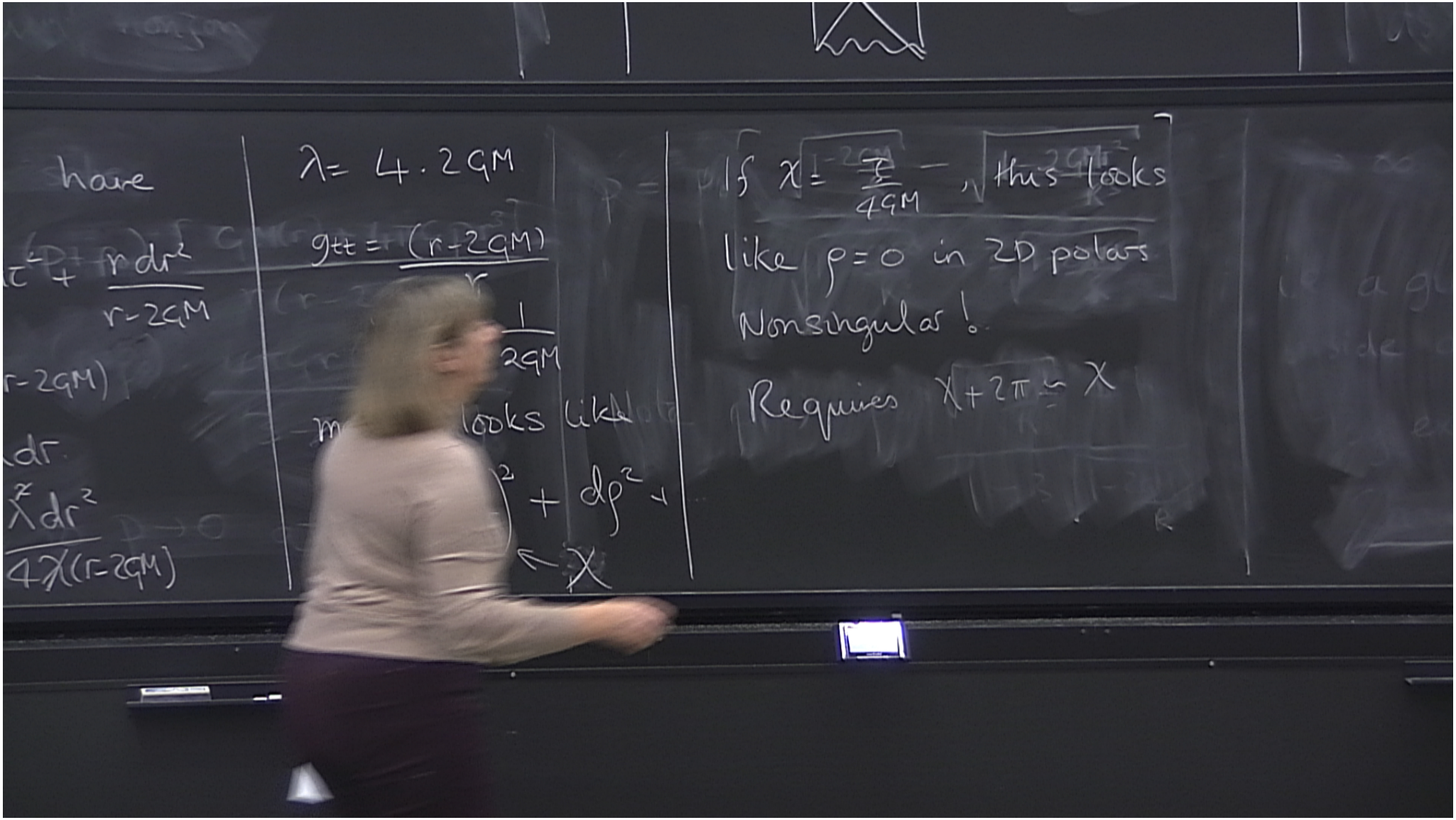
Require nonsingularity

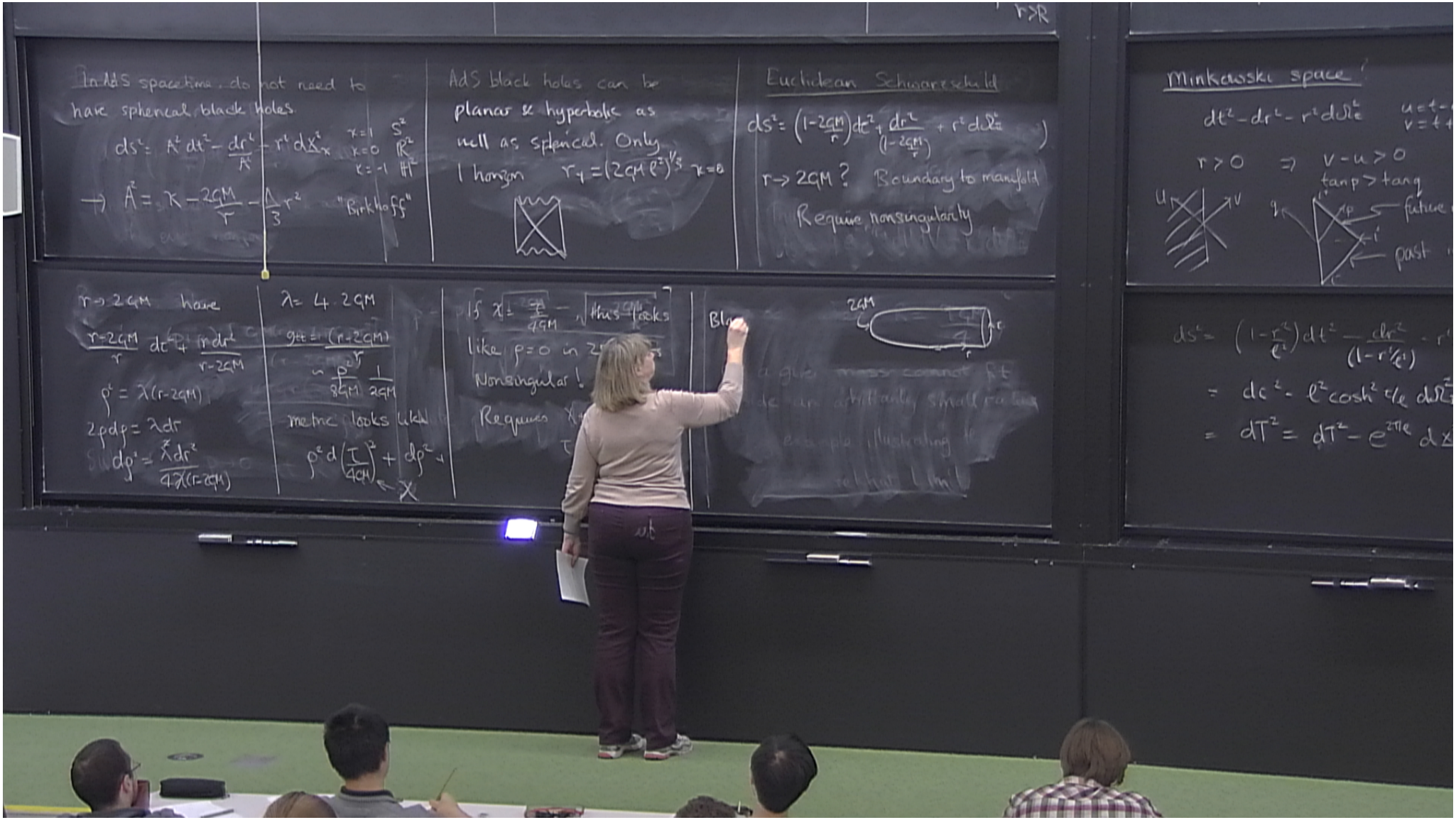

r → 2GM hare
r=2GM
dr
Note

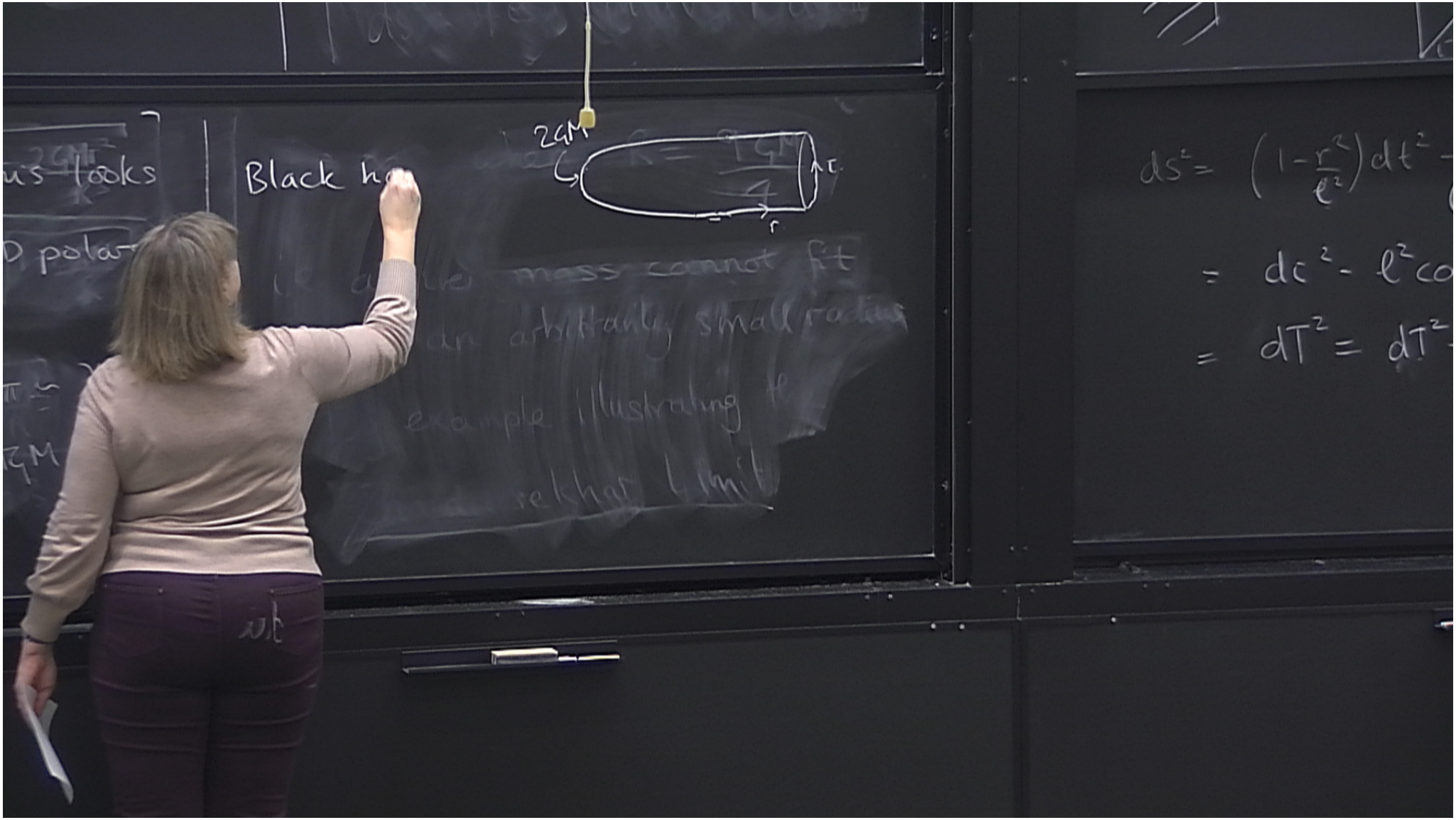

r → 2GM horizon
$\frac{r-2GM}{r}dt^2 + \frac{r\,dr^2}{r-2GM}$
$\rho^2 = \lambda(r-2GM)$
$2\rho\,d\rho = \lambda\,dr$
$d\rho^2 = \frac{\lambda^2 dr^2}{4\lambda(r-2GM)}$
$\lambda = 4\cdot 2GM$
$g_{tt} = \frac{(r-2GM)}{r}$
$\sim \frac{\rho^2}{8GM}\,\frac{r}{2GM}$

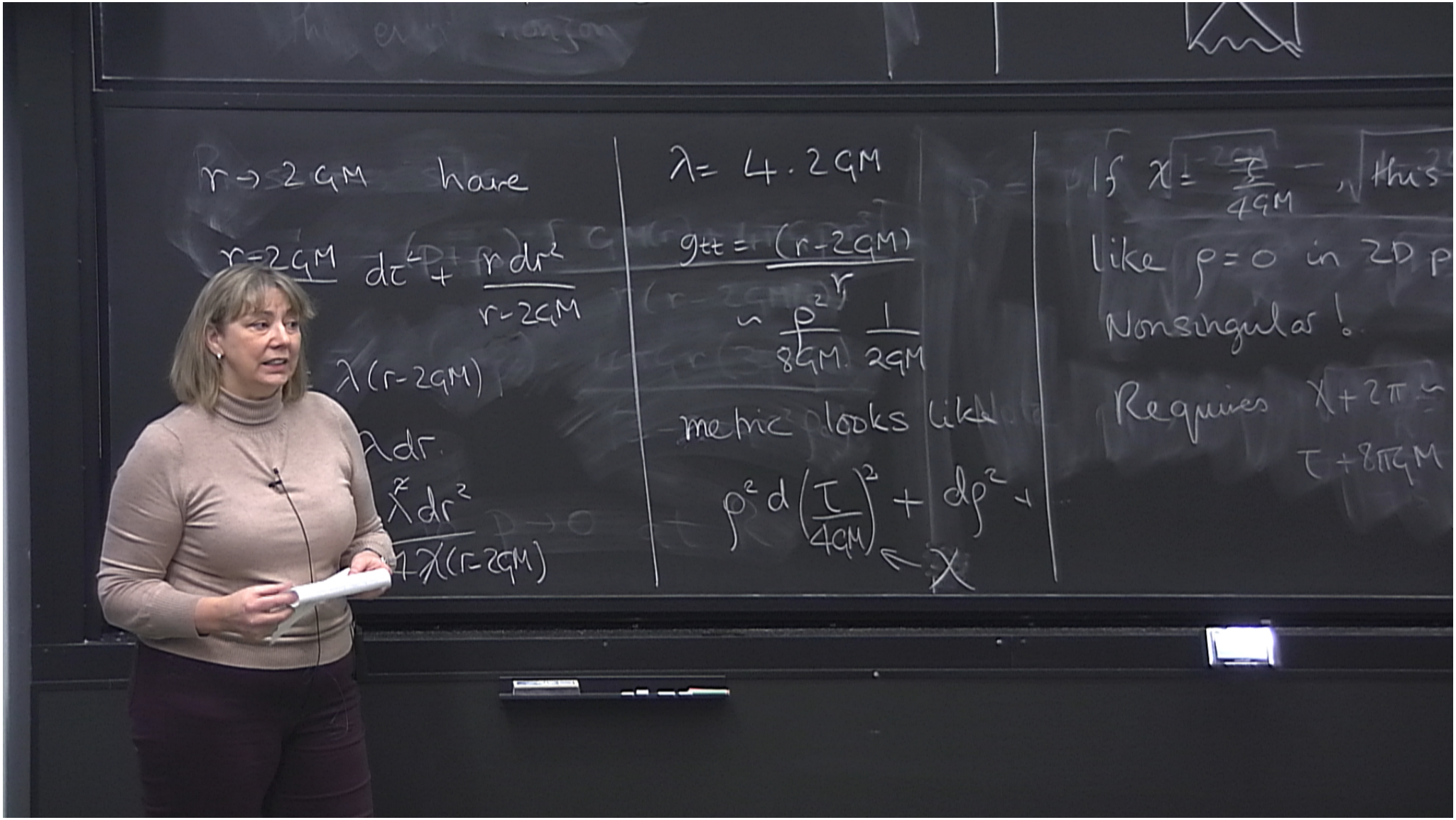

$r \to 2GM$ horizon
$\frac{r-2GM}{r} dt^2 + \frac{r\,dr^2}{r-2GM}$
$\rho^2 = \lambda(r-2GM)$
$2\rho d\rho = \lambda dr$
$d\rho^2 = \frac{\lambda^2 dr^2}{4\lambda(r-2GM)}$
$\lambda = 4 \cdot 2GM$
$g_{tt} = \frac{(r-2GM)}{r}$
$\sim \frac{\rho^2}{8GM} \frac{1}{2GM}$
metric looks like
$\rho^2 d\left(\frac{T}{4GM}\right)^2 + d\rho^2$

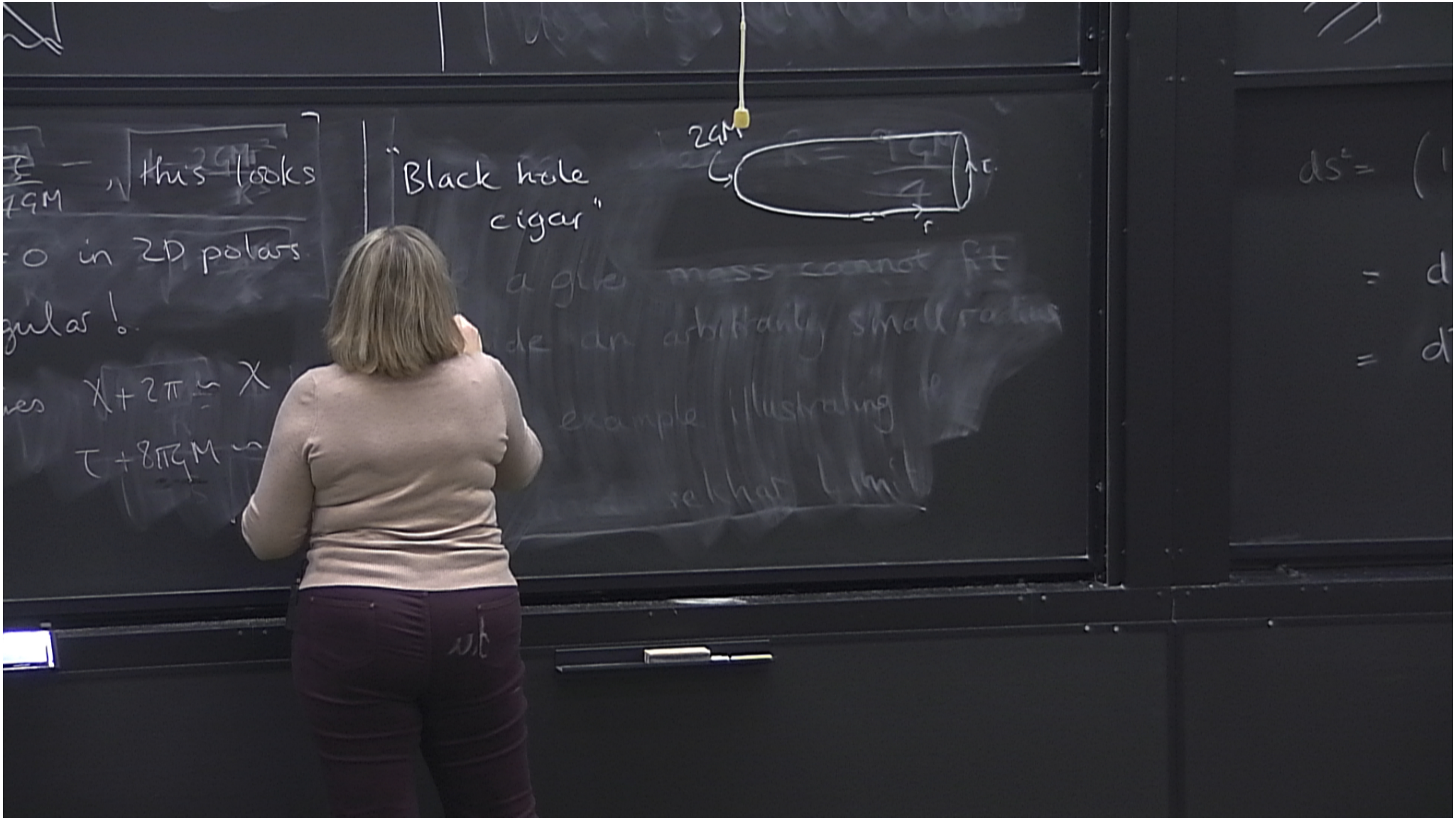

$\lambda = 4 \cdot 2GM$
$g_{tt} = \frac{(r-2GM)}{r}$
$\sim \frac{\rho^2}{8GM} \frac{r}{2GM}$
metric looks
$\rho^2 d\left(\frac{I}{4GM}\right)^2 +$
$\frac{r\,dr^2}{r-2GM}$
$\frac{\chi^2 dr^2}{4\chi(r-2GM)}$
if $\chi = \frac{I}{4GM}$
thus looks

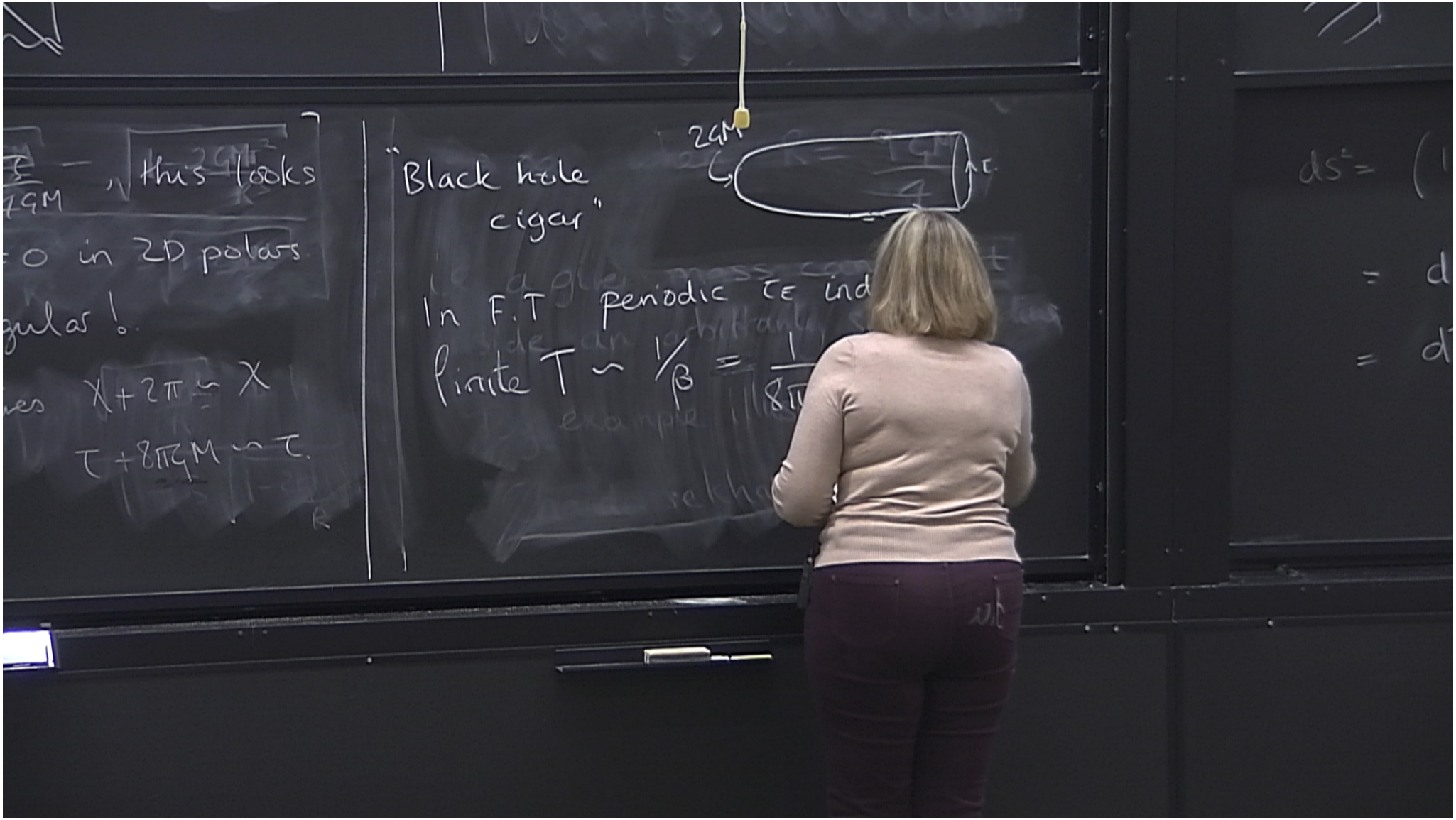

hare
$\lambda = 4 \cdot 2GM$
$g_{tt} = \frac{(r-2GM)}{r}$
looks like
$d\rho^2 +$
χ
If $\chi = \frac{r}{4GM}$, this looks
like $\rho = 0$ in 2D polars
Nonsingular!
Requires $\chi + 2\pi \sim \chi$

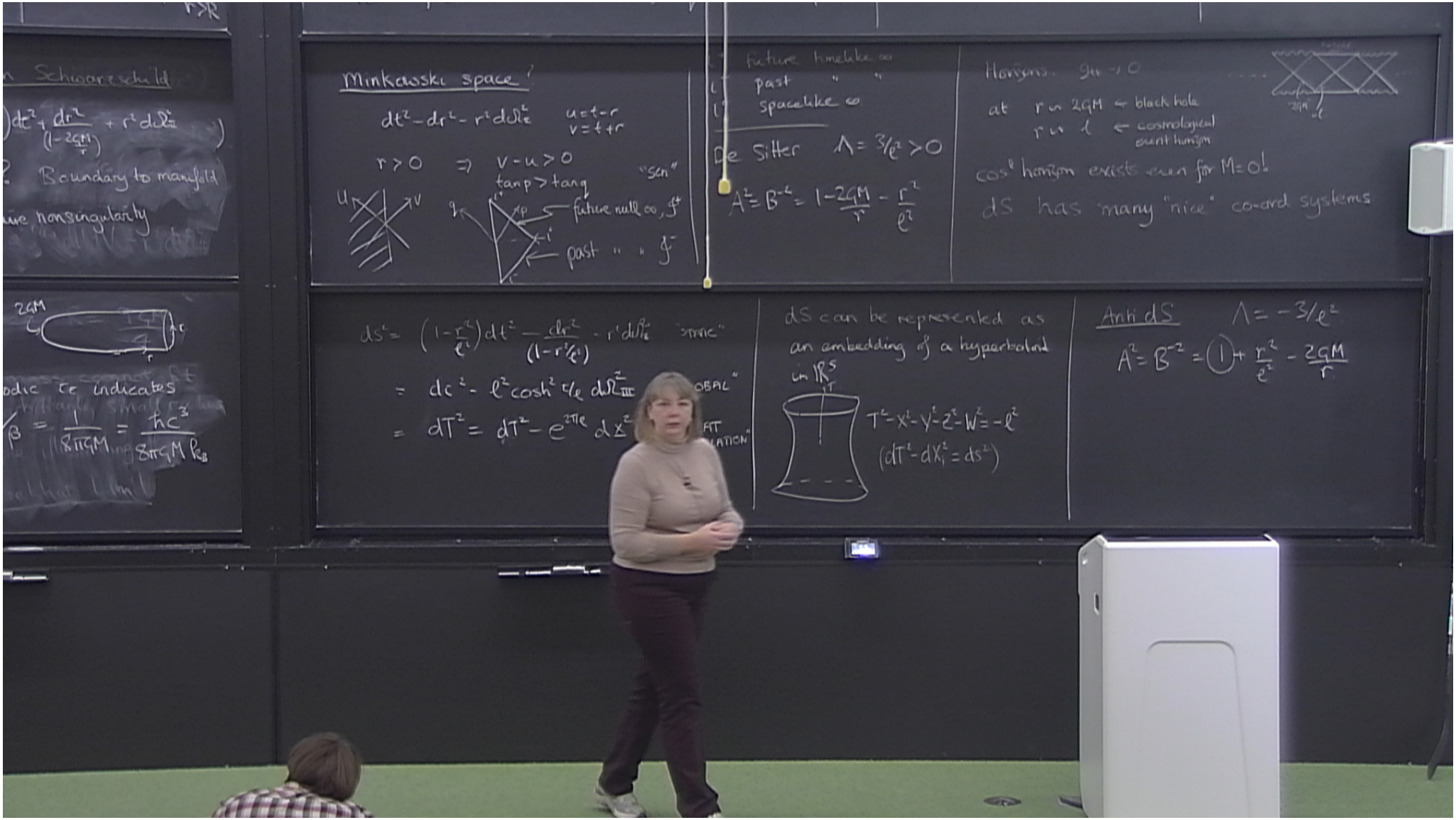

In AdS spacetime, do not need to have spherical black holes.
AdS black holes can be planar & hyperbolic as well as spherical. Only
Euclidean Schwarzschild
Boundary to manifold
Require nonsingularity
Minkowski space
future
past
This looks like ρ=0 in
Nonsingular!
metric looks like


Black h
2GM
r
r
$ds^2 = \left(1 - \frac{r^2}{\ell^2}\right) dt^2$
$= dc^2 - \ell^2 co$
$= dT^2 = dT^2$


$r \to 2GM$ have
$\frac{r-2GM}{r-2GM} dt^2 + \frac{r\,dr^2}{r-2GM}$
$\lambda(r-2GM)$
$\lambda = 4 \cdot 2GM$
$g_{tt} = \frac{(r-2GM)}{r}$
$\sim \frac{\rho^2}{8GM} \frac{1}{2GM}$
metric looks like
$\rho^2 d\left(\frac{\tau}{4GM}\right)^2 + d\rho^2 +$
$\chi = \frac{\tau}{4GM}$
like $\rho = 0$ in 2D p
Nonsingular!
Requires $\chi + 2\pi \sim$
$\tau + 8\pi GM$


this looks
in 2D polars
gular!
"Black hole
cigar"
2GM
r
a glo
mass cannot fit
arbitrarily small ra
example illustrating
relativ Limit
$ds^2 =$
$=$
$=$


this looks
in 2D polars
gular!
$\chi + 2\pi \sim \chi$
$\tau + 8\pi GM \sim \tau$.
"Black hole cigar"
2GM
In F.T periodic τE ind
finite $T \sim 1/\beta = \frac{1}{8\pi}$
$ds^2 = ($
$= d$
$= d$


Schwarzschild
Boundary to manifold
nonsingularity
Minkowski space
future timelike ∞
past " "
spacelike ∞
de Sitter Λ = 3/ℓ² > 0
Horizons. g_tt → 0
at r = 2GM ← black hole
r = ℓ ← cosmological event horizon
cosmological horizon exists even for M = 0!
dS has many "nice" co-ord systems
"static"
future null ∞, ℐ⁺
past " " ℐ⁻
dS can be represented as an embedding of a hyperboloid in ℝ^{4,1}
T² − X² − Y² − Z² − W² = −ℓ²
(dT² − dX_i² = ds²)
Anti dS Λ = −3/ℓ²
A² = B^{-2} = 1 + r²/ℓ² − 2GM/r