Title: Resurgence in quantum field theory: handling the Devil's invention

Date: Dec 09, 2014 02:00 PM

URL: http://pirsa.org/14120045

Abstract: <p>Renormalized perturbation theory for QFTs typically produces divergent series, even if the coupling constant is small, because the series coefficients grow factorially at high order. A natural, but historically difficult, challenge has been how to make sense of the asymptotic nature of perturbative series. In what sense do such series capture the physics of a QFT, even for weak coupling? I will discuss a recent conjecture that the semiclassical expansion of path integrals for asymptotically free QFTs - that is, perturbation theory - yields well-defined answers once the implications of resurgence theory are taken into account. Resurgence theory relates expansions around different saddle points of a path integral to each other, and has the striking practical implication that the high-order divergences of perturbative series encode precise information about the non-perturbative physics of a theory. These ideas will be discussed in the context of a QCD-like toy model theory, the two-dimensional principal chiral model, where resurgence theory appears to be capable of dealing with the most difficult types of divergences, the renormalons. Fitting a conjecture by â€™t Hooft, understanding the origin of renormalon divergences allows us to see the microscopic origin of the mass gap of the theory in the semiclassical domain.</p>

Resurgence in quantum field theory: dealing with the Devil's invention

Aleksey Cherman
FTPI, University of Minnesota

with various linear combinations of
Daniele Dorigoni (DAMTP, Cambridge U.),
Gerald Dunne (Connecticut U.),
Peter Koroteev (Perimeter Institute),
and Mithat Unsal (North Carolina State U.)

arXiv:1308.0127, 1403.1277, 1410.0388, …

The dark side of perturbation theory

In QFTs with small coupling λ, observables computable as

$$\mathcal{O}(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots$$

ask an undergrad | graduate student | postdoc, faculty | computers?

But in interesting QFTs like QCD, $c_n \sim n!$ for large n

Dyson 1952

Perturbation theory yields divergent series!

If perturbative expansions are divergent, then why do they work so well?

Why does the divergence happen?

Historically, this caused a lot of confusion…

The dark side of perturbation theory

In QFTs with small coupling λ, observables computable as

$$\mathcal{O}(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots$$

ask an undergrad — graduate student — postdoc, faculty — computers?

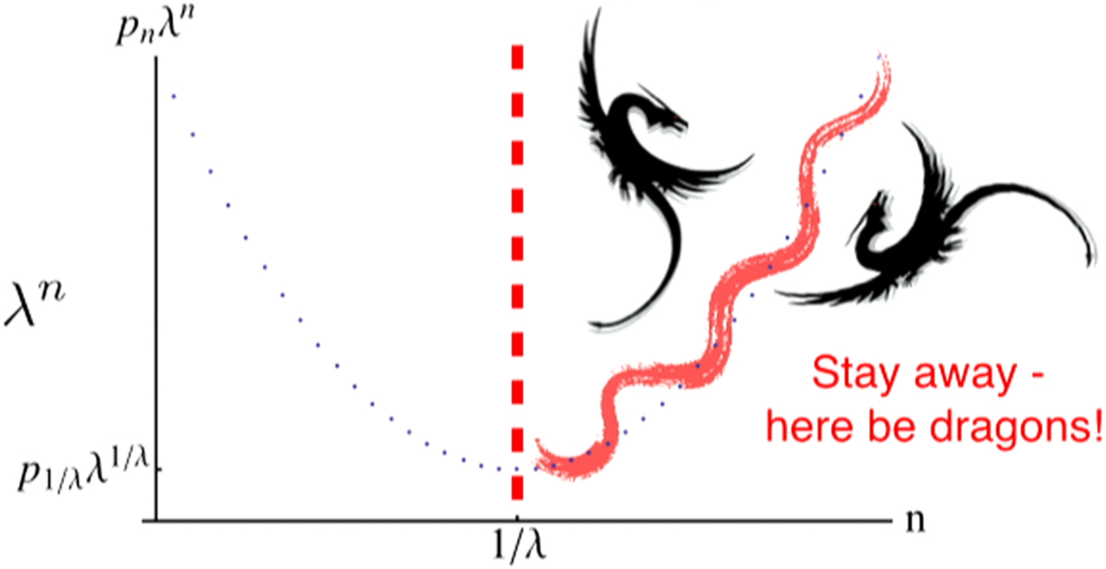
But in interesting QFTs like QCD, $c_n \sim n!$ for large n

Dyson 1952

"Divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever… Yet for the most part, the results [from using them] are valid… I am looking for the reason, a most interesting problem." Niels Henrik Abel 1828

Traditional view on asymptotic series

$$\sum_{n=1}^{\infty} p_n \lambda^n$$



Poincare 1886

Can argue that `mistake' made is order $e^{-1/\lambda}$

Exponentially small - so is it uninteresting?

$e^{-1/\lambda}$ is precisely scale of non-perturbative effects in e.g. QCD

In asymptotically-free theories, at least, non-perturbative effects drive the most interesting part of the physics!

A more systematic approach is called for…

Perturbation theory as a semiclassical expansion

$$\langle \mathcal{O}[\lambda] \rangle = Z[\lambda]^{-1} \int d[U]\, e^{-S(U;\lambda)} \mathcal{O}$$

regularized path integral

For small λ tempting to use saddle-point approximation

$$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} p_n \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,k} \lambda^k$$

Usually *all* of these series are sick, suffer from divergences!

Traditional view is that semiclassical expansions have an inherent and irreducible 'vagueness' of order $e^{-1/\lambda}$

Modern approach, based on resurgence theory:

'transseries' expansions are faithful and unambiguous (but subtle) representations of observables.

Perturbation theory as a semiclassical expansion

$$\langle \mathcal{O}[\lambda] \rangle = Z[\lambda]^{-1} \int d[U]\, e^{-S(U;\lambda)} \mathcal{O}$$

regularized path integral

For small λ tempting to use saddle-point approximation

$$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} p_n \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,k} \lambda^k$$

Usually *all* of these series are sick, suffer from divergences!

If above `transseries' is to encode well-defined smooth function of λ, need intricate relations connecting $p_{c,n}$ for different saddles

⇒ quantitative relations between perturbative and non-perturbative physics

Vainshtein, 1964; Bender+Wu 1969; Lipatov 1977

Resurgence theory is the detailed implementation of this idea

Dingle, Berry 1960+...

Ecalle: 1980s

Argyres, Dunne, Unsal… QFT

Aniceto, Marino, Schiappa… strings

How to think about asymptotic series?

original formal series

$$\mathcal{O} = \sum_{n=1}^{\infty} p_n \lambda^n, \quad p_n \sim n!$$

`Borel transform'

$$B[\mathcal{O}](t) \equiv \sum_{n=1}^{\infty} \frac{p_n}{(n-1)!} t^{n-1}$$

BO(t) defines function analytic within finite radius around t=0

Borel sum

$$\mathcal{SO}(\lambda) = \frac{1}{\lambda} \int_0^{\infty} dt \, e^{-t/\lambda} B[\mathcal{O}](t)$$

SO(λ) has same power expansion as O(λ)

Should think about *SO*(λ) as a useful representation of data in formal series with $|p_n| \leq n! \, c^n$

But the integral — and hence sum — doesn't always exist!

How to think about asymptotic series?

Borel sum: $$\mathcal{S}\mathcal{O}(\lambda) = \frac{1}{\lambda}\int_0^\infty dt\, e^{-t/\lambda} B[\mathcal{O}](t)$$

Working case: $$E(\lambda) = \sum_{n=0}^{\infty} (-1)^n\, n!\, \lambda^{n+1} \;\Rightarrow\; B[E(\lambda)] = \frac{1}{1+t}$$

No pole on R⁺ contour, Borel integral exists, resummation unambiguous

Failing case: $$E(\lambda) = \sum_{n=0}^{\infty} (+1)^n\, n!\, \lambda^{n+1} \;\Rightarrow\; B[E(\lambda)] = \frac{1}{1-t}$$ singularity on R⁺ !

Singularity on R⁺ contour, Borel sum does not exist.

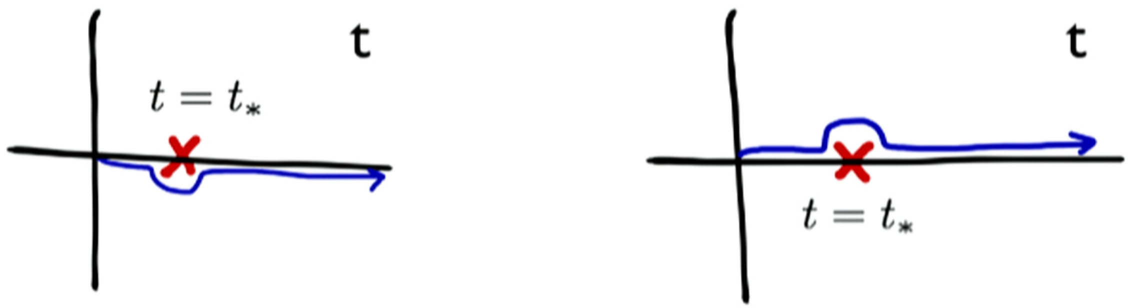
This is the typical situation in series coming from QFT

Why is this happening?

And what should we do about it?

How to treat non-Borel summable series?

Can deform contour, above or below real axis.



Amounts to analytic continuation of path integral $\lambda \to \lambda(1 \pm i\epsilon)$

Imaginary non-perturbative (NP) ambiguity in resummation, depending on direction of continuation

$$\mathcal{S}_{\pm}\mathcal{O}(\lambda) = \mathrm{Re}\left[\tilde{\mathcal{O}}(\lambda)\right] \pm 2\pi i\, e^{-t_*/\lambda}$$

Form of ambiguity points to the guilty party:

Contribution from NP saddle with action $S = t_*/\lambda$

Dingle, Berry +Howls…

Conspiracies between P and NP data

For such resummation ambiguities to cancel, perturbation theory must contain quantitative data about NP physics

$$\mathcal{O}(\lambda) \simeq \sum_n p_{n,P} \lambda^n + e^{-\frac{S_{NP}}{\lambda}} \sum_n p_{n,NP} \lambda^n + \cdots$$

Resurgence theory gives tools to decode the NP data hidden in an asymptotic P series, and vice-versa.

Example of an implication of resurgence theory (and origin of name!)

$$p_{n,P} \longrightarrow \frac{(n-1)!}{\pi \left(S_{NP}\right)^n} \left(p_{0,NP} + \frac{p_{1,NP} S_{NP}}{(n-1)} + \frac{p_{2,NP} S_{NP}^2}{(n-1)(n-2)} + \cdots \right) + \cdots$$

Through the lens of resurgence, we see that P and NP data are not independent, and must be treated together to get unambiguous results!

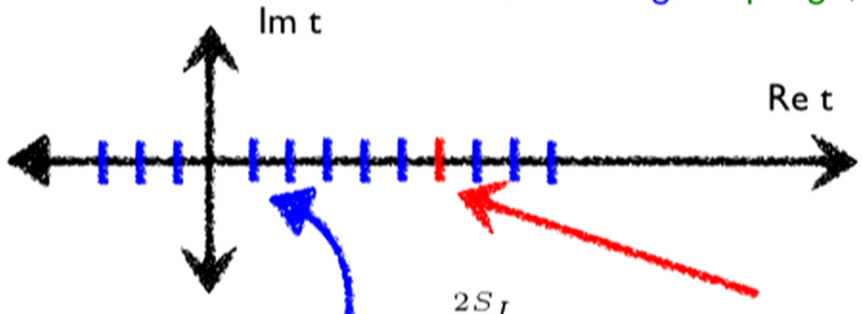
So how does all this work in QCD-like theories?

Borel plane singularities in QCD

't Hooft, 1979

(1) Combinatorial singularities — related to the number of diagrams at order n growing as n!

(2) Renormalon singularities — related to `single' planar diagrams with n running couplings, scale as n!



$t = 2S_I/|\beta_0| \Rightarrow e^{-\frac{2S_I}{|\beta_0| g^2}}$ $\qquad$ $t = 2S_I \Rightarrow e^{-2S_I/g^2}$

$\beta_0 = 11N/3$ so renormalon ambiguity >> 'instanton' ambiguity

Not just a formal problem!

Renormalons arise in pQCD calculations relevant for e.g. collider physics

Resulting ignorance parametrized by introducing 'power corrections' $(\Lambda/Q)^{\#}$

Borel singularities for QCD and its relatives

Inspiration: In QM, perturbation theory is also asymptotic.

Bogomolny; Zinn-Justin early 1980s

Perturbative ambiguities cancel precisely against ambiguities of instanton-anti-instanton events in QM

't Hooft's dream: QFT renormalons associated to some kind of fractional instantons, related to confinement

But no such configurations known in QCD on R^4, or in other asymptotically-free theories

Moreover, many asymptotically-free theories don't have instantons at all, let alone `fractional instantons'!

Argyres, Dunne, Unsal 2012-13

Key idea: find smooth compactification which preserves confinement, while driving theory to weak coupling.

Desired fractional instantons emerge, allow application of resurgence theory, yield systematic ambiguity cancellations.

SU(N) Principal Chiral Model

Focus for the rest of the talk:

$$S = \frac{1}{2g^2} \int_M d^2x \, \mathrm{Tr} \; \partial_\mu U \partial^\mu U^\dagger, \qquad U \in SU(N)$$

Why is it interesting?

Asymptotically free, like QCD
Dynamically generated mass gap, like QCD
Matrix-like large N limit, like QCD
Large N confinement-deconfinement transition, like QCD
Perturbation theory suffers from combinatorial and renormalon ambiguities, just like QCD
Integrable, $M = R^2$ S-matrix known, so easier than QCD — Fateev, Kazakov, Wiegmann

But $\pi_2[SU(N)] = 0$, so no instantons, unlike QCD!

Lack of known NP saddles seems like big difference from QCD.

Almost a nice toy model for QCD

Dealing with strong coupling

'Coupling is small' assumption for saddle-point expansion doesn't make sense in PCM: $\beta < 0$

Need a weakly coupled limit, while keeping mass gap etc, with physics adiabatically connected to original theory

Our approach is to put the theory on $M = R^{time} \times S^1(L)$

For small enough L, weak coupling guaranteed by asymptotic freedom

But with periodic boundary conditions, looks like a thermal circle!

small L | large L
$F/N^2 \sim 1$ | $F/N^2 \sim 0$

Resembles confinement/deconfinement transition in 4D YM!

In PCM, large N phase transition, finite N cross-over

Twisted boundary conditions

PCM has an $SU(N)_L \times SU(N)_R$ symmetry

$$U \to \Omega_L U \Omega_R^\dagger$$

Wide variety of sensible spatial boundary conditions:

$$U(x_1, x_2 + L) = e^{iL^{-1}H_L} U(x_1, x_2) e^{-iL^{-1}H_R}$$

Working with a gapped theory - when $L \gg \Lambda^{-1}$, choice of BCs doesn't matter

But at small L, dialing H_L, H_R parametrizes a wide family of distinct theories

Claim: unique choice of H_L, H_R such that physics appears to be adiabatically connected to large L limit

Twisted boundary conditions

Convenient to trade fields with twisted BCs for background gauge fields + fields with periodic BCs

$$\partial_\mu U \to \partial_\mu \tilde{U} - i\delta_{\mu,x_2} \left([H_V, \tilde{U}] + \{H_A, \tilde{U}\} \right)$$

$\tilde{U}$ is periodic

$$2H_{V,A} = H_L \pm H_R$$

Essentially 'chemical potentials' for spatial $SU(N)_{L,R}$ currents

$$J^L_\mu = i\, U^\dagger \partial_\mu U, \qquad J^R_\mu = i\, \partial_\mu U\, U^\dagger$$

Partition function now depends on $H_{V,A}$

$$Z \to Z(L; H_V, H_A)$$

What are the desirable 'adiabaticity conditions' in terms of Z?

(A) A free energy scaling as $F/N^2 \sim 0$ at large N

(B) Insensitivity of theory to changes in BCs

Adiabaticity conditions

At small L, complete insensitivity to BCs is not possible. Closest we can come is to pick H_V, H_A such that

1

$$\frac{\partial\left[\mathcal{V}^{-1}\log Z(L)\right]}{\partial H_V} = \langle J_x^V\rangle_{H_V,H_A} = 0$$

$$\frac{\partial\left[\mathcal{V}^{-1}\log Z(L)\right]}{\partial H_A} = \langle J_x^A\rangle_{H_V,H_A} = 0$$

Picks out BCs which extremize the free energy F

2

$$\frac{L^2}{N^2}\mathcal{F}\big|_{H_V,H_A} \sim 0$$

Make sure we stay in 'confining' phase

Our task: compute F(L; H_A, H_V) at small L, where theory is weakly coupled, and look at large N scaling of extrema

Small L Free Energy

$$V_{1-\text{loop}}(\Omega = \Omega_V, \Omega_A = 1) = \frac{-1}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(|\mathrm{Tr}\, \Omega^n|^2 - 1 \right)$$

One extremum corresponds to $H_V=0$, $H_A=0$

$$\Omega = \Omega_T \equiv 1_N$$ broken Z_N symmetry

$$F = -\frac{\pi}{6L^2}(N^2 - 1) = \mathcal{O}(N^2)$$

This is a deconfined small L limit.

Indeed, this are exactly the thermal BCs, and L = 1/T !

Clearly not what we want…

Small L Free Energy

$$V_{1-\text{loop}}(\Omega) = \frac{-1}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (|\text{Tr } \Omega^n|^2 - 1)$$

The only other (non-degenerate, Z_N preserving) extremum:

$$\Omega = \Omega_S \equiv e^{i\frac{\pi}{N}\nu} \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix}$$

v = 0,1 for
N odd, even

$$\log Z = \frac{-1}{\pi L^2} \times \frac{\pi^2}{6} = \mathcal{O}(N^0)$$

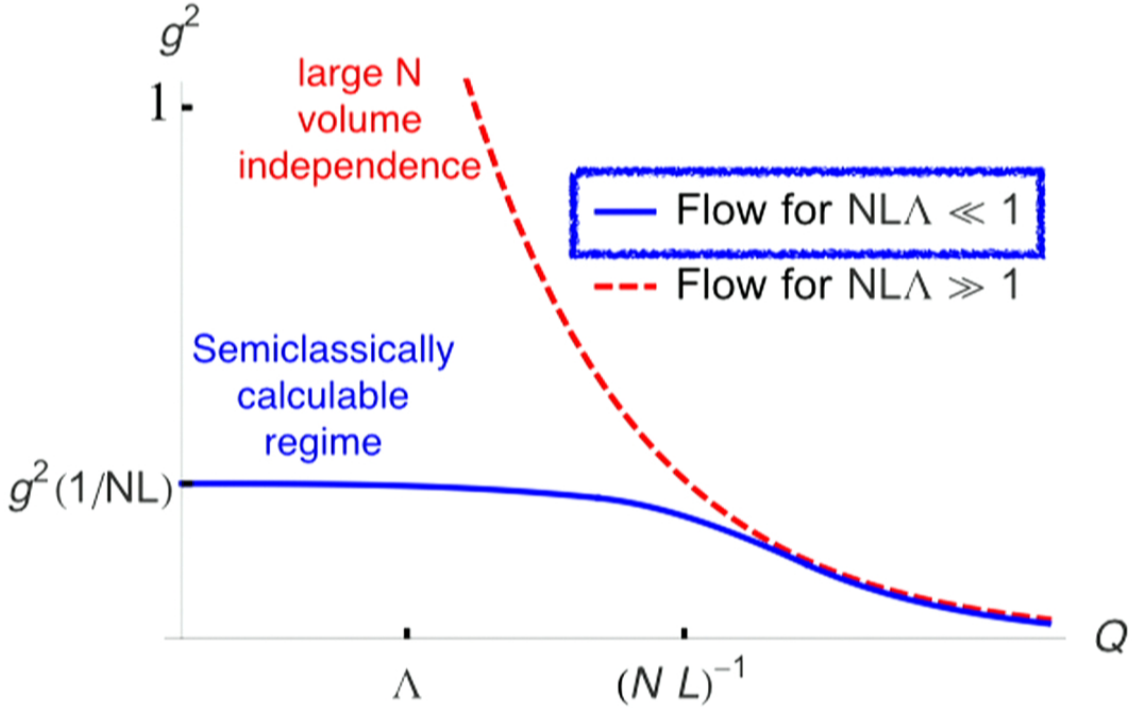
'Confinement' even at small L

Z_N-symmetric BCs give desired adiabatic small-volume limit.

Related construction of an adiabatic small L limit known for 4D YM theories

Unsal, Yaffe;
Shifman, Unsal; ...

Flow of coupling constant in Z_N-twisted PCM



Scale NL appears due to Z_N-symmetric form of H_V

We focus on $NL\Lambda \ll 1$ to get a weakly-coupled theory

Physics is very rich - mass gap, renormalons present at small N L!

Perturbation theory at small L

Pick dependence of ground state energy on λ as an observable

For small L, 2D PCM describable via a 1D EFT: quantum mechanics with a Z_N-symmetric background gauge field

Are renormalons still present?

In PCM, $|\beta| = N$. Renormalon means an ambiguity in perturbation theory of order

$$\sim \pm i e^{-\frac{\#}{g^2 N}}$$

On R^2, integrability calculations of Kazakov, Fateev, Wiegmann give:

$$\sim \pm i e^{-\frac{8\pi}{g^2 N}}$$

If small-L limit is adiabatic, expect size of renormalon ambiguity to move by order-1 amount as L goes from large to small.

But result should still involve $\#/g^2 N$

Perturbation theory at small L

Pick dependence of ground state energy on λ as an observable

For small L, 2D PCM describable via a 1D EFT: quantum mechanics with a Z_N-symmetric background gauge field

Are renormalons still present?

In PCM, $|\beta| = N$. Renormalon means an ambiguity in perturbation theory of order

$$\sim \pm i e^{-\frac{\#}{g^2 N}}$$

On R^2, integrability calculations of Kazakov, Fateev, Wiegmann give:

$$\sim \pm i e^{-\frac{8\pi}{g^2 N}}$$

If small-L limit is adiabatic, expect size of renormalon ambiguity to move by order-1 amount as L goes from large to small.

But result should still involve $\#/g^2N$

Perturbation theory at small L

SU(2) Example

$$U = \begin{pmatrix} \cos\theta e^{i\phi_1} & i\sin\theta e^{i\phi_2} \\ i\sin\theta e^{-i\phi_2} & \cos\theta e^{-i\phi_1} \end{pmatrix}$$

Hopf parametrization

$$S = \frac{1}{g^2}\int_{\mathbb{R}\times S^1} dt\, dx\, \left[(\partial_\mu\theta)^2 + \cos^2\theta(\partial_\mu\phi_1)^2 + \sin^2\theta(\partial_\mu\phi_2 + \xi\delta_{\mu,x})^2\right]$$

KK reduction

Imprint of Z_N-sym. twist $\xi=2\pi/(NL)=\pi/L$

$$S = \frac{L}{g^2}\int dt\left[\dot\theta^2 + \cos^2\theta\dot\phi_1^2 + \sin^2\theta\dot\phi_2^2 + \xi^2\sin^2\theta\right]$$

Compute perturbative expansion for ground state energy:

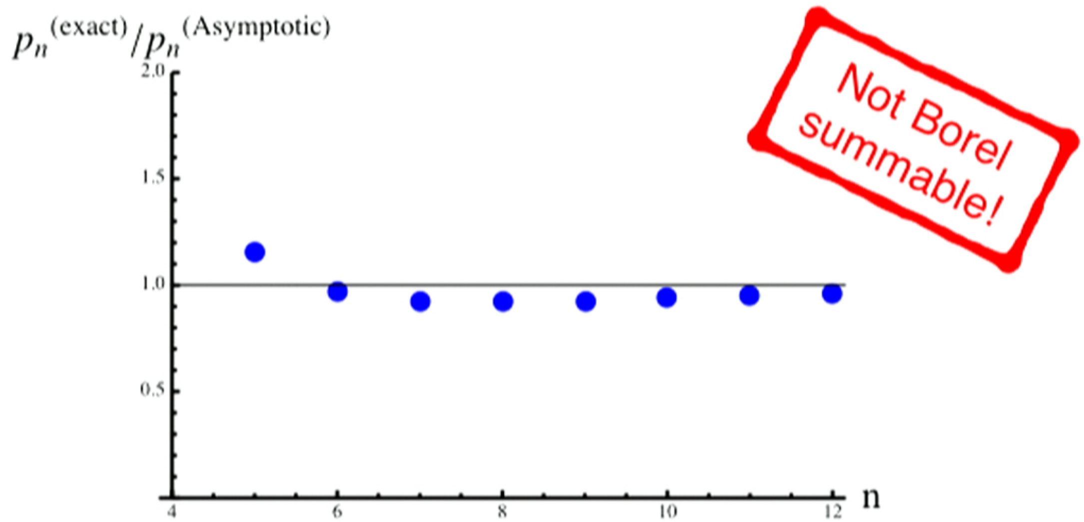
$$\mathcal{E}(g^2) = E\xi^{-1} = \sum_{n=0}^{\infty} p_n (g^2)^n$$

Large order structure of perturbation theory

Large-order behavior can be shown to be

Stone, Reeve
1978

$$p_n \sim -\frac{2}{\pi}\left(\frac{1}{\frac{16\pi}{N}}\right)^n n!\left[1-\frac{5}{2n}+\mathcal{O}(n^{-2})\right]$$



Factorially growing and non-alternating series!

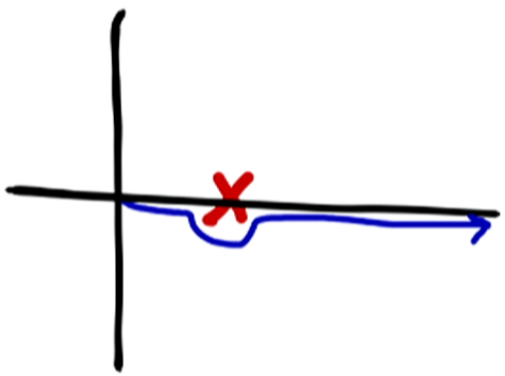
Non-perturbative ambiguity

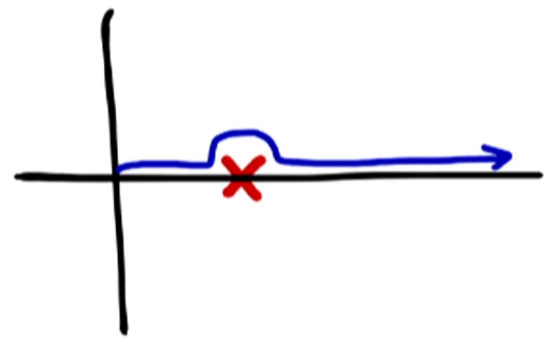
Borel transform of leading n! piece is

$$B\mathcal{E}(t) \sim \text{polynomial} + \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{t}{\left[\frac{16\pi}{N}\right]}\right)^n = \text{polynomial} - \frac{2}{\pi}\frac{1}{1-\frac{t}{\left[\frac{16\pi}{N}\right]}}$$

$$\mathcal{S}\mathcal{E}(g^2) = \int_0^\infty dt e^{-t/g^2} B\mathcal{E}(t)$$

Singularity on $C=R^+$ at $t = 16\pi/N$, Borel sum does not exist!





Non-perturbative ambiguity

$$\mathcal{S}_{\pm}\mathcal{E}(\lambda) = \int_{C_{\pm}} dt e^{-t/g^2} B\mathcal{E}(t)$$

$$= \mathrm{Re}\mathcal{S}\mathcal{E}(\lambda) \mp i \frac{32\pi}{\lambda} e^{-16\pi/\lambda}$$

λ = g²N
renormalon!

What to make of the red term?

(1) System is stable, ground state energy must be real!

(2) E must be well-defined - no sign-ambiguous parts allowed!

If E is a `resurgent function', perturbation ambiguities must cancel against ambiguities of some non-perturbative saddle F

$$\mathrm{Im}\left[\mathcal{S}_{\pm}\mathcal{E}(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0, \text{ up to } \mathcal{O}\left(e^{-4S_F}\right)$$

plus more intricate relations between P and NP physics at higher orders

But what are the relevant saddle points in the PCM?

Recall $\pi_2[SU(N)] = 0$…

Uhlenbeck 1985...

Non-topological saddle points

Finite-action `uniton' solutions of PCM EoMs are known

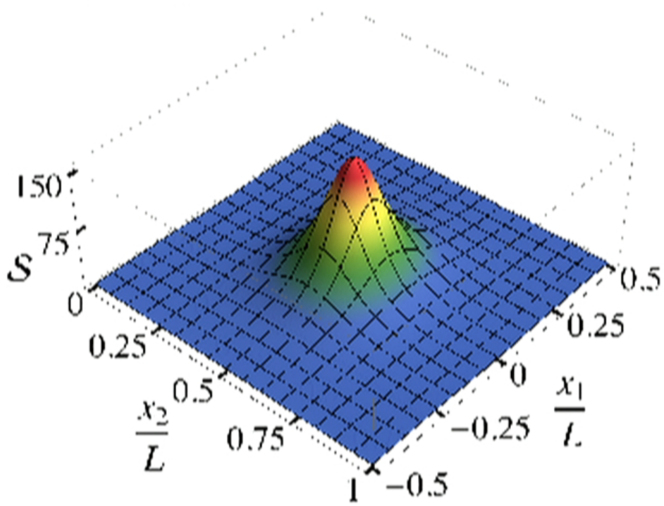
Based on observation that CP^{N-1} is a geodesic submanifold of SU(N)

CP^{N-1} instantons lift to uniton solutions in SU(N) PCM

Stable solutions within CP^{N-1} submanifold, but not in the full SU(N) manifold!

$$U(z,\bar{z}) = e^{i\pi/N}(1-2\mathbb{P}) \qquad \mathbb{P} = \frac{v\cdot v^\dagger}{v^\dagger \cdot v}$$

v(z), $z = x_1 + i\, x_2$ is the CP^{N-1} instanton in homogeneous coordinates



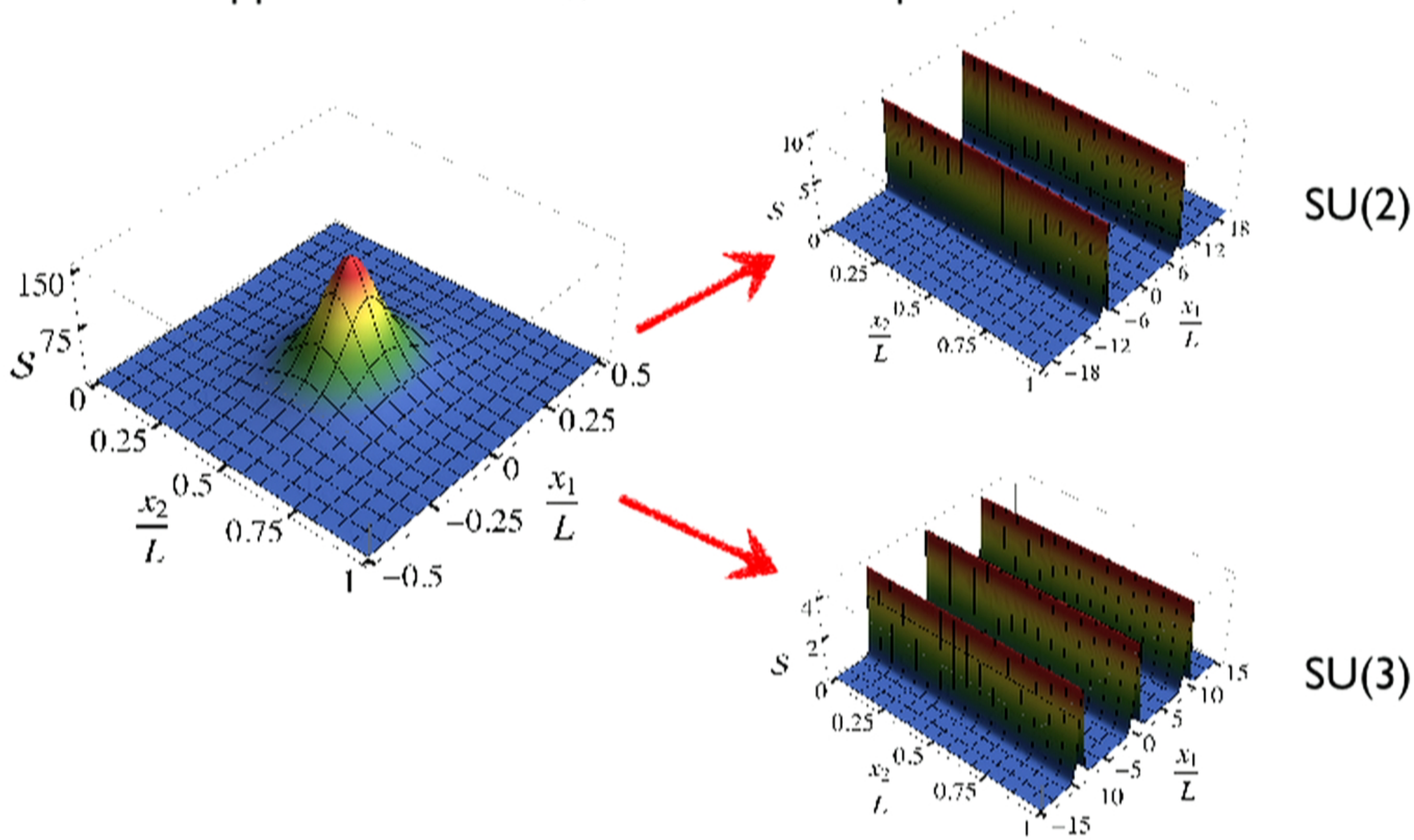
$$S_{\rm uniton} = \frac{8\pi}{g^2}$$

see also Smilga, Shifman in Schwinger model, 1994

Fractons

AC, Dorigoni, Dunne, Unsal

Uniton appearance with Z_N-twisted BCs depends on size modulus



Unitons fractionalize into N `fracton' constituents on small S^1

Fractons

AC, Dorigoni, Dunne, Unsal

SU(2) Example, small L effective theory:

$$S = \frac{L}{g^2} \int dt \left[\dot{\theta}^2 + \cos^2\theta \dot{\phi}_1^2 + \sin^2\theta \dot{\phi}_2^2 + \xi^2 \sin^2\theta \right]$$

Explicit solutions:

$$\theta(t; t_0) = 2\,\mathrm{arcCot}\left[e^{-\xi(t-t_0)} \right]$$

$$\bar{\theta}(t; t_0) = \pi - 2\,\mathrm{arcCot}\left[e^{-\xi(t-t_0)} \right]$$

ϕ_1 = const

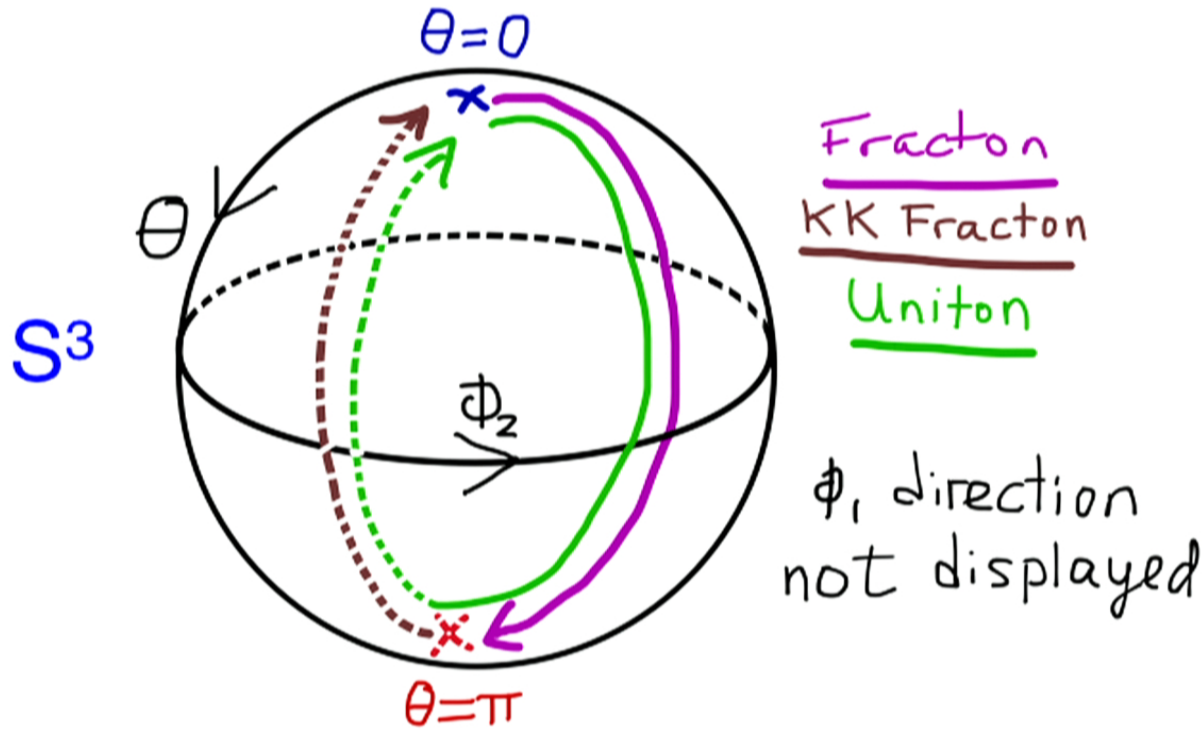
ϕ_2 = const

$$S_{\mathrm{fracton}} = \frac{8\pi}{g^2 N} = \frac{S_{\mathrm{uniton}}}{N}$$

N types of minimal-action fractons in SU(N)

N-1 fractons associated to N-1 simple roots of su(N)
The other - called KK fracton - associated to `affine root'

Unitons, Fractons, and KK fractons in SU(2) ~ S^3



SU(2) Uniton = fracton + KK fracton

The sum over finite-action configurations

$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n} \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n} \lambda^n$$

How can NP saddles give ambiguous contributions to path integral?

Small-L theory weakly coupled, dilute fracton gas approximation is valid

Contributions entering NP sum:

(1) Arbitrarily widely separated 'fundamental' fracton events

Within small-L EFT, individual fractons are just instantons, and are stable - 1-fracton events have unambiguous amplitudes

(2) Correlated multi-fracton events

Fluctuation sum includes zero modes, perturbative modes, and quasi-zero modes such as constituent separation

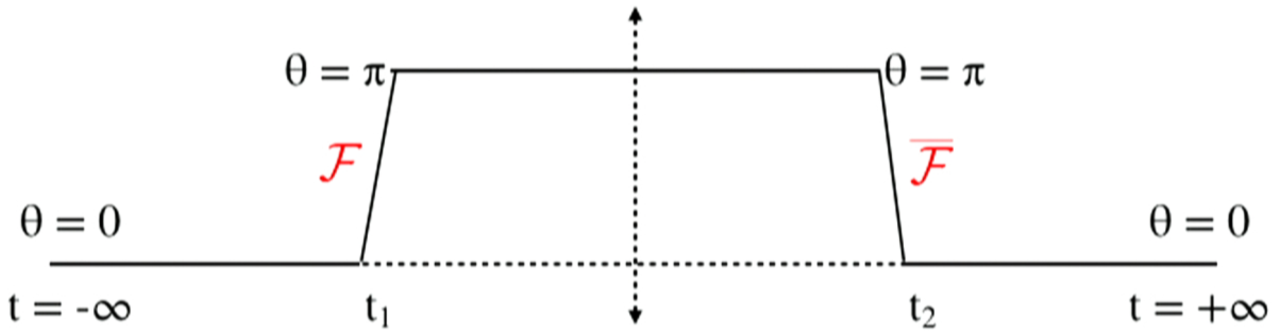
Gives rise to `correlated' events, some of which are ambiguous!

Contribution from fracton-anti-fracton events

Fracton size ~ $LN = \xi^{-1}$

Typical uncorrelated fracton separation ~ $LNe^{+8\pi/\lambda}$

But sometimes there are events which are closer than this!



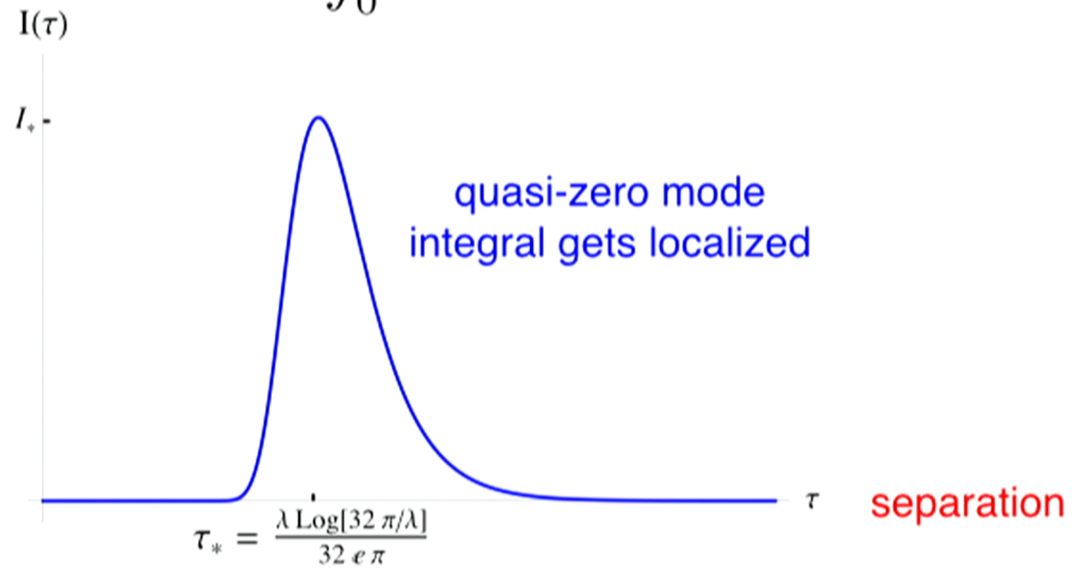
For large t_1 - t_2 this is a (quasi) saddle-point of the path integral

$t_1 + t_2$ is a zero mode, while $t_1 - t_2$ is a quasi-zero mode

Correlated multi-fracton events

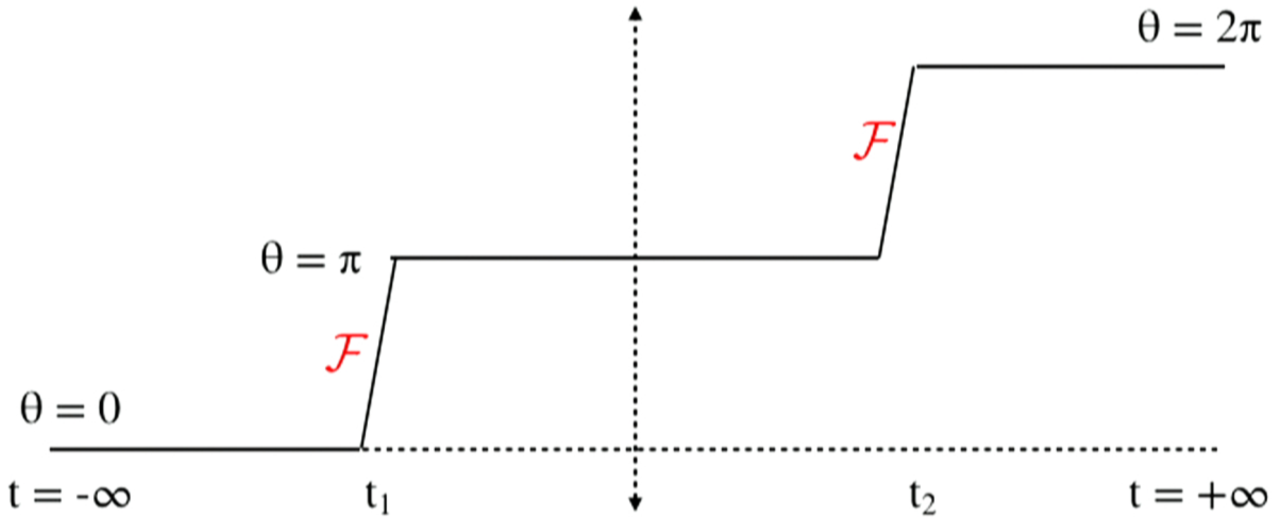
Correlated fracton-fracton events are unambiguous

$$I_{\mathcal{F}\mathcal{F}} \sim e^{-2S_F} \int_0^\infty d\tau \tau e^{-\left(\frac{32\pi}{\lambda} e^{-\tau} + \tau\right)} \qquad \tau = t_1 - t_2$$



amplitude: $$[\mathcal{F}\mathcal{F}] = \left(-\log\left[\frac{32\pi}{\lambda}\right] - \gamma\right) \frac{16}{\lambda} e^{-2S_F}$$

Contribution from fracton-fracton events

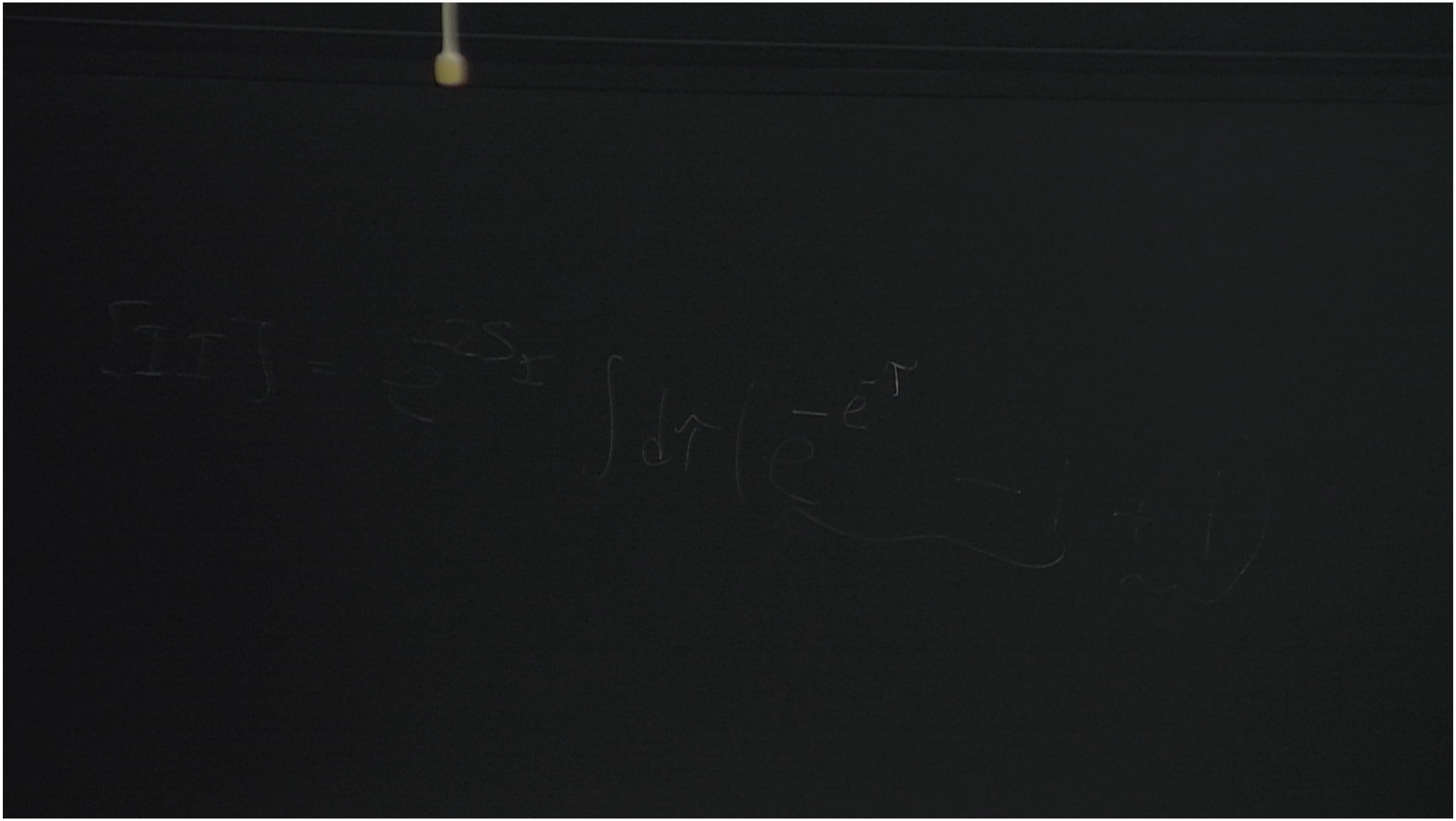


For large t_1 - t_2 this is a (quasi) saddle-point of the path integral

$t_1 + t_2$ is a zero mode, while $t_1 - t_2$ is a quasi-zero mode

Quasi-zero fluctuation mode sum gives another scale!

Correlated fluctuation size ~ $LN \log\left(\frac{32\pi}{\lambda}\right)$


$[II] = e^{-2S_I} \int d\tau \left(e^{-e^{-\tau}} \right.$

Correlated fracton-anti-fracton events

Correlated fracton-anti-fracton events are ambiguous

$$I_{\mathcal{F}\mathcal{F}} \sim e^{-2S_F} \int_0^{\infty} d\tau \tau e^{-\left(-1 \times \frac{32\pi}{\lambda} e^{-\tau} + \tau\right)}$$

The anti-fracton-fracton interaction is `attractive'!

Fracton-anti-fractons `want' to get close to annihilate

Since dilute gas approximation means all fractons must be widely separated, we should expect subtleties….

Making sense of fracton-anti-fracton events

Quasi-zero-mode integrals dominated by
$\tau=0$ region, do not make sense as written

Making sense of fracton-anti-fracton events

Quasi-zero-mode integrals dominated by
τ=0 region, do not make sense as written

This is a feature, not a bug.

Bogomolny,
Zinn-Justin

Analytically continue $g^2 \to g^2(1 \pm i\epsilon)$

Remember, we had to this for perturbation theory too!

Away from Im[g^2]=0, integral dominated by well-separated fractons

$$\begin{array}{ccccccc} r_{\mathcal{F}} & \ll & r_{[\mathcal{F}\overline{\mathcal{F}}]_\pm} & \ll & d_{[\mathcal{F}][\overline{\mathcal{F}}]} & \ll & d_{[\mathcal{F}\overline{\mathcal{F}}][\mathcal{F}\overline{\mathcal{F}}]} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \frac{1}{\xi} & \ll & \frac{1}{\xi}\log(4S_F) & \ll & \frac{1}{\xi}e^{+S_F} & \ll & \frac{1}{\xi}e^{2S_F} \end{array}$$

Analytic continuation back to positive g^2 is ambiguous!

$$[\mathcal{F}_j\bar{\mathcal{F}}_j]_\pm = \left(-\log\left[\frac{32\pi}{g^2 N}\right] - \gamma\right)\frac{16}{g^2 N}e^{-\frac{16\pi}{g^2 N}} \pm i\frac{16\pi}{g^2 N}e^{-\frac{16\pi}{g^2 N}}$$

Cancellation of ambiguities

Contribution from P saddle is ambiguous. So are some from NP saddles.

Neither is directly physical, only sum is. Resurgence predicts:

$$\mathrm{Im}\left[\mathcal{S}_{\pm}\mathcal{E}(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0, \text{ up to } \mathcal{O}\left(e^{-4S_F}\right)$$

Preceding result implies that this works in PCM

Systematic demonstration that leading renormalon ambiguities of perturbation theory cancel against ambiguities in saddle-point sum

Illustrates that exact information about NP physics is present in perturbation theory, albeit in coded form!

At higher order resurgence implies more intricate relations:

$$F(\lambda) = \mathrm{Re}\mathcal{S}P_0 + \mathrm{Re}[\mathcal{F}\bar{\mathcal{F}}]\mathrm{Re}\mathcal{S}P_{\mathcal{F}\bar{\mathcal{F}}} + \mathrm{Im}[\mathcal{F}\bar{\mathcal{F}}]_{\pm}\mathrm{Im}\mathcal{S}_{\pm}P_{\mathcal{F}\bar{\mathcal{F}}} + \mathrm{Re}[\mathcal{F}_2\bar{\mathcal{F}}_2]\mathrm{Re}\mathcal{S}P_{\mathcal{F}_2\bar{\mathcal{F}}_2} + \mathcal{O}(e^{-6S_F})$$

Mass gap at small L

Mass gap = splitting between ground state and 1st excited state

The splitting driven by one-fracton amplitude

$$\text{renormalon} \sim e^{-\frac{2\times 8\pi}{g^2 N}} = e^{-\frac{2\times 8\pi}{\lambda}}$$

Gap between ground state and first excited state is

$$\Delta_{\mathrm{SU(N)\ PCM}} \sim \frac{1}{NL}\frac{8\pi}{\sqrt{\lambda}}e^{-\frac{8\pi}{\lambda}}$$

Same relation in all small-L cases checked so far: PCM, CP^N, YM

$$\Delta \sim \text{renormalon}^{1/2}$$

Relation also holds when massless fermions are added,
which changes size of both Δ and the renormalon ambiguity!

Mass gap at small L

Mass gap = splitting between ground state and 1st excited state

The splitting driven by one-fracton amplitude

$$\text{renormalon} \sim e^{-\frac{2\times 8\pi}{g^2 N}} = e^{-\frac{2\times 8\pi}{\lambda}}$$

Gap between ground state and first excited state is

$$\Delta_{\text{SU(N) PCM}} \sim \frac{1}{NL}\frac{8\pi}{\sqrt{\lambda}}e^{-\frac{8\pi}{\lambda}}$$

Same relation in all small-L cases checked so far: PCM, CP^N, YM

$$\Delta \sim \text{renormalon}^{1/2}$$

Relation also holds when massless fermions are added,
which changes size of both Δ and the renormalon ambiguity!

What we learned so far…

Even when there's no topology, resurgence predicts existence of NP saddle points with specific properties, which can then be found.

In semiclassical domain, renormalon ambiguities systematically cancel against contributions of non-BPS NP saddles

Renormalons closely related to mass gap, as 't Hooft dreamt

All results so far fit conjecture of resurgent nature of observables in QFTs with weak-coupling limits

Lots left to do!

Now exploring relations to analytic continuation of path integrals

Lefshetz thimble decomposition of integration cycles appears to geometrize resurgence — Witten 2010; AC, Dorigoni, Unsal 2014

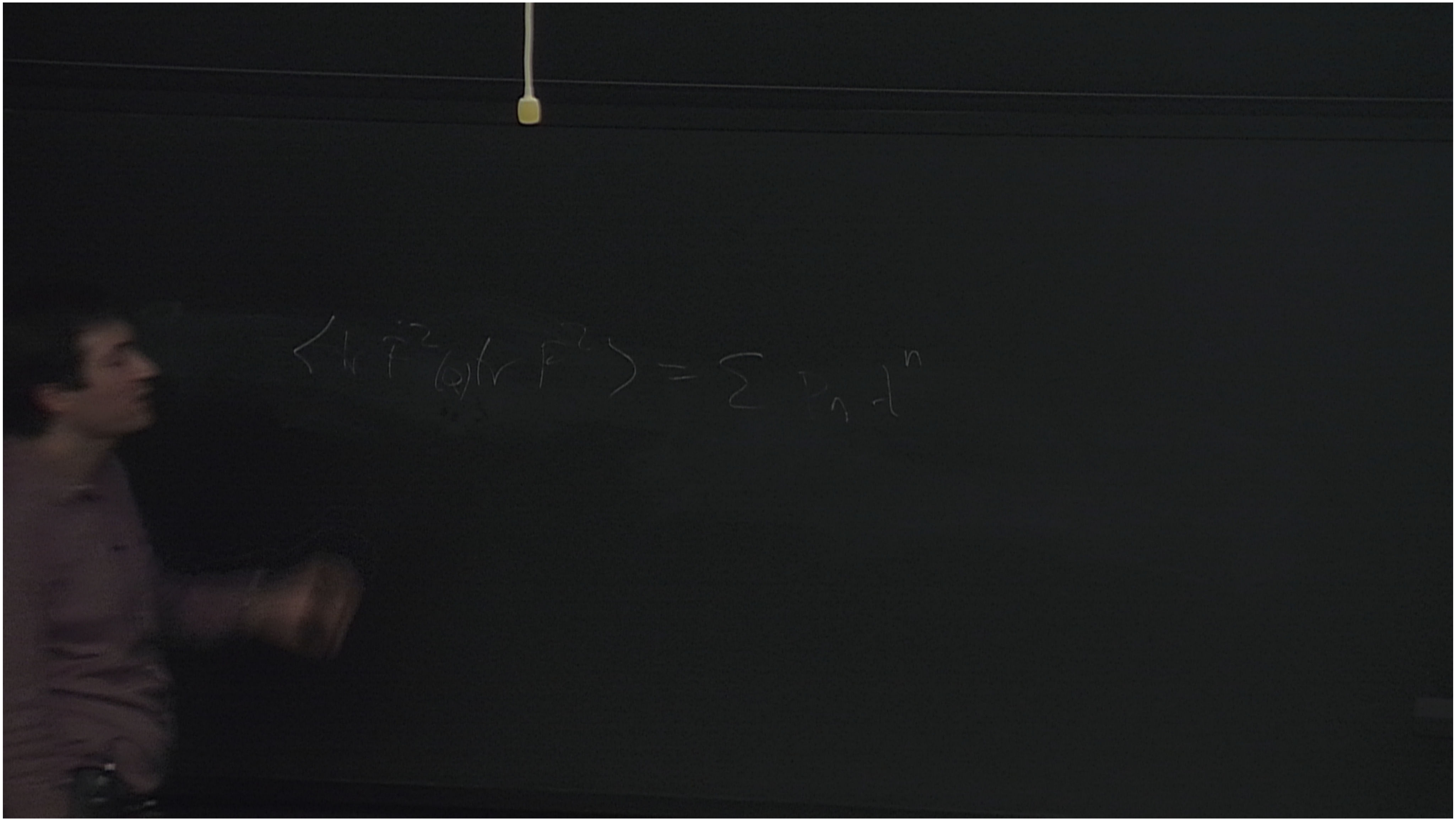
There are likely to be many practical implications!

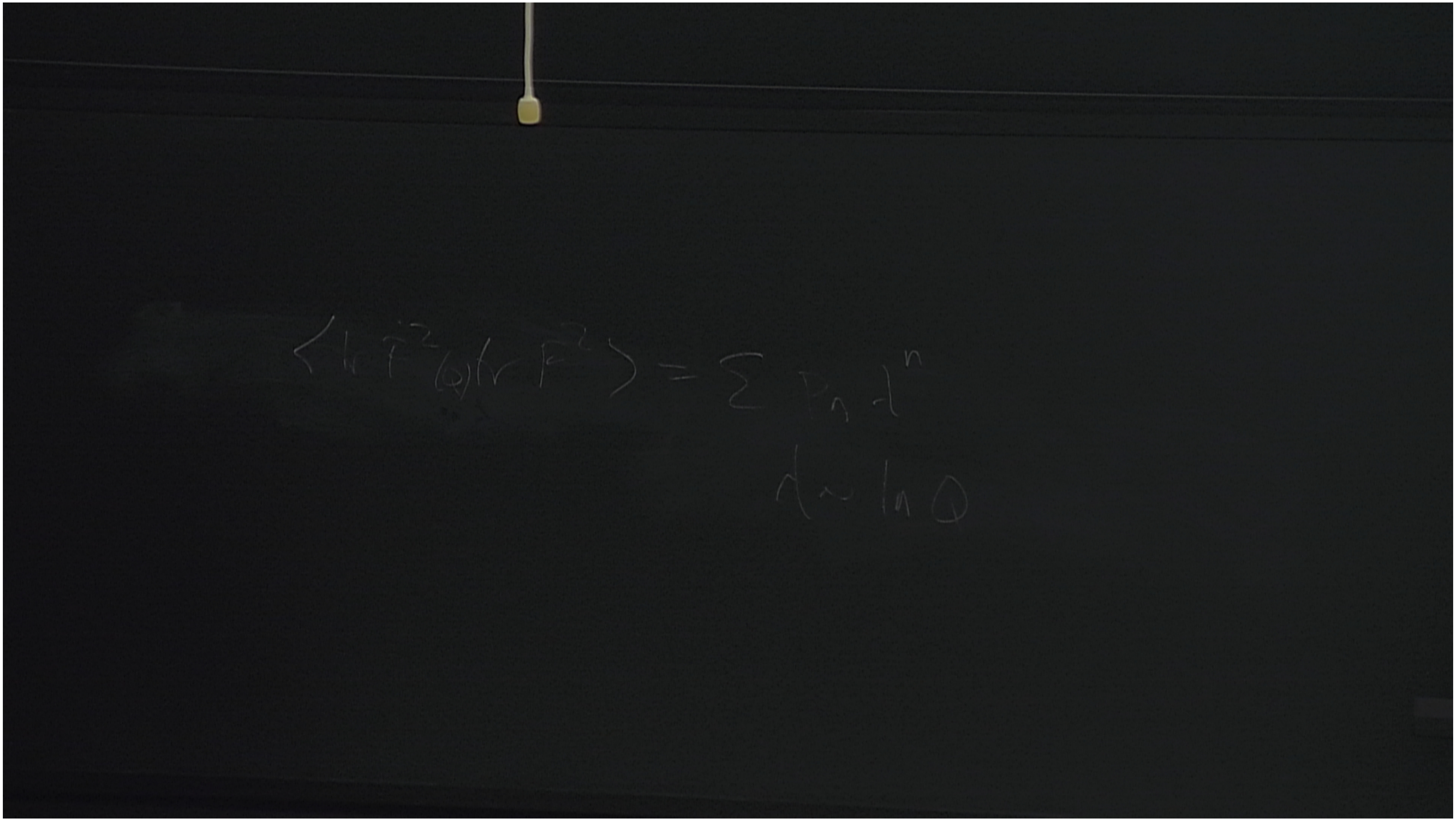
Better understanding of QFTs with complex actions

Resurgence theory and Lefshetz thimble technology play vital role in seeing how instantons appear in real-time Feynman path integrals. — AC, Unsal 2014

Improved understanding of connections between strong and weak coupling regimes — AC, Koroteev, Unsal 2014

Applications of resurgence in SUSY QFTs — Aniceto, Russo, Schiappa, 2014


$\langle \mathrm{tr} F^2(x) \, \mathrm{tr} F^2 \rangle = \sum P_n \lambda^n$


$\langle \mathrm{tr} F^2(x)\, \mathrm{tr} F^2 \rangle = \sum P_n \alpha^n$
$\alpha \sim \ln Q$