Title: TCSA: solving RG flows using Conformal Field Theory?

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Abstract: The Truncated Conformal Space Approach (TCSA) is a method that allows for controlled calculations in strongly coupled quantum field theories. It can be used whenever a theory is conformally invariant at short distances, and allows infrared quantities to be calculated in terms of the CFT data of this UV theory.

As an example, I will discuss how the TCSA applies to phi^4 theory in D dimensions and I will show that the method reliably reproduces several nonperturbative phenomena.

Motivation

- In QFT, think of Lagrangian field theories with

$$\mathcal{L} = \text{kinetic terms} + \sum_I g^I(\mu)\,\mathcal{O}_I(x)$$

  (or even non-Lagrangian theories.)
- If any $\mathcal{O}_I$ relevant, couplings are large at low energies:

$$g^I(\Lambda_{\text{IR}}) = O(1)$$

- We can only avoid this problem if many extra symmetries are present.
- Fall-back option is lattice: quantitative measurement with error bars but expensive and not always possible.

Motivation (II)

- Ex: $SU(N_c)$ Yang-Mills in $d \leq 4$ dimensions + massless fermions. Phase structure depends on two parameters: N_c, N_f.
- Border between confinement and IR CFT?

$$? < N_f < 11/2\, N_c$$

- Source of both fundamental and practical questions.

Role of Conformal Field Theory

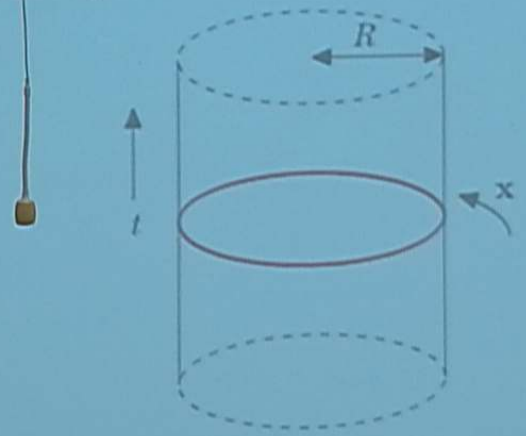
- Turn 'problem' around: relevant perturbations are well-behaved at high energies, $g^I \to 0$ in UV.
- Symmetry enhancement: Poincaré $\to$ Poincaré + dilatations. Generically full conformal group $= SO(d+1,1)$. RG flow = deformation away from CFT/critical point
- Many examples where UV theory is free, but not necessarily so: think of minimal model flows.
- Is it possible to get mileage out of this strategy?

Radial quantization (I)

- A CFT on $\mathbb{R}^d$ can be put on any conformally flat space $\mathcal{M}$.
- Natural choice: $\mathcal{M} = \mathbb{R} \times S^{d-1}$ of radius R – will become RG scale. "Time" on cylinder $\leftrightarrow$ radius r in flat space through

$$t = R \ln r$$

Dilatations map to time translations.



Radial quantization (II)

- Operator-state correspondence:

$$\mathcal{O}_n(x) \quad \leftrightarrow \quad |n\rangle = \mathcal{O}_n(t = -\infty)|0\rangle.$$

- Every operator has a scaling dimension

$$\mathcal{O}_n(\lambda x) = \lambda^{-\Delta_n}\mathcal{O}_n(x).$$

Hamiltonian H_{CFT} is generator of dilatations:

$$H_{\mathrm{CFT}}|n\rangle = \frac{\Delta_n}{R}|n\rangle.$$

Few states with $\Delta = O(1-10)$, exponentially many as $\Delta \to \infty$.

Radial quantization with interactions

- How about adding a relevant operator, e.g.

$$\mathcal{L} = \mathcal{L}_0 + g\,\mathcal{O}(x)?$$

Maps to

$$H = H_{\text{CFT}} + g\,R^{d-\Delta}\int_{\text{sphere}} \mathcal{O}(\mathbf{n}, t=0).$$

This is quantum mechanics with a potential

$$V_{ij} = \int_{\text{sphere}} \langle i|\mathcal{O}(\mathbf{n})|j\rangle.$$

- If $g(R) = g\,R^{d-\Delta}$ small $\rightarrow$ old-fashioned perturbation theory.
- What about $g(R)$ large?

Truncation

- Proposal [Yurov-Zamolodchikov 1989]: truncate Hilbert space i.e. keep only states with scaling dimension $\Delta \leq \Delta_{\text{max}}$.
- Physically speaking, this is a UV cutoff

$$\Lambda_{\text{UV}} = \Delta_{\text{max}}/R.$$

  Physics at energies $E \lesssim \Lambda_{\text{UV}}$ should be safe.
- Main advantage: Hilbert space becomes finite
  $\rightarrow$ do calculations on computer.
  Can extract interacting spectrum E_i directly via Hamiltonian

Setup

Which ingredients needed to do a TCSA calculation?

- Truncated Hilbert space

$$\mathcal{H}_{\text{low}} = \{\mathcal{O} \,|\, \Delta[\mathcal{O}] \leq \Delta_{\text{max}}\}$$

- A finite number of three-point functions

$$c_{123} \sim \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle$$

in the UV CFT.

Both are part of the definition of the UV CFT.
No loop calculations etc. needed

Example: massive boson (I)

- Consider the massive boson in d dimensions

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m^2 \phi^2$$

  Has $\mathbb{Z}_2$ symmetry $\phi \leftrightarrow -\phi$.
- Spectrum and Casimir energy on S^{d-1} provide check for method.
- Keep d arbitrary when possible (but $d > 2$).
- Hilbert space $\mathcal{H}_{\text{low}}$ contains all composite operators up to $\Delta_{\text{max}} = O(10-20)$

$$|0\rangle, |\phi\rangle, |\phi^2\rangle, \ldots, |\phi^2 \partial_\mu \phi \partial^\mu \phi\rangle, \ldots \qquad (\text{mod } \partial^2 \phi = 0)$$

  (for simplicity, take only spin-0 sector).

Example: massive boson (II)

- Matrix elements given by

$$V_{ij} = \frac{1}{R}\int_{\text{sphere}} \langle i|\phi^2(\mathbf{n})|j\rangle$$

  using Wick's theorem (algorithmically).
- Full Hamiltonian is now explicitly known

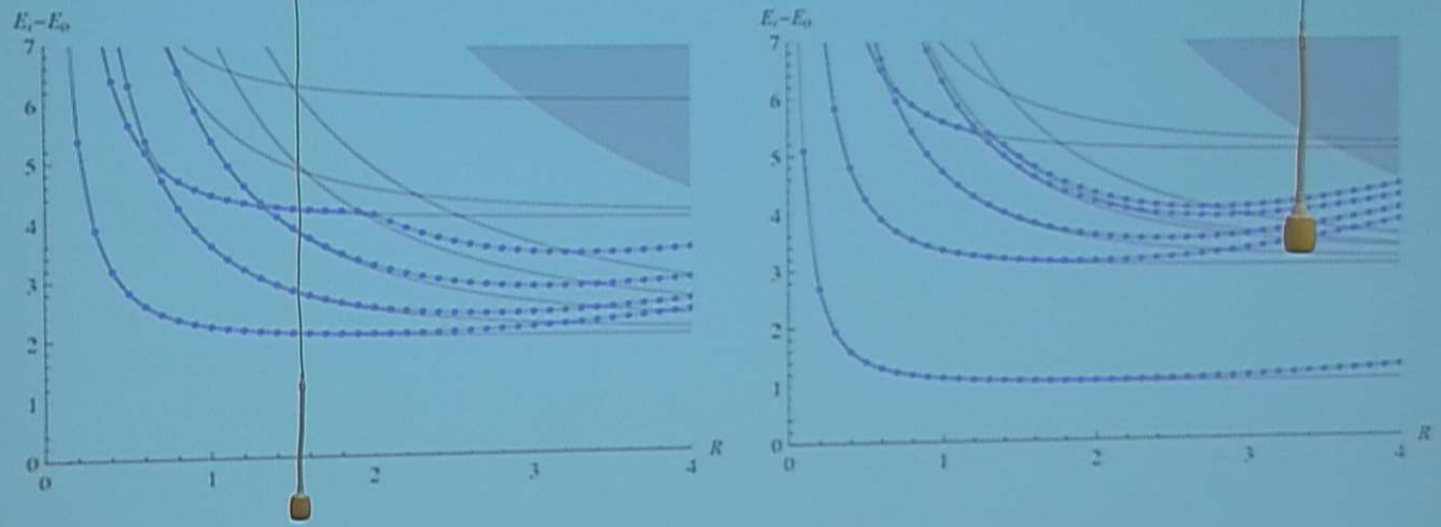
$$H_{ij} = (H_{\text{CFT}})_{ij} + g(R)V_{ij}, \qquad g(R) = m^2R^2.$$

- Scaling:

$$E_i \sim \Delta_i/R \qquad R \to 0,$$
$$E_i \sim m_i, \qquad R \to \infty.$$

Example: massive boson (III)

Results for $\mathbb{Z}_2$ even (left) and odd (right) – $d = 3$:



- Good match up to $mR \simeq 3$, beyond perturbative regime. Cf. $E \lesssim \Lambda_{UV}$
- Casimir energy agrees with same precision.

Scaling: $E_i \sim \Delta_i / R$ as $R \to 0$, $E_i \sim m_i$ as $R \to \infty$

Cutoff dependence (I)

- Hilbert space decomposes as $\mathcal{H} = \mathcal{H}_{\text{low}} \oplus \mathcal{H}_{\text{high}}$; in this basis

$$H = \begin{pmatrix} H_{ll} & H_{hl} \\ H_{lh} & H_{hh} \end{pmatrix}$$

- So far, we solve

$$H_{ll}|\tilde{\Psi}_i\rangle = \tilde{E}_i|\tilde{\Psi}_i\rangle \qquad \text{(bare TCSA)}$$

- The **exact** Schrödinger equation for $\mathcal{H}_{\text{low}}$ is

$$\left[H_{ll} - H_{lh} \cdot (H_{hh} - E_i)^{-1} \cdot H_{hl}\right] |\Psi_i\rangle = E_i|\Psi_i\rangle.$$

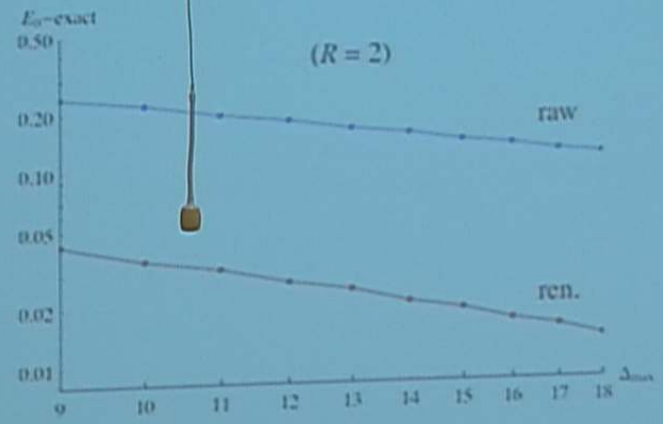
Not directly useful, but can extract several leading and subleading contributions.

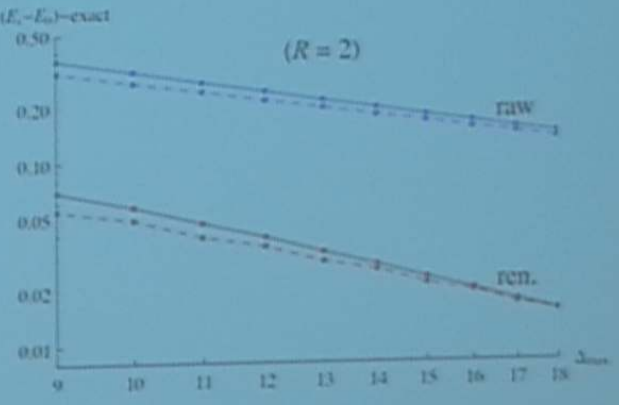
Cutoff dependence (II)

- Take-home message: at fixed R, truncation effects are power-law suppressed:

$$\tilde{E}_i = E_i + O\left(\frac{1}{\Lambda_{\mathrm{UV}}^{\#}}\right).$$

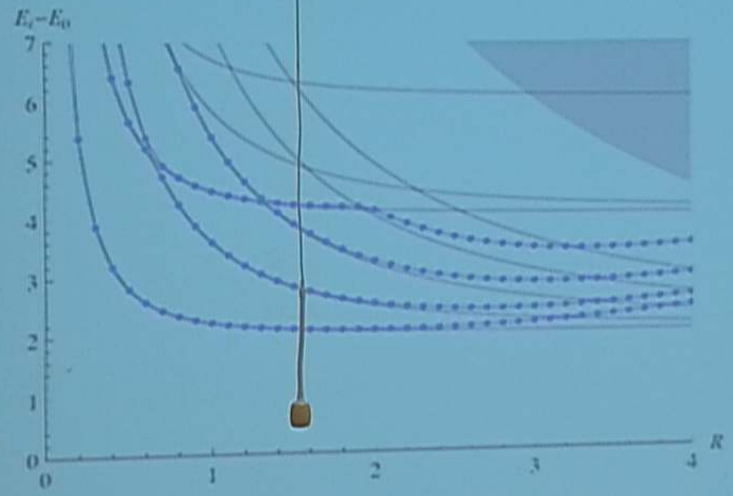
By adding suitable counterterms, the dependence on Λ_{UV} can be improved.

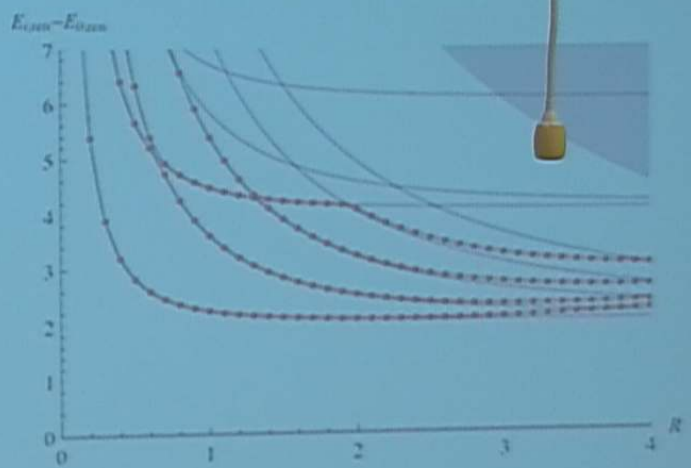




Cutoff dependence (III)

Including these effects extends the domain where TCSA is useful:





Cutoff dependence (IV)

- So far: considered improved Hamiltonians (à la Symanzik).
- Continuum limit $\Lambda_{UV} \to \infty$ not always naively well-defined.
- Ex. when perturbation is not very relevant, $d/2 < [\mathcal{O}] < d$. Can be seen in perturbation theory:

$$F_\Lambda = F_0 + \#g^2 \Lambda_{UV}^{2[\mathcal{O}]-d} + O(g^3).$$

- Counterterms that need to be added in UV depend on theory.

ϕ^4 theory (I)

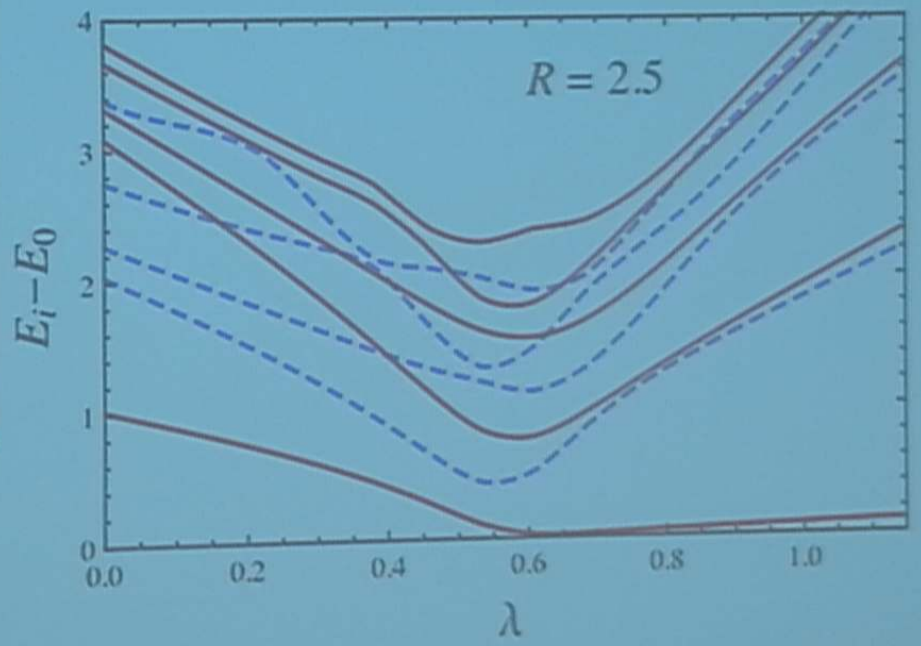
- Consider now ϕ^4 theory in $2 \leq d \leq 4$ dimensions

$$\mathcal{L} = (\partial_\mu \phi)^2 + m_0^2 \phi^2 + \lambda \phi^4$$

  Landau-Ginzburg model for the Ising model ($m_0^2 \sim T$).
  Again $\mathbb{Z}_2$-symmetric.
- $m_0^2 > m_*^2(\lambda)$: $\mathbb{Z}_2$ unbroken.
  $m_0^2 < m_*^2$: $\mathbb{Z}_2$ broken (double well).
  $m_0^2 = m_*^2(\lambda)$, Ising critical point.
- For $d < 8/3$, TCSA is fully finite.

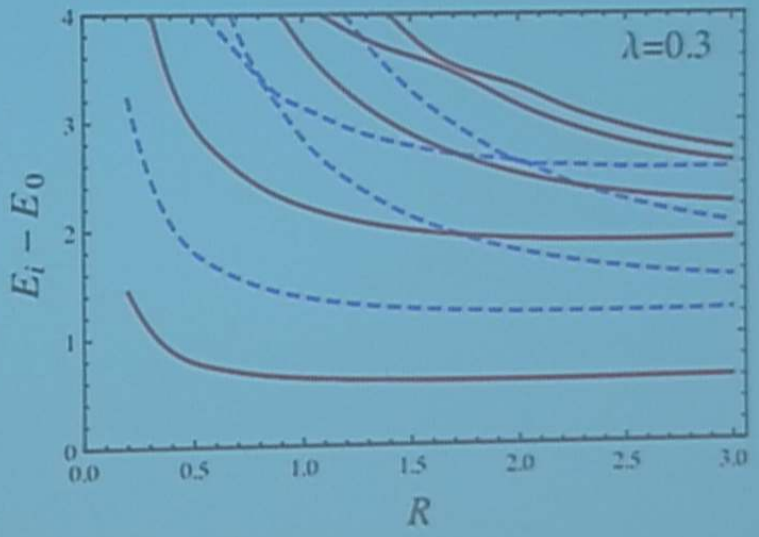
ϕ^4 theory (II)

- Rather than varying m_0^2, we set $m_0^2 \equiv 1$ and vary λ.
- Move away from the UV and plot how the spectrum changes:



ϕ^4 theory: unbroken phase

- For $0 < \lambda < \lambda_*$, theory is massive but does not correspond to free spectrum (dressed by $\lambda\phi^4$ interaction):

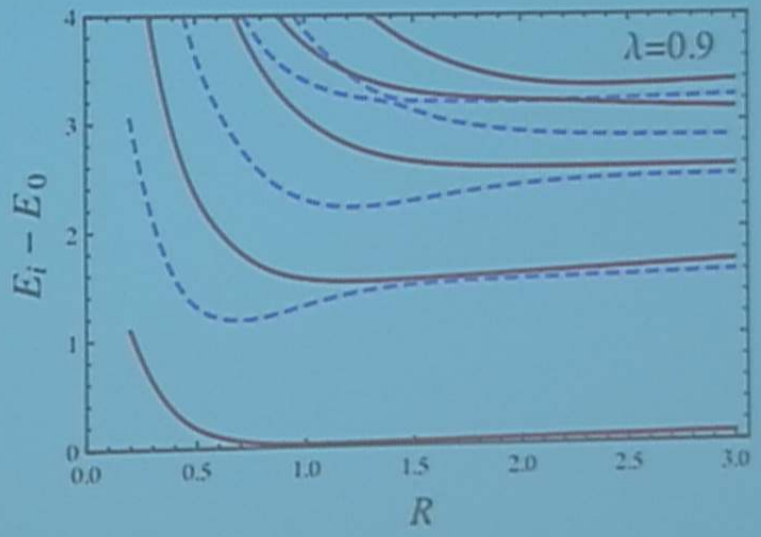


Level crossings avoided!

- These are all predictions. Match to other data for $d = 3$?

ϕ^4 theory: broken phase

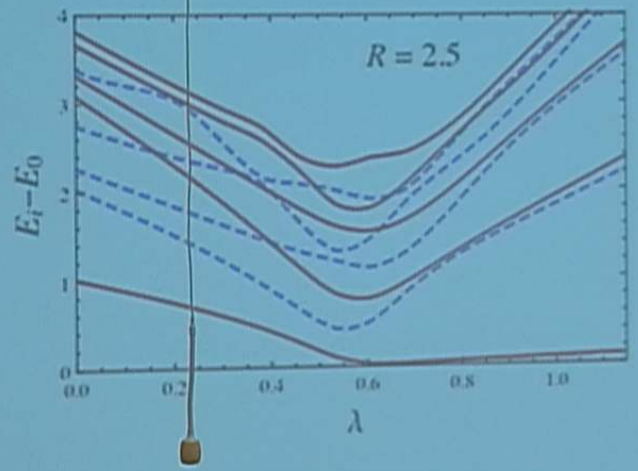
- For $\lambda > \lambda_*$, theory is massive but $\mathbb{Z}_2$ is broken.
- Tunneling suppressed as $\exp(-\text{cst} \times \text{volume})$.
  All states should come in pairs with small degeneracy:

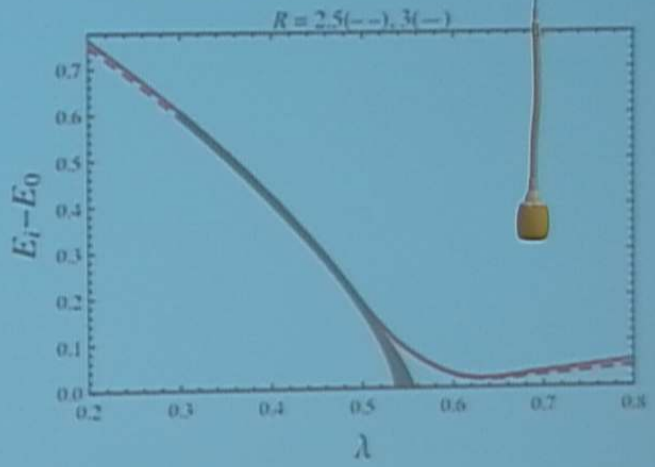


- Lowest excitations $m_1 \simeq 1.6$, $m_2 \simeq 2.5 \Rightarrow$ bound state.
  Cf. to MC result [Caselle-Hasenbusch-Provero] $m_2/m_1 \simeq 1.83$ for $d = 3$

ϕ^4 theory: tuning to critical point

- Around $\lambda = \lambda_* \simeq 0.55$, the mass gap becomes small:





- Approach to criticality: $E_1 \simeq |\lambda - \lambda_*|^\nu$.
- Bootstrap* in $d = 2.5$: $\nu = 0.755$, TCSA: $\nu = 0.65 - 0.8$.
- Errors due to finite R and finite cutoff.

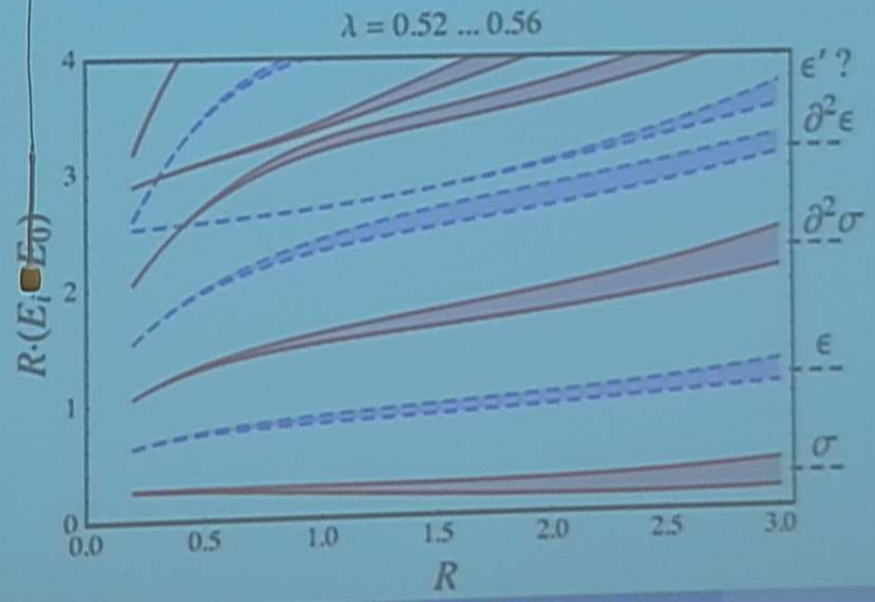
ϕ^4 theory: critical theory (II)

- Fix $\lambda = \lambda_*$. For a flow that interpolates between two CFTs we expect the scaling

$$E_i \sim \Delta_i^{\mathrm{UV}}/R \quad (R \to 0), \qquad E_i \sim \Delta_i^{\mathrm{IR}}/R \quad (R \to \infty)$$

- For d-dim Ising, good predictions for lowest scaling dimensions

$$[\sigma], \quad [\varepsilon].$$



Null states...

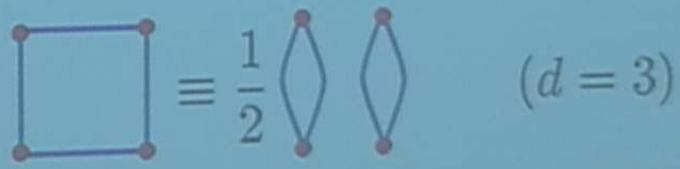
- The basis $|i\rangle = |\mathcal{O}_i\rangle$ we use is **not** orthonormal (not even organized in representations of conformal group).
- Existence of orthonormal basis $\leftrightarrow$ positive definite Gram matrix

$$G_{ij} = \langle i|j\rangle.$$

- For integer d some states $\{|\chi\rangle\}$ become null:

$$\langle i|\chi\rangle = 0 \quad \forall\langle i|$$

(consequence of trace relations/Cayley-Hamilton). Ex:



. . . and nonunitarity

- In $d = \text{integer} \pm \varepsilon$, some of these states have negative norm.
  Free boson in fractional d is nonunitary theory.
- $\lambda > 0$ in TCSA: presence of imaginary energies (high in spectrum)
  $\rightarrow$ implies that Ising in fractional d is also nonunitary.
- Work in progress: show same result in the ε expansion.
- No inconsistency with the fact that Ising in 2d, 3d is unitary
  (transfer matrix argument).
- Violations of unitarity seem tiny. Other probes?

Summary

- TCSA: nonperturbative method that allows for controlled calculations.
- Has passed several checks in 2d in the past; now also for general d.
- Very clean results for massive boson.
- In Landau-Ginzburg theory, we reliably reproduce
  IR phase diagram: strong coupling works.
  Results consistent with Ising critical exponents.
- Truncation effects are power-law suppressed; improved actions.

Outlook

- Generalization to fermions.
- **Renormalization** not completely under control (i.e. to take ϕ^4 in $d = 3$). Can even be tested in massive theories (scalar at high d, Dirac fermion).
- High-precision for $d = 2, 3$?
- SUSY (e.g. 2d superconformal minimal models)?
- **Gauge theories**: what is the best strategy? Can gauge invariance be kept manifest?