Title: Braiding statistics of loops in three spatial dimensions

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Abstract: <p>In two spatial dimensions, it is well known that particle-like excitations can come with fractional statistics, beyond the usual dichotomy of Bose versus Fermi statistics. In this talk, I move one dimension higher to three spatial dimensions, and study loop-like objects instead of point-like particles. Just like particles in 2D, loops can exhibit interesting fractional braiding statistics in 3D. I will talk about loop braiding statistics in the context of symmetry protected topological phases, which is a generalization of topological insulators.</p>

CW and M. Levin, PRL 2014
CW and M. Levin, in preparation

Braiding statistics of loops in three dimensions

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Acknowledgements: Michael Levin, Chien-Hung Lin

❖ Perimeter Institute for Theoretical Physics, Dec. 2, 2014

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Braiding statistics in 2D

- Fermions and bosons

$$\theta_{\alpha\beta} = 0$$

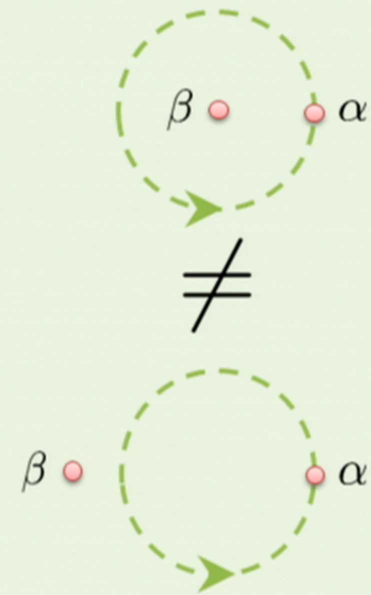
- Abelian anyons

$$\Psi(\alpha, \beta, \ldots) \to e^{i\theta_{\alpha\beta}} \Psi(\alpha, \beta, \ldots)$$

$$\theta_{\alpha\beta} \neq 0$$

- Non-Abelian anyons

$$\begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix} \to U_{\alpha\beta} \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}$$



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Examples: topologically ordered phases

- Fractional quantum Hall states
  - Laughlin states (Laughlin 1983)
  - Moore-Read state (Moore and Read 1991)

1/3 Laughlin State

Excitations:

$$0, \frac{e}{3}, \frac{2e}{3}, e, \frac{4e}{3}, \frac{5e}{3}$$

Statistics:

$$\theta_{\frac{1}{3},\frac{1}{3}} = \frac{2\pi}{3}$$

$$\theta_{\frac{1}{3},\frac{2}{3}} = \frac{4\pi}{3}$$

...

- p+ip superconductors (Majorana zero modes)

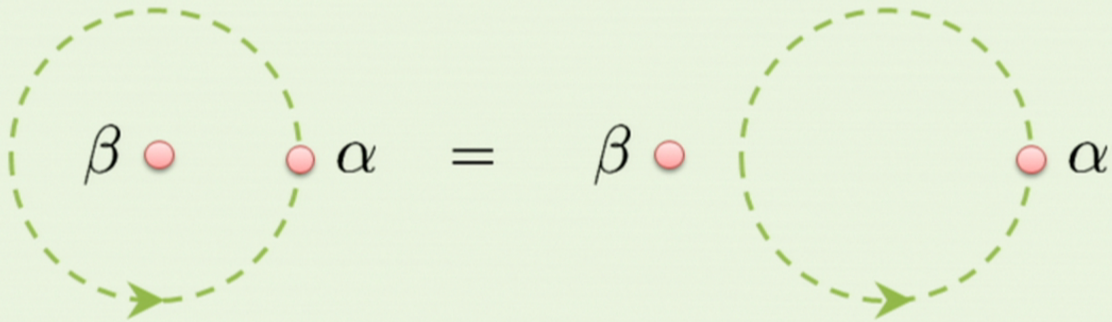
  (Read and Green 2000, Ivanov 2001)

- Toric code and many other exactly soluble models

  (Kitaev 1997, Levin and Wen 2005)

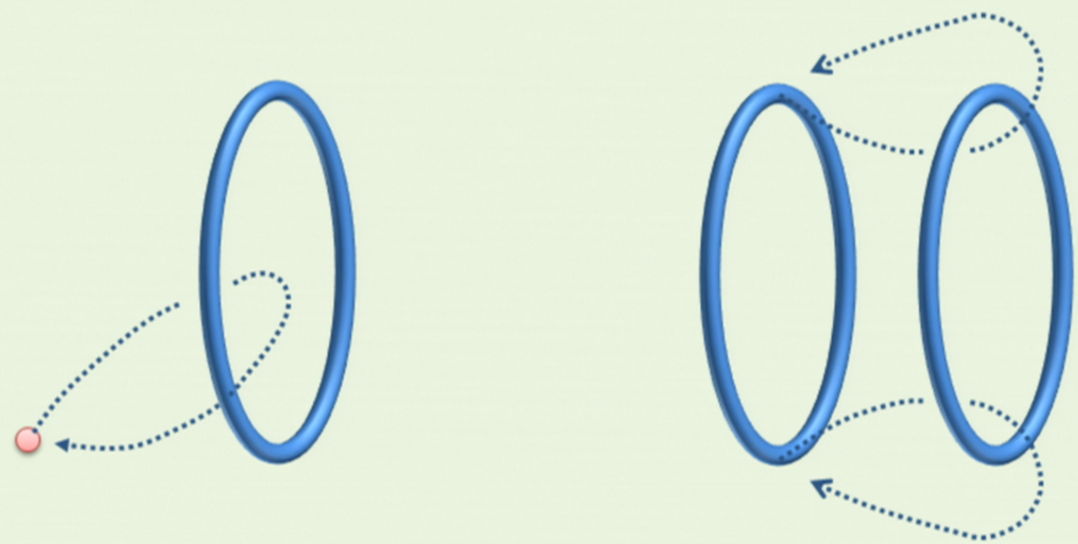
Fractional statistics in 3D

- Particles ? ---- No!



Fractional statistics in 3D

- Loops ? ---- Yes!



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Fractional statistics in 3D

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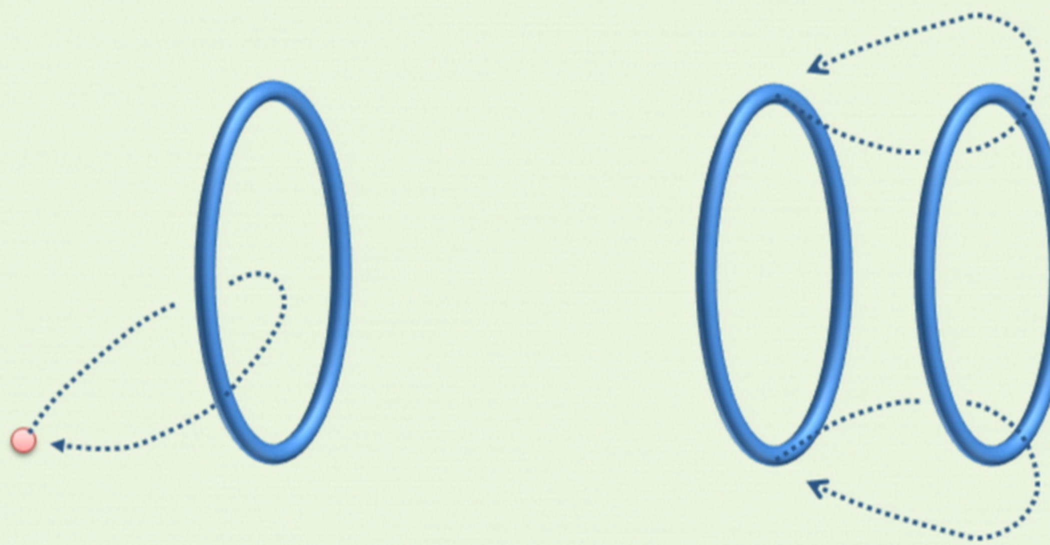

Loops are ubiquitous in 3D

From Wikipedia

- Vortices in classical fluids
- Vortices in superfluids and superconductors
- Line-like topological defects in broken symmetry phases (Mermin, RMP 1979)
- Flux loops in gauge theories

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Symmetry protected topological phases

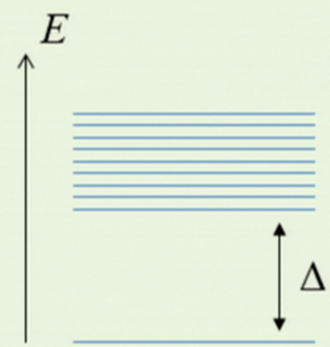
- 2D Topological insulators --- Protected by $U(1) \times TRS$ (Hasan&Kane, RMP 2010)

Symmetry protected topological phases

General symmetry

- "Boring" bulk: gapped, no symmetry breaking, short-range entangled (no fractional statistics)

  - non-example: FQHE (fractional statistics, long-range entanglement)



- Interesting boundary: protected gapless modes

  - Trivial phase: no protected gapless boundary modes

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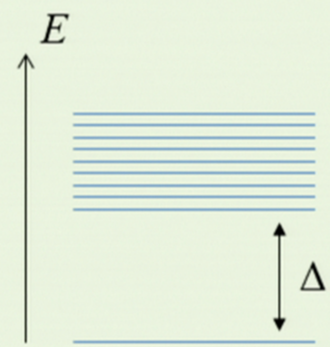

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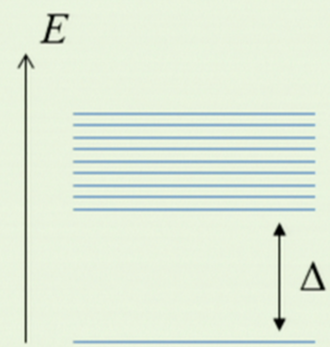

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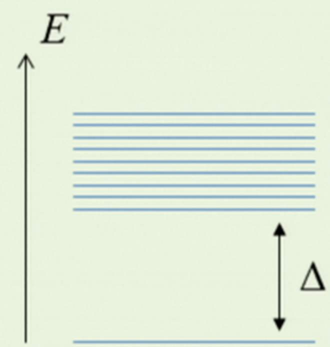

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Questions in SPT phases

- Classification: *How may distinct phases for a given symmetry?*
  - Non-interacting fermions (Kitaev 2008, Schnyder et al 2008)
  - General Boson systems:

    Conjecture: classified by group cohomology $H^{d+1}[G, U(1)]$

    (Chen, Gu, Liu & Wen 2011)

- Characterization: *What properties distinguish different phases?*
  - Boundary modes (edge modes in 2D, surface modes in 3D, etc)
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What is gauging a symmetry?

- Minimally couple the "matter field" to a "gauge field" (continuum or lattice gauge field)
- Example: 2D s-wave superconductor

- BCS theory, BdG equation

- Z_2 fermion parity

- Excitations: $1, \epsilon$

Gauging fermion parity →

- Z_2 gauge theory
- Excitations:

$$1, \epsilon, m_a, m_b$$

- Statistics:

$$\theta_{\epsilon a} = \theta_{\epsilon b} = \pi$$
$$\theta_\epsilon = \pi, \theta_a = \theta_b = 0$$

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Conjecture:

Gauging the symmetry in different SPT phases leads to different types of braiding statistics

It is verified by various examples of 2D SPT phases.

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Outlines

- 2D braiding and 3D loop braiding
- Symmetry protected topological (SPT) phases
- Loop braiding statistics in gauged $(Z_N)^K$ 3D SPT phases

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Model

- 3D lattice boson model with $(Z_N)^K$ symmetry

- Gapped, no symmetry breaking, short-range-entangled

- Couple to a $(Z_N)^K$ lattice gauge field

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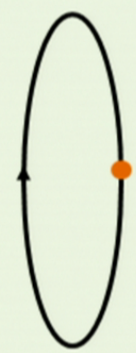
Excitations

- Assuming Abelian statistics
- Charges

$$q = (q_1, q_2, \ldots, q_K)$$
$$q_i = 0, 1, \ldots, N-1$$

- Vortex loops

$$\phi = \frac{2\pi}{N}(c_1, c_2, \ldots, c_K)$$
$$c_i = 0, 1, \ldots, N-1$$



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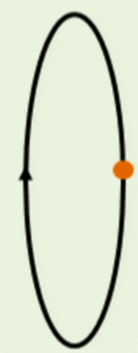

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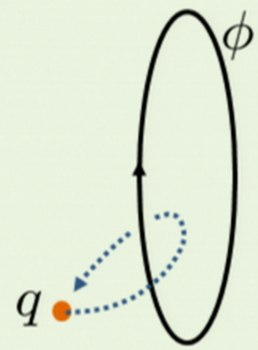
Braiding statistics

- charge-charge

Charges are bosons

- charge-loop



Aharonov-Bohm law

$$\theta = q \cdot \phi$$

Braiding statistics

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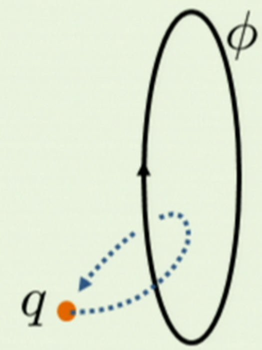

Aharonov-Bohm law

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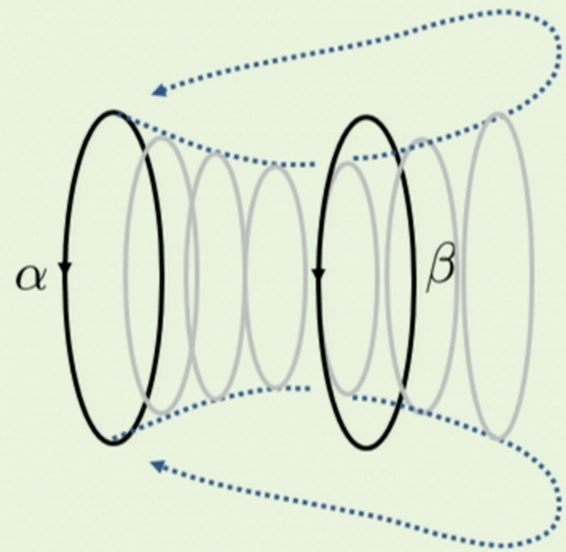
Braiding statistics

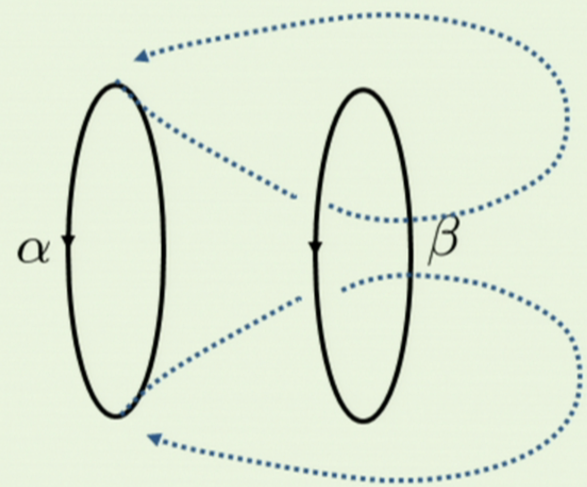
- Two loops



$$\theta_{\alpha\beta} = ?$$

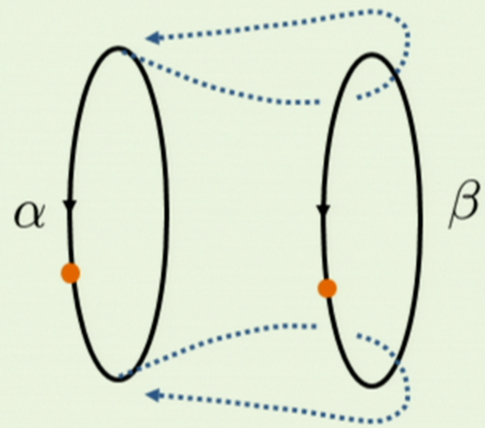
Braiding statistics

- If the loops are "neutral"



Braiding statistics

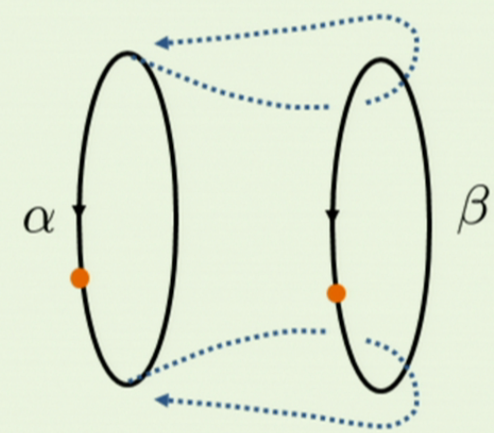
- General two-loop braiding



$$\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha = \frac{2\pi}{N} \times \text{integer}$$

Braiding statistics

- ❑ General two-loop braiding



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Braiding statistics

- Charge-charge: 0
- Charge-loop: $\theta = q \cdot \phi$
- Loop-loop: $\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha$

No features of the underlying SPT phases are reflected!

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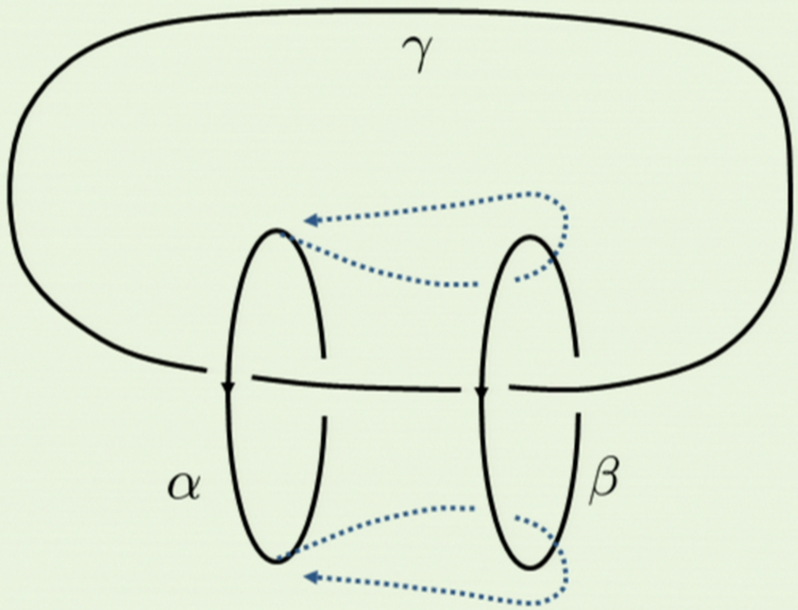
Three-loop braiding process



$$\theta_{\alpha\beta,\phi_\gamma} = ?$$

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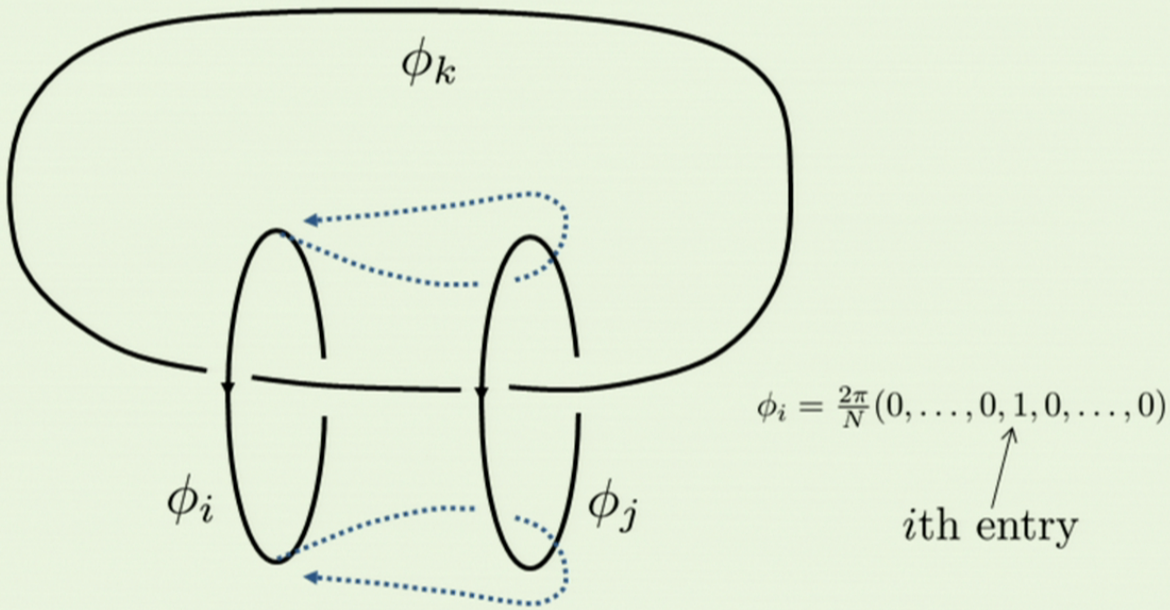
Statistics of unit fluxes



- Define: $\Theta_{ij,k} = N\theta_{ij,k}$, independent of charge attachment

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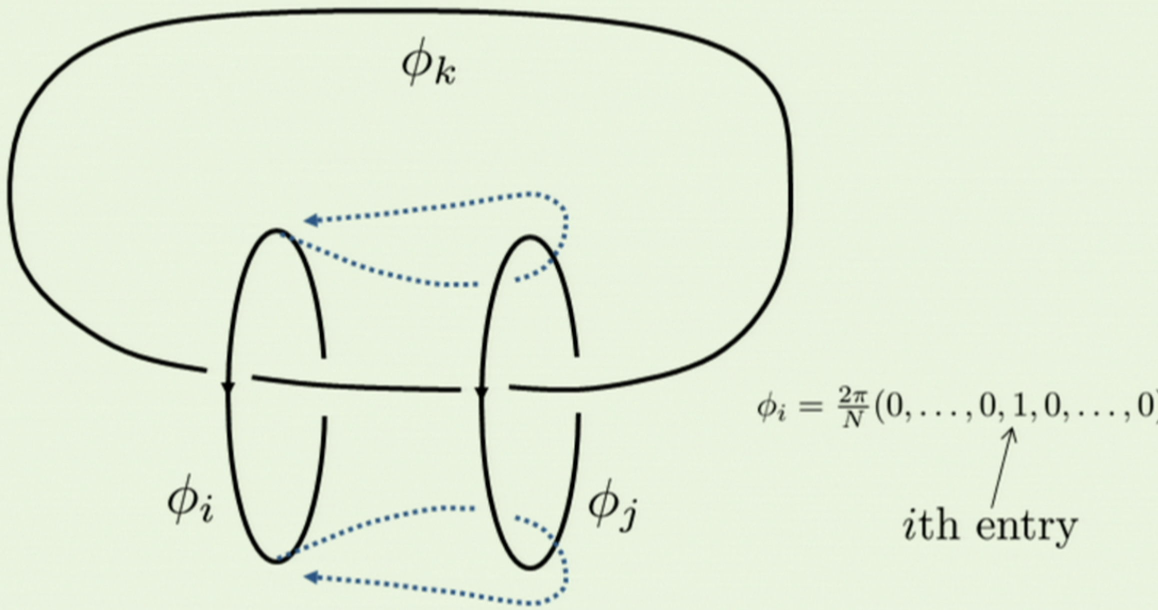
Statistics of unit fluxes



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Constraints

- Symmetric: $\Theta_{ij,k} = \Theta_{ji,k}$
- Quantized: $\Theta_{ij,k} = \frac{2\pi}{N} \times \text{integer}$
- Cyclic: $\Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} = 0$

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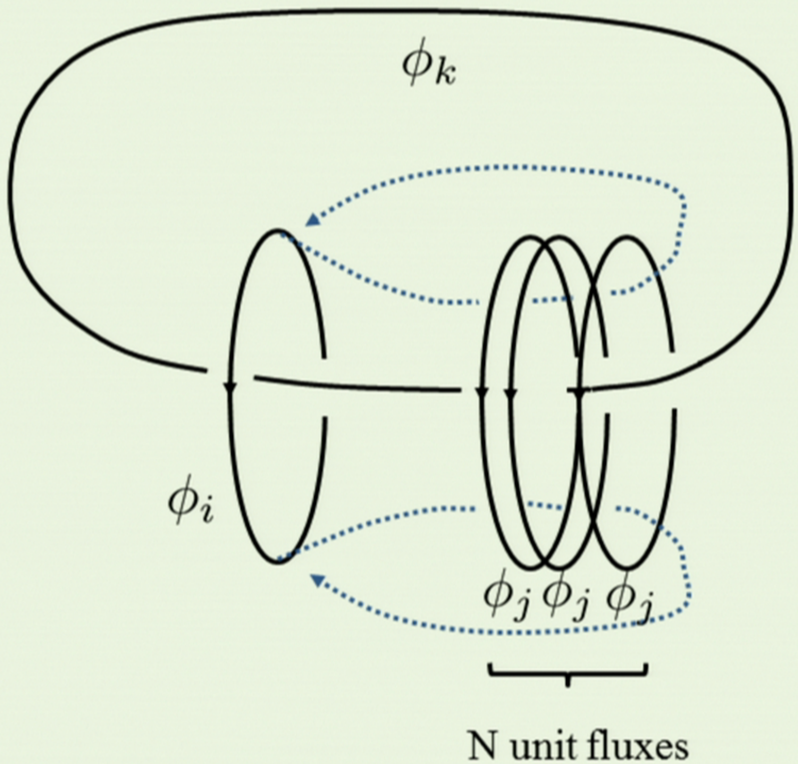
Proving $\Theta_{ij,k} = \frac{2\pi}{N}(\text{integer})$



1. Total phase is $N\theta_{ij,k}$
2. N unit fluxes fuses to a charge

   $$q \cdot \phi_i = \frac{2\pi}{N} \times \text{integer}$$

3. Combine

   $$N\theta_{ij,k} = \frac{2\pi}{N} \times \text{integer}$$

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❖ Additional constraints on exchange statistics of loops

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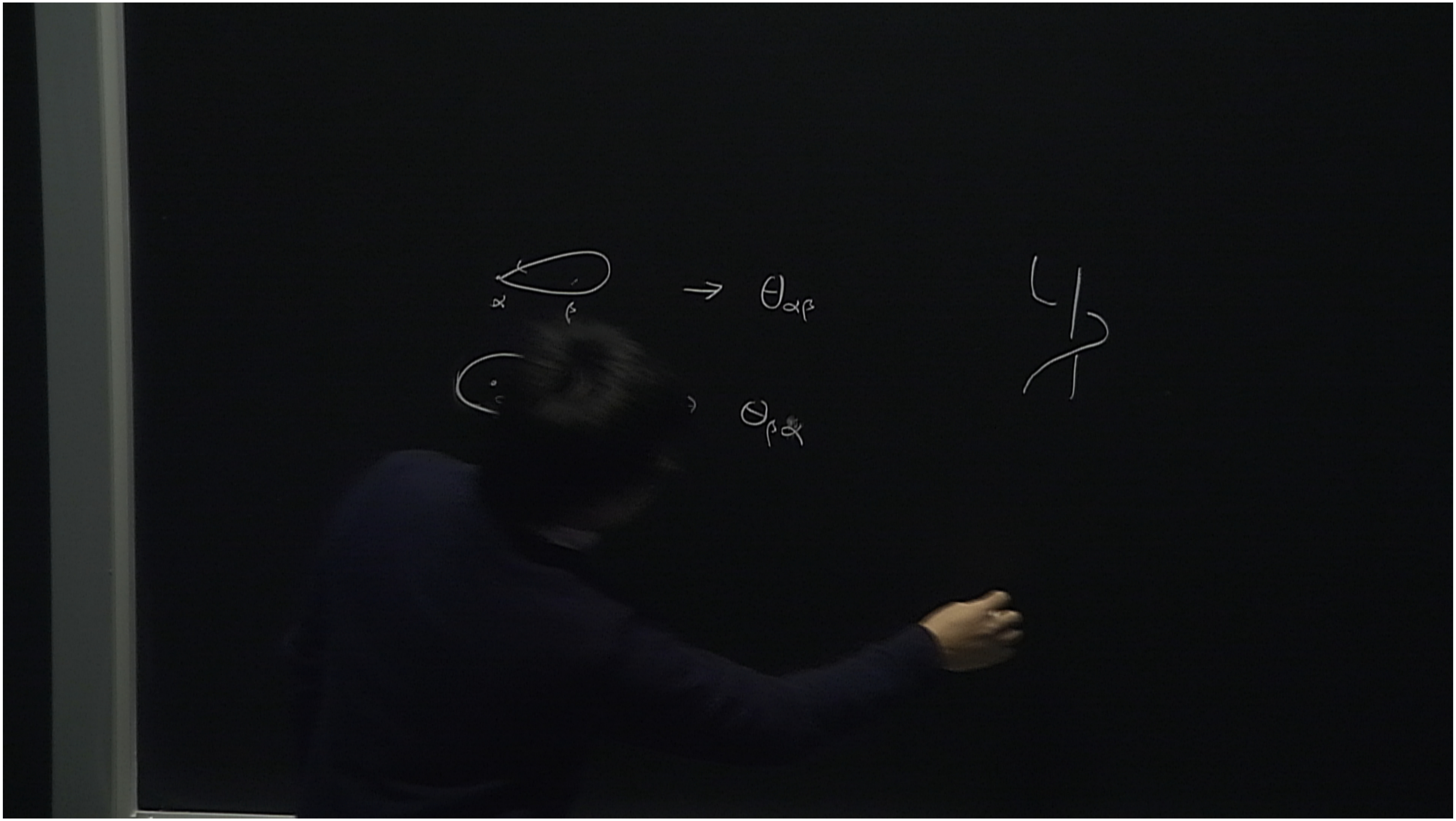
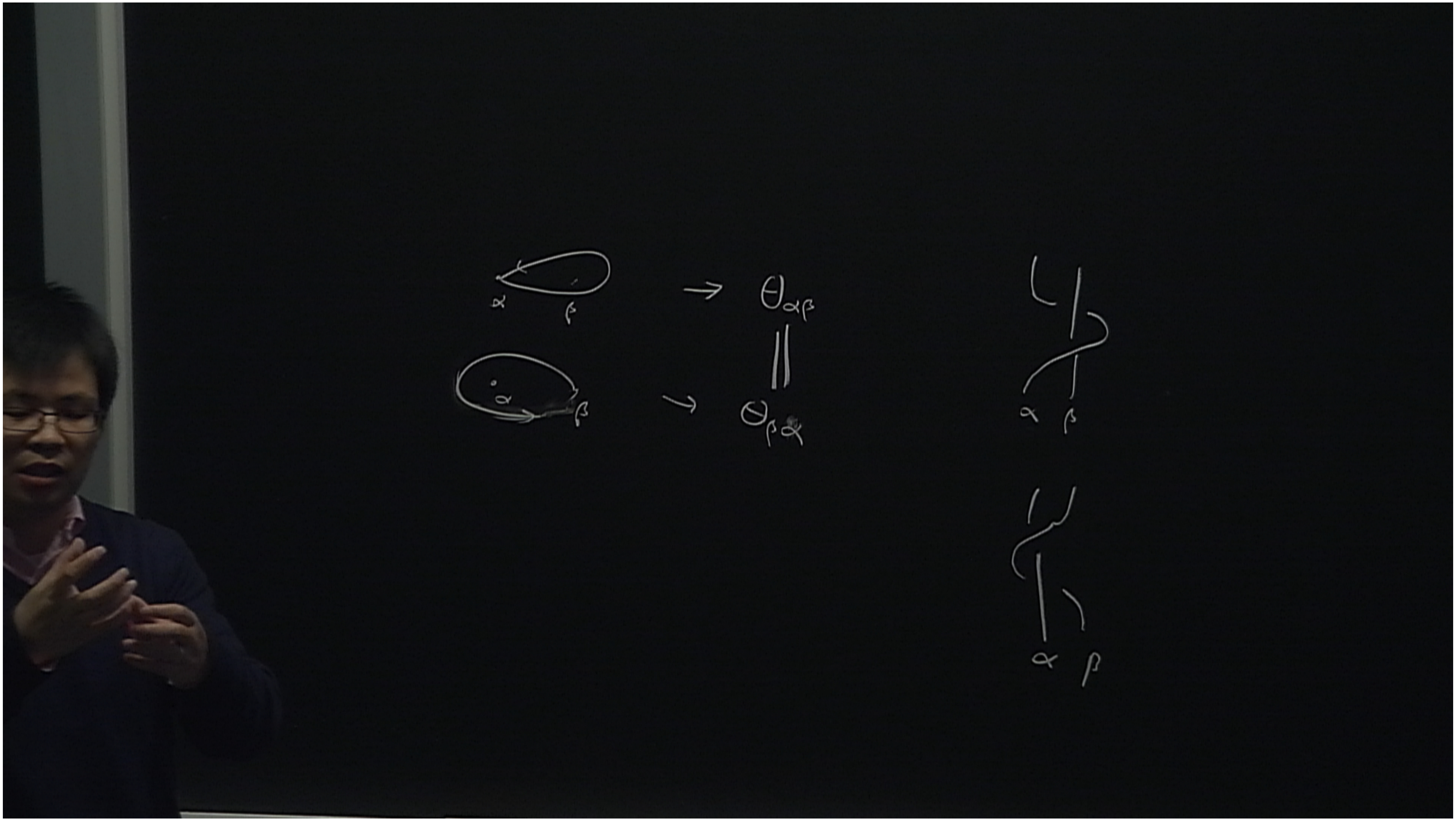
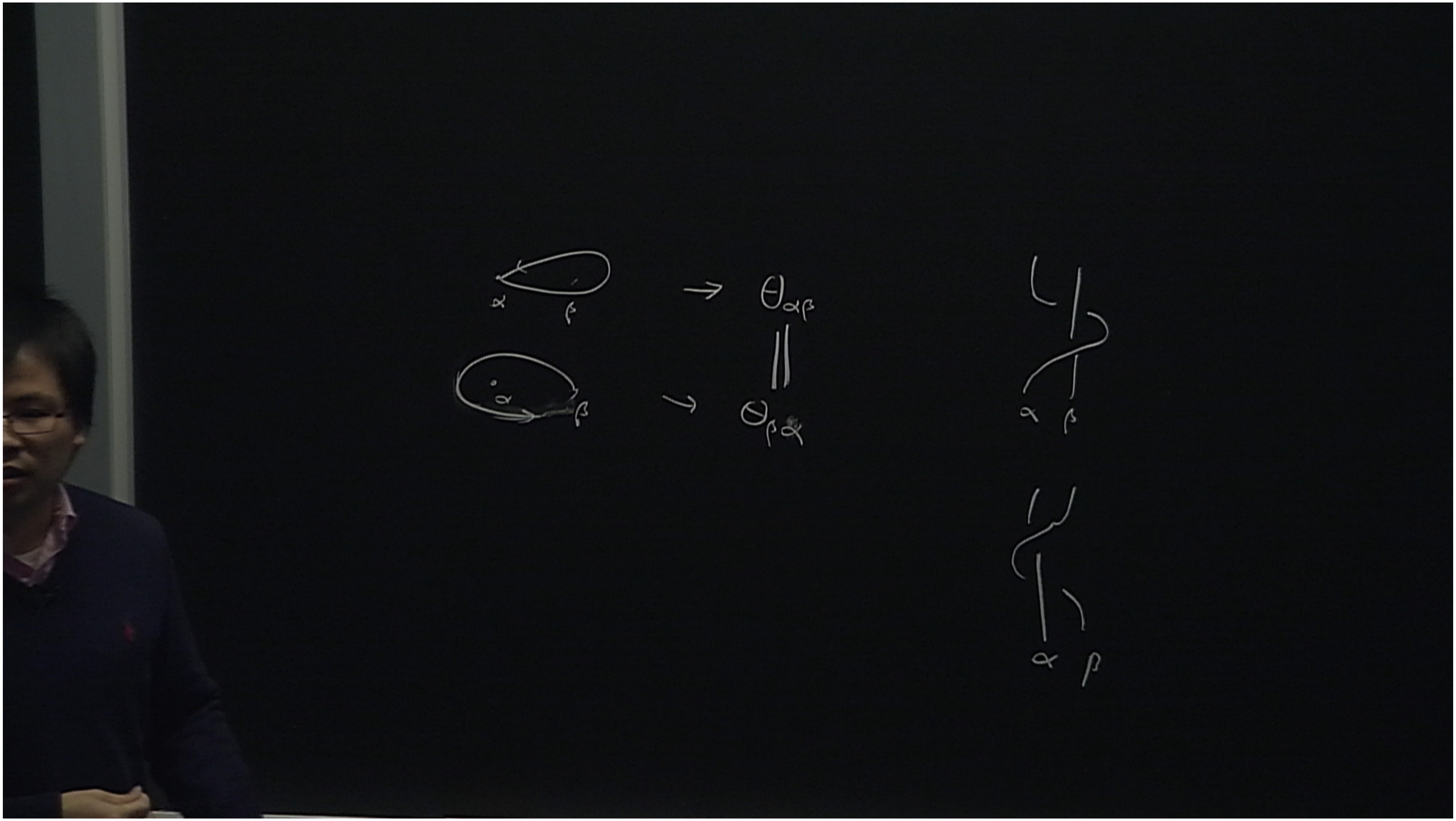
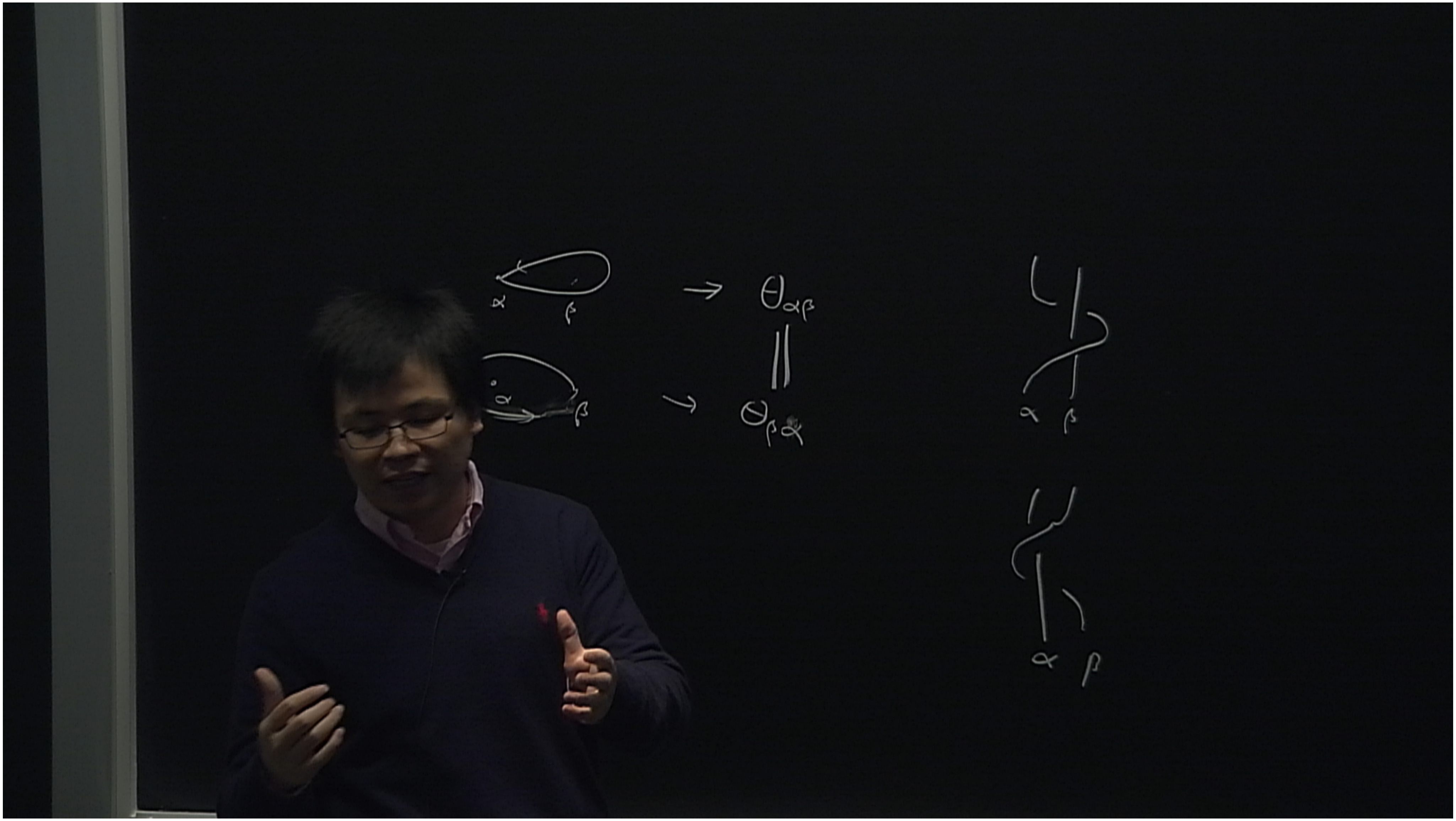

α
β
$\theta_{\alpha\beta}$
$\theta_{\beta\alpha}$





α
β
→ Θ_{αβ}
||
→ Θ_{βα}
α β
α β


α
β
$\Theta_{\alpha\beta}$
α
β
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α β
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$Z_N \times Z_N$ SPT phases

- Take the exactly soluble lattice model from [Chen, Gu, Liu and Wen, 2011] with $(Z_N)^2$ symmetry
  - Group cohomology models are labeled by

$$H^4[(Z_N)^2, U(1)] = (Z_N)^2$$

  Conjecture: they belong to distinct SPT phases

- Couple to $(Z_N)^2$ lattice gauge field

- Extract the three-loop statistics

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$Z_N \times Z_N$ SPT phases

- The three-loop statistics

$$\Theta_{11,1} = 0$$

$$\Theta_{12,1} = \Theta_{21,1} = \frac{2\pi}{N} p_1$$

$$\Theta_{22,1} = -\frac{4\pi}{N} p_2$$

$$\Theta_{11,2} = -\frac{4\pi}{N} p_1$$

$$\Theta_{12,2} = \Theta_{21,2} = \frac{2\pi}{N} p_2$$

$$\Theta_{22,2} = 0$$

$$\Theta_{1,1} = \Theta_{2,2} = 0$$

$$\Theta_{1,2} = -\frac{2\pi}{N} p_1$$

$$\Theta_{2,1} = -\frac{2\pi}{N} p_2$$

p_1, p_2 are integers

CW and M. Levin, PRL 2014

General $(Z_N)^K$ SPT phases

- Repeat the previous procedure
  - Take the group cohomology models with $(Z_N)^K$ symmetry
  - Couple to $(Z_N)^K$ lattice gauge field
  - Extract the three-loop statistics

- However, there is a complication: non-Abelian statistics when $K \geq 4$

CW and M. Levin, in preparation

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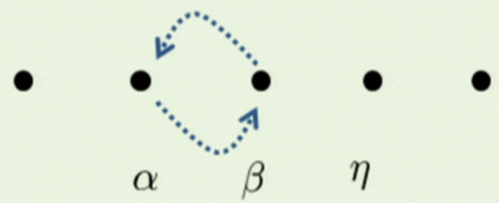
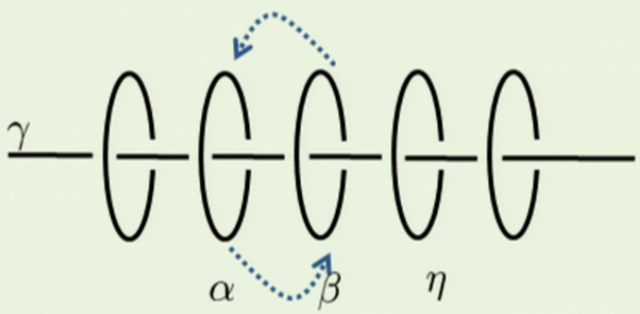
Parallel between 2D and 3D braiding



- Fusion Space: $\mathbb{V}^{\delta}_{\alpha\beta}$
- R-symbol:

$$R^{\delta}_{\alpha\beta} : \mathbb{V}^{\delta}_{\alpha\beta} \rightarrow \mathbb{V}^{\delta}_{\beta\alpha}$$



- Fusion Space: $\mathbb{V}^{\delta}_{\alpha\beta,\phi_\gamma}$
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R-matrices are too complicated. Instead, we define the *topological invariants:*

$$\Theta_{i,j}, \Theta_{ij,k}, \Theta_{ijk,l}$$

CW and M. Levin, in preparation

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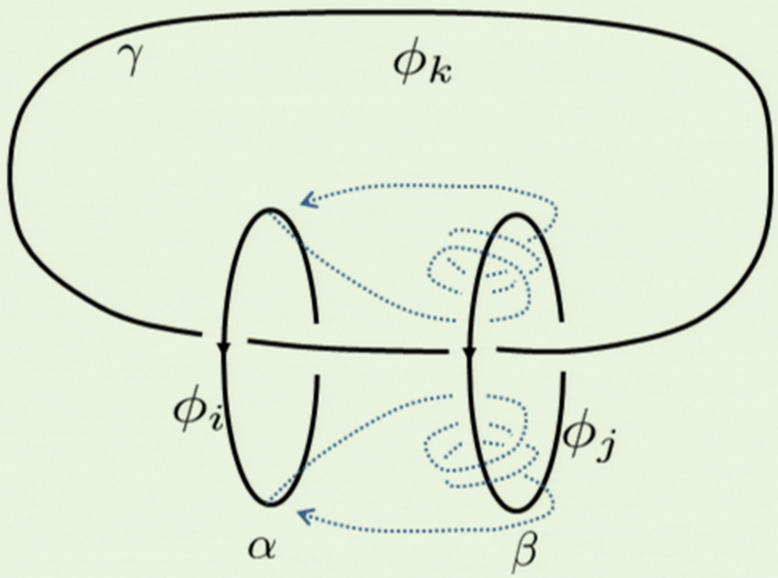
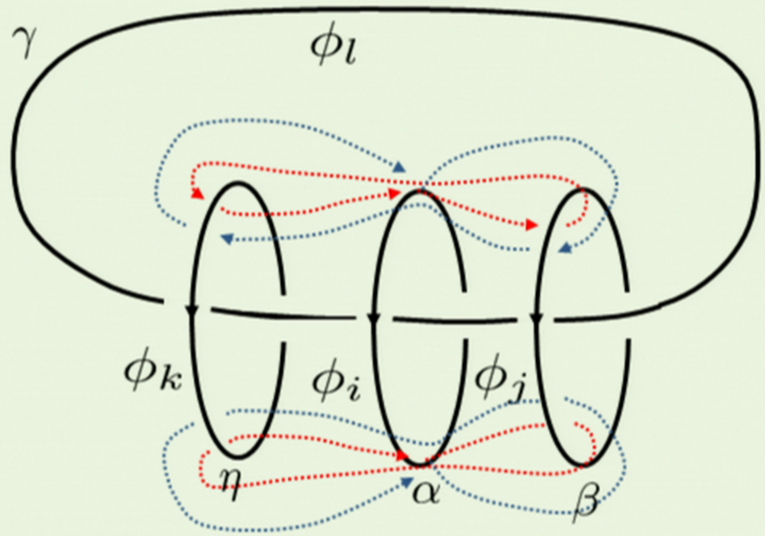
The invariant $\Theta_{ij,k}$



- Abelian phase
- Function of ϕ_i, ϕ_j, ϕ_k
- In the case of Abelian statistics
  $\Theta_{ij,k} = N\theta_{ij,k}$

$$(B_{\alpha\beta,\phi_k})^N = e^{i\Theta_{ij,k}} \cdot \mathrm{id}_{\mathbb{V}^{\delta}_{\alpha\beta,\phi_k}}$$

The invariant $\Theta_{ijk,l}$



- Abelian phase
- Function of $\phi_i, \phi_j, \phi_k, \phi_l$
- Vanishes for Abelian statistics

$$B^{-1}_{\alpha\eta,\phi_l} B^{-1}_{\alpha\beta,\phi_l} B_{\alpha\eta,\phi_l} B_{\alpha\beta,\phi_l} = e^{i\Theta_{ijk,l}} \cdot \mathrm{id}_{\mathbb{V}}$$

Example: $(Z_N)^4$ models

- number of group cohomology models:

$$H^4[(Z_N)^4, U(1)] = (Z_N)^{19}$$

- Independent entries: 12 for $\Theta_{i,j}$, 6 for $\Theta_{ij,k}$, 1 for $\Theta_{ijk,l}$.

Each takes value in

$$0, \frac{2\pi}{N}, \frac{2\pi}{N}2, \ldots, \frac{2\pi}{N}(N-1)$$

So, N^{19} combinations in total.

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- Independent entries: 12 for $\Theta_{i,j}$, 6 for $\Theta_{ij,k}$, 1 for $\Theta_{ijk,l}$.

Each takes value in

$$0, \frac{2\pi}{N}, \frac{2\pi}{N}2, \ldots, \frac{2\pi}{N}(N-1)$$

So, N^{19} combinations in total.

Conclusion

- Three-loop braiding statistics distinguish the group cohomology models. In particular, the topological invariants take different values in each of these models.

- Confirm that each of the group cohomology models indeed belongs to a distinct SPT phase (in the case of finite abelian group)

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Open questions

- Loop braiding statistics in non-Abelian gauge theories
- Beyond gauge theories
- Algebraic theories of loops fusion and braiding
- Loop braiding in fermionic systems
- Field theory description
- Surface theories and surface-bulk correspondence
- Experimental realization and detection
- ……