

Title: Gravitational Physics-2

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Abstract:

Lecture 2 : Differentiation on Manifolds

(i) Forms & exterior derivative

Recall basic defn of derivative.

$$\frac{df}{dt} = \lim_{\delta t \rightarrow 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

On a manifold
might expect

$$t \rightarrow P$$

$$\& \text{ so } t + \delta t \rightarrow P + \delta P$$

What is δP ?

ation on Manifolds

derivative

derivative.

$f(t) - f(t)$

δt

On a manifold
might expect

$$t \rightarrow P$$

$$\text{so } t + \delta t \rightarrow P + \delta P$$

What is δP ?

On the manifold can move in different
directions, so an analogue of

$$\underline{D} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

This is specific to a chart, but
can write

$$\frac{\partial f}{\partial x^\mu} dx^\mu \quad - \text{covector, "df"}$$

like ∇f in $\mathbb{R}^3$.

Have succeeded for $C^\infty(M)$, but
the defn cannot be extended beyond for

$$\frac{\partial}{\partial X^\mu} \left(\frac{\partial f}{\partial X^\nu} \right) \rightarrow \frac{\partial X'^{\mu'}}{\partial X^\mu} \frac{\partial X'^{\nu'}}{\partial X^\nu} \frac{\partial^2 f}{\partial X'^{\mu'} \partial X'^{\nu'}} + \frac{\partial X'^{\mu'}}{\partial X^\mu} \frac{\partial^2 X'^{\nu'}}{\partial X^\mu \partial X^\nu} \frac{\partial f}{\partial X'^{\nu'}}$$

- $\delta p \mapsto \delta X^\mu$ is not geometric int
- $T_p^*(M) \neq T_{p+SP}^*(M)$

but

Have succeeded for $C^\infty(M)$, but
the defn cannot be extended beyond P_{ns}

"df"

$$\frac{\partial}{\partial X^\mu} \left(\frac{\partial f}{\partial X^\nu} \right) \rightarrow \frac{\partial X^{\mu'}}{\partial X^\mu} \frac{\partial X^{\nu'}}{\partial X^\nu} \frac{\partial^2 f}{\partial X^{\mu'} \partial X^{\nu'}} + \frac{\partial X^{\nu'}}{\partial X^\mu} \frac{\partial^2 X^{\nu'}}{\partial X^\mu \partial X^\nu} \frac{\partial f}{\partial X^{\nu'}}$$

- $\delta P \leftrightarrow \delta X^\mu$ is not geometric covt
- $T_p^*(M) \neq T_{P+SP}^*(M)$ - cannot subtract (co) vectors from different spaces!

Have succeeded for $C^\infty(M)$, but
the defn cannot be extended beyond for

$$\frac{\partial}{\partial X^m} \left(\frac{\partial f}{\partial X^v} \right) \rightarrow \frac{\partial X'^{m'}}{\partial X^m} \frac{\partial X'^{v'}}{\partial X^v} \frac{\partial^2 f}{\partial X'^{m'} \partial X'^{v'}} + \boxed{\frac{\partial X'^{m'}}{\partial X^m} \frac{\partial^2 X'^{v'}}{\partial X'^m \partial X'^v} \frac{\partial f}{\partial X'^{v'}}$$

- $\delta p \mapsto \delta X^m$ is not geometric (not)
- $T_p^*(M) \neq T_{p+sp}^*(M)$ - cannot subtract (co)vectors from different spaces!

3 possible get-outs

The connection provides a
link between $T_p(M)$ &
 $T_{p+sp}(M)$

The exterior derivative eliminates
unwanted $\partial^2 X^m$ term by taking
antisymmetric derivatives

The Lie derivative is a derivative
w.r.t. a vector field. No extra
structure required, but limited to a deriv
along a
vector field

Have seen that df is
geometric, it maps fns to
covectors or 1-forms. df

$$\underline{d} : C^\infty(M) \rightarrow C^\infty(M)$$

$$\text{st. at } p \quad \langle \underline{d}f | T \rangle = T f$$

$$\forall T \in T_p(M)$$

To extend d we apply to covariant
anti-symmetric tensors.

Defn: A p-form is an antisymmetric
rank p covariant tensor. Form by
antisymm tensor product - $\wedge$ "wedge"

$$\underline{B} \otimes \underline{A}$$

$\underline{B}$ is a 2-form.

$$\underline{A} = A_{a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}}$$

not commutative.

$$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-1)^{pq} \underline{B} \wedge \underline{A}$$

Can only have $p \leq n$ forms

The n -form is unique up to a factor, & is a multiple of the alternating symbol.

If we have a metric, then can
define the ϵ -tensor

$$\epsilon_{abcd} = \sqrt{|\det g|} \epsilon_{abcd}$$

Under a coord transfm:

$$g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}$$

det

Hence

$\epsilon_{\mu\nu\lambda\rho}$

can

$$\det g = \left| \frac{\partial x'}{\partial x} \right|^2 \det g'$$

Hence

$$\epsilon_{\mu\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \sqrt{|\det g|} \left| \frac{\partial x}{\partial x'} \right| \epsilon_{\mu'\nu'\lambda'\rho'} \dots$$

$$= \sqrt{|\det g|} \epsilon_{\mu'\nu'\lambda'\rho'} \dots$$

$$= \epsilon_{\mu'\nu'\lambda'\rho'} dx^{\mu'} \wedge \dots \wedge dx^{\rho'}$$

Can also use g to define
a map (Hodge dual).

$$* : \Lambda^{(p)} \rightarrow \Lambda^{(n-p)}$$

$$(* A^{(p)})_{a_1 \dots a_{n-p}} = \frac{1}{p!} \epsilon_{a_1 \dots a_{n-p}}{}^{b_1 \dots b_p} A_{b_1 \dots b_p}$$

$$\sqrt{|\det g|} \left| \frac{\partial x}{\partial x'} \right| \epsilon_{\mu' \nu' \dots \rho'}$$

$$\epsilon_{\mu' \nu' \dots \rho'}$$

$$dx^{\mu'} = \Lambda^{\mu'}_{\nu'} dx^{\nu'}$$

$$\underline{d} : \Lambda^{(p)} \rightarrow \Lambda^{(p+1)}$$

$$A_{a_1 \dots a_p} \mapsto \frac{(p+1)!}{p!} \partial_{[a_1} A_{a_2 \dots a_{p+1}]}$$

$\underline{d}$ reduces to partial diff on fns.

$$\underline{d} \text{ is "Leibnitzian"} \quad \underline{d}(A^{(p)} \wedge B^{(q)}) = \underline{d}A^{(p)} \wedge B + (-1)^p A \wedge \underline{d}B$$

$$\underline{d}^2 = 0$$

$$(-1)^p A \wedge \underline{d}B$$

Using
a "..."

Using $\star$ can define
a "dual" derivation.

$$\delta = \star d \star : \Lambda^p \rightarrow \Lambda^{p-1}$$

$$A_{a_1 \dots a_p} \mapsto (-1)^p \nabla^{a_1} A_{a_1 \dots a_p}$$

$$= dA^{(p)} \wedge B$$

$$(-1)^p A \wedge dB$$

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In 3D

$$\text{grad } f \leftrightarrow d$$

$$\text{div } \underline{v} \leftrightarrow \star$$

$$\text{curl } \underline{v} \leftrightarrow \star$$

$$= dA^{(p)} \wedge B$$

$$(-)^p A \wedge dB$$

define

$$\rightarrow \Lambda^{p-1}$$

$$\nabla^{a_1} A_{a_1 \dots a_p}$$

In 3D

$$\text{grad } f \leftrightarrow \underline{d} f$$

$$\text{div } \underline{v} \leftrightarrow \underline{\delta} \underline{v}$$

$$\text{curl } \underline{v} \leftrightarrow \underline{*} \underline{d} \underline{v}$$

e.g. Electromagnetism

Gauge theory:

$$A_\mu = (\phi, -\underline{A}) \quad \text{1-form}$$

Field strength

$$\boxed{\underline{F} = d\underline{A}}$$

↑
1-form.

$$= \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

We automatically get

$$\underline{dF} = 0 \quad (\underline{d} \underline{d} \underline{A})$$

Gauge invariance:

$$\underline{A} \rightarrow \underline{A} + \underline{d}\lambda$$

$$\underline{F} \rightarrow \underline{F} + \cancel{d^2\lambda}$$

Can get

(in h...)

e

from different spaces:

antisymmetric

ally get
 dA

Can generalise to higher forms
(in higher dims')

e.g. $B_{\mu\nu}$ 2-form gauge potential

$$\underline{B} \rightarrow \underline{B} + d\underline{A} \quad (\underline{A} \text{ is 1-form})$$

$$\underline{H} = d\underline{B} \quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$$

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antisymmetric derivatives

$$\frac{\partial^2 X^{\mu'}}{\partial X^\mu \partial X^\nu} \frac{\partial f}{\partial X^{\mu'}}$$

(co) vectors
aces!

In 4D $\underline{H} \leftrightarrow \underline{d} \varphi.$

2-form physics $\leftrightarrow$ scalar physics [axion]

$B_{\mu\nu}$ comes up in string theory

$B_{\mu\nu}$ couples to a line source.

$$\underline{A} = \begin{pmatrix} -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad \underline{F} \rightarrow \underline{F} + d\underline{A} \quad \underline{H} = d\underline{B} \quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda}$$

scalar
physics [axion]

theory

source

Particles trace out worldlines $X^\mu(t)$
(1D submanifold of M).

String has extension in space & moves in
time $X^\mu(t, s)$ (2D submanifold of M)

3 possible

The connection
link between
 $T_{\text{ptsp}}(M)$

The exterior derivative
Unwanted ∂^2
antisymmetric

$$\underline{A} = \begin{pmatrix} -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

form

$$\underline{F} \rightarrow \underline{F} + d\underline{A}$$

$$\underline{H} = d\underline{B} \quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda}$$

scalar
physics [axion]

ing theory

ource

Particles trace out worldlines $X^\mu(t)$
(1D submanifold of M).

String has extension in space & moves in
time $X^\mu(t, s)$ (2D submanifold of M)

$A_\mu \rightarrow$ point charge $\int q \dot{X}^\mu A_\mu$



$B_{\mu\nu}$

$$\begin{pmatrix} -B_3 & B_2 \\ 0 & -B_1 \\ B_2 & B_1 & 0 \end{pmatrix}$$

$$F \rightarrow F + dA$$

$$H = dB$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$$

Particles trace out worldlines $X^\mu(t)$
(1D submanifold of M).

String has extension in space & moves in time $X^\mu(t,s)$ (2D submanifold of M)

$A_\mu \leftrightarrow$ point charge $\int q \dot{X}^\mu A_\mu$



Charged worldsheet?

$$B_{\mu\nu} X'^\mu X'^\nu$$

$$\begin{pmatrix} -B_3 & B_2 \\ 0 & -B_1 \\ B_2 & B_1 & 0 \end{pmatrix}$$

$$\underline{F} \rightarrow \underline{F} + d\underline{A}$$

$$\underline{H} = d\underline{B}$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$$

Particles trace out worldlines $X^\mu(t)$
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String has extension in space & moves in time $X^\mu(t, s)$ (2D submanifold of M)

$A_\mu \leftrightarrow$ point charge $\int q \dot{X}^\mu A_\mu$



Charged worldsheet?

$$B_{\mu\nu} \dot{X}^\mu \dot{X}^\nu$$



