Title: Stochastic Inflation Revisited: A Self-Consistent Recursive Approach and Applications

Date: Dec 16, 2014 11:00 AM

URL: http://pirsa.org/14120053

Abstract: <p>In this talk, I will review the main ideas underlying stochastic inflation, by introducing the formalism in two independent ways. First I will start from the intuitive picture stemming from the equations of motion of the system. I will then introduce a more rigorous approach based on the in-in formalism, and show how the usual set of Langevin equations can emerge from a path integral formulation. With this understanding, I will then formulate a new, recursive method which allows to solve consistently both in slow-roll parameters and in quantum corrections. I will then discuss examples of how this method can be applied to derive corrected predictions for cosmological observables in the case of hybrid inflation, multi-field inflation, and inflation on modulated potentials.</p>

Outline

Stochastic Inflation Formalism

- Heuristically and some intuition;
- Motivating a recursive method;

Microphysics justification

- CPT (in-in) formalism & rederivation of the Langevin eqns
- Perturbative expansion

Applications

- Hybrid Inflation
- Multi-fields
- Modulated potentials

Slow-Roll Inflation

The quasi-exponential expansion of $a(t)$ is driven by the slow roll of a scalar field $\hat{\Phi}$ down the slope of a flat potential.

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0$$

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{\dot{\varphi}^2}{2} + V\right]$$

Assume:

$$\left.\begin{array}{l} \hat{\Phi} \approx \varphi \\ \ddot{\varphi} \ll 3H\dot{\varphi} \\ \rho \approx V \sim cst \end{array}\right. \quad \Rightarrow \quad \left.\begin{array}{l} H \approx cst \\ a(t) \sim e^{Ht} \end{array}\right\} \text{quasi-de Sitter}$$

$\Rightarrow$ Physical lengths grow quasi-exponentially

Split: $\hat{\Phi} = \varphi + \delta\phi$

Classical Background (fixed)	Small quantum perturbations
homogeneous $\Rightarrow$ acceleration	Structure

Quantum fluctuations: $\delta\phi$ are created on small scales, are stretched by the inflating space beyond the Hubble radius where they freeze out (when $k/aH \sim 1$), get squeezed and undergo classicalisation. (they later re-enter the Hubble, and seed the fluctuations of the CMB and the LLS of the Universe)

Is this split accurate/good?

Idea: we are interested in the *classical* theory, beyond the Hubble radius, since these are the range of scales that are observable in the CMB.

$\Rightarrow$ Write an effective classical theory for these modes, by coarse-graining, or averaging, over scales $\sim H^{-1}$

«Problem»: Modes smaller than the coarse-graining scale, that is quantum-fluctuating modes, are constantly escaping the coarse-grained region and sourcing the classical theory. From this perspective, they act as a *noise* for the classical theory

Stochastic inflation describes how to perform this averaging, and how quantum fluctuation give rise to a classical noise term in the effective coarse-grained classical equation.

Why does this even matter?

Shouldn't the constant contribution of incoming quantum modes into the coarse grained theory be negligible anyway?

- Matters a lot, *e.g.* when the classical trajectory in field space is constrained to small fields values, quantum dispersion may dominates

- Also, in eternal inflation, quantum corrections must dominate over the classical trajectory

In general,

allows to constantly «renormalise» the background trajectory, *i.e.* re-sums the incoming quantum modes in the background.
so *e.g.* $H(t)$ assumes it physical values at all t

$\Rightarrow$ **Powerful non-perturbative method**

How does it work?
(Heuristically)

Consider a set of 2 quantum fields $\left\{\hat{\Phi}, \hat{\Psi}\right\}$ (generalization to larger numbers easy)

Split each one into long and short wavelengths at a coarse graining scale using a window function

$$\boxed{\Phi = \varphi + \phi_> \,, \;\; \Psi = \chi + \psi_> \,,}$$

$$\phi_> \,, \; \psi_> \text{ correspond to } \; k > H(t)a(t) \,,$$
$$\varphi \,, \; \chi \text{ correspond to } \; H(t)a(t) > k > 0 \,,$$

How Does it Work

(Heuristically) continued...

Expand $\phi_>, \psi_>$ in creation/annihilation opts on a time-dept background:

$$\phi_>(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_H(k,t) \left[\phi_\mathbf{k} \hat{a}_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{x}} + \phi^*_\mathbf{k} \hat{a}^\dagger_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}\right] ,$$

$$\psi_>(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_H(k,t) \left[\psi_\mathbf{k} \hat{b}_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{x}} + \psi^*_\mathbf{k} \hat{b}^\dagger_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}\right] ,$$

$W_H(k,t)$ is the time-dependent window function filtering only the sub-Hubble modes.

Simplest choice: $W_H(k,t) = \theta(k/\epsilon a H - 1)$

Also, only choice to make $\phi_>, \psi_>$ appear as white noise to φ, χ

$\Rightarrow \varphi, \chi$ become Markovian processes (memoryless)

BUT: not very physical...

Winitzki & Vilenkin, 2000; Matarrese et al. 2004

How Does it Work

(Heuristically) continued...

Plug this expansion back in the KG equations

$$
\begin{aligned}
&-\Box\varphi + m_\Phi^2\varphi + V_{\mathrm{pert},\Phi}(\varphi,\chi) + \left[-\Box\phi_> + m_\Phi^2\phi_> + V^{\mathrm{pert}}_{,\Phi\Phi}(\varphi,\chi)\phi_> + V^{\mathrm{pert}}_{,\Phi\Psi}(\varphi,\chi)\psi_>\right] \\
&= -V^{\mathrm{pert}}_{,\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2 + \dots,
\end{aligned}
$$

HOW DOES IT WORK
(HEURISTICALLY) CONTINUED...

Plug this expansion back in the KG equations
subtract the linearized quantum fields EoM. Left with:

$$-\Box\varphi + m_\Phi^2\varphi + V_{\mathrm{pert},\Phi}(\varphi,\chi) = \delta S_{\phi_>}$$

$$-V^{\mathrm{pert}}_{,\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V^{\mathrm{pert}}_{,\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2 + \dots .$$

Where:

$$\delta S_{\phi_>} = 3H\xi_1^\phi + \dot{\xi}_1^\phi - \xi_2^\phi$$

With:

$$\xi_1^\phi = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{W}_H\left(\frac{k}{\epsilon a(t)H(t)}\right)\left[\phi_\mathbf{k}\hat{a}_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}} + \phi_\mathbf{k}^*\hat{a}_\mathbf{k}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right],$$

$$\xi_2^\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\dot{W}_H\left(\frac{k}{\epsilon a(t)H(t)}\right)\left[\dot{\phi}_\mathbf{k}\hat{a}_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}} + \dot{\phi}_\mathbf{k}^*\hat{a}_\mathbf{k}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right].$$

Stochastic equations of motion:

$$\boxed{\Box\varphi + m_{\Phi}^2\varphi + V_{\mathrm{pert},\Phi}\,\varphi = 3H\xi_1^{\phi} + \dot{\xi}_1^{\phi} - \xi_2^{\phi}}$$

$$- V_{\mathrm{pert},\Phi\Phi\Psi}(\varphi,\chi)\phi_>\psi_> - \frac{1}{2}V_{\mathrm{pert},\Phi\Phi\Phi}(\varphi,\chi)\phi_>^2 - \frac{1}{2}V_{\mathrm{pert},\Phi\Psi\Psi}(\varphi,\chi)\psi_>^2\,,$$

Mode coupling terms

Slow roll-suppressed

$$\boxed{\Box\chi + m_{\psi}^2\chi + V_{\mathrm{pert},\Psi}\,\chi = 3H\xi_1^{\psi} + \dot{\xi}_1^{\psi} - \xi_2^{\psi}}$$

$$- V_{\mathrm{pert},\Psi\Psi\Phi}(\varphi,\chi)\psi_>\phi_> - \frac{1}{2}V_{\mathrm{pert},\Psi\Psi\Psi}(\varphi,\chi)\psi_>^2 - \frac{1}{2}V_{\mathrm{pert},\Psi\Phi\Phi}(\varphi,\chi)\phi_>^2\,.$$

These form a **system of *classical* Langevin equations**, sourced by random gaussian noise terms (which are completely determined by their 2-pt functions). **They describe a stochastic process.**

A Recursive Method?

We now have two coupled systems of 2 equations each:
the **classical stochastic system** and the **quantum system**

In order to solve it consistently, we must solve both *to the same order of accuracy in the slow-roll parameters and in* $\hbar$

$\Rightarrow$ We use a recursive approach!

Outline of the Recursive Approach

1. Solve for the quantum fields $\phi_>, \chi_>$ mode functions to zeroth order in slow-roll, that is, as if they were free, massless fields in dS space. Get the zeroth order noise:

$$\langle \xi_1^{\phi,\psi}(\mathbf{x},t), \xi_1^{\phi,\psi}(\mathbf{x},t)\rangle = \frac{H^3}{4\pi^2}\frac{\sin(\epsilon a H r)}{\epsilon a H r}\delta(t-t')$$

2. Use this noise to find the classical fields φ, χ to leading order in slow-roll and their corresponding PDFs. *i.e.* we need to solve:

$$3H^2\frac{\mathrm{d}\varphi}{\mathrm{d}N} = -V_{,\Phi} + 3H\xi_\phi(N)$$

$$3H^2\frac{\mathrm{d}\chi}{\mathrm{d}N} = -V_{,\Psi} + 3H\xi_\psi(N)$$

where we changed the time variable to the e-fold number: $N \equiv \ln(a/a_\mathrm{i})$

Recursive Approach

continued...

3. Go back to the linearized mode functions for the quantum fields and **replace all occurrences of the coarse-grained fields** by their average values, variances, and higher momenta:

$$\begin{aligned} \varphi, \chi &\rightarrow \langle\varphi\rangle, \langle\chi\rangle, \\ \varphi^2, \chi^2 &\rightarrow \langle\varphi^2\rangle, \langle\chi^2\rangle, \\ \varphi^p\chi^q &\rightarrow \langle\varphi^p\chi^q\rangle, \end{aligned}$$

solve the corrected linearized equations for $\phi_>, \psi_>$, this time expanding to leading order in slow-roll.

i.e. **solve for the full linearized mode functions**

How does it work?

(Actually)

To understand why this is a sensible thing to do and in general why the stochastic approach of coarse-graining the full quantum EoM makes sense to derive a classical theory, look at the microphysics of the process.

This is very similar to quantum Brownian motion

$\Rightarrow$ use similar techniques, *i.e.*

the *in-in* (or CPT) Schwinger-Keldysh formalism

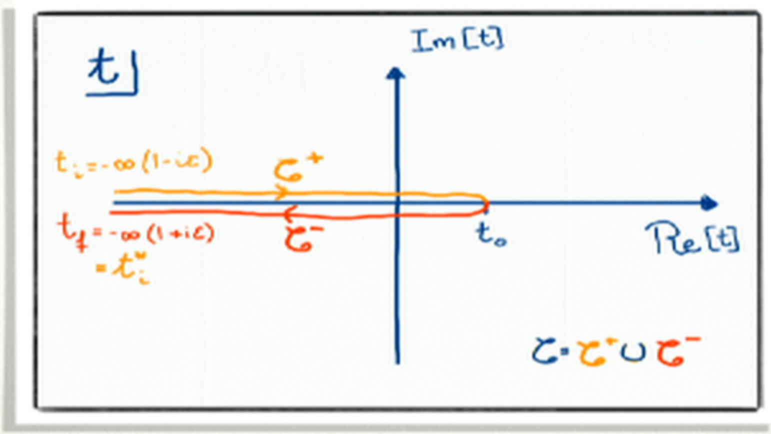
As opposed to the *in-out* formalism where:

- one calculates *S*-matrix elements,
- for transition amplitudes between *in* and *out* asymptotic states,
- with one-particles states defined in the ∞-distant past and future.

In the *in-in* formalism:

- one calculates expectation values of operators at a fixed time t_0, (EEVs for quantum statistical mechanics)
- with one-particles states defined in the ∞-distant past only.

In-In Formalism



- Split the fields into: - a bath $\phi_>, \psi_>$ (same k-mode exp. as before)
  - a system $\varphi = \Phi - \phi_>, \ \chi = \Psi - \psi_>$
- Split each of the bath & system fields into: part $\epsilon\ \mathcal{C}^+$ & part $\epsilon\ \mathcal{C}^-$
  get: $\varphi^+, \varphi^-, \phi_>^+, \phi_>^-$ & similarly for Ψ
- The *in* state, at $-\infty$, is taken to be the Bunch-Davis vacuum,
- Evaluate operators at fixed t_0

Goal: Integrate out the bath degrees of freedom.

- In the same spirit as Wilsonian renormalisation, we want to get a V_{eff} for the system fields once the bath has been integrated out.

- Because assume Bunch-Davis vacuum, the initial density matrix factorizes:

$$\hat{\rho}(t=t_i) = \hat{\rho}_{sys}(t_i) \times \hat{\rho}_{bath}(t_i)$$

can write the reduced evolution operator for the system fields as a functional representation, so the effective action can be written as:

$$\int_{\varphi_i^\pm}^{\varphi_f^\pm} \mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^\pm} \mathcal{D}\chi^\pm \exp\left\{\frac{i}{\hbar} S_{eff}[\varphi^\pm, \chi^\pm]\right\}$$
$$\equiv \int_{\varphi_i^\pm}^{\varphi_f^\pm} \mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^\pm} \mathcal{D}\chi^\pm \exp\left(\frac{i}{\hbar}\left\{S_{sys}[\varphi^+, \chi^+] - S_{sys}[\varphi^-, \chi^-]\right\}\right) F[\varphi^\pm, \chi^\pm],$$

$F[\varphi^\pm, \chi^\pm]$ is known as the *influence functional*. In general, it is a non-local, non-trivial object: -depends on the time history, mixes the forward and backward histories along the CTP in an irreducible manner.

INFLUENCE FUNCTIONAL

It can be written explicitly in the bilinear form when $V_{pert} = 0$

$$F\left[\varphi^{\pm}, \chi^{\pm}\right] =$$

$$\int_{-\infty}^{\infty} d\phi^{+}_{>_f} d\psi^{+}_{>_f} \int^{\phi^{+}_{>_f}} \mathcal{D}\phi^{\pm}_{>} \int^{\psi^{+}_{>_f}} \mathcal{D}\psi^{\pm}_{>} e^{\frac{i}{\hbar}\int d^4x\left[\left(\frac{1}{2}\bar{\phi}^T_{>}\tilde{\Lambda}_{\phi}\bar{\phi}_{>} + \bar{\varphi}^T\tilde{\Lambda}_{\phi}\bar{\phi}_{>}\right) + \left(\frac{1}{2}\bar{\psi}^T_{>}\tilde{\Lambda}_{\psi}\bar{\psi}_{>} + \bar{\chi}^T\tilde{\Lambda}_{\psi}\bar{\psi}_{>}\right)\right]}$$

$$\equiv \exp\left[\frac{i}{\hbar} S_{IA}[\varphi^{\pm}, \chi^{\pm}]\right] .$$

where S_{IA} is the influence action and we used the vector notation:

$$\bar{\phi}_{>} = \begin{pmatrix} \phi^{+}_{>} \\ \phi^{-}_{>} \end{pmatrix} \quad \bar{\psi}_{>} = \begin{pmatrix} \psi^{+}_{>} \\ \psi^{-}_{>} \end{pmatrix} \quad \bar{\varphi} = \begin{pmatrix} \varphi^{+} \\ \varphi^{-} \end{pmatrix} \quad \bar{\chi} = \begin{pmatrix} \chi^{+} \\ \chi^{-}_{>} \end{pmatrix} \quad \tilde{\Lambda}_{\phi} = \begin{pmatrix} \Lambda_{\phi} & 0 \\ 0 & -\Lambda_{\phi} \end{pmatrix} \quad \tilde{\Lambda}_{\psi} = \begin{pmatrix} \Lambda_{\psi} & 0 \\ 0 & -\Lambda_{\psi} \end{pmatrix},$$

$$\Lambda_{\phi} = -a^3(t)\left[\partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2(t)} + m^2_{\Phi}\right]; \qquad \Lambda_{\psi} = -a^3(t)\left[\partial_t^2 + 3H\partial_t - \frac{\nabla^2}{a^2(t)} + m^2_{\Psi}\right].$$

Integrating Out the Bath DOFs

- In flat space, the term linear in $\tilde{\phi}_>$ or $\tilde{\psi}_>$ are set to zero to ensure that ϕ_k and ψ_k are indeed solutions to the linearized mode eqn. (*c.f.* the tadpole method Weinberg 74, Boyanovsky et al. 94 - to ensure we are expanding around the right background)

- However, because of the time-dependence of $W_H(k/\epsilon a H)$, the time derivative in the $\Lambda_{\phi,\psi}$ operators act on the window function, giving a non-zero result. This is *precisely* the effect of the modes leaving the quantum theory and joining the coarse-grained theory.
(else, system & bath are orthogonal in *k*-space in $W_H \to \theta(\frac{k}{\epsilon a H} - 1)$ limit)

- We can perform this Gaussian integral over $\tilde{\phi}_>, \tilde{\psi}_>$

Integrating Out the Bath DOFs (continued)

- Performing the path integral over the bath fields, we obtain:

$$S_{IA}^{(1)} = \frac{i}{2\hbar}\int d^4x d^4x' \varphi_q(x)\mathrm{Re}\left[\Pi_\phi(x,x')\right]\varphi_q(x') - \frac{2}{\hbar}\int d^4x d^4x' \theta(t-t')\varphi_q \mathrm{Im}\left[\Pi_\phi(x,x')\right]\varphi_c + (\chi \leftrightarrow \varphi)\ ,$$

Morikawa,
Matarrese et al.

- Where:

$$\Pi_\phi(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^3(t)\left[P_t \phi_\mathbf{k}(t)\right] e^{-i\mathbf{k}\cdot\mathbf{x}} a^3(t')\left[P_{t'}\phi^*_\mathbf{k}(t')\right] e^{i\mathbf{k}\cdot\mathbf{x}'}\ ,$$

$$P_t = \left[\ddot{W}_H(t) + 3H\dot{W}_H(t) + 2\dot{W}_H(t)\partial_t\right]\ ;$$

& defined the *quantum* and *classical* fields, rotating to the Keldysh basis:

$$\begin{pmatrix}\varphi_c\\ \varphi_q\end{pmatrix} \equiv \begin{pmatrix}\frac{\varphi^+ + \varphi^-}{2}\\ \varphi^+ - \varphi^-\end{pmatrix}\ , \qquad \begin{pmatrix}\chi_c\\ \chi_q\end{pmatrix} \equiv \begin{pmatrix}\frac{\chi^+ + \chi^-}{2}\\ \chi^+ - \chi^-\end{pmatrix}.$$

Integrating Out the Bath DOFs (continued)

- Performing the path integral over the bath fields, we obtain:

$$S^{(1)}_{IA} = \frac{i}{2\hbar}\int d^4x\, d^4x'\, \varphi_q(x)\,\mathrm{Re}\left[\Pi_\phi(x,x')\right]\varphi_q(x') - \frac{2}{\hbar}\int d^4x\, d^4x'\,\theta(t-t')\,\varphi_q\,\mathrm{Im}\left[\Pi_\phi(x,x')\right]\varphi_c + (\chi \leftrightarrow \varphi)\,,$$

Imaginary term

Real term

Morikawa, Matarrese et al.

- Where:

$$\Pi_\phi(x,x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a^3(t)\left[P_t\phi_{\mathbf{k}}(t)\right]e^{-i\mathbf{k}\cdot\mathbf{x}}a^3(t')\left[P_{t'}\phi^*_{\mathbf{k}}(t')\right]e^{i\mathbf{k}\cdot\mathbf{x}'}\,,$$

$$P_t = \left[\ddot{W}_H(t) + 3H\dot{W}_H(t) + 2\dot{W}_H(t)\partial_t\right]\;;$$

& defined the *quantum* and *classical* fields, rotating to the Keldysh basis:

$$\begin{pmatrix}\varphi_c\\ \varphi_q\end{pmatrix} \equiv \begin{pmatrix}\frac{\varphi^+ + \varphi^-}{2}\\ \varphi^+ - \varphi^-\end{pmatrix}, \qquad \begin{pmatrix}\chi_c\\ \chi_q\end{pmatrix} \equiv \begin{pmatrix}\frac{\chi^+ + \chi^-}{2}\\ \chi^+ - \chi^-\end{pmatrix}.$$

Fluctuation-Dissipation Theorem

- Leading order influence action splits into a real and an imaginary part -> they represent dissipation and noise, respectively.

- The kernels $\mathrm{Im}\,[\Pi_{\phi,\psi}(x,x')]$ are the dissipation kernels. *i.e.* their non-symmetric part add a non-local extra term in the classical fields EoM, proportional to $\dot{\varphi}_c$ and $\dot{\chi}_c$ $\Rightarrow$ friction, or dissipation.

  Slow-roll $\Rightarrow$ negligible compared to H-friction

  Not negligible $\Rightarrow$ *e.g.* warm inflation Berera et al. 2009

- The kernels $\mathrm{Re}\,[\Pi_{\phi,\psi}(x,x')]$ each give an imaginary part to the effective action. Interpret them as a result of a weighted average over configurations of stochastic noise terms, representing the «coupling» btw φ and $\phi_>$ and χ and $\psi_>$ $\Rightarrow$ reintroduce these noises w/ right PDF Morikawa

- Fluctuation-dissipation thm: they are linked since they come from the same underlying dofs $\Rightarrow$ real and imaginary part of same kernel!

- To interpret the imaginary part as noise, introduce two real classical random fields per field in the system ξ_1^ϕ, ξ_2^ϕ and ξ_1^ψ, ξ_2^ψ, each obeying the Gaussian pdf: Stratonovich, Hubbard

$$\mathcal{P}\left[\xi_1^{\phi,\psi}, \xi_2^{\phi,\psi}\right] = \exp\left\{-\frac{1}{2}\int d^4x d^4x' [\xi_1^{\phi,\psi}(x), \xi_2^{\phi,\psi}(x)]\mathbf{A}^{-1}(x,x') \left[\begin{array}{c} \xi_1^{\phi,\psi}(x') \\ \xi_2^{\phi,\psi}(x') \end{array}\right]\right\},$$

$$\int_{\varphi_i^\pm}^{\varphi_f^+} \mathcal{D}\varphi^\pm \int_{\chi_i^\pm}^{\chi_f^+} \mathcal{D}\chi^\pm \exp\left[\frac{i}{\hbar} S_{eff}^{(1)}\right]$$

$$= \int \mathcal{D}\varphi^{q,c}\mathcal{D}\chi^{q,c} \int \mathcal{D}\xi_1 \mathcal{D}\xi_2 \mathcal{P}[\xi_1, \xi_2] \exp\Big(i\int d^4x a^3(t) \{\varphi_q \left[\left(\Box - m_\Phi^2\right)\varphi_c - V_{\text{pert},\Phi}(\varphi_c, \chi_c)\right]$$

$$+\chi_q\left[\left(\Box - m_\Psi^2\right)\chi_c - V_{\text{pert},\Psi}(\varphi_c,\chi_c)\right] + \varphi_q\left[p_\phi(t)\xi_1^\phi + \xi_2^\phi\right] - \dot{\varphi}_q\xi_1^\phi + \chi_q\left[p_\psi(t)\xi_1^\psi + \xi_2^\psi\right] - \dot{\chi}_q\xi_1^\psi\}\Big),$$

- To take the classical limit of the action: rescale $\varphi_q, \chi_q \to \hbar\varphi_q, \hbar\chi_q$ and expand in powers of $\hbar$. EoM in the classical limit are given by:

$$\left.\frac{\delta S_{eff}^{(1)}}{\delta\varphi_q}\right|_{\varphi_q=0} = 0 \quad ; \quad \left.\frac{\delta S_{eff}^{(1)}}{\delta\chi_q}\right|_{\chi_q=0} = 0.$$

We obtain:

$$(-\Box + m_\Phi^2)\varphi_c + \tilde{V}_{,\Phi}(\varphi_c, \chi_c) = p_\phi(t)\xi_1^\phi + \xi_2^\phi + \dot{\xi}_1^\phi + 3H\xi_1^\phi,$$
$$(-\Box + m_\Psi^2)\chi_c + \tilde{V}_{,\Psi}(\varphi_c, \chi_c) = p_\psi(t)\xi_1^\psi + \xi_2^\psi + \dot{\xi}_1^\psi + 3H\xi_1^\psi.$$

The noise correlations are found by solving the linearized mode functions:

$$\Lambda_\phi \phi_k = 0$$
$$\Lambda_\psi \psi_k = 0$$

These are indeed two coupled systems

These are the same as in the heuristic approach, provided we perform a simple redefinition of $\xi_2^{\phi,\psi}$

$$\xi_2^\phi \to -p_\phi(t)\xi_1^\phi - \xi_2^\phi$$

Perturbative Expansion

- Easy to extend this formalism to include non-trivial interacting potential;

- Introduce a current per branch of the CPT contour, J^+ and J^- and define a diagrammatic expansion of V, integrate order by order the bath fields, and derive a similar influence action;

Morikawa, Hu et al., Boyanovsky

- Quadratic terms in the bath fields are considered as part of the free bath propagator, $e.g.\ \phi_>^2\varphi^2$ coming from $V_{pert} \supset \Phi^4$

$\Rightarrow$ To solve for the noise variance using the full linearized mode function EoM, we obtain 2 coupled system, which justifies a recursive approach

- For every loop correction, we obtain an extra noise term, dissipation term (real and imaginary part of the same kernel), and mass-renormalisation term

Example: $V_{pert} = g^2\Phi^2\Psi^2 + \frac{\lambda}{4!}\Phi^4$

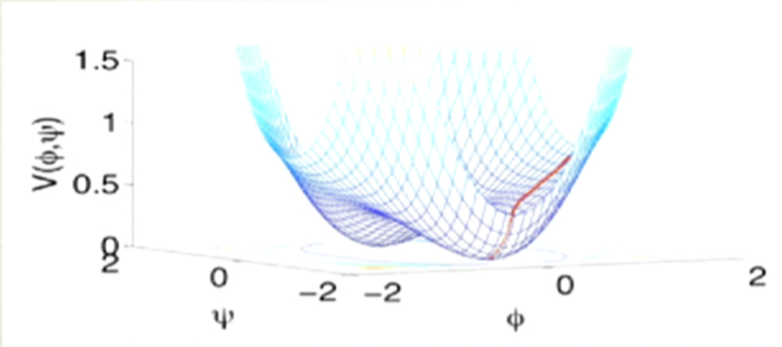
↳ leading order noise: 

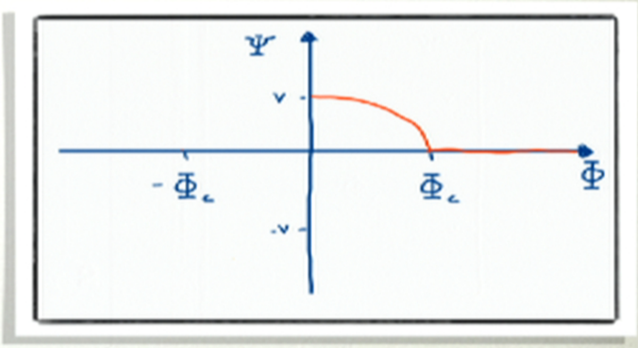
bath self-interactions correct the bath propagator, eg: $\lambda\, \phi_>^4 \Rightarrow$ 

↳ Higher order noise: eg. $\frac{\lambda}{3!}\varphi\,\phi_>^3 \Rightarrow$ 

or $g\,\varphi\,\chi\,\phi_>\psi_> \Rightarrow$ 

Example 1: Hybrid Inflation





- Two scalar fields inflation: the inflaton Φ and the waterfall field Ψ
- Inflation takes place when ϕ is slowly rolling for $\Phi > \Phi_c$
- The energy density is dominated by the mass of ψ
- For $\Phi < \Phi_c$, the $\Psi \to -\Psi$ symmetry is broken and ψ develop a tachyonic instability, which trigger its rapid rolling toward a true ground state

Dynamics and Steps 1-2

- Potential:

$$V(\Phi,\Psi)=\frac{1}{2}m^2\Phi^2+\frac{\lambda}{4}(\Psi^2-v^2)^2+\frac{g^2}{2}\Phi^2\Psi^2$$

Recursive Solution:

- Step 1: Free, massless dS noise:

$$\langle\xi_1^{\phi,\psi}(\mathbf{x},t),\xi_1^{\phi,\psi}(\mathbf{x},t)\rangle=\frac{H^3}{4\pi^2}\frac{\sin(\epsilon aHr)}{\epsilon aHr}\delta(t-t')$$

- Step 2: Zeroth order stochastic equations:

$$3H^2\frac{\mathrm{d}\varphi}{\mathrm{d}N}=-m^2\varphi\left(1+\frac{g^2\chi^2}{m^2}\right)+3H\xi_\phi(N)\ ,$$

$$3H^2\frac{\mathrm{d}\chi}{\mathrm{d}N}=-\lambda v^2\chi\left(\frac{\varphi^2-\Phi_c^2}{\Phi_c^2}+\frac{\chi^2}{v^2}\right)+3H\xi_\psi(N)$$

- Step 2 (continued): Leading order coarse-grained stochastic solutions:

Martin & Vennin 2011

$$\langle \chi^2 \rangle = \frac{1}{384\pi^2} \frac{\lambda^2 v^8}{m^2 M_{pl}^4} \left(\frac{m^2 e^x}{\lambda v^2 x} \right)^{\frac{\lambda v^2}{m^2}} \Gamma \left(\frac{\lambda v^2}{m^2}, \frac{\lambda v^2}{m^2} x \right) ,$$

$$\varphi = \exp\left[-4 \frac{m^2 M_{pl}^2}{\lambda v^4} (N - N_{in}) \right] \left[\varphi_{in} + 2\sqrt{\frac{3}{\lambda}} \frac{M_{pl}}{v^2} \int_{N_{in}}^{N} \exp\left(4 \frac{m^2 M_{pl}^2}{\lambda v^4} n \right) \xi_\phi(n)\, dn \right] ,$$

Zeroth order dispersions:

$$\sigma_\varphi = \frac{\lambda v^4}{8\sqrt{6}\pi m M_{pl}^2} .$$

$$\sigma_\chi \equiv \sqrt{\langle \chi^2 \rangle - \langle \chi \rangle^2} = \frac{\lambda v^4}{8\sqrt{6}\pi m M_{pl}^2} \left(\frac{m^2 e^x}{\lambda v^2 x} \right)^{\frac{\lambda v^2}{2m^2}} \Gamma^{\frac{1}{2}} \left(\frac{\lambda v^2}{m^2}, \frac{\lambda v^2}{m^2} x \right) ,$$

$$\sigma_{\chi_c} \simeq \left(\frac{\lambda}{2\pi} \right)^{3/4} \left(\frac{v}{3m} \right)^{1/2} \frac{v^3}{8 M_{pl}^2} .$$

STEP 3

Linearized quantum perturbations on a stochastically shifted background:
- Replace coarse-grained quantities with their stochastic mean:

$$F\left[\varphi^{(0)}, \chi^{(0)}\right] \to \left\langle F\left[\varphi^{(0)}, \chi^{(0)}\right]\right\rangle .$$

- We work in the spatially flat gauge;
- The EoM & solution for the canonically normalized field, $\delta\dot{\phi}_k^{(1)} = a^{-1} v_{\mathbf{k}}$

$$v_{\mathbf{k}}'' + \left[k^2 - \frac{2 - m^2/H^2 - g^2\sigma_\chi^2/H^2 + 9\varepsilon_1}{\tau^2}\right] v_{\mathbf{k}} = 0$$

$$v_{\mathbf{k}} \to \begin{cases} -e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \frac{2^{\nu-1}}{\sqrt{\pi}} \Gamma(\nu) \frac{(-\tau)^{-\nu+1/2}}{k^\nu} & 0 < \nu \leq 3/2 \\ e^{i\frac{\pi}{2}} (-\tau)^{1/2} \ln(-k\tau) & \nu = 0 \end{cases}$$

$$\nu^2 = 9/4 - (m^2 + g\sigma_\chi^2)/H^2 + 9\varepsilon_1$$

Step 3 (continued...)

Similarly, for $\delta\psi_k^{(1)} = a^{-1} u_{\mathbf{k}}$

$$u_{\mathbf{k}}'' + \left[k^2 - m_u^2(\tau)\right] u_{\mathbf{k}} = 0 \,, \quad m_u^2(\tau) \equiv \frac{2 - m_\psi^2/H^2}{\tau^2}$$
$$= \frac{1}{\tau^2}\left[2 + 15\varepsilon_1 - 3\frac{\lambda\sigma_\chi^2}{H^2} - \frac{12M_{pl}^2}{v^2}\left(\frac{\varphi^{(0)^2}}{\varphi_c^2} - 1\right)\right]$$

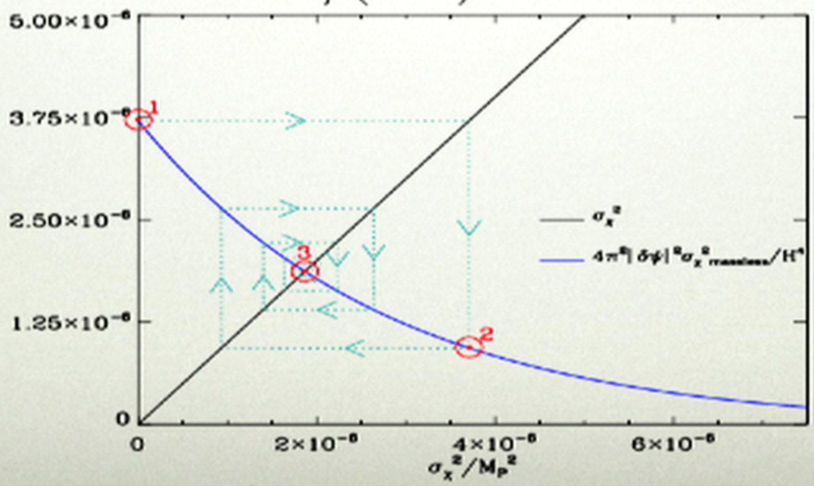
Solution in terms of Airy functions, but not very enlightening to write down...

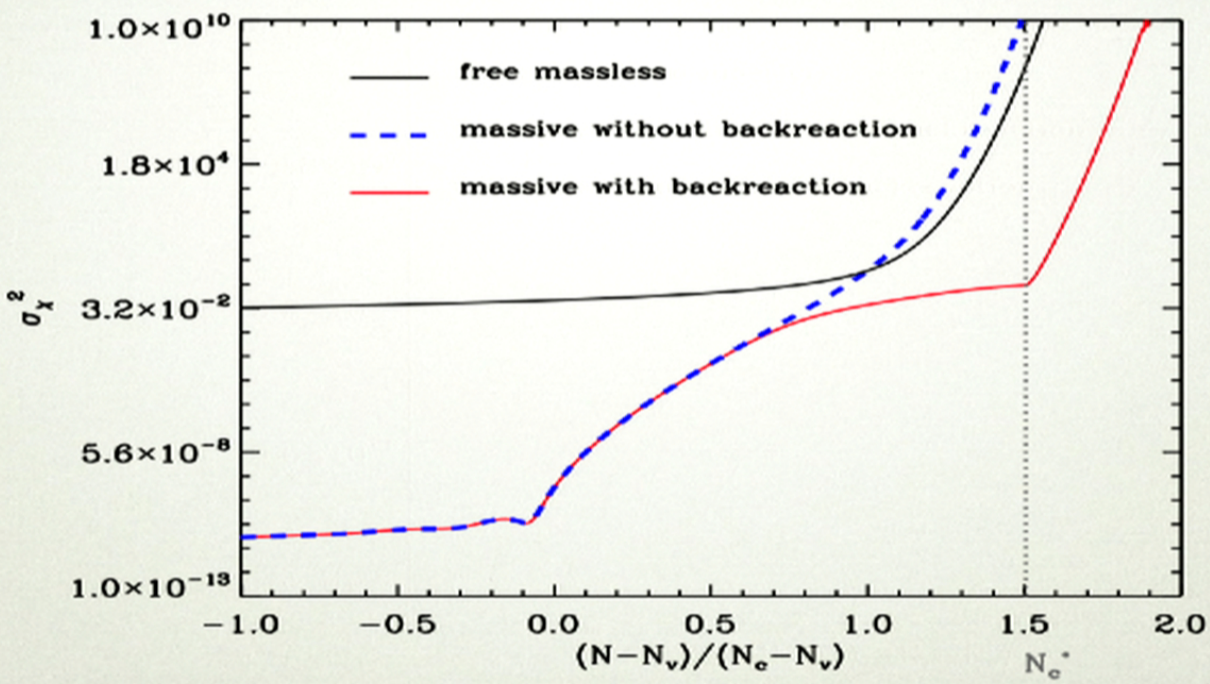
Step 4: Results

Under the quasi-static approximation, i.e. the relaxation time for the χ distribution is very small, and it swiftly acquires its "stationary" local dispersion.

$$\sigma_\chi^2 / \left.\sigma_\chi^2\right|_{\rm massless} \simeq \langle \xi_\psi^2 \rangle / \langle \xi_\psi^2 \rangle_{\rm massless} = |\delta\psi^{(1)}|^2 / |\delta\psi^{(1)}|^2_{\rm massless}$$

$$\Rightarrow \sigma_\chi^2 \simeq \frac{\left|\delta\psi^{(1)}\right|^2}{H^4/(4\pi^2)} \left.\sigma_\chi^2\right|_{\rm massless}$$




1.0×10^10
1.8×10^4
3.2×10^-2
5.6×10^-8
1.0×10^-13
σ_χ^2
free massless
massive without backreaction
massive with backreaction
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
(N−N_v)/(N_c−N_v)
N_c^*

For the inflaton:

- Noise amplitude

$$\langle \xi_\phi(N)\,\xi_\phi(N')\rangle = \frac{H^4}{4\pi^2}\delta(N-N')\left[1+\frac{2}{3}\frac{m^2+g\sigma_\chi^2}{H^2}(\ln 2\epsilon+\gamma-2)\right],$$

- Classical perturbations

$$\langle(\delta\varphi^{(1)})^2\rangle \approx \frac{3H^4\varphi_0^2}{8\pi^2\tilde{m}^2}\left(1-\frac{\varphi_0^2}{(\varphi_0)_{in}^2}\right)\left(1+\frac{2}{3}\frac{A}{H^2}\right)$$

$$\Rightarrow \left|\delta\varphi_k^{(1)}\right|^2 \approx \left(\frac{k}{aH}\right)^{\frac{2\tilde{m}^2}{3H^2}-\frac{4}{9}\frac{\tilde{m}^4}{H^4}(\ln 2\epsilon+\gamma-2)}$$

with:

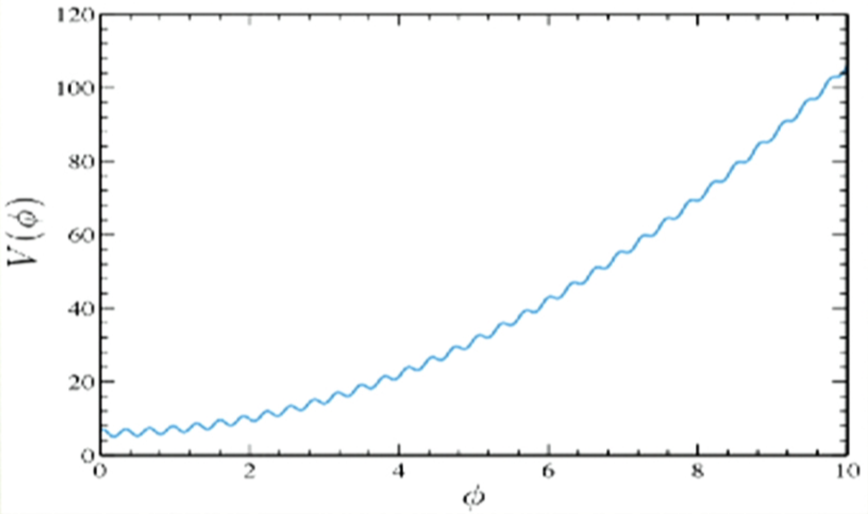
$$A = \tilde{m}^2(\ln 2\epsilon+\gamma-2) \qquad \tilde{m}^2 = (m^2+g^2\sigma_\chi^2)$$

Generalization to Inflation with Heavy Fields

- If multiple fields are present during inflation, it is easy to generalize this result to capture the effect on the inflaton
- $g^2\Phi^2\Psi^2$-type of couplings always make the inflaton *heavier*

=> in chaotic-type models, makes the tilt *redder*

Example 2: Modulated Potential



$$V(\Phi) = V_{sr}(\Phi) + \Lambda^4 \sin\left(\frac{\Phi}{f}\right)$$

$$v_k'' + \left\{ k^2 - \frac{1}{\tau^2}\left[2 + 9\epsilon_1 - \left[m^2 - \frac{\Lambda^4}{f^2}\cos\left(\frac{\varphi}{f}\right) \right] \frac{1}{H^2} + \frac{4\epsilon_1}{3H^2}\frac{\Lambda^4}{f^2}\cos\left(\frac{\varphi}{f}\right) \right] \right\} v_k = 0$$

Resonance

- Frequency of the driving force: $\omega \sim -\dfrac{m^2}{3H^2}\dfrac{\varphi_0}{f}$

- 'Fuzziness' of the coarse-grained field over one period:

$$\frac{\sqrt{3f}H}{m\sqrt{\phi_0}}\xi_1$$

Conclusion

- Reviewed stochastic inflation starting from the EoM
- Proposed a recursive approach
- Showed how this is motivated from the microphysics of stochastic inflation
- Applied the recursive to derive new results in hybrid inflation (tilt, dispersion of the waterfall field…), multi-field inflation, modulated potentials…