Title: Gravitational Physics-2

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Abstract:

# Lecture 2 : Differentiation on Manifolds

(1) Forms & exterior derivative

Recall basic defn of derivative.

$$\frac{df}{dt} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

On a manifold might expect

$$t \to P$$

& so $t+\delta t \to P+\delta P$

What is $\delta P$?

On

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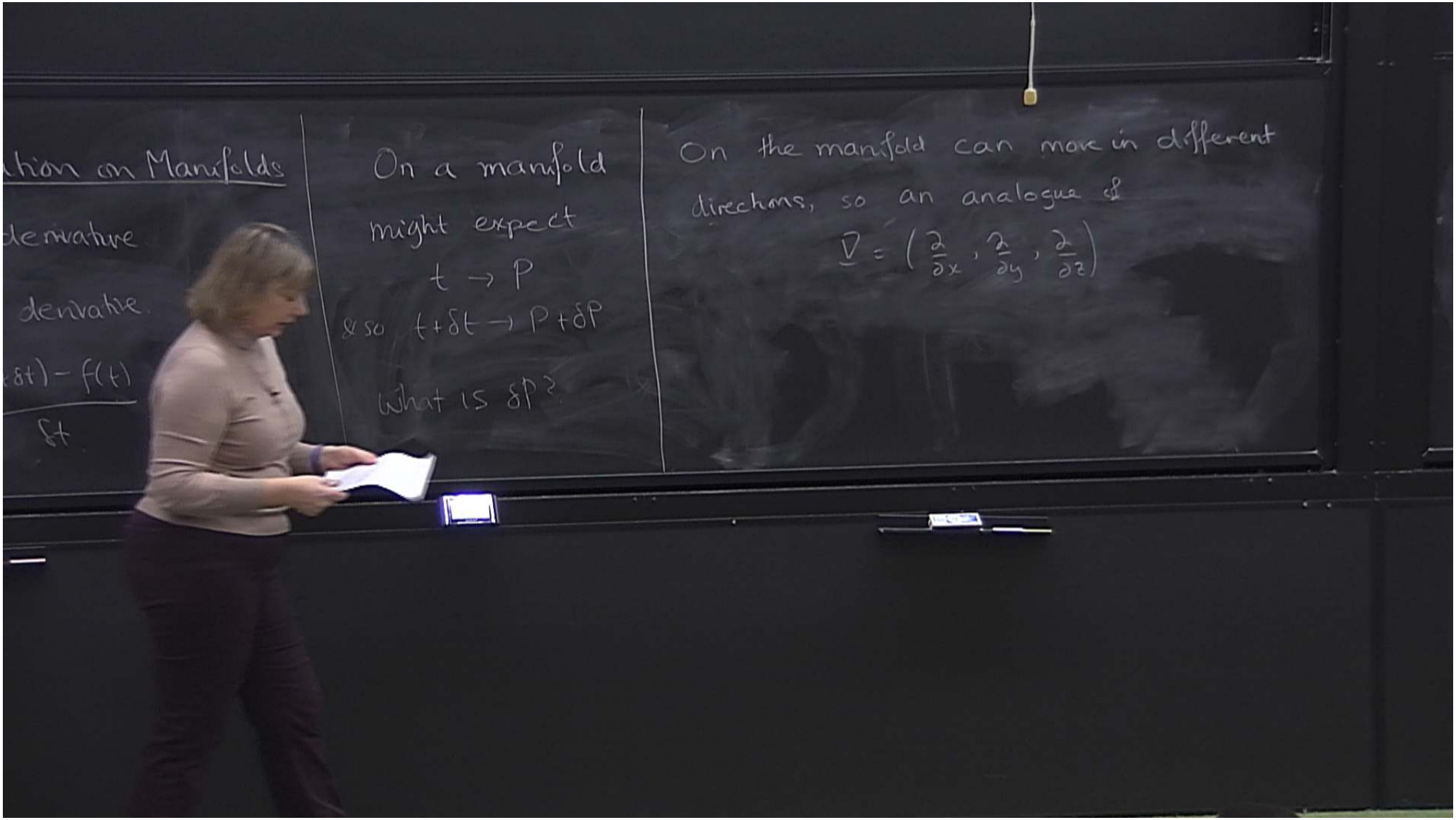
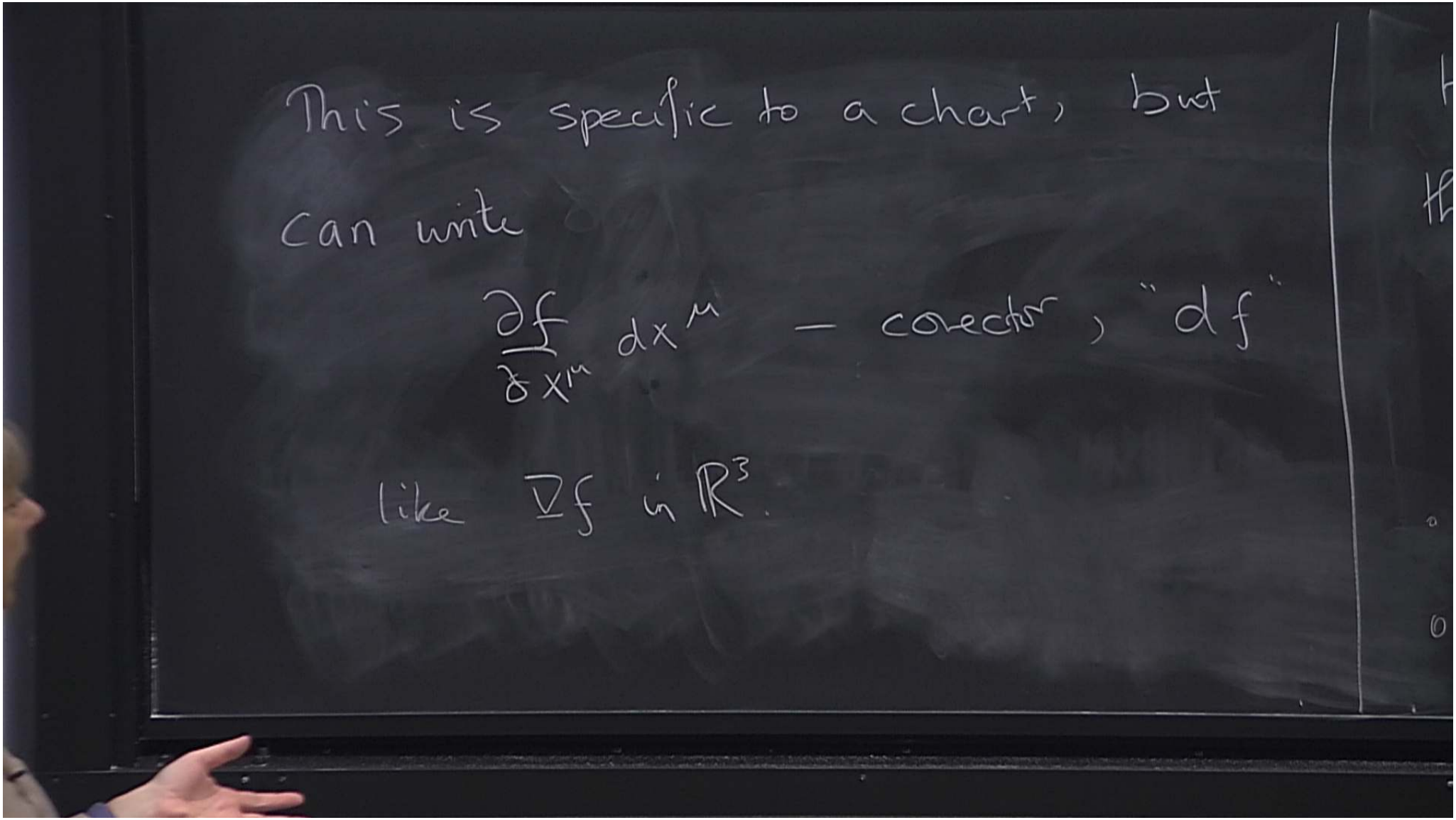

tion on Manifolds
derivative
derivative
δt) − f(t)
δt
On a manifold might expect
t → P
& so t+δt → P+δP
What is δP?
On the manifold can move in different directions, so an analogue of
∇ = (∂/∂x, ∂/∂y, ∂/∂z)


This is specific to a chart, but
can write
$\frac{\partial f}{\partial x^\mu} dx^\mu$ — covector, "df"
like $\nabla f$ in $\mathbb{R}^3$.

Have succeeded for $C^\infty(M)$, but

the defn cannot be extended beyond $p_0$

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial f}{\partial x^{\nu}}\right) \to \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \frac{\partial^2 f}{\partial x'^{\mu'} \partial x'^{\nu'}} + \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x'^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial f}{\partial x'^{\nu'}}$$

- $\delta p \hookrightarrow \delta x^{\mu}$ is <u>not</u> geometric cut
- $T^*_{p_0}(M) \neq T^*_{p_0+\delta p}(M)$

but

"df"

Have succeeded for $C^\infty(M)$, but

the defn cannot be extended beyond fns

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial f}{\partial x^\nu}\right) \to \frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial^2 f}{\partial x'^{\mu'}\partial x'^{\nu'}} + \frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial^2 x'^{\nu'}}{\partial x^\mu \partial x^\nu}\frac{\partial f}{\partial x'^{\nu'}}$$

- $\delta p \leftrightarrow \delta x^\mu$ is <u>not</u> geometric invt
- $T_p^*(M) \neq T^*_{p+\delta p}(M)$ – cannot subtract (co) vectors from different spaces!

Have succeeded for $C^\infty(M)$, but

the defn cannot be extended beyond fns

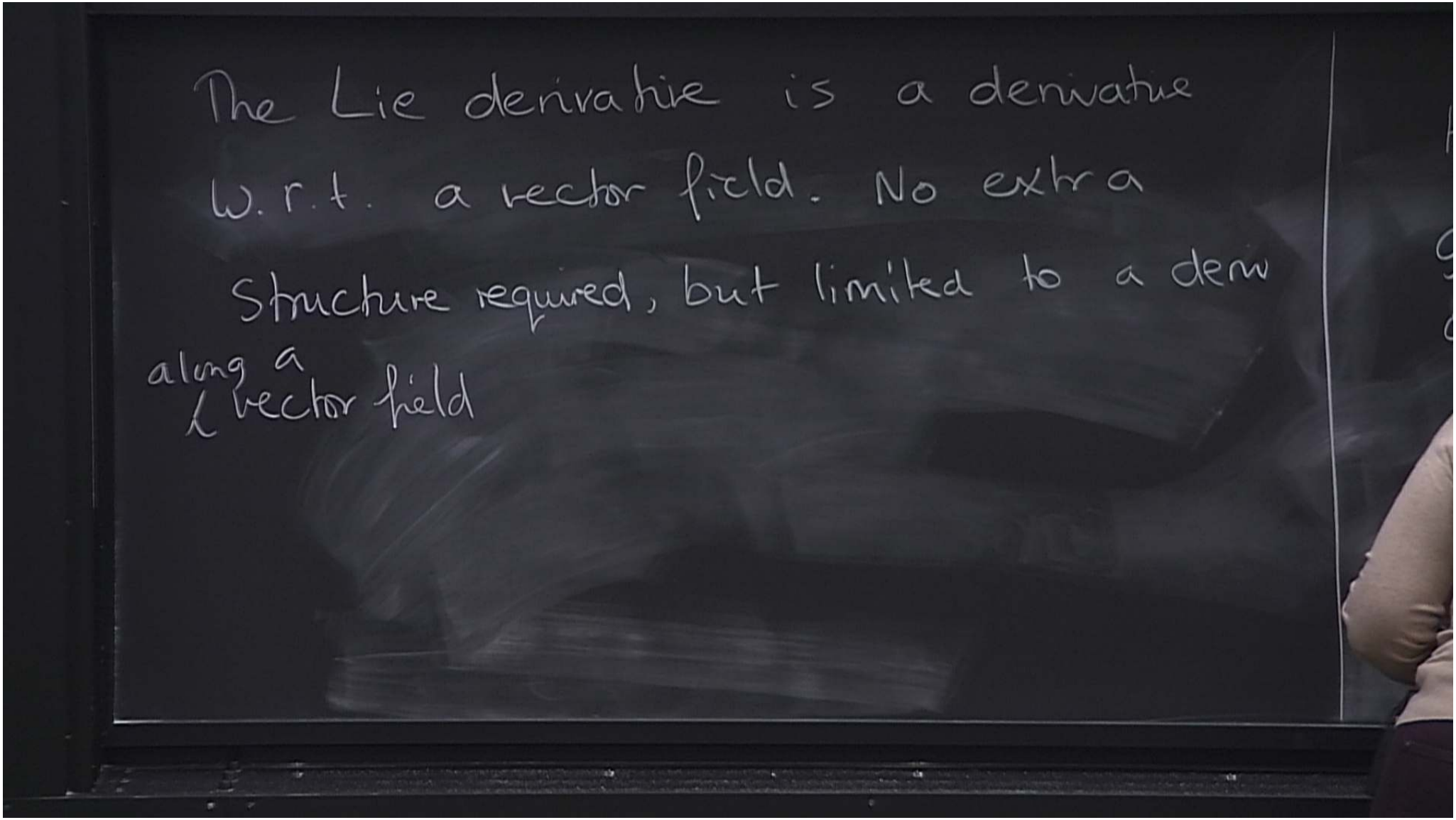
$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial f}{\partial x^\nu}\right) \to \frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial^2 f}{\partial x'^{\mu'}\partial x'^{\nu'}} + \boxed{\frac{\partial x'^{\mu'}}{\partial x^\mu}\frac{\partial^2 x'^{\nu'}}{\partial x^\mu \partial x^\nu}\frac{\partial f}{\partial x'^{\nu'}}}$$

- $\delta p \leftrightarrow \delta x^\mu$ is not geometric inv't
- $T_p^*(M) \neq T^*_{p+\delta p}(M)$ – cannot subtract (co)vectors from different spaces!

3 possible get-outs

The connection provides a link between $T_p(M)$ & $T_{p+\delta p}(M)$

The exterior derivative eliminates unwanted $\partial^2 x^\mu$ term by taking antisymmetric derivatives

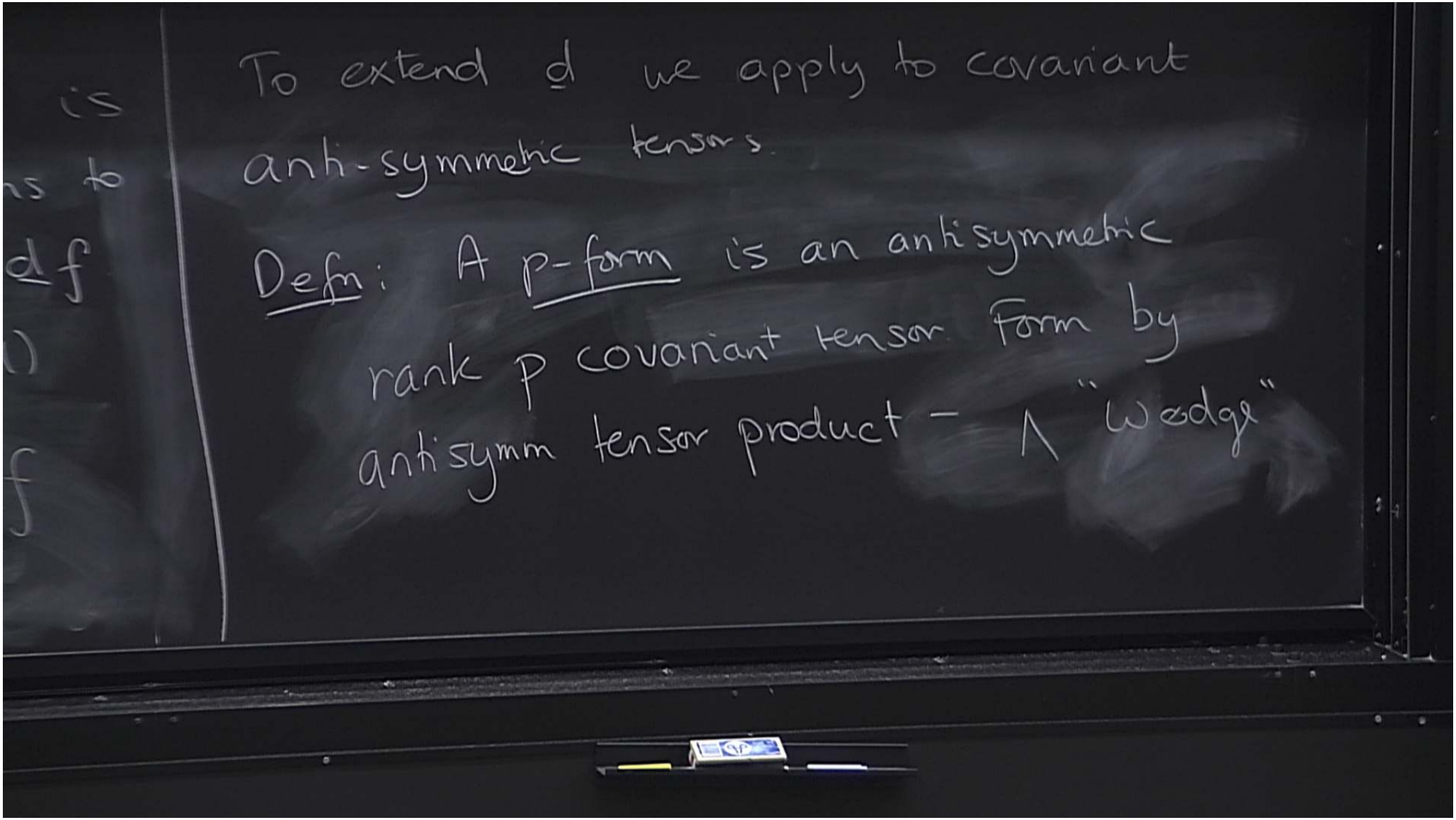

The Lie derivative is a derivative
w.r.t. a vector field. No extra
structure required, but limited to a dens
along a
vector field

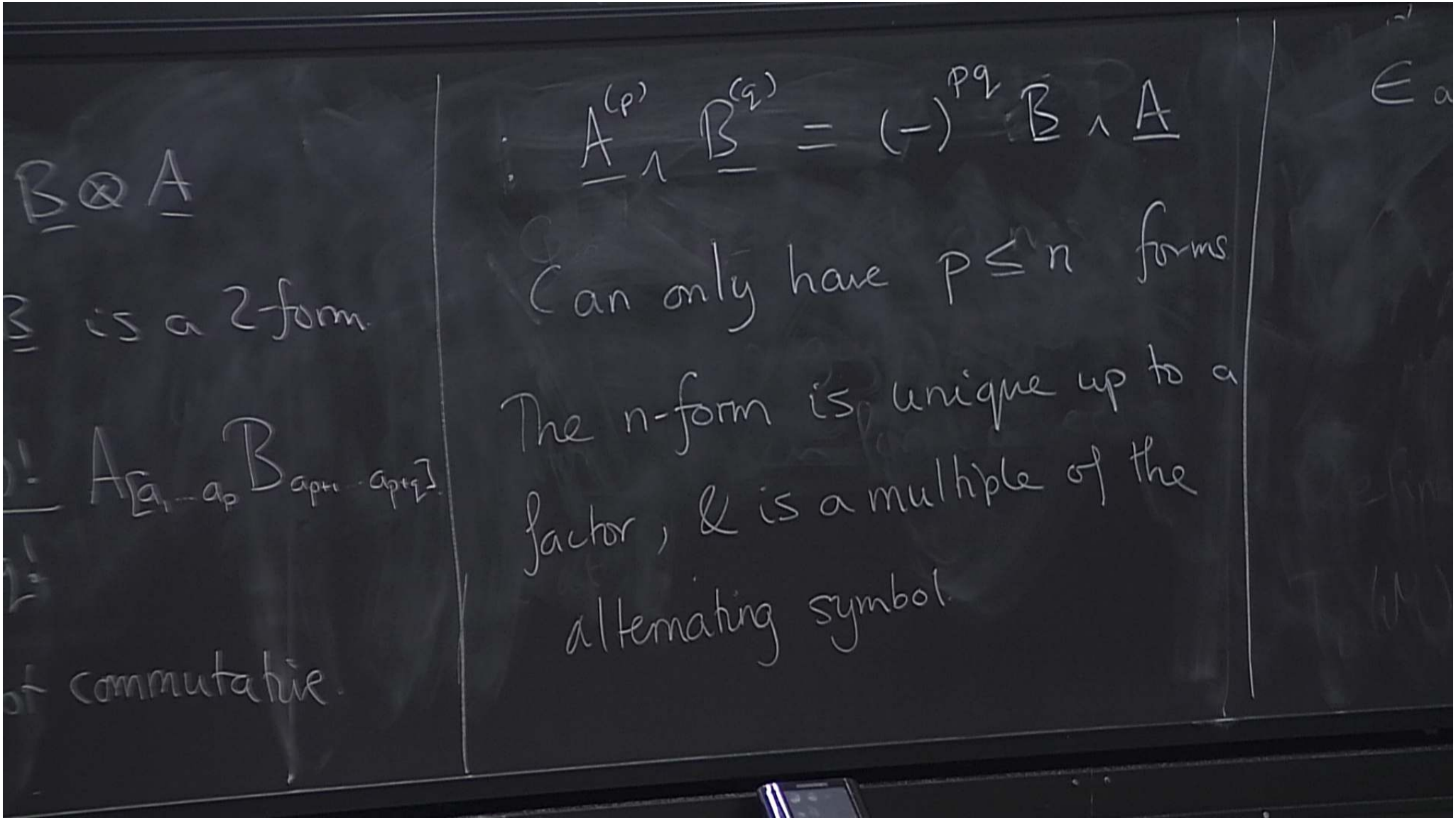
Have seen that df is geometric, it maps fns to covectors or 1-forms. $\underline{d}f$

$$\underline{d} \quad C^\infty(M) \to C^\infty(M)$$

- st. at $p$ $\langle \underline{d}f | \underline{T} \rangle = \underline{T}f$

$\forall T \in T_p(M)$


To extend d we apply to covariant
anti-symmetric tensors.
Defn: A p-form is an antisymmetric
rank p covariant tensor Form by
antisymm tensor product - ∧ "wedge"



$\underline{B} \otimes \underline{A}$

$\underline{B}$ is a 2-form.

$\underline{\ }\, A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}$

$q!$

not commutative.

$\therefore \underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq}\, \underline{B} \wedge \underline{A}$

Can only have $p \leq n$ forms

The n-form is unique up to a factor, & is a multiple of the alternating symbol.

$\in a$

If we have a metric, then can define the $\underline{\varepsilon\text{-tensor}}$

$$\varepsilon_{abcd} = \sqrt{|\det g|}\, \epsilon_{abcd}$$

Under a coord transfm:

$$g_{\mu\nu} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}$$

det

Hence

$\varepsilon_{\mu\nu\lambda}$

can

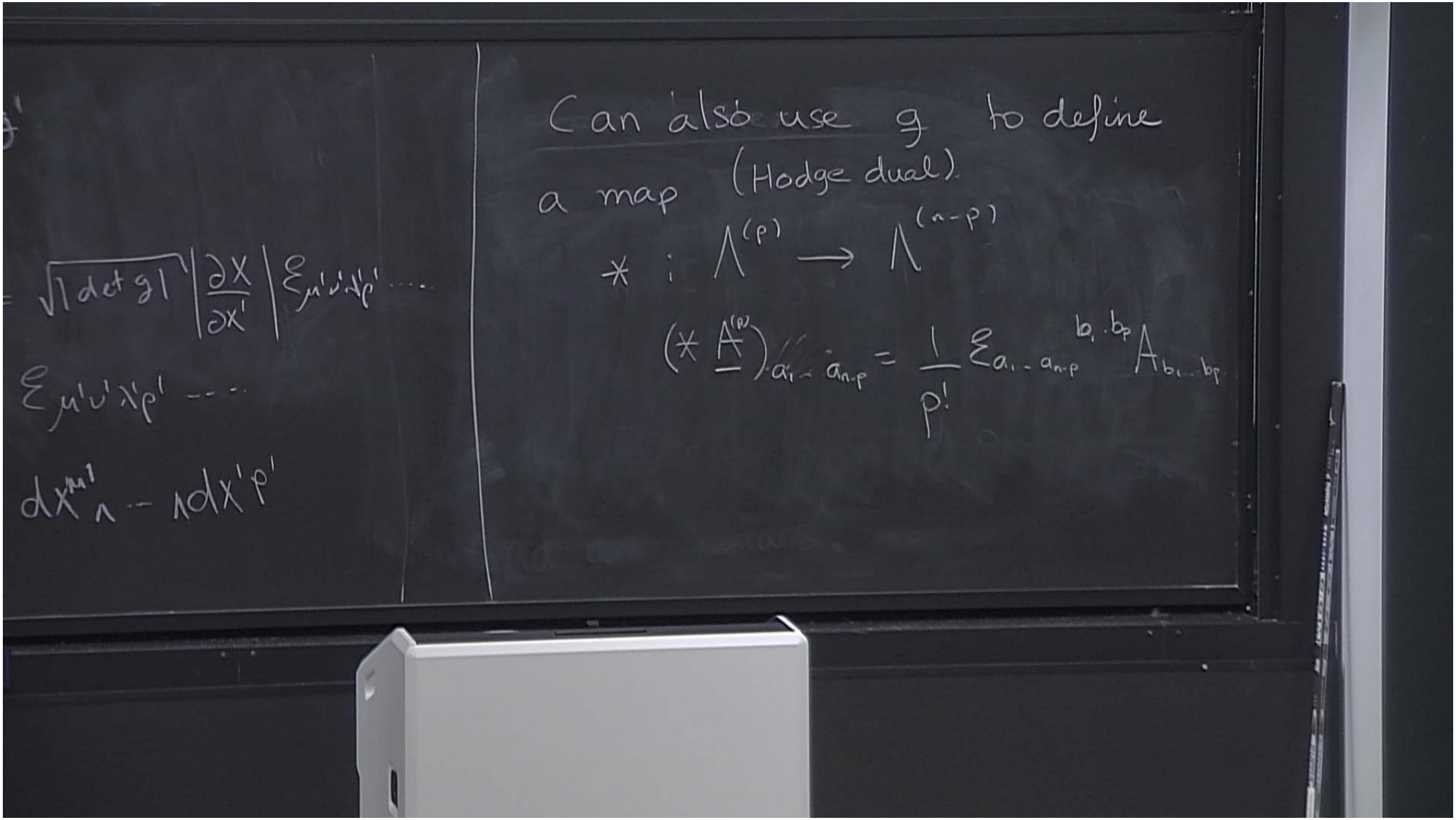
$$\det g = \left|\frac{\partial x'}{\partial x}\right|^2 \det g'$$

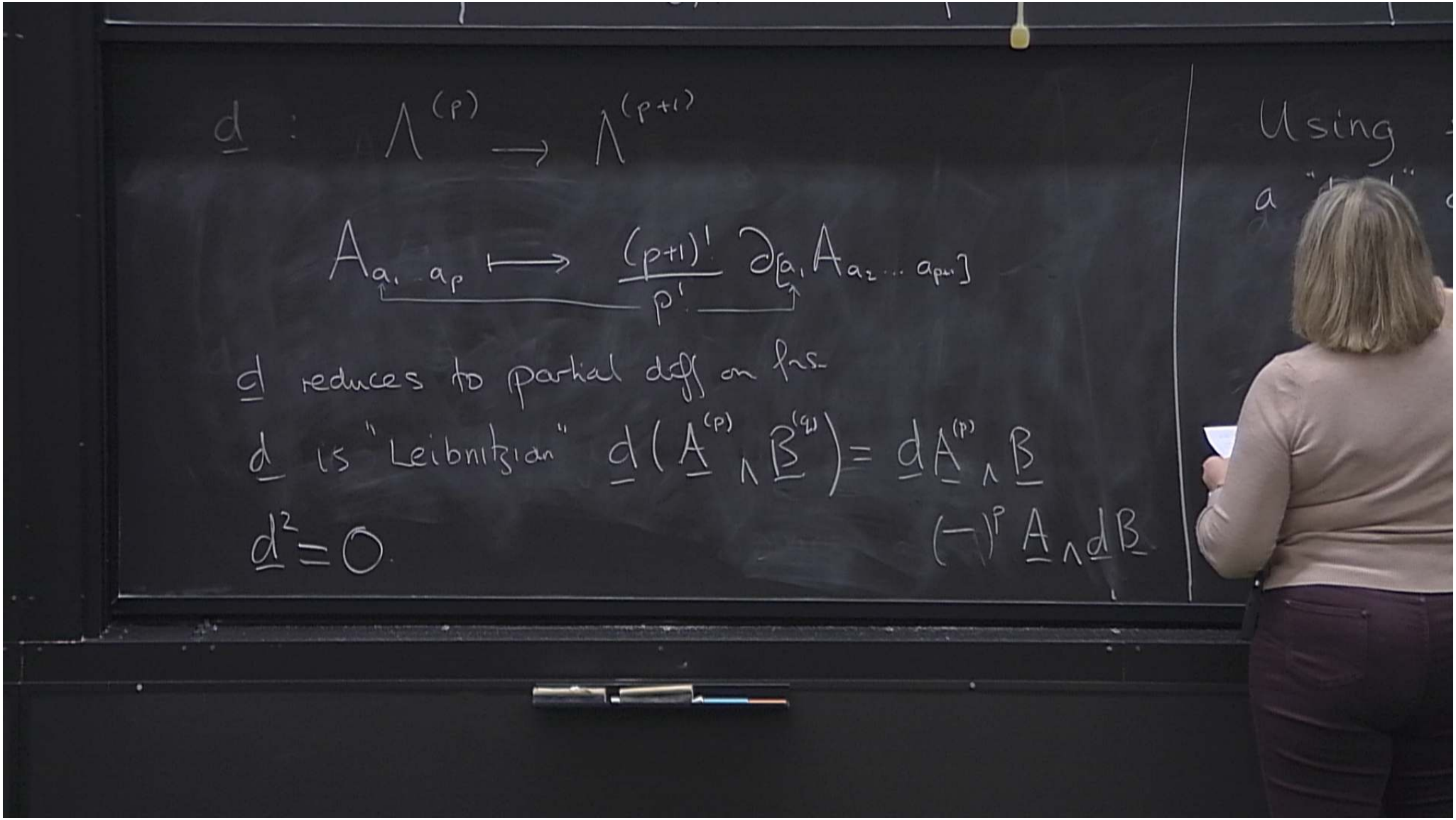
Hence

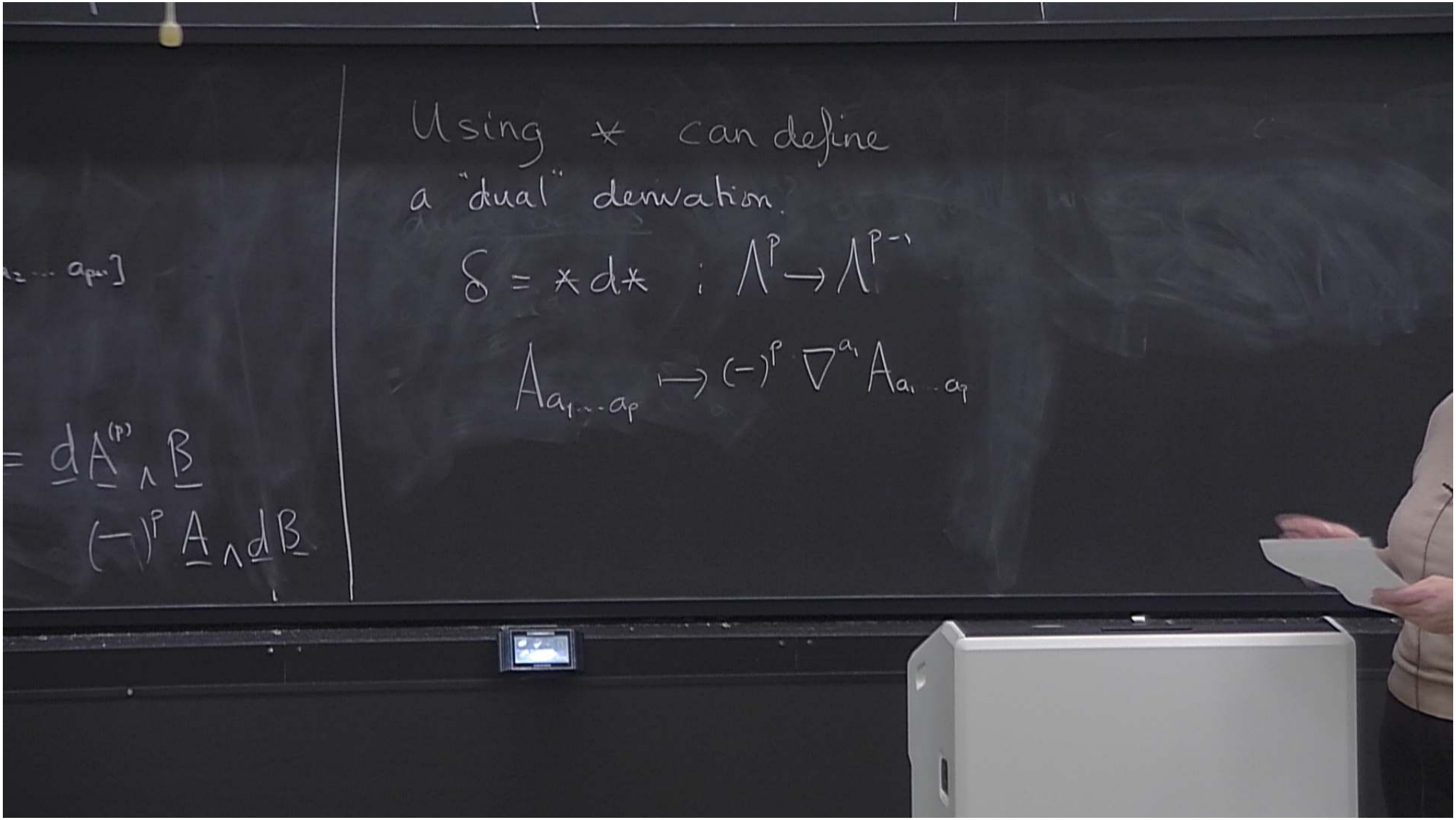
$$\varepsilon_{\mu\nu\lambda\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \sqrt{|\det g|}\left|\frac{\partial x}{\partial x'}\right| \varepsilon_{\mu'\nu'\lambda'\rho'} \cdots$$

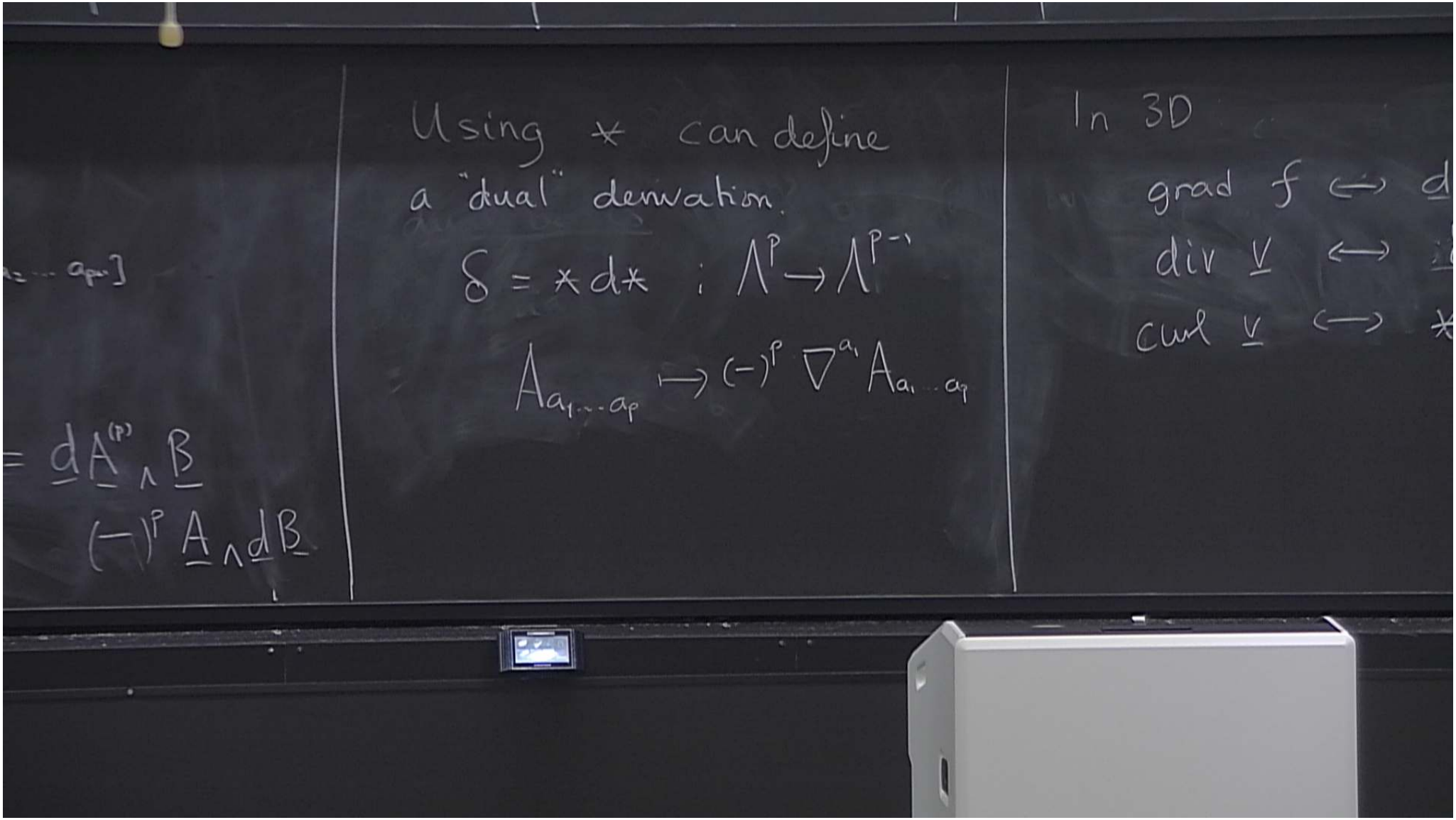
$$= \sqrt{|\det g'|}\, \varepsilon_{\mu'\nu'\lambda'\rho'} \cdots$$

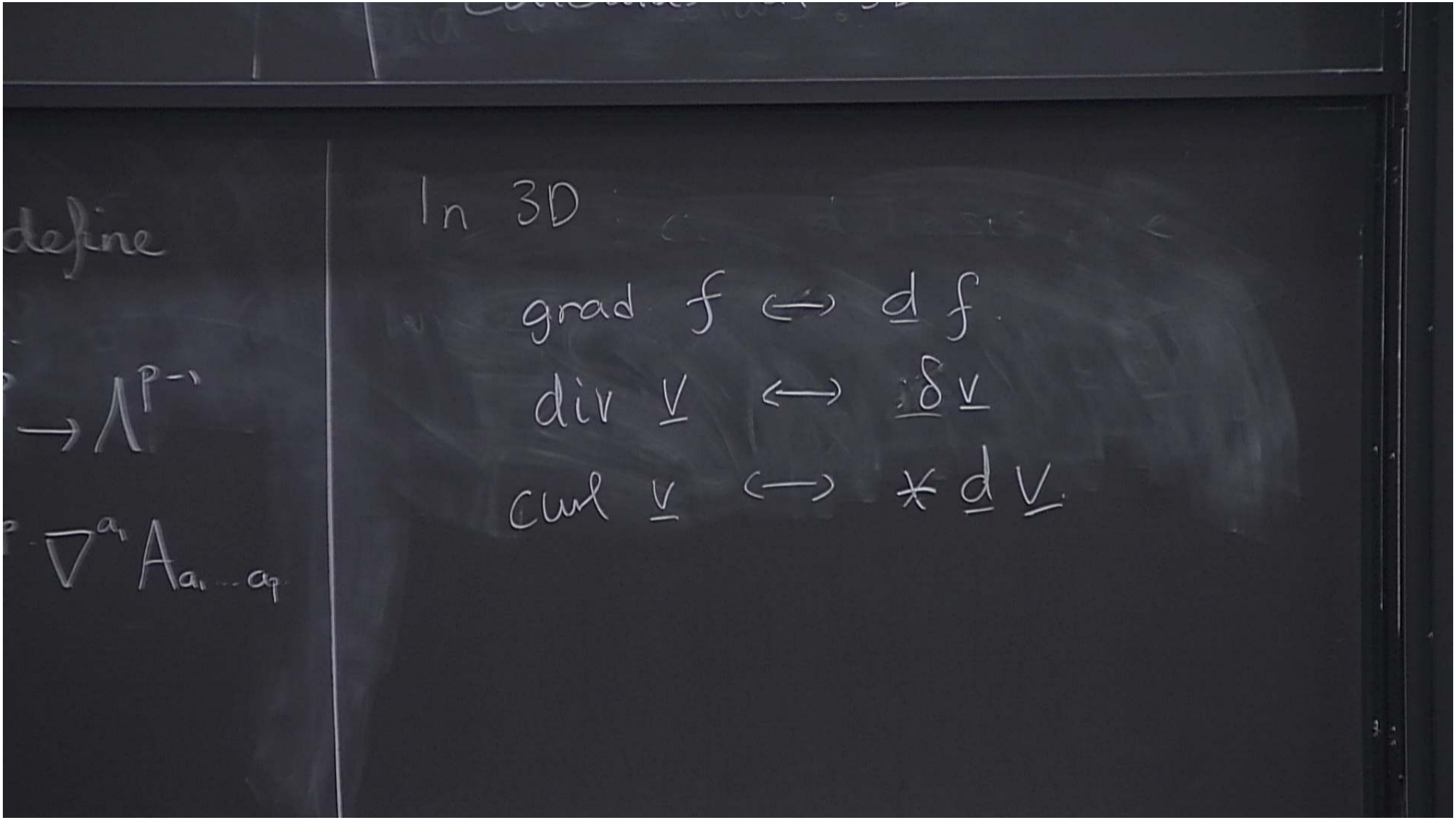
$$= \varepsilon_{\mu'\nu'\lambda'\rho'}\, dx^{\mu'} \wedge \cdots \wedge dx^{\rho'}$$


$\sqrt{|\det g|}\left|\frac{\partial x}{\partial x'}\right| \varepsilon_{\mu'\nu'\lambda'\rho'} \cdots$
$\varepsilon_{\mu'\nu'\lambda'\rho'} \cdots$
$dx^{\mu'} \wedge \cdots \wedge dx^{\rho'}$
Can also use $g$ to define a map (Hodge dual).
$* : \Lambda^{(p)} \longrightarrow \Lambda^{(n-p)}$
$(* \underline{A}^{(p)})_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_{n-p}}{}^{b_1 \cdots b_p} A_{b_1 \cdots b_p}$


$\underline{d} : \Lambda^{(p)} \longrightarrow \Lambda^{(p+1)}$
$A_{a_1 \ldots a_p} \longmapsto \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \ldots a_{p+1}]}$
$\underline{d}$ reduces to partial diff on fns.
$\underline{d}$ is "Leibnitzian" $\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}\underline{A}^{(p)} \wedge \underline{B}$
$(-)^p \underline{A} \wedge \underline{d}\underline{B}$
$\underline{d}^2 = 0$
Using


Using * can define
a "dual" derivation.
$\delta = *d*$ ; $\Lambda^p \to \Lambda^{p-1}$
$A_{a_1 \dots a_p} \mapsto (-)^p \nabla^{a_1} A_{a_1 \dots a_p}$
$= \underline{d}\underline{A}^{(p)} \wedge \underline{B}$
$(-)^p \underline{A} \wedge \underline{d}\underline{B}$


Using * can define a "dual" derivation.
$\delta = *d*$ ; $\Lambda^p \to \Lambda^{p-1}$
$A_{a_1 \dots a_p} \mapsto (-)^p \nabla^{a_1} A_{a_1 \dots a_p}$
In 3D
grad $f$ ⟷ $d$
div $\underline{V}$ ⟷
curl $\underline{V}$ ⟷ *
$= \underline{d}\underline{A}^{(p)} \wedge \underline{B}$
$(-)^p \underline{A} \wedge \underline{d}\underline{B}$


define
$\to \Lambda^{p-1}$
$\nabla^{a_1} A_{a_1 \dots a_p}$
In 3D
grad $f \longleftrightarrow \underline{d} f$
div $\underline{v} \longleftrightarrow \underline{\delta} \underline{v}$
curl $\underline{v} \longleftrightarrow * \underline{d} \underline{v}$

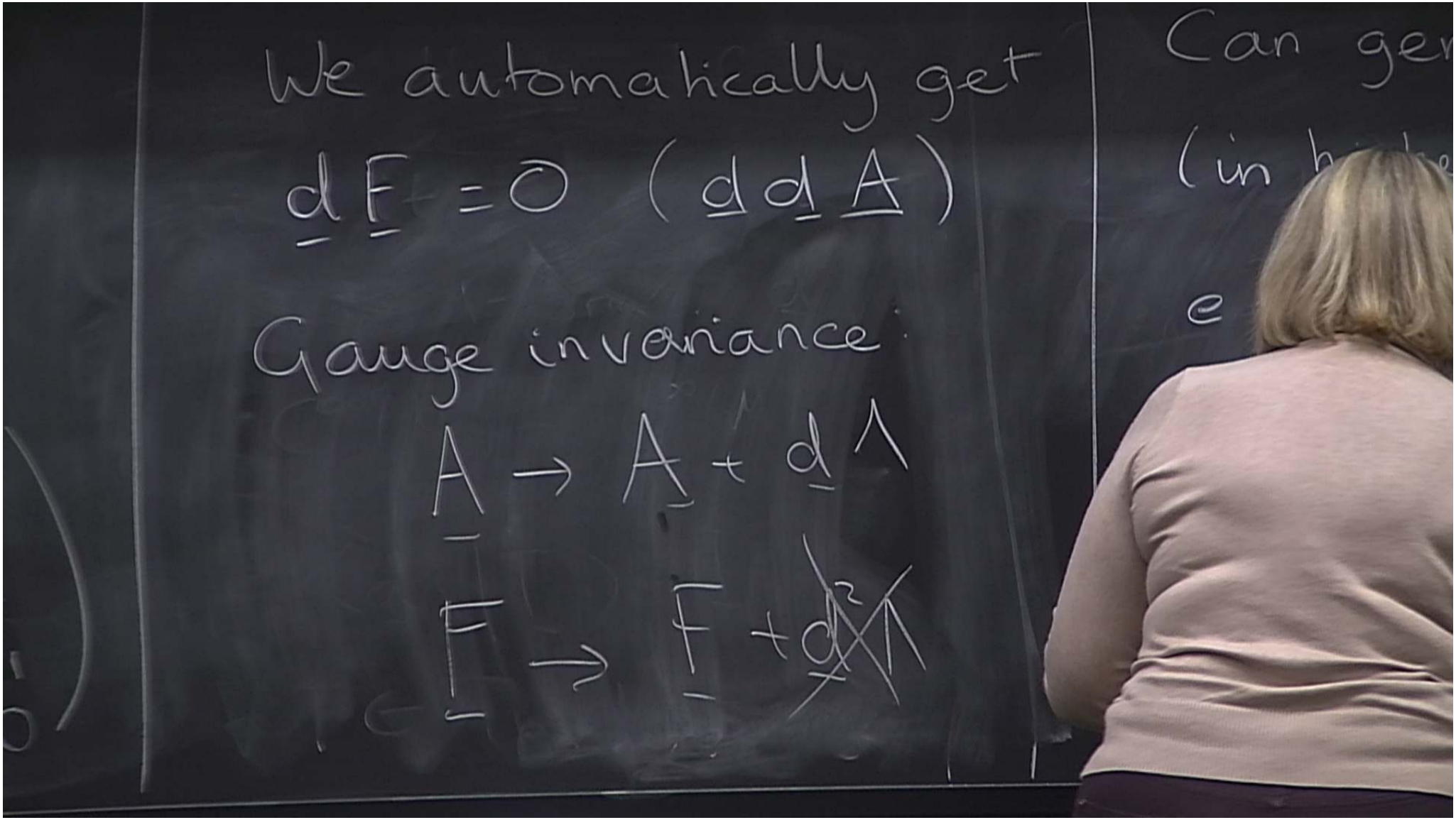
e.g. Electromagnetism

Gauge theory:

$A_\mu = (\phi, -\underline{A})$ 1-form

Field strength $\boxed{\underline{F} = d\underline{A}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$

↑ 1-form.


We automatically get
$\underline{d}\,\underline{F} = 0 \quad (\underline{d}\,\underline{d}\,\underline{A})$
Gauge invariance:
$\underline{A} \to \underline{A} + \underline{d}\,\Lambda$
$\underline{F} \to \underline{F} + \underline{d}^2\Lambda$
Can ge
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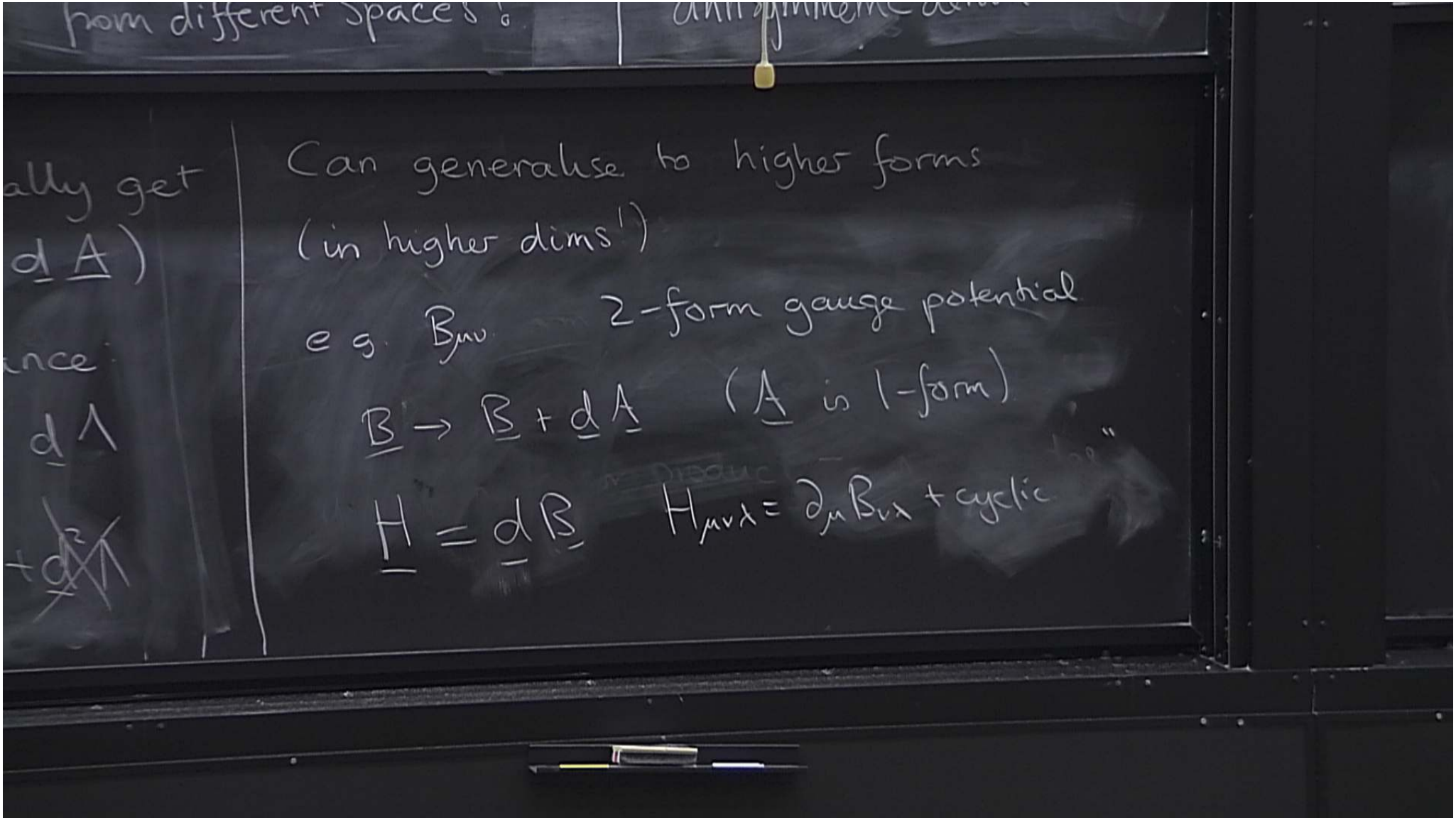
e


from different spaces!
ally get
$\underline{d}\,\underline{A}$)
nce:
$\underline{d}\,\Lambda$
$+\underline{d}^2\Lambda$
Can generalise to higher forms
(in higher dims!)
e.g. $B_{\mu\nu}$ 2-form gauge potential
$\underline{B} \to \underline{B} + \underline{d}\,\underline{\Lambda}$ ($\underline{\Lambda}$ is 1-form)
$\underline{H} = \underline{d}\,\underline{B}$ $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$

$\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} \frac{\partial f}{\partial x^{\nu'}}$

(co) vectors

aces!

3 possible get-outs:

The connection provides a link between $T_p(M)$ & $T_{p+\delta p}(M)$

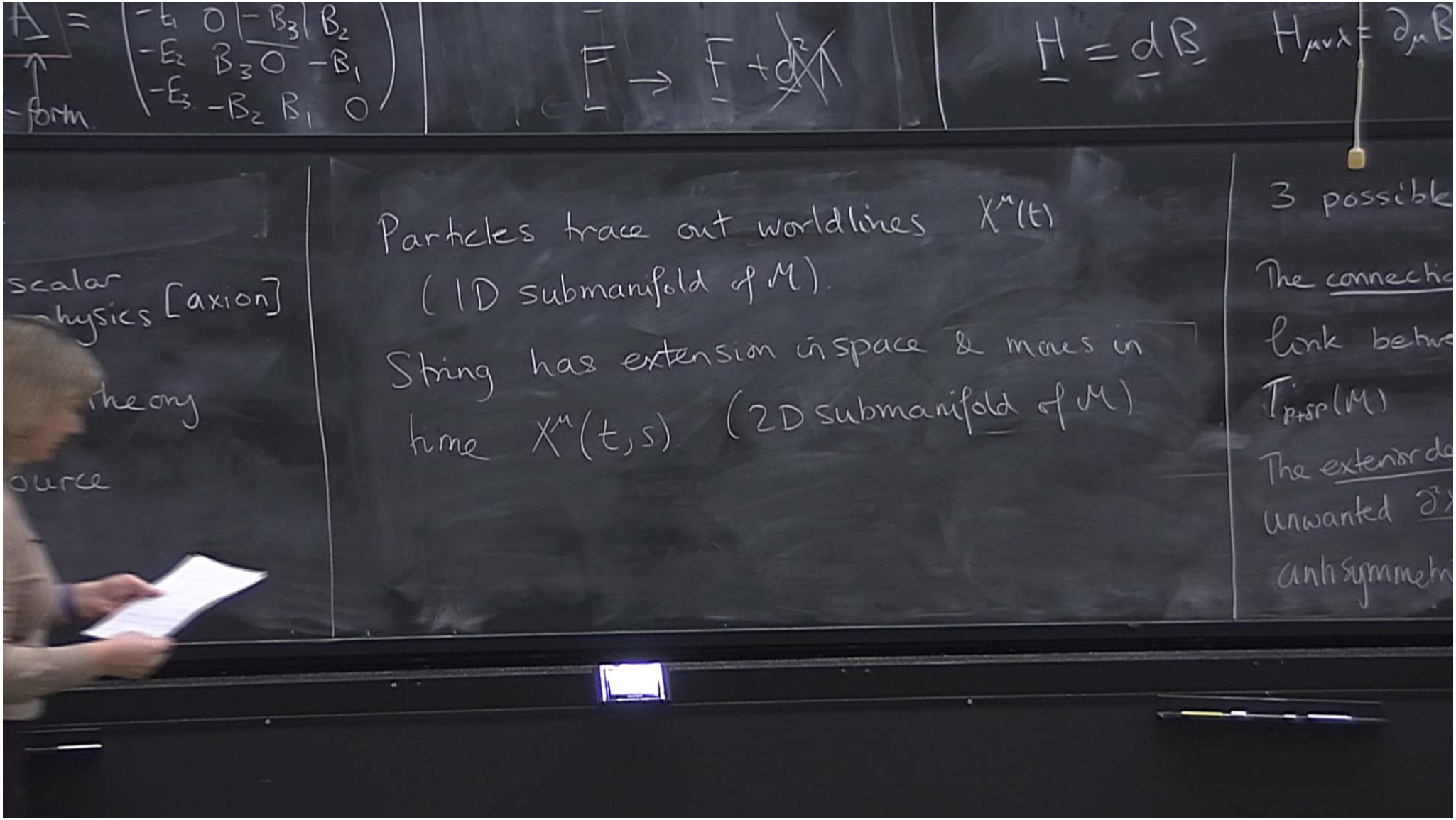
The exterior derivative eliminates unwanted $\partial^2 x^\mu$ term by taking anti-symmetric derivatives

In 4D $\underline{H} \leftrightarrow \underline{d}\varphi$.

2-form physics $\leftrightarrow$ scalar physics [axion]

$B_{\mu\nu}$ comes up in string theory

$B_{\mu\nu}$ couples to a line source


$\underline{A} = \begin{pmatrix} -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$
-form.
$\underline{F} \to \underline{F} + \underline{d}\underline{A}$
$\underline{H} = \underline{d}\underline{B}$
$H_{\mu\nu\lambda} = \partial_\mu B$
scalar
physics [axion]
theory
ource
Particles trace out worldlines $X^\mu(t)$
(1D submanifold of $M$).
String has extension in space & moves in
time $X^\mu(t,s)$ (2D submanifold of $M$)
3 possible
The connectio
link betwe
$T_{P+SP}(M)$
The exterior de
unwanted $\partial^2$
anhsymmetr

$\underline{A} = \begin{pmatrix} -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$

1-form

$\underline{F} \to \underline{F} + d\underline{A}$

$\underline{H} = d\underline{B}$ $\quad H_{\mu\nu\lambda} = \partial_\mu B$

scalar physics [axion]

ng theory

ource

Particles trace out worldlines $X^\mu(t)$
(1D submanifold of $\mathcal{M}$)

String has extension in space & moves in time $X^\mu(t,s)$ (2D submanifold of $\mathcal{M}$)

$A_\mu$ ⟶ point charge $\int_{X^\mu(t)} q \dot{X}^\mu A_\mu$.

$B_{\mu\nu}$

$\begin{pmatrix} \cdot & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}$

$\underline{F} \to \underline{F} + \underline{d}\underline{A}$

$\underline{H} = \underline{d}\underline{B}$ $\quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$

Particles trace out worldlines $X^\mu(t)$
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String has extension in space & moves in time $X^\mu(t,s)$ (2D submanifold of $M$)

$A_\mu \hookrightarrow$ point charge $\int_{}^{X^\mu(t)} q \dot{X}^\mu A_\mu$.

Charged worldsheet?

$B_{\mu\nu} \ldots X^\nu$

$\begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}$

$\underline{F} \to \underline{F} + \cancel{d\underline{A}}$

$\underline{H} = d\underline{B}$ $\quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$

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Charged worldsheet?

$B_{\mu\nu} \dot{X}^\mu X'^\nu$