Title: Tomography from Entanglement

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Abstract: <p>The Ryu-Takayanagi formula relates the entanglement entropy in a conformal field theory to the area of a minimal surface in its holographic dual. I will show that this relation can be inverted to reconstruct the bulk stress-energy tensor near the boundary of the bulk spacetime, from the entanglement on the boundary. I will also show that the positivity and monotonicity of the relative entropy for small spherical domains between the reduced density matrices of an excited state and of the ground state of the CFT, translate to energy conditions in the bulk.</p>


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Based on 1412.1879 with M. Marcolli, H. Ooguri and B. Stoica

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Motivation

- In gauge/gravity duality, how do gravitational dynamics emerge from CFT dynamics?
- Recent work (e.g. Ryu and Takayanagi) suggests that entanglement is key.
- Entanglement dynamics of CFT are dual to linearized Einstein equations around pure AdS.
  [Lashkari, McDermott, van Raamsdonk] [Faulkner, Guica, Hartman, Myers, van Raamsdonk]

Main Idea

- Today I will discuss some holographic consequences of other universal properties of entanglement in CFT's (positivity of the relative entropy and monotonicity under increase in the size of the entangling domain)

- I will show that these properties imply (an integrated form of) positivity of the bulk stress tensor in the linearized near-AdS region for a holographic dual to an excited state of a CFT.

- Moreover, I will show that the bulk stress tensor in this near-AdS region can be reconstructed point by point from the entanglement on the boundary.

Relative Entropy

- Given density matrices ρ_0 and ρ_1, the relative entropy is defined as

$$S(\rho_1|\rho_0) = \mathrm{tr}(\rho_1 \log \rho_1) - \mathrm{tr}(\rho_1 \log \rho_0)\,.$$

- Relative entropy is a measure of distinguishability between quantum states.

- It is positive, $S(\rho_1|\rho_0) \geq 0\,.$

- It increases with system size,

$$S(\rho_1^V|\rho_0^V) \leq S(\rho_1^W|\rho_0^W)\,, V \subseteq W\,.$$

- We'll specialize to the case where the density matrices are reduced density matrices over a spatial domain for two states of a QFT.

- When ρ_0 and ρ_1 are reduced density matrices for two states across a family of domains with characteristic size R, the second property implies

$$\partial_R S(\rho_1|\rho_0) \geq 0\,.$$

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- When ρ_0 and ρ_1 are reduced density matrices for two states across a family of domains with characteristic size R, the second property implies

$$\partial_R S(\rho_1|\rho_0) \geq 0 \,.$$

- In terms of the modular Hamiltonian of ρ_0 which is its normalized log

$$\rho_0 = \frac{e^{-H_{mod}}}{\mathrm{tr}(e^{-H_{mod}})} \,,$$

the positivity of the relative entropy can be expressed as

$$S(\rho_1|\rho_0) = \Delta\langle H_{mod}\rangle - \Delta S_{EE} \geq 0$$

where

$$\Delta\langle H_{mod}\rangle = \mathrm{tr}(\rho_1 H_{mod}) - \mathrm{tr}(\rho_0 H_{mod}),$$

$$\Delta S_{EE} = -\mathrm{tr}(\rho_1 \log \rho_1) + \mathrm{tr}(\rho_0 \log \rho_0) \,.$$

Modular Hamiltonian

- For most density matrices, the modular Hamiltonian has no simple form in terms of local operators of the theory. There are a few special cases where it is known.

- For example, for a global thermal state of temperature T,

$$H_{mod} = H/T \,.$$

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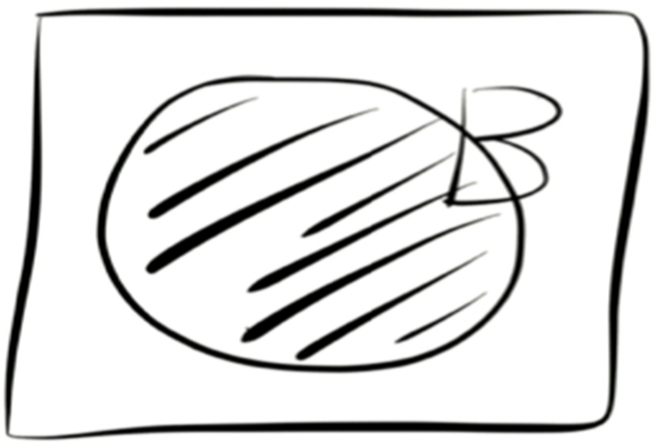
$$H_{mod} = H/T\ .$$

- Yet another example where the modular Hamiltonian is known is a ball-shaped entangling domain B in the vacuum state of a CFT.

- This is because a conformal transformation maps the Rindler wedge to the domain of dependence of the ball.



- Explicitly, the modular Hamiltonian for the reduced density matrix of a ball shaped domain, in the vacuum state of a CFT, is

$$H_{mod} = 2\pi \int_B d^d x \frac{R^2 - r^2}{2R} T_{00}$$

where T is the CFT stress tensor.

Entanglement First Law

- Going back to the relative entropy

$$S(\rho_1|\rho_0) = \Delta\langle H_{mod}\rangle - \Delta S_{EE} \geq 0\,,$$

when the density matrices are close, the inequality is saturated to linear order [Blanco, Casini, Hung, Myers]:

$$\delta\langle H_{mod}\rangle - \delta S_{EE} = 0\,.$$

- I.e. $S(\rho(\lambda)|\rho_0)$ at linear order in λ, where $\rho(\lambda) = (1-\lambda)\rho_0 + \lambda\rho_1\,.$

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Assumptions

- CFT vacuum $|0\rangle$ is dual to AdS_{d+1} .
- To talk about relative entropy, we need an excited state of the CFT. I'll assume it has a holographic bulk dual whose near-boundary geometry is expanded in Fefferman-Graham form,

$$g_{AdS} = \frac{\ell^2}{z^2}\left[dz^2 + (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu\right] \ .$$

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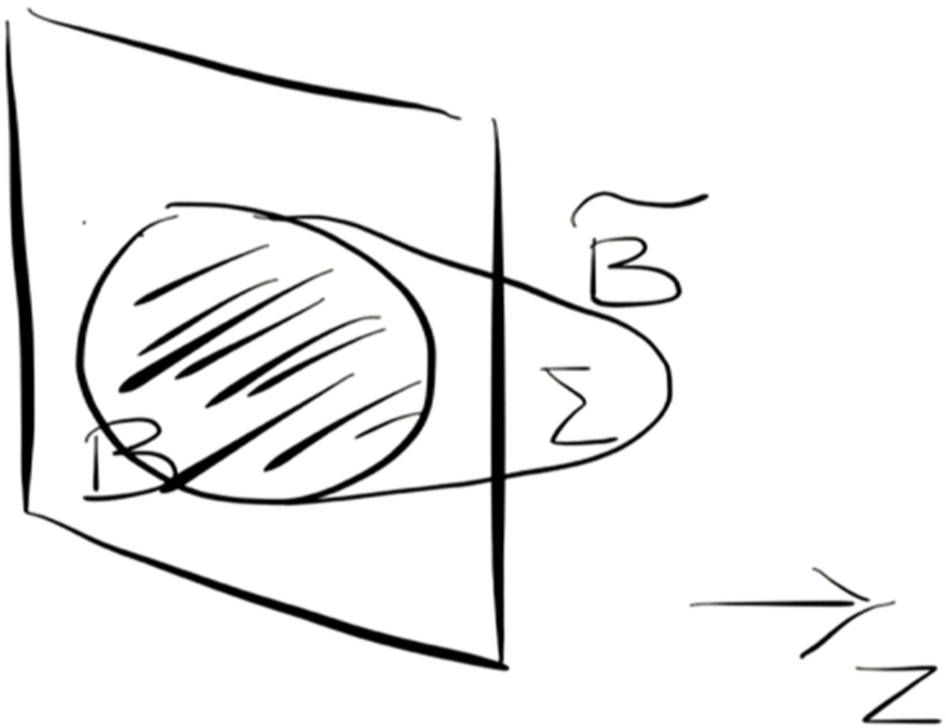
Holographic Entanglement Entropy



- The entanglement entropy of the CFT across B is equal to the area of the minimal bulk surface $\tilde{B}$.
- I.e.: $S_B = \dfrac{A(\tilde{B})}{4G_N}$

Review: linearized EFE's from entanglement 1st law

- Consider a family of states in the CFT

$$|\Psi(\lambda)\rangle = (1-\lambda)|0\rangle + \lambda|\Psi\rangle \, .$$

- At linear order in λ, these saturate the inequality,

$$2\pi \int_B d^d x \frac{R^2 - r^2}{2R} \delta\langle T_{00}\rangle = \delta S_{EE} \, .$$

- Mapping this to the bulk gives roughly one equation for each bulk spacetime point, that can be inverted to give the linearized Einstein equations for the dual to the excited state. [Lashkari, McDermott, van Raamsdonk] [Faulkner, Guica, Hartman, Myers, van Raamsdonk]

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- Explicitly

$$\langle T_{00}(x_0)\rangle = \frac{d^2-1}{2\pi\Omega_{d-2}} \lim_{R\to 0}\left(\frac{1}{R^d}\Delta S_{EE}(R,x_0)\right) \propto h^{(d)}_{00}\,.$$

- This result does not rely on the excited CFT state being parametrically close to the vacuum state.

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$$\langle T_{00}(x_0)\rangle = \frac{d^2-1}{2\pi\Omega_{d-2}} \lim_{R\to 0}\left(\frac{1}{R^d}\Delta S_{EE}(R,x_0)\right) \propto h^{(d)}_{00}\,.$$

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$\langle T_{\mu\nu} \rangle$ from holography

- One can compute the boundary stress tensor using holographic renormalization. This can be tedious.

- Short-cut [Faulkner et al.]: Look at the relative entropy for a small ball

$$lim_{R\to 0}\Delta\langle H_{mod}\rangle = lim_{R\to 0}\Delta S_{EE}\ .$$

- Translating this to holography gives an expression for the boundary stress tensor as a function of the bulk metric.

- Explicitly

$$\langle T_{00}(x_0)\rangle = \frac{d^2-1}{2\pi\Omega_{d-2}} \lim_{R\to 0}\left(\frac{1}{R^d}\Delta S_{EE}(R,x_0)\right) \propto h^{(d)}_{00}\,.$$

- This result does not rely on the excited CFT state being parametrically close to the vacuum state.

- Suppose we perturb the bulk geometry away from AdS by h_{ab} which is parametrically small.

- Because the original surface was extremal, the shape of the surface is unchanged from the half-sphere to leading order in h_{ab} .

- The leading variation in the holographic EE comes from evaluating that shape on the perturbed area functional:

$$\Delta S_{EE} = \frac{1}{8G_N R} \int_{|x|<R} d^{d-1}x z^{-d} (R^2 \eta^{ij} - x^i x^j) h_{ij} \, .$$

- The shape of the surface changes at order h^2. This does rely on analyzing a state near the vacuum.

- There exists a d-1 form $\boldsymbol{\chi}[h_{ab}]$ with the properties

$$\int_B \boldsymbol{\chi} = \Delta\langle H_{mod}\rangle, \qquad \int_{\tilde{B}} \boldsymbol{\chi} = \Delta S_{EE} \,,$$

and moreover, $\mathrm{d}\boldsymbol{\chi} = f(x_0, R)\delta E^g_{tt}[h]\sqrt{g_\Sigma}\,.$

- By the Stokes theorem,

$$0 = \Delta S_{EE} - \Delta\langle H_{mod}\rangle = \int_{\tilde{B}} \boldsymbol{\chi} - \int_B \boldsymbol{\chi} = \int_{\partial\Sigma} \boldsymbol{\chi} = \int_\Sigma \mathrm{d}\boldsymbol{\chi}\,.$$

- Considering this on every ball on a spatial slice at fixed time t=0 gives $\delta E^{g}_{tt} = 0$. One can show that actually this gives vanishing of all components of the linearized Einstein tensor.

- Adding the leading 1/N correction to the Ryu-Takayanagi formula [Faulkner, Lewkowycz, Maldacena], the same derivation yields the linearized Einstein equations coupled to the bulk stress tensor [Swingle, van Raamsdonk].

Beyond the linearized EFE's

- We use the same technique to translate the positivity of relative entropy in the CFT to holography. There is an obvious problem. A Stokes theorem argument relies on having the same profile for the minimal surface in the vacuum and excited state.

- To maintain analytic control for a generic excited CFT state, we take the radius of the entangling domain to be small, so that the Ryu-Takayanagi surface only penetrates the asymptotically AdS region of the bulk.

- Now consider ball-shaped entangling domains in the boundary CFT with radius R bounded above so that the bulk metric fluctuation are parametrically small.

- Suppose there is one energy scale in the CFT set by μ so that $\langle T_{\mu\nu}\rangle \sim \mu^d$, $\langle \mathcal{O}\rangle \sim \mu^\Delta$ and consider the dimensionless combination $\epsilon = R\mu$.

- Then, nonlinear gravity couplings that would change the profile of the Ryu-Takayanagi minimal surface and invalidate the Stokes theorem technique start at order ϵ^{2d} .

- By truncating at that order we can study the positivity of the relative entropy $\Delta\langle H_{mod}\rangle - \Delta S_{EE} \geq 0$ holographically while keeping the bulk profile for the extremal surface fixed.

- But at order $\epsilon^{2\Delta} < \epsilon^{2d}$, the vev of the relevant operator sources a sub-leading contribution to h_{ab}. This leads to a strict inequality above.

Bulk Energy Condition from Relative Entropy

- So for an arbitrary CFT state with classical holographic dual, we can take the d-1 form $\boldsymbol{\chi}[h]$ and evaluate it on the bulk metric fluctuation in the interior of minimal surfaces for ball-like entangling domains, whose radii satisfy $R^d \langle T_{\mu\nu} \rangle \ll 1$.

$$\int_B \boldsymbol{\chi} = \Delta \langle H_{mod} \rangle, \qquad \int_{\tilde{B}} \boldsymbol{\chi} = \Delta S_{EE} ,$$

$$\mathrm{d}\boldsymbol{\chi} = f(x_0, R) \delta E^g_{tt}[h] \sqrt{g_\Sigma} .$$

- The Stokes theorem now implies that

$$\Delta S_{EE} - \Delta\langle H_{mod}\rangle = \int_{\tilde{B}} \boldsymbol{\chi} - \int_{B} \boldsymbol{\chi} = \int_{\Sigma} \mathrm{d}\boldsymbol{\chi} \leq 0\,.$$

- Meanwhile, the linearized Einstein tensor appearing in $\mathrm{d}\boldsymbol{\chi}$ couples to bulk matter in the form of the classical bulk stress tensor,

$$\delta E^{g}_{ab}[h] = 8\pi G_N T_{ab}\,.$$

- Plugging in, we find

$$S(\rho_1|\rho_0) = 8\pi^2 G_N \int_V \frac{R^2 - (z^2 - x^2)}{R} \varepsilon \sqrt{g_V} \geq 0$$

where ε is the classical bulk energy density.

- Moreover,

$$\partial_R S(\rho_1|\rho_0) = 8\pi^2 G_N \int_V \left(1 + \frac{z^2 + x^2}{R^2}\right) \varepsilon \sqrt{g_V} \geq 0\,.$$

- Positivity and monotonicity of the relative entropy for ball shaped entangling domains maps to (integrated) positivity of the bulk stress tensor.

Inverting the bulk integral

- We can invert this relation to compute the energy density ε point-by-point in the bulk using the relative entropy:

$$\left[\partial_R + \frac{1}{R}\right] S(\rho_1|\rho_0) = 16\pi^2 G_N \int_V \varepsilon\sqrt{g_V}$$
$$\left[\partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\right] S(\rho_1|\rho_0) = 16\pi^2 G_N \int_\Sigma \varepsilon\sqrt{g_\Sigma}\,.$$

- The right-hand side is still non-negative if we assume positivity of the bulk stress tensor, thus

$$\left[\partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\right] S(\rho_1|\rho_0) \geq 0\,.$$

Inverse Radon Transform

- Let's look more closely at the RHS of

$$\left[\partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2}\right] S(\rho_1|\rho_0) = 16\pi^2 G_N \int_\Sigma \varepsilon \sqrt{g_\Sigma}\,.$$

- The surfaces Σ are totally geodesic in AdS, and this integral is the Radon transform, whose inverse is known.

- Given a smooth function f in hyperbolic space, its Radon transform $\mathcal{R}f[\Sigma]$ is a function on the space of totally geodesic manifolds, defined by the integral of f over Σ.

- The dual Radon transform $\mathcal{R}^*\mathcal{R}f(z,x)$ gives back a function in hyperbolic space in the following way: Pick a point in hyperbolic space and integrate $\mathcal{R}f[\Sigma]$ over all Σ's passing through it.

- f can be extracted by applying a differential operator to $\mathcal{R}^*\mathcal{R}f$ [Helgason]. Ex. for d odd,

$$f = \frac{(-4\pi)^{(d-1)/2}\Gamma[d/2]}{\sqrt{\pi}} Q(\boldsymbol{\Delta})\mathcal{R}^*\mathcal{R}f$$

$$Q(\boldsymbol{\Delta}) = (\boldsymbol{\Delta} + 1\cdot(d-2))(\boldsymbol{\Delta} + 2\cdot(d-3))\dots(\boldsymbol{\Delta} + (d-2)\cdot 1)\,.$$

  There is a similar formula for d even.

- So we get the point-by-point bulk stress tensor from the relative entropy on spheres in the CFT.

Summary

- We showed that positivity and monotonicity of relative entropy on small spherical entangling regions in CFT's is holographically related to positivity of the bulk stress tensor in the linearized near-AdS region of its dual.
- This can be inverted to obtain the bulk stress tensor in the near-AdS region.

Related Questions

- Can we derive the nonlinear EFE's from CFT entropy + the Ryu-Takayanagi formula?
- Can we connect a bulk energy condition to the properties of the relative entropy for non-spherical domains?