Title: Completeness Results for Graphical Quantum Process Languages

Date: Dec 10, 2014 04:00 PM

URL: http://pirsa.org/14120040

Abstract: <p>From Feynman diagrams via Penrose graphical notation to quantum circuits, graphical languages are widely used in quantum theory and other areas of theoretical physics. The category-theoretical approach to quantum mechanics yields a new set of graphical languages, which allow rigorous pictorial reasoning about quantum systems and processes. One such language is the ZX-calculus, which is built up of elements corresponding to maps in the computational and the Hadamard basis. This calculus is universal for pure state qubit quantum mechanics, meaning any pure state, unitary operation, and post-selected pure projective measurement can be represented. It is also sound, meaning any graphical rewrite corresponds to a valid equality when translated into matrices.</p>

<p>While the calculus is not complete for general quantum mechanics, I show that it is complete for stabilizer quantum mechanics and for the single-qubit Clifford+T group. This means that within those subtheories, any equality that can be derived using matrices can also be derived graphically.</p>

<p>The ZX-calculus can thus be applied to a wide range of problems in quantum information and quantum foundations, from the analysis of quantum non-locality to the verification of measurement-based quantum computation and error-correcting codes. I also show how to construct a ZX-like graphical calculus for Spekkens' toy bit theory and give its associated completeness proof.</p>

Completeness Results for Graphical Quantum Process Languages

Miriam Backens

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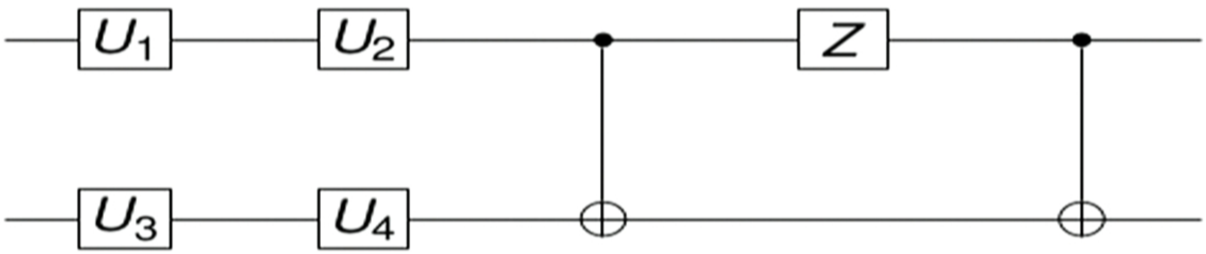
Motivation: graphical languages and rewriting



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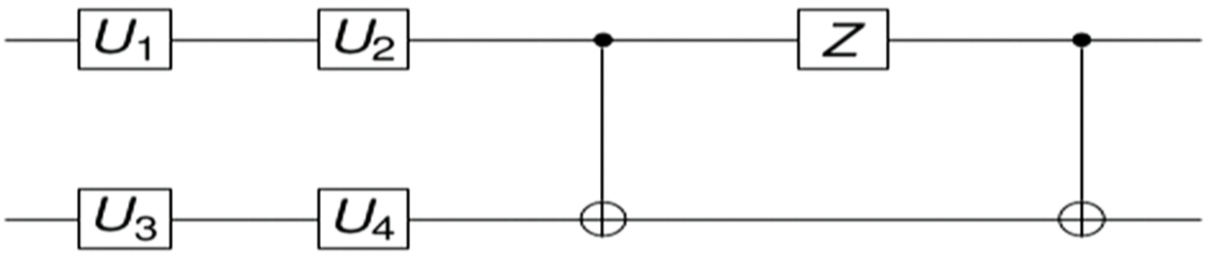

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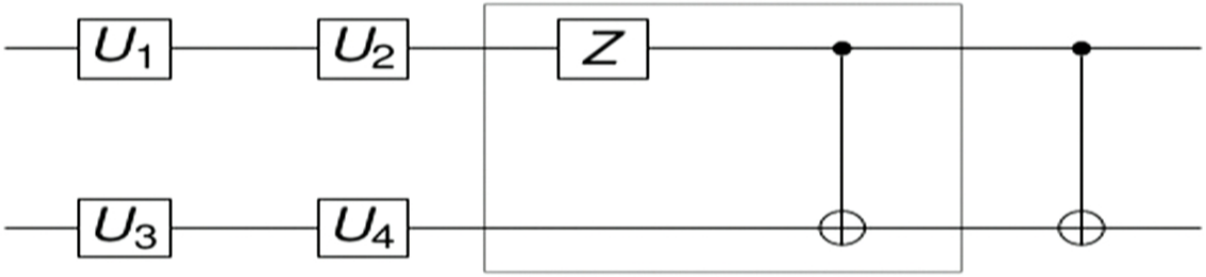
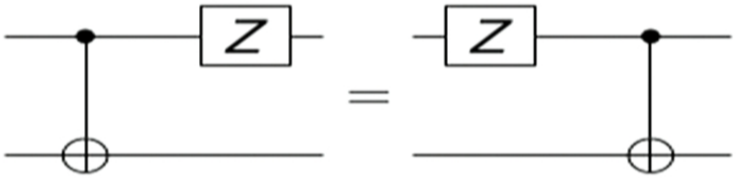


$$\Lambda X_{1,2} \circ (Z \otimes I) = (Z \otimes I) \circ \Lambda X_{1,2}$$



Outline

- Background
- Completeness results
  - Stabilizer quantum mechanics
  - Comparison: Spekkens' toy bit theory
  - The single-qubit Clifford+T group
- Outlook and further work

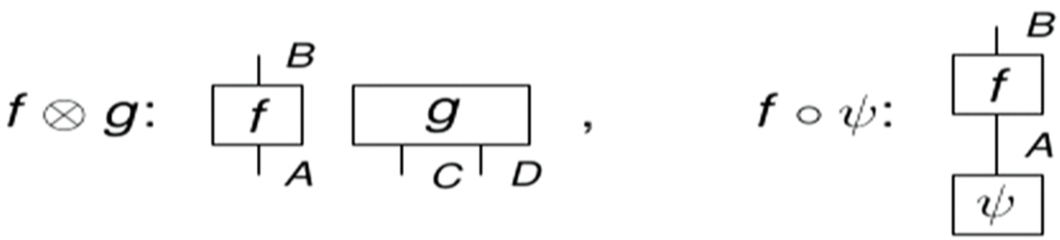
Categorical quantum mechanics and graphical languages

- focus is study of systems (A, B) and processes ($f : A \to B, \psi : I \to A$), and how they compose

Categorical quantum mechanics and graphical languages

- focus is study of systems (A, B) and processes ($f : A \to B, \psi : I \to A$), and how they compose
- graphically: systems are lines, processes are boxes, except for the "trivial system" I which isn't drawn at all

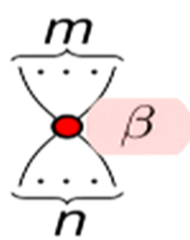
- composition: if $f : A \to B, g : C \otimes D \to I, \psi : I \to A$, then



The ZX-calculus for pure state qubit quantum mechanics

- green nodes with n inputs and m outputs, $\alpha \in (-\pi, \pi]$

- red nodes with n inputs and m outputs, $\beta \in (-\pi, \pi]$



- Hadamard nodes with one input and one output

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⟦  ⟧ $:= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n}$,

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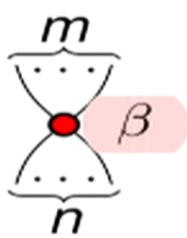
⟦  ⟧ $:= |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\beta} |-\rangle^{\otimes m} \langle -|^{\otimes n}$,

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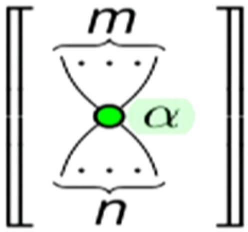

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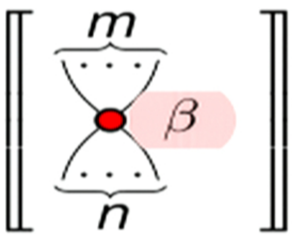

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- only the topology matters
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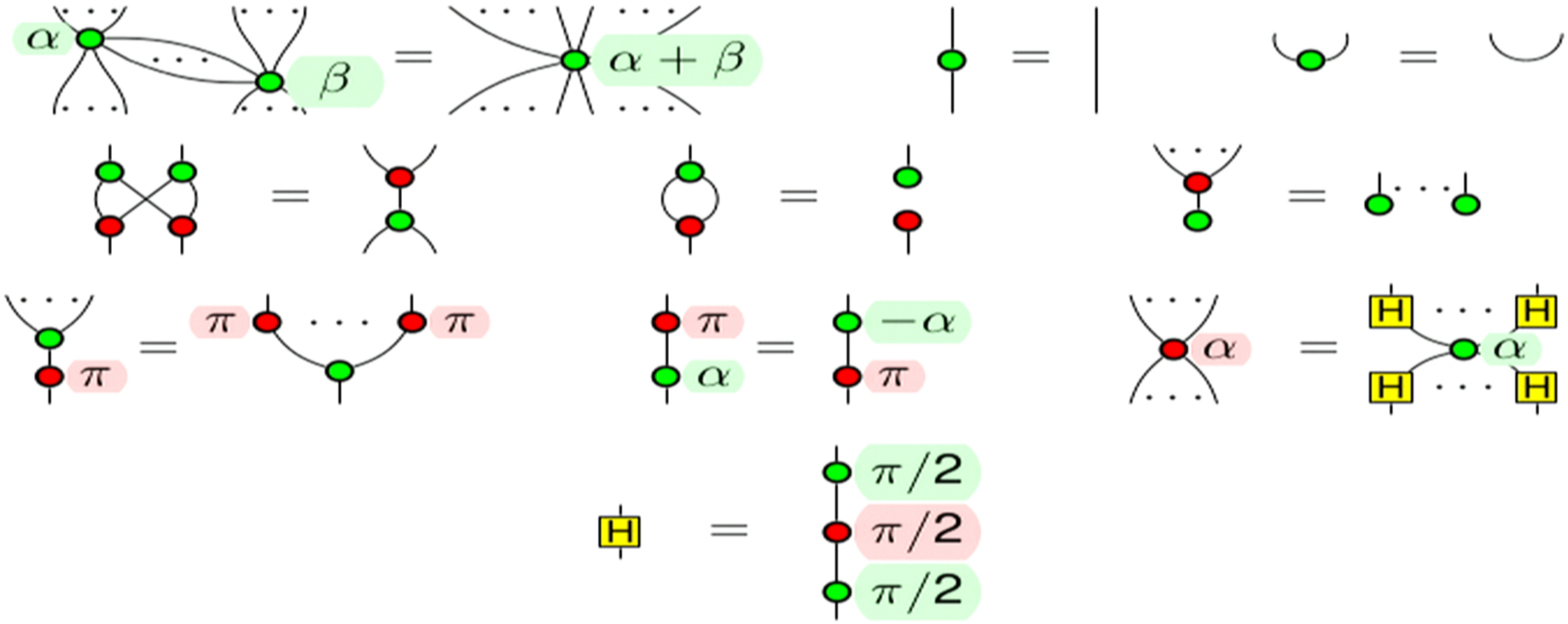
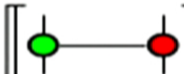


details e.g. in arXiv:0908.1787

Universality and soundness

- *universality*: any pure state, post-selected pure projective measurement, and unitary operation can be expressed: to see this note that $[\![$  $]\!] \doteq |0\rangle$, $[\![$  $]\!] \doteq \langle 0|$, $[\![$  $]\!]$ is the usual Hadamard operator, and

$[\![$  $]\!] \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$, $[\![$  $]\!] \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

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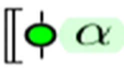
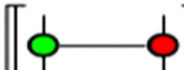
$$[\![$$$$]\!] \doteq \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \quad [\![$$$$]\!] \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- *soundness*: any equality that can be derived graphically can also be derived using matrices; to see this note that

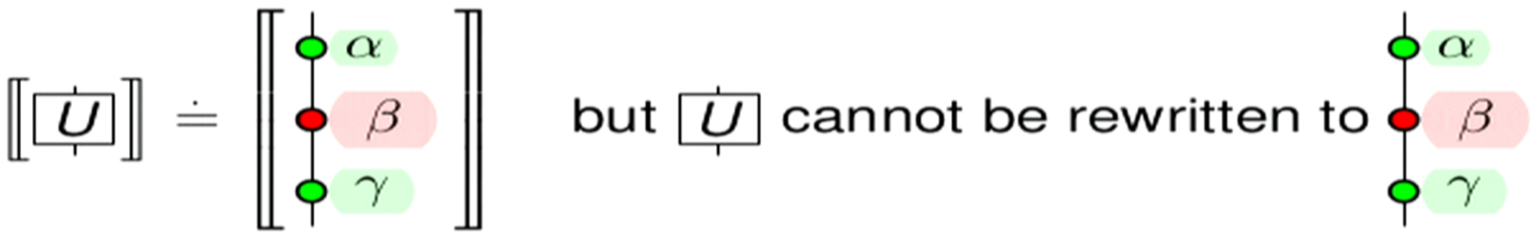
$$[\![\mathrm{LHS}]\!] \doteq [\![\mathrm{RHS}]\!] \quad \text{for all rewrite rules}$$

(In)completeness

Completeness means any equality that can be derived using matrices can also be derived graphically, i.e.

$$[\![D_1]\!] \doteq [\![D_2]\!] \text{ implies } D_1 = D_2.$$

- ZX-calculus is not complete for general pure state qubit QM: can find unitary U and phases α, β, γ s.t.



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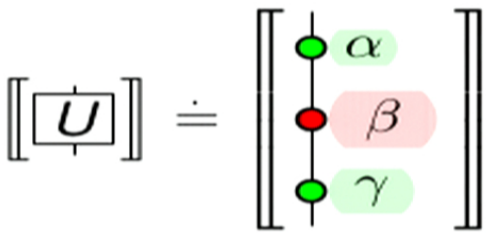

but 

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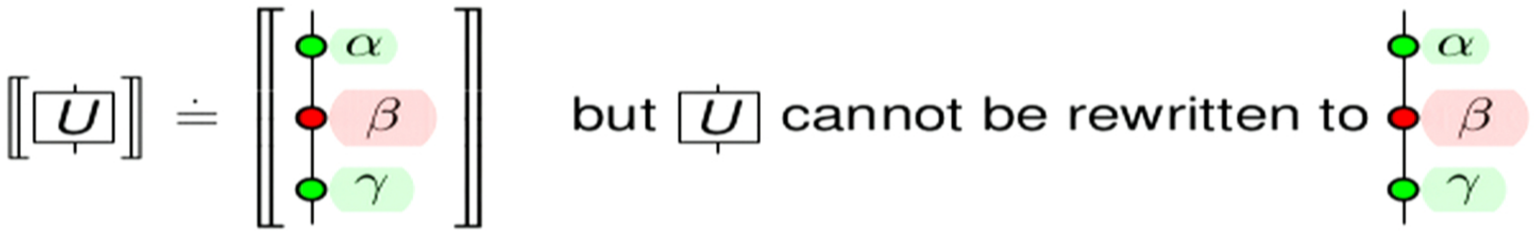
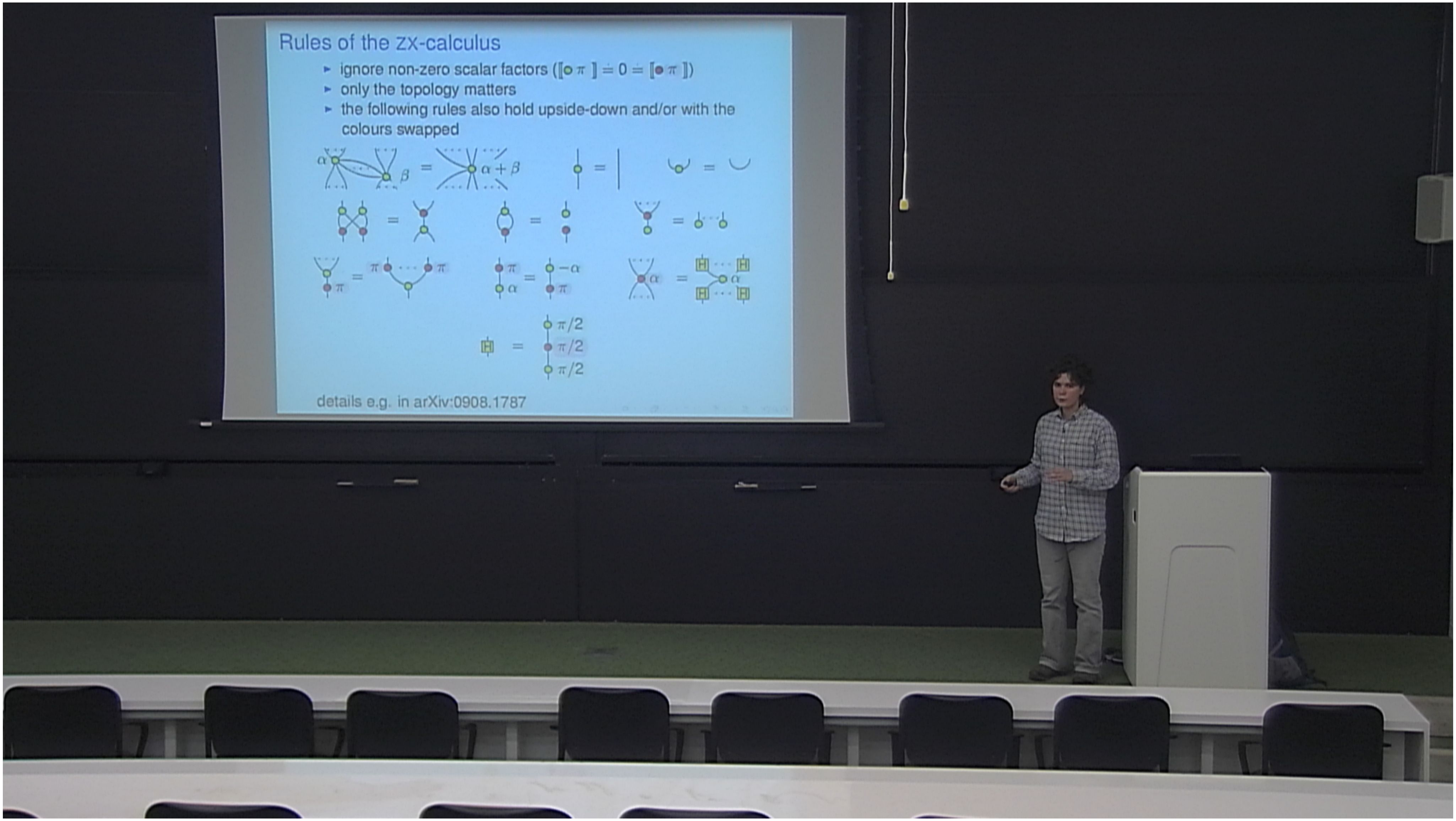


- if phases are restricted, Euler decomposition rule does no longer hold
- thus, can still derive completeness results for fragments of the calculus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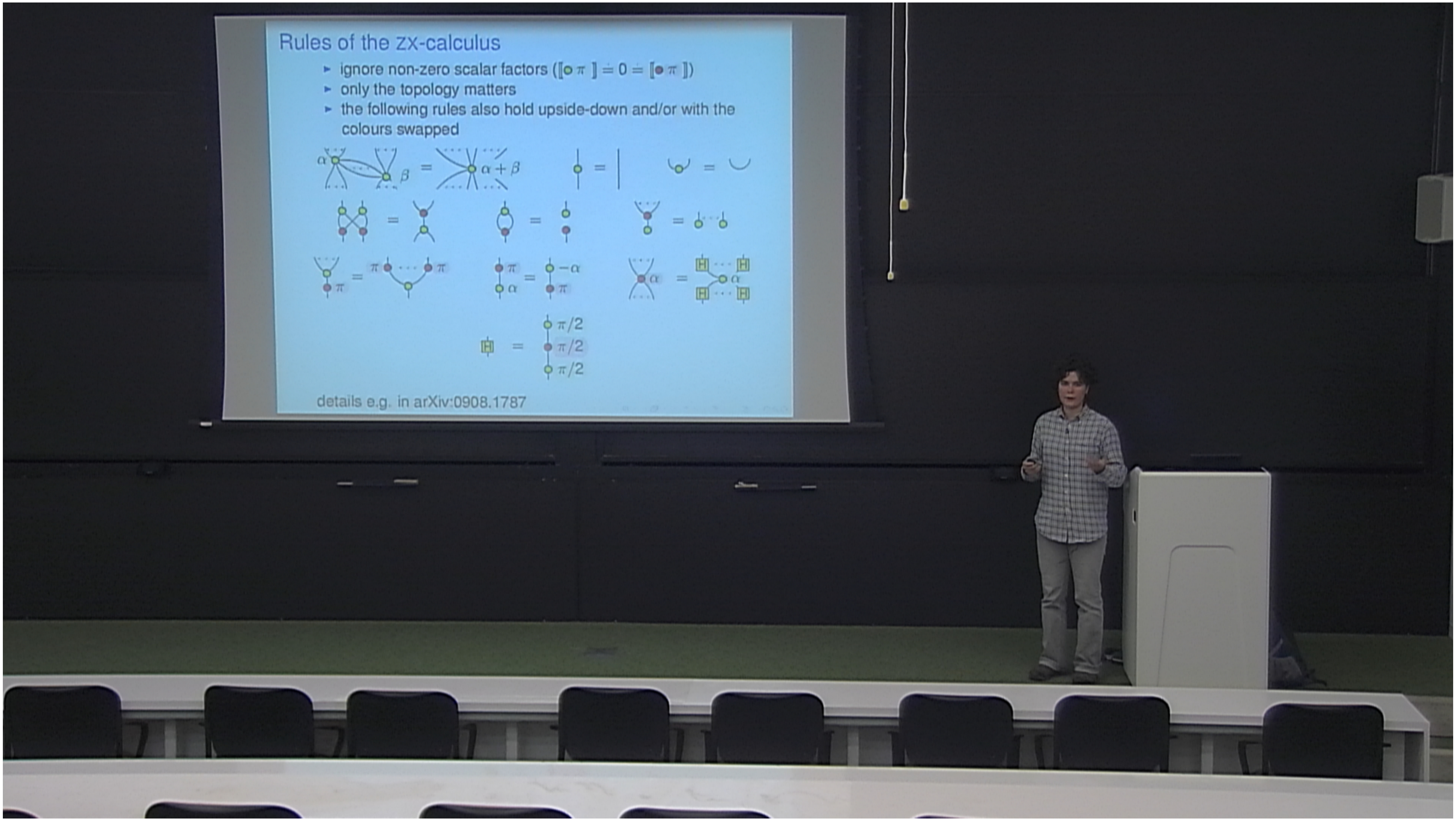
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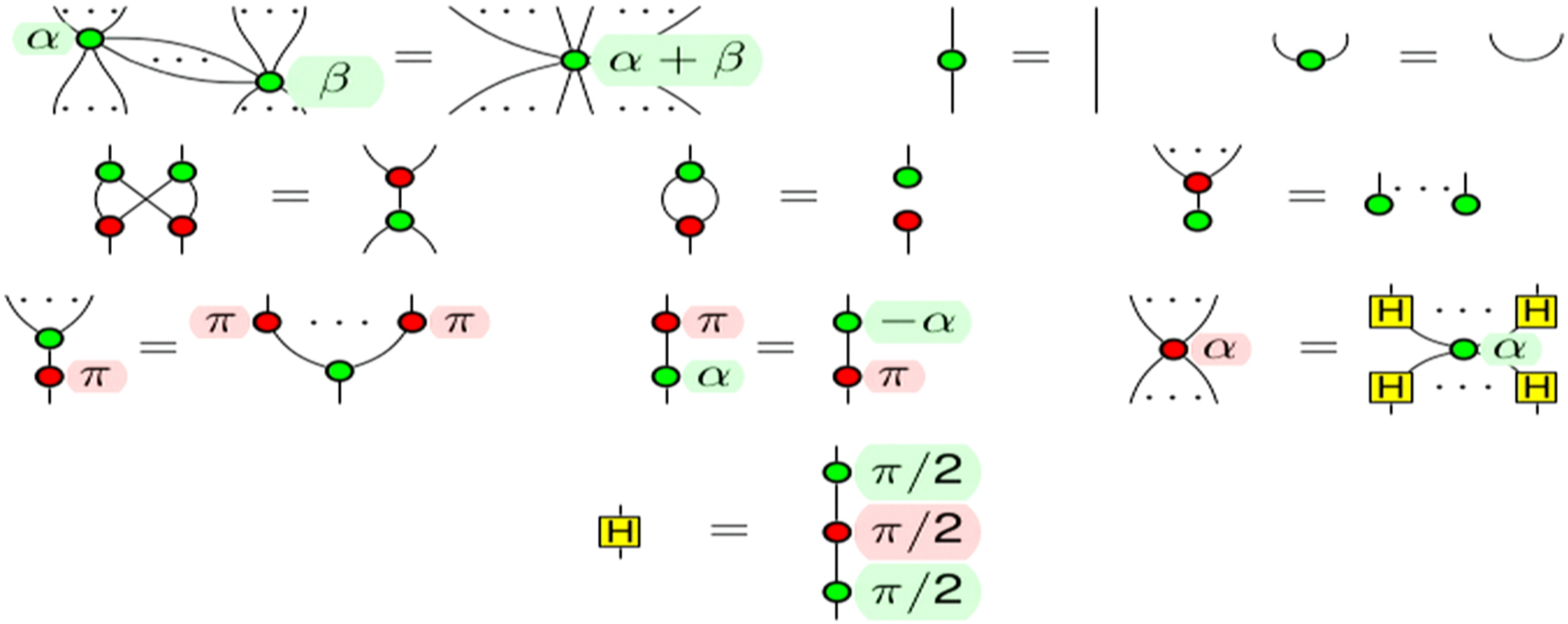

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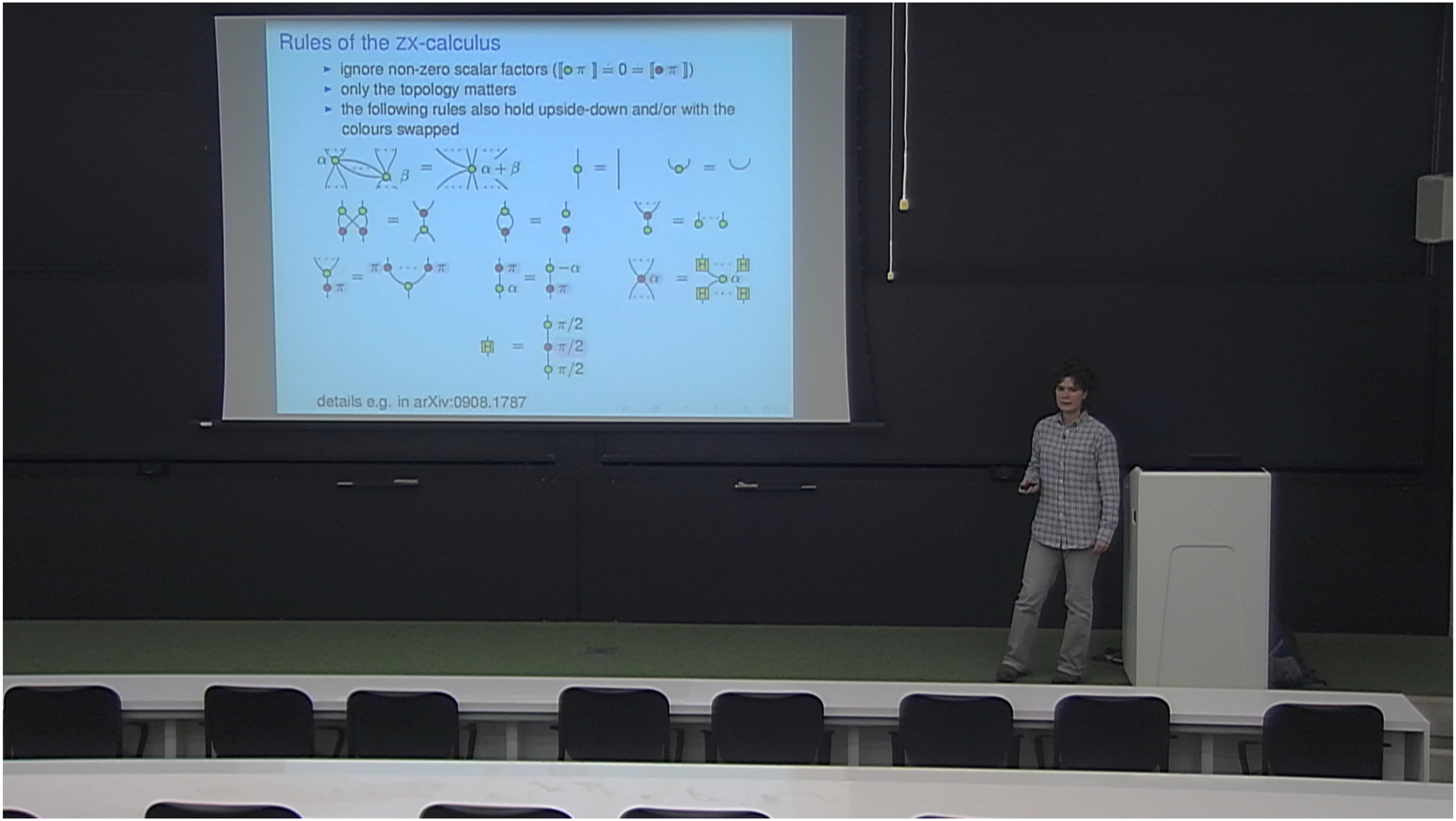
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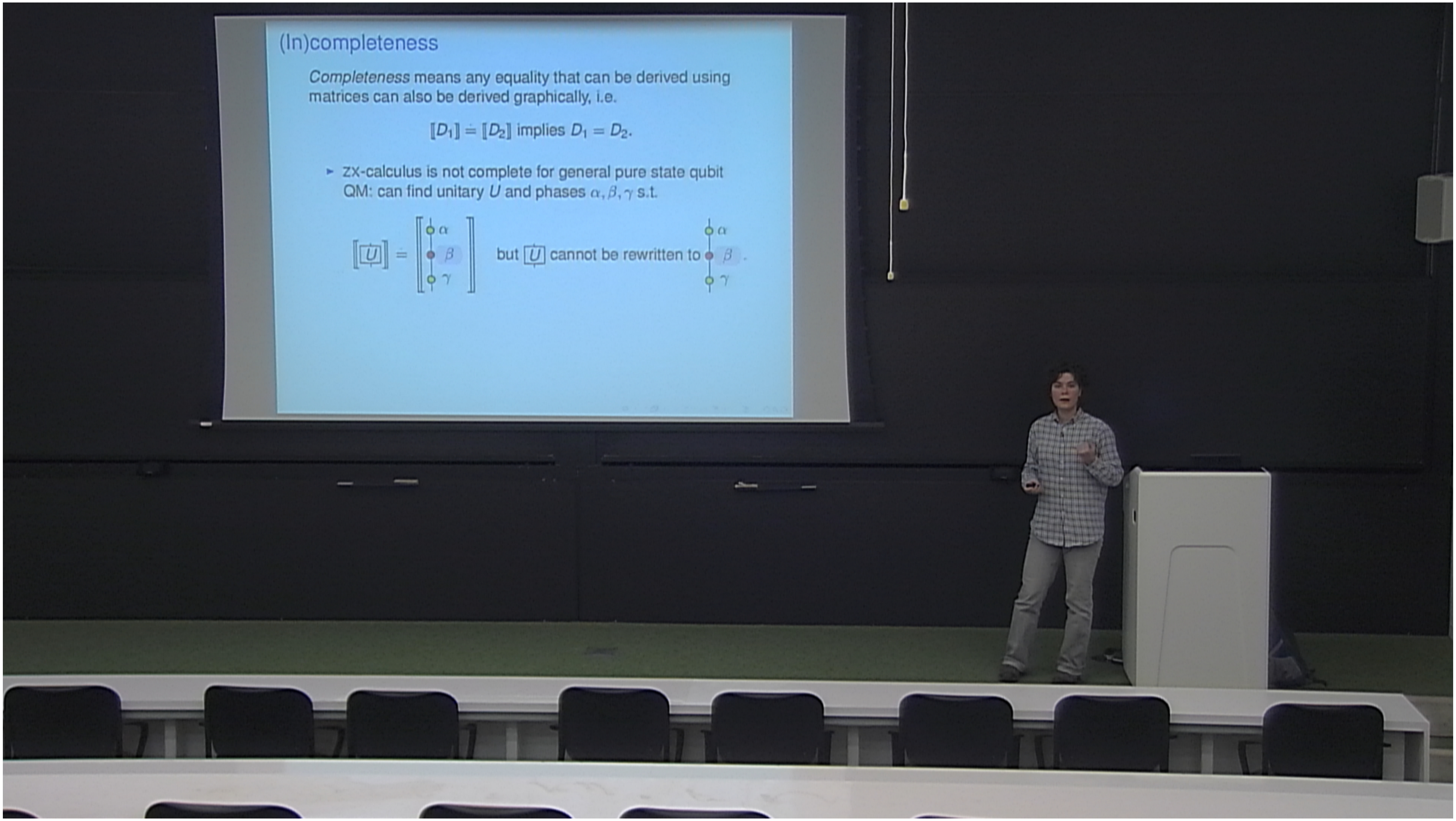

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Comparison: Spekkens' toy bit theory

The single-qubit Clifford+T group

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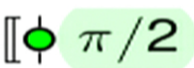
Stabilizer quantum mechanics

Stabilizer operations:

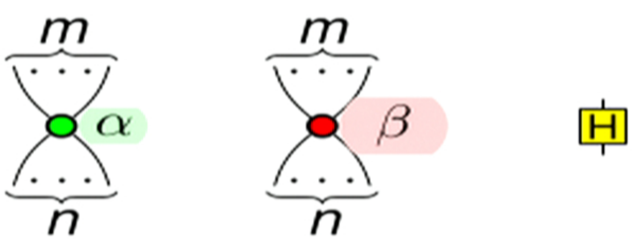
- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- measurements in computational basis

ZX-calculus: $S \doteq [\![$  $]\!]$, $H \doteq [\![$  $]\!]$, $\wedge X \doteq [\![$  $]\!]$

i.e. need to restrict phases to integer multiples of $\pi/2$



where $\alpha, \beta \in \{-\pi/2, 0, \pi/2, \pi\}$

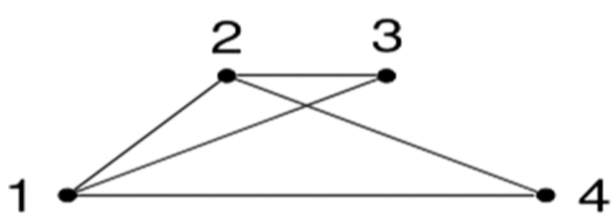
Graph states: a special class of stabilizer states

Definition

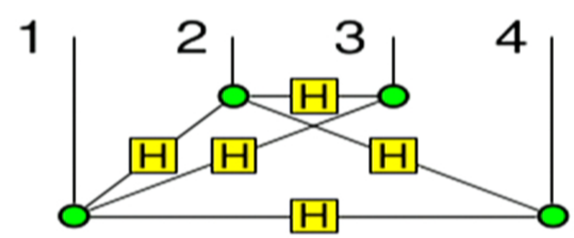
Let G be a finite simple undirected graph. The ZX-calculus diagram for the corresponding graph state consists of:

- for each node in G, a green node with one output, and
- for each edge in G, an edge with a Hadamard node on it.

E.g.



$\mapsto$



Results about graph states

The local Clifford group consists of all tensor products of the single-qubit Clifford operators $\langle S, H \rangle$.

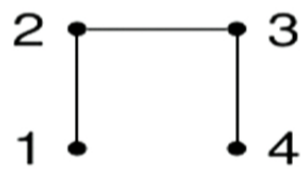
Theorem (Van den Nest et. al, 2004)

Any stabilizer state is local Clifford-equivalent to some graph state.

Theorem (Van den Nest et. al, 2004)

Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.

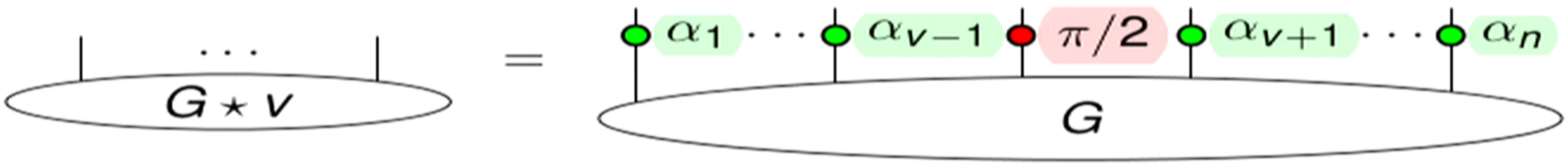
A local complementation about a vertex v inverts the subgraph generated by the neighbourhood of v: e.g.



Local complementations in the ZX-calculus

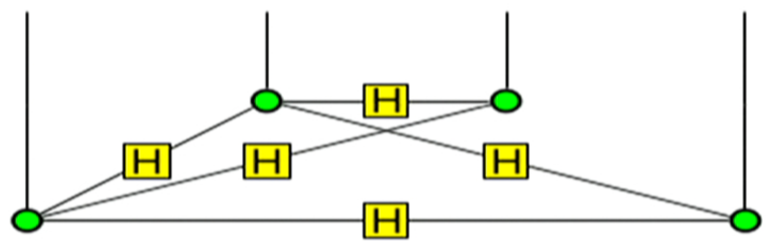
Theorem (Duncan & Perdrix, 2009)

Denote the result of a local complementation about the vertex v in the graph G by $G \star v$. The graph state diagrams for G and $G \star v$ satisfy



where $\alpha_u = -\pi/2$ if $\{u, v\}$ is an edge, $\alpha_u = 0$ otherwise.

E.g. a local complementation about the 3rd qubit of the previous graph state:



Every state diagram is equal to some GS-LC diagram

Theorem

Any stabilizer ZX-calculus diagram with no inputs and at least one output can be rewritten to a GS-LC diagram.

Proof.

- Decompose the diagram into basic spiders

  and single-qubit Clifford unitaries.
- Diagrams with no inputs must contain at least one copy of , this is a GS-LC diagram.
- For each basic element, applying it to a GS-LC diagram yields a diagram that can be rewritten into GS-LC form.
- Thus, by induction, the theorem holds. □

Equalities between GS-LC diagrams

All equalities between GS-LC diagrams are generated by local complementations.

Theorem (inspired by Elliott et al., 2008)

There exists a terminating algorithm that, given a pair of GS-LC diagrams on the same number of qubits, rewrites them to a pair of identical diagrams if and only if the two diagrams represent the same state.

The Choi-Jamiołkowski isomorphism

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For any operator A from n to m qubits and for any n + m-qubit state B,

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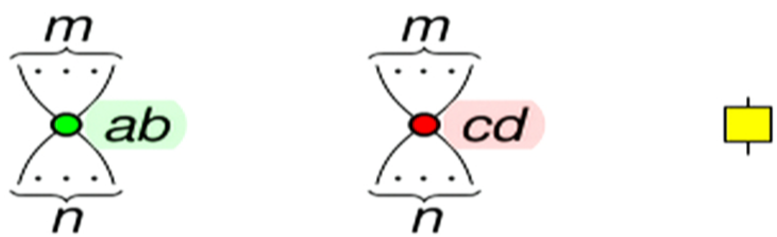
Brief introduction to Spekkens' toy bit theory

- local hidden variable theory exhibiting many properties that are generally considered "quantum"
- structurally very similar to stabilizer quantum mechanics
- observer's knowledge is restricted: distinction between ontic and epistemic states
- in categorical formulation of pure fragment, *phase group* $\{$  $\}_\alpha$ is the only difference between toy theory and stabilizer QM [Coecke et al. 2011]
  - stabilizer QM: $\mathbb{Z}_4$, represented as $\{0, \pi/2, \pi, -\pi/2\}$ under addition

A graphical calculus for the pure fragment of Spekkens' toy bit theory

[joint work with Ali Nabi Duman, arXiv:1411.1618]

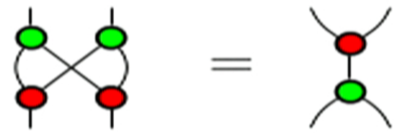
- represent $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\{00, 01, 10, 11\}$ under bitwise addition, then the toy theory graphical calculus elements are



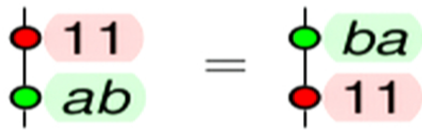
  with $a, b, c, d \in \{0, 1\}$
- red and green nodes are maps in mutually unbiased orthonormal bases
-  maps between bases

The toy theory graphical calculus is complete

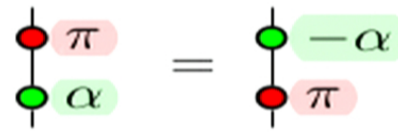
- rewrite rules not involving phase group are identical to ZX-calculus, e.g.



- rules involving phase group still have equivalents, e.g.



instead of



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Completeness for the single-qubit Clifford+T group

- generated by the single-qubit Clifford group $\mathcal{C}_1 = \langle S, H \rangle$ and the T gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

- approximately universal for single-qubit unitaries

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ZX-calculus: $T \doteq [\![\pi/4]\!]$ and $H \doteq [\![\boxed{H}]\!]$

- diagrams are restricted to line graphs (each node has one input and one output)
- phases are restricted to integer multiples of $\pi/4$

Theorem

There exists a unique normal form for single-qubit Clifford+T diagrams, hence the ZX-calculus is complete for this fragment.

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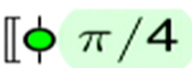
ZX-calculus: $T \doteq [\![$  $]\!]$ and $H \doteq [\![$  $]\!]$

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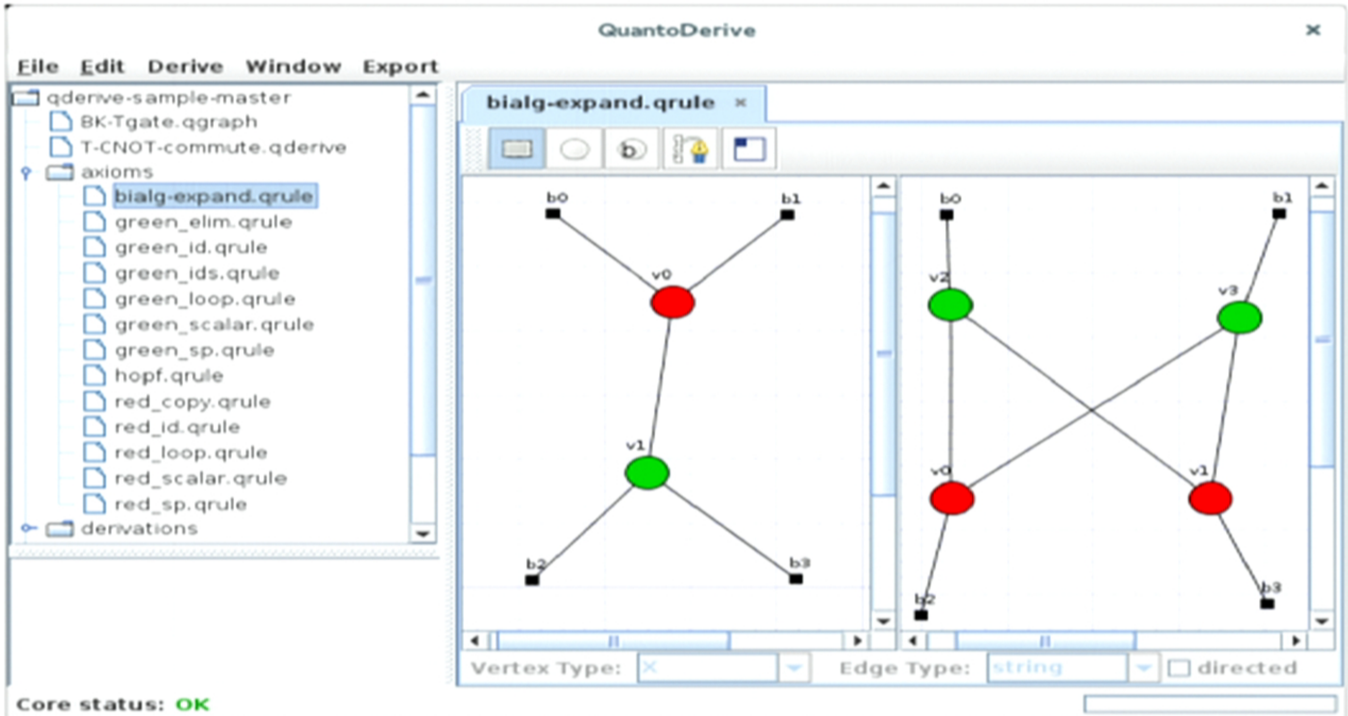
Further work on Spekkens' toy theory using graphical methods

- extend work to full theory, including mixed states
- construct similar graphical calculi for higher-dimensional stabilizer theory and toy theory

Completeness and automated graph rewriting

- automate computations in stabilizer QM, the single-qubit Clifford+T group, or any other fragment that can be made complete [joint work with Aleks Kissinger]



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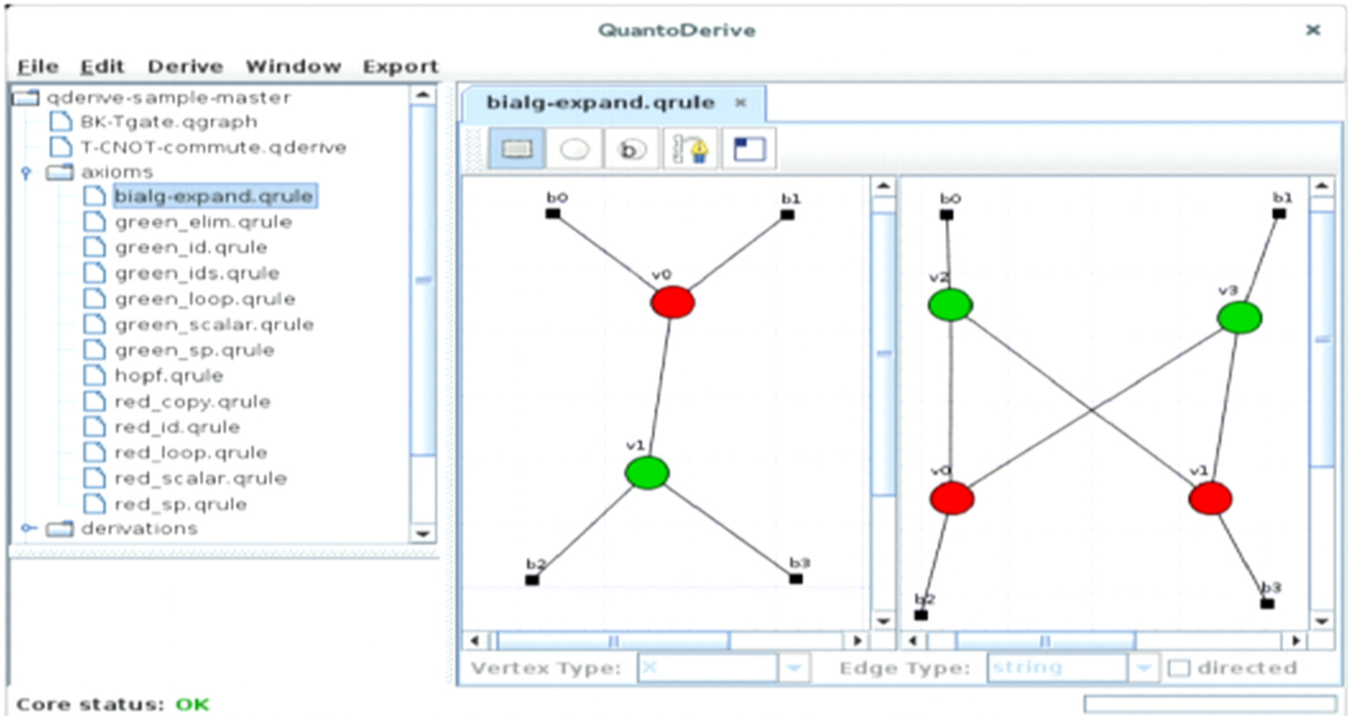
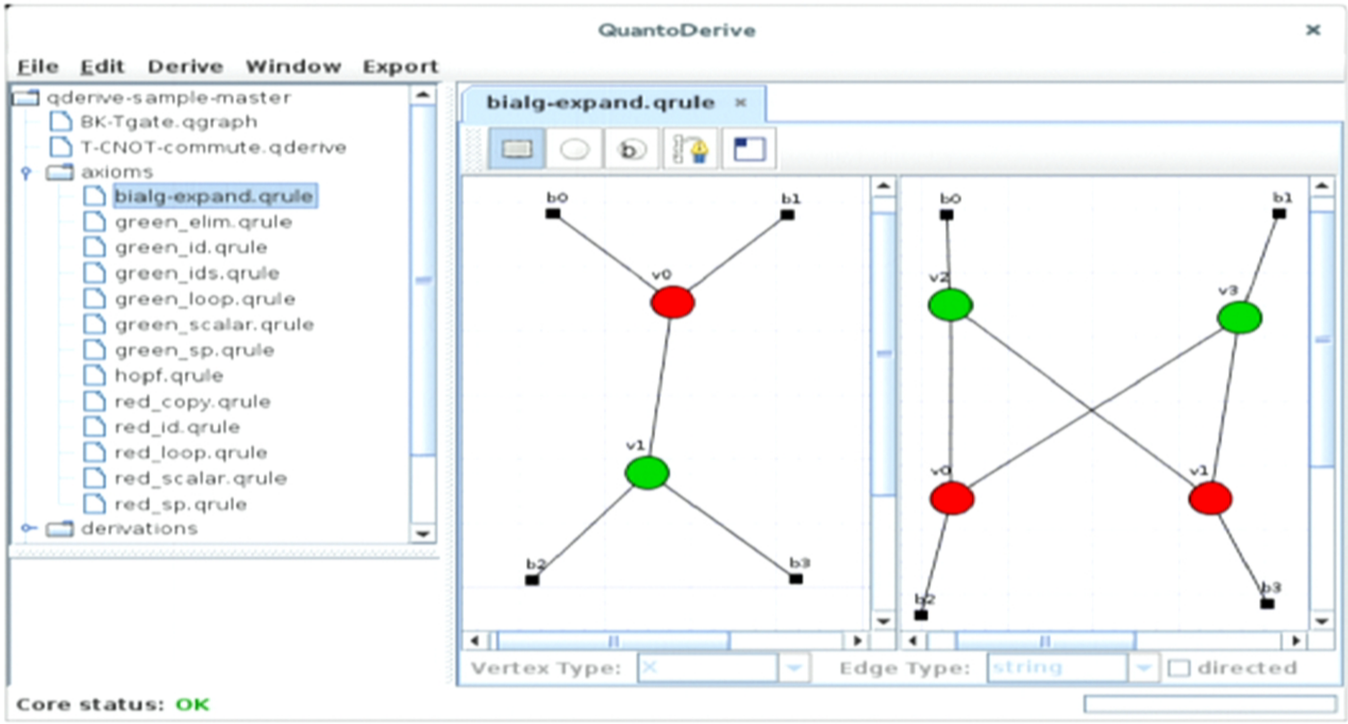


Completeness and automated graph rewriting

- automate computations in stabilizer QM, the single-qubit Clifford+T group, or any other fragment that can be made complete [joint work with Aleks Kissinger]

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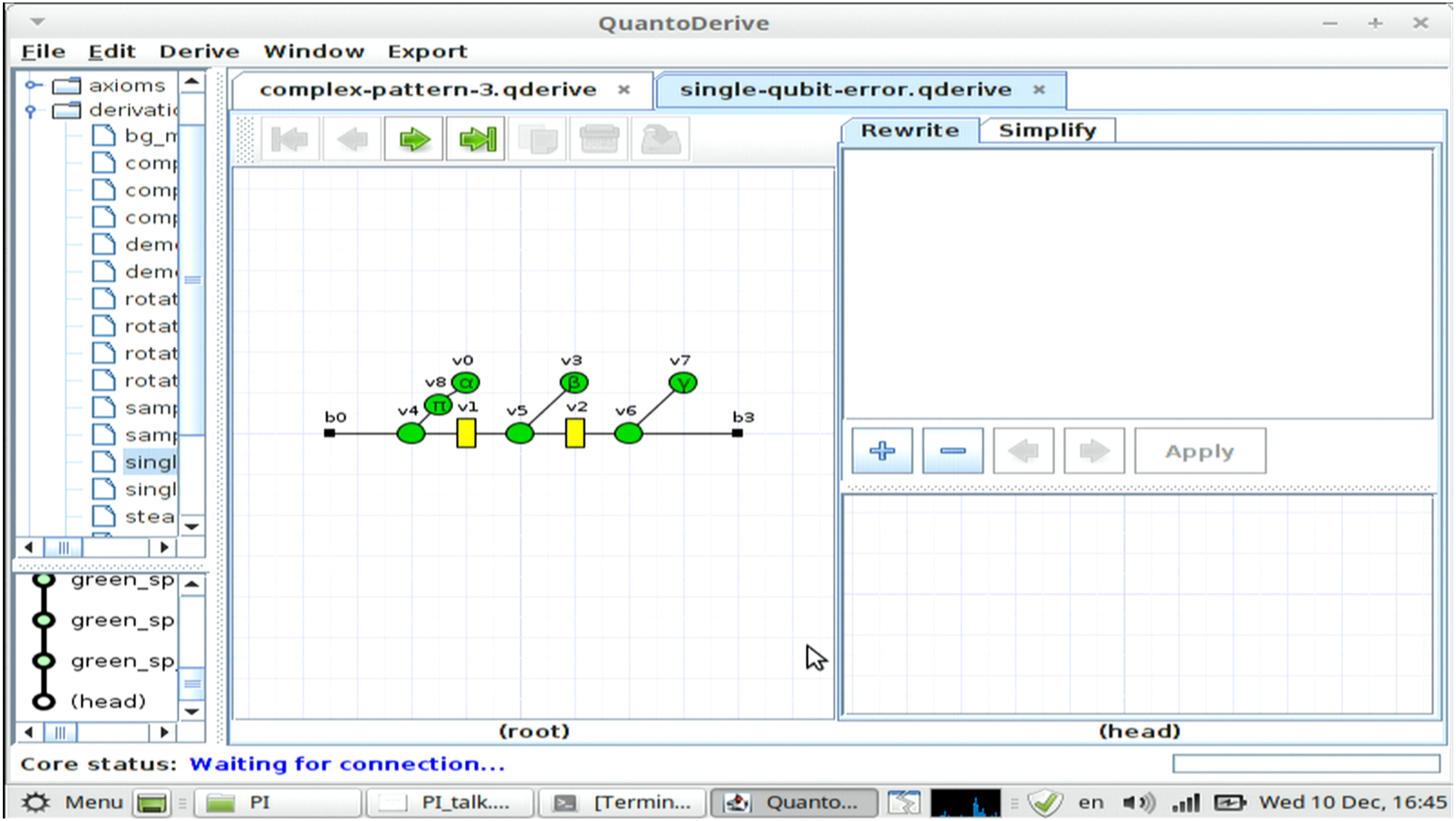
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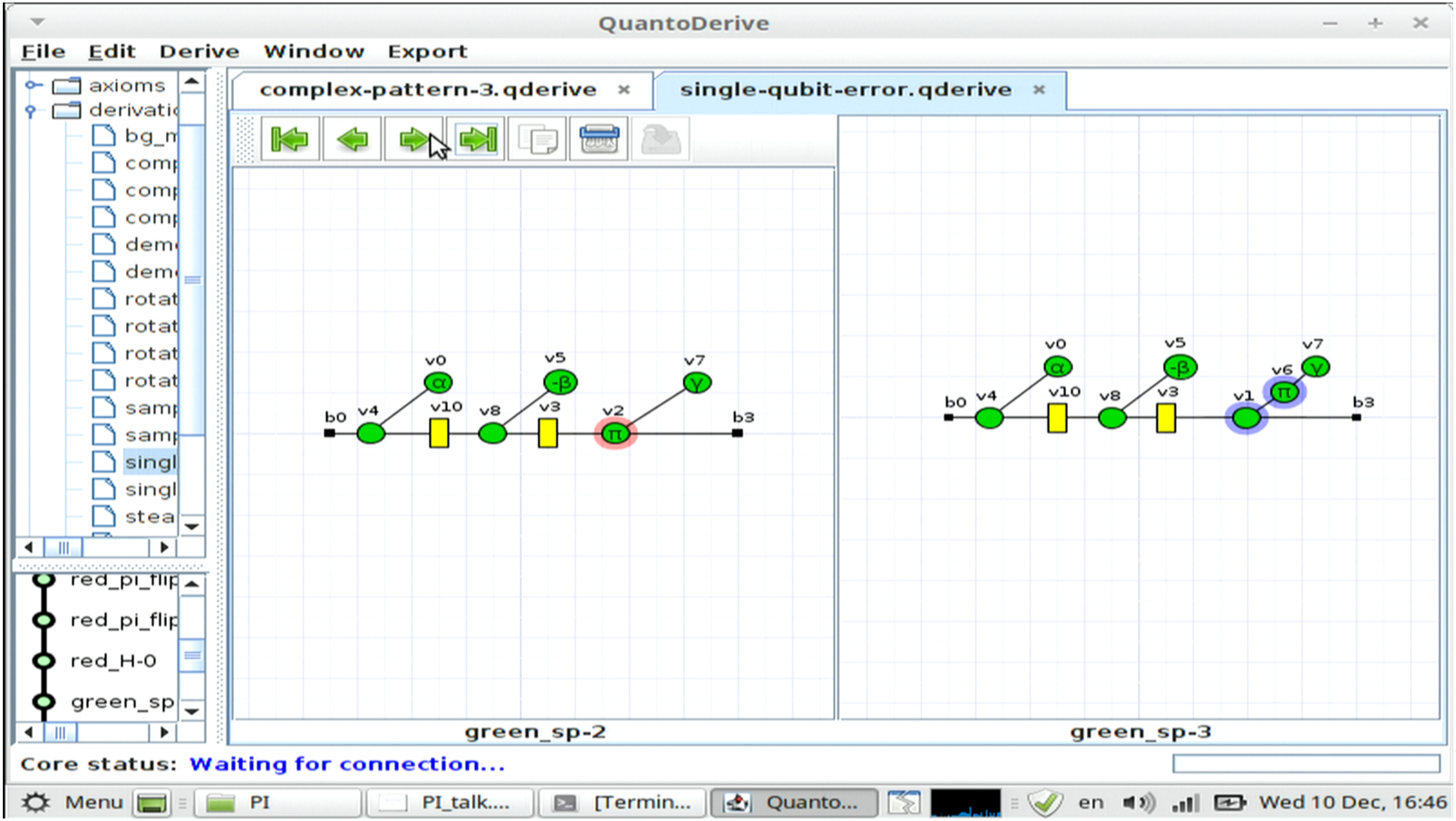
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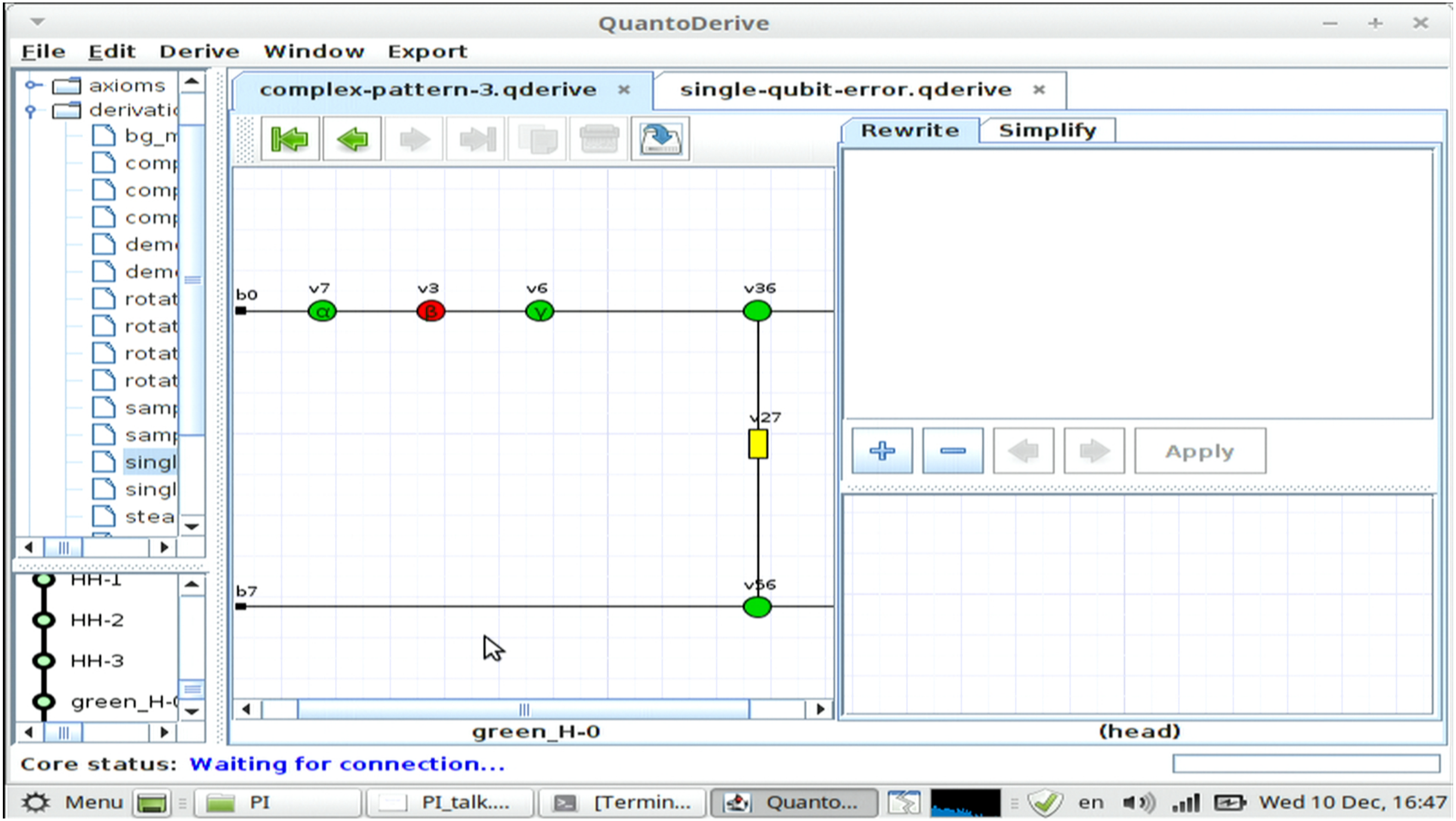
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Core status: Waiting for connection...
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Summary

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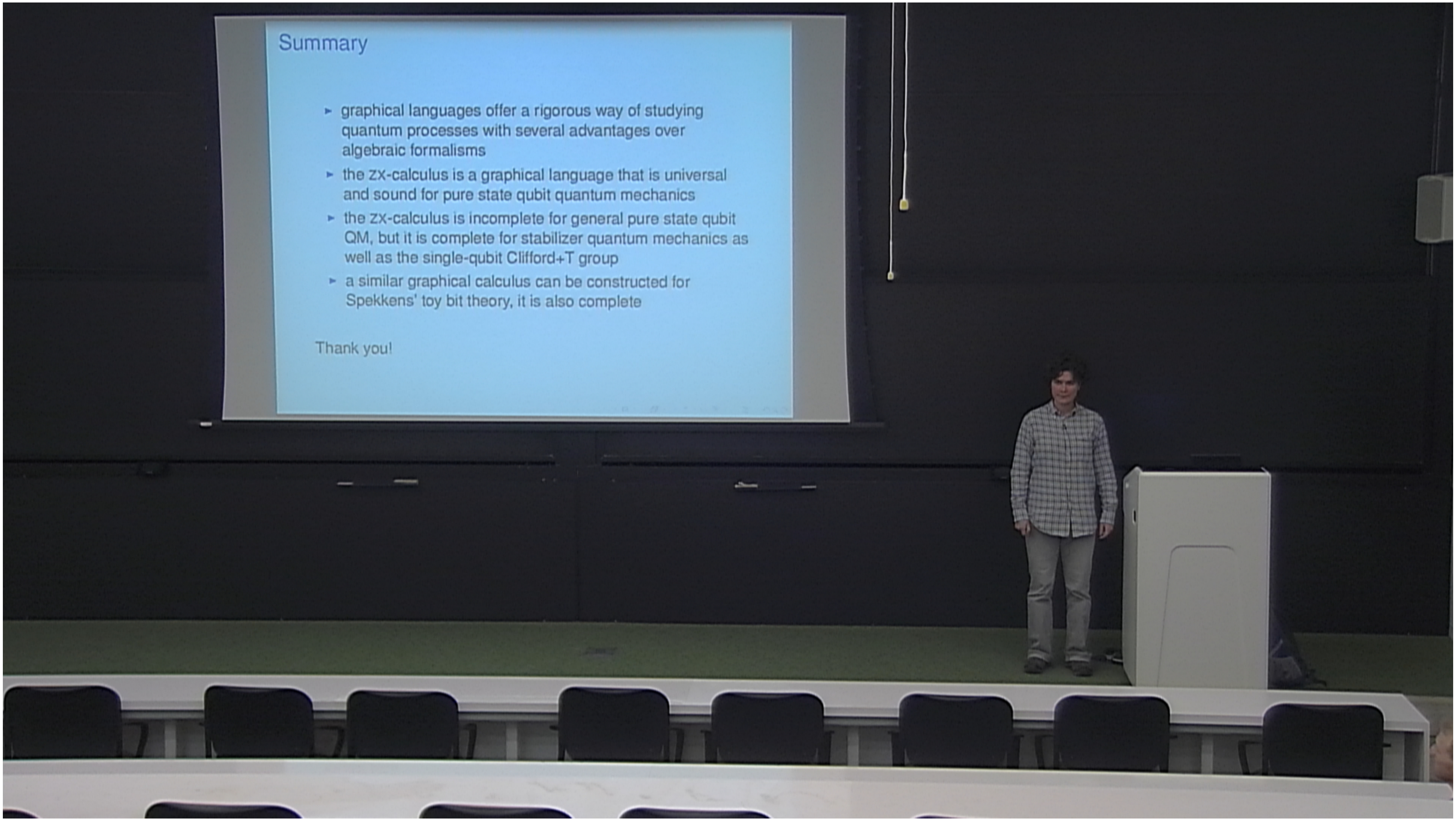
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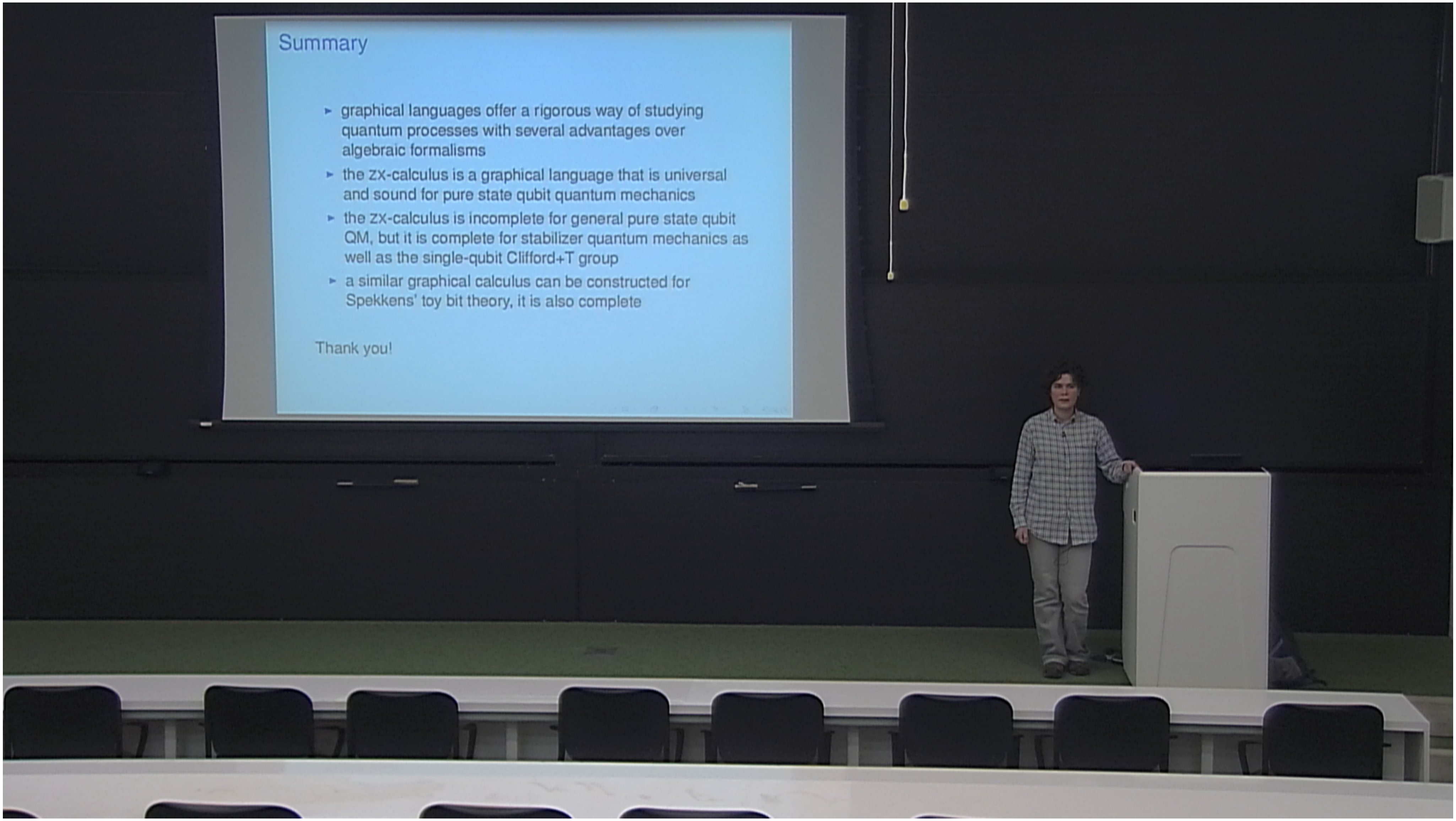
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