Title: Givental J-functions, Quantum integrable systems, AGT relation with surface operator

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Abstract: <p>I will talk about 4d N=2 gauge theories with a co-dimension-two full surface operator, which exhibit a fascinating interplay of supersymmetric gauge theories, equivariant Gromov-Witten theory and geometric representation theory. For pure Yang-Mills and N=2* theory, a full surface operator can be described as the 4d gauge theory coupled to a 2d N=(2,2) gauge theory. By supersymmetric localizations, we present the exact partition functions of both 4d and 2d theories which satisfy integrable equations. In addition, I will show the validity of the orbifold method in one-loop computations when a full surface operator is inserted, and the form of the structure constants with a semi-degenerate field in SL(N,R) WZNW model is predicted from one-loop determinants.</p>

Givental J-functions, Quantum integrable systems, AGT relation with surface operator

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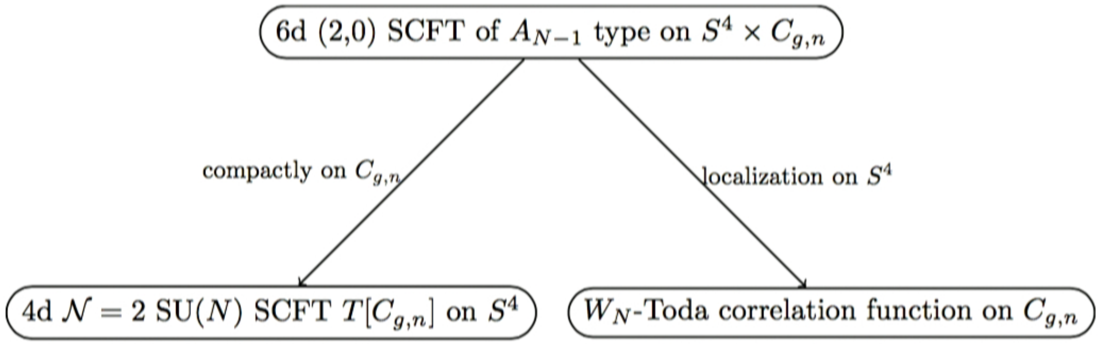
AGT relation



4d partition function is equivalent to Today correlation function [Alday-Gaiotto-Tachikawa, Wyllard]

$$
\begin{aligned}
Z^{4d}[T[C_{g,n}]] &= \langle V_{\beta_1}(z_1)\cdots V_{\beta_n}(z_n)\rangle_{C_{g,n}} \\
\int da\; Z_{\text{1-loop}}(a,m,\epsilon_1,\epsilon_2)\, |Z_{\text{inst}}(a,m,\epsilon_1,\epsilon_2;z)|^2 &= \int d\alpha\; C(\alpha,\beta,b)\, |F(\alpha,\beta,b:z)|^2
\end{aligned}
$$

Defects in 6d perspective

- co-dimension four defects: M2-branes attach to 2d submanifold of M5-branes
- co-dimension two defects: intersection of M5-branes supported on 4d submanifold

	4d	2d	4d	2d
codim 4	2	0	surface op.	deg. field
codim 4	1	1	line op.	Verlinde loop op.
codim 4	0	2	?	change theory
codim 2	4	0	change theory	?
codim 2	3	1	domain wall	loop op.
codim 2	2	2	surface op.	change theory

Defects in 6d perspective

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codim 2	2	2	surface op.	change theory

Co-dimension two surface operator

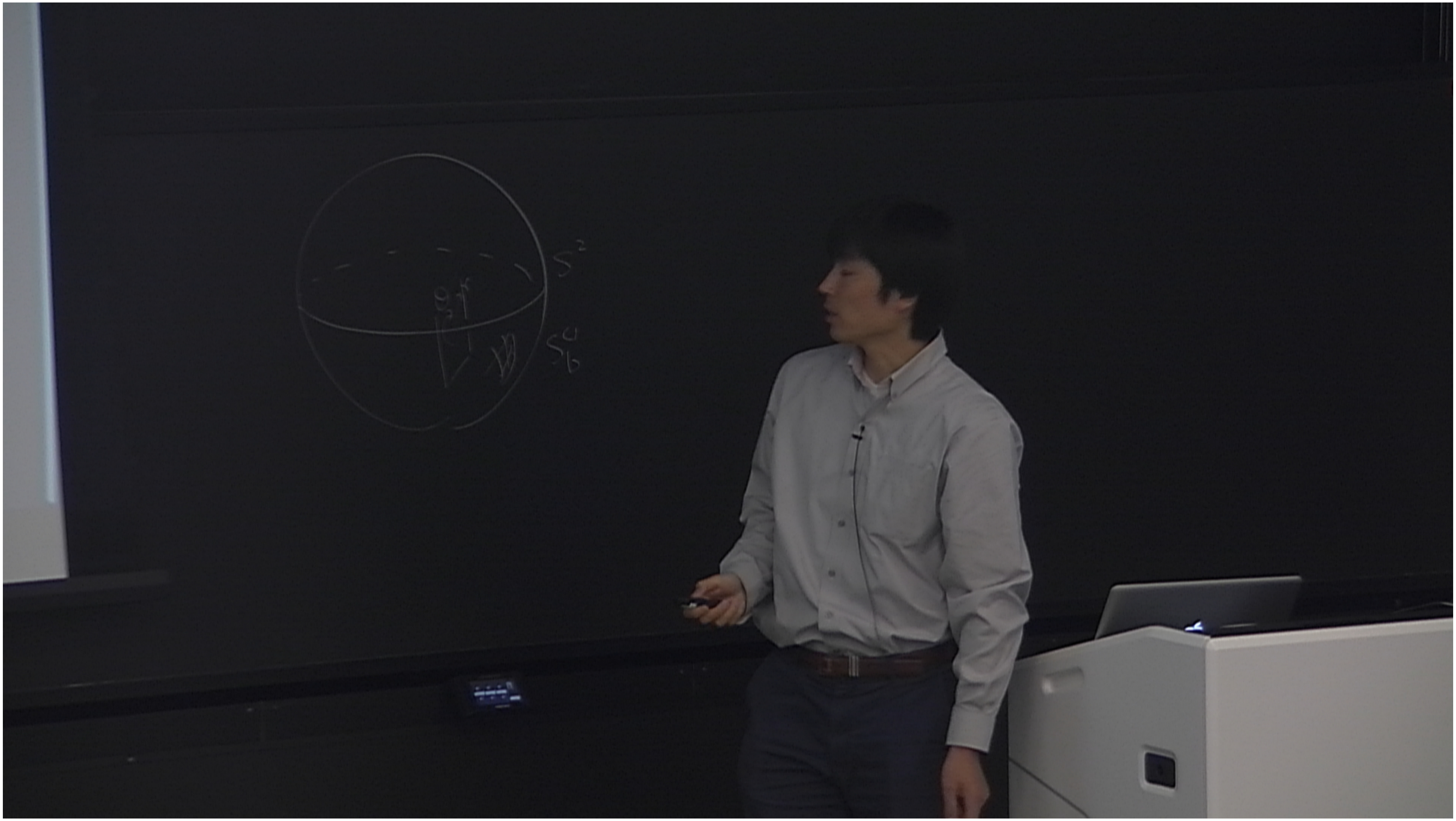
surface operator can be described in two ways [Gukov-Witten]

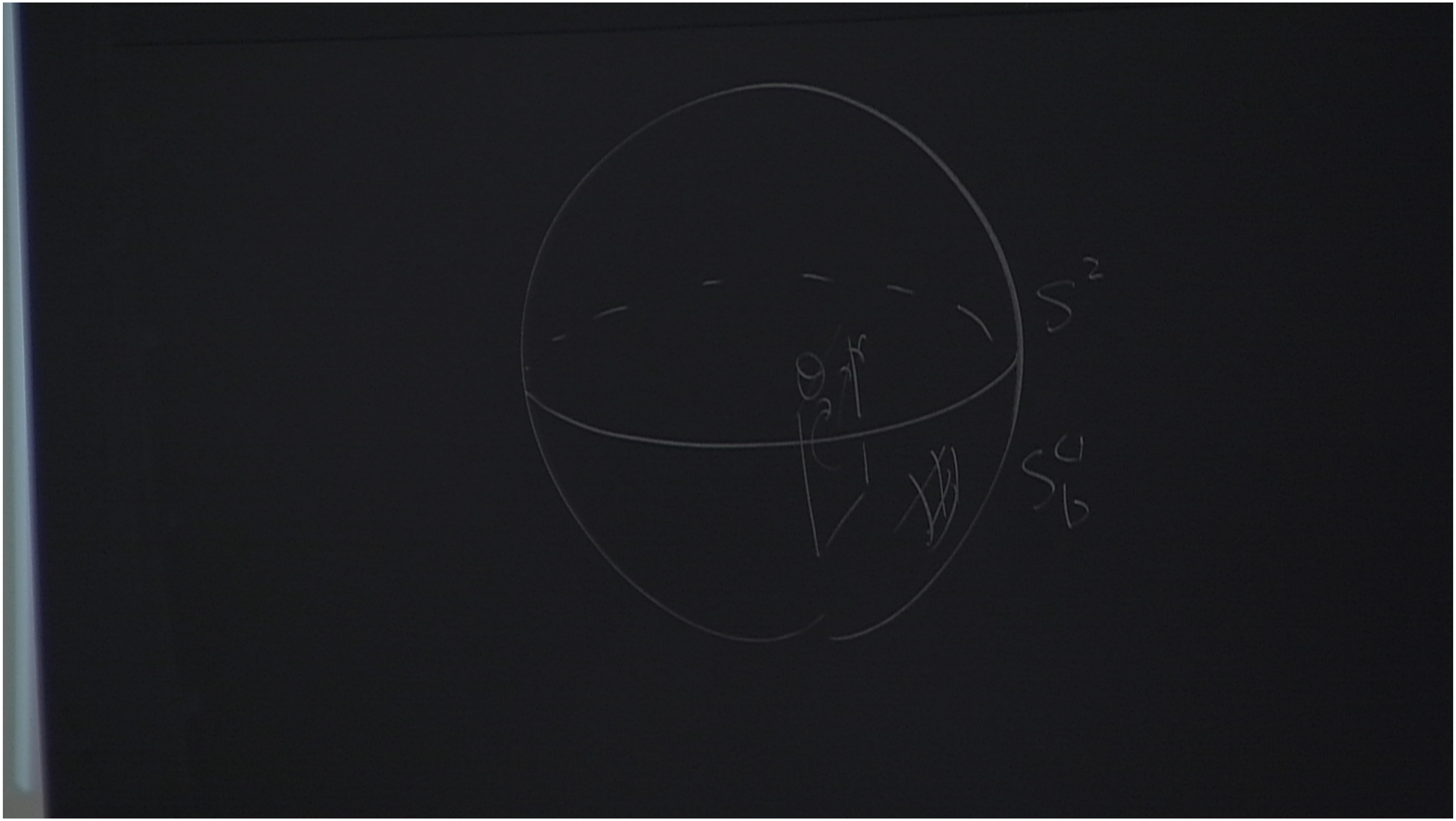
- imposing singular behavior of gauge field

$$A_\mu dx^\mu \sim \mathrm{diag}(\alpha_1, \ldots, \alpha_N) i d\theta \ ,$$

  and inserting the phase factor $\exp(i\eta_l \mathfrak{m}^l)$ in the path integral
- 4d $\mathcal{N} = 2$ gauge theory coupled to 2d (2,2) gauge theory on the surface


S^2
S_b^u


S^2
S^4_b



S^2
S^4_b

Co-dimension two surface operator

- If the parameter $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ is specified by

$$\vec{\alpha} = (\underbrace{\alpha_{(1)}, \ldots, \alpha_{(1)}}_{N_1 \text{ times}}, \underbrace{\alpha_{(2)}, \ldots, \alpha_{(2)}}_{N_2 \text{ times}}, \ldots, \underbrace{\alpha_{(M)}, \ldots, \alpha_{(M)}}_{N_M \text{ times}}) ,$$

the gauge group is broken to the commutant of $\vec{\alpha}$ on the surface $\mathcal{C}$:

$$\mathbb{L} = \mathrm{S}[\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_M)] ,$$

$\mathbb{L}$ is the Levi part of a parabolic subgroup $\mathcal{P}$ of the complexified Lie group $G_{\mathbb{C}}$

  - $\vec{\alpha} = (\alpha, \cdots, \alpha, (N-1)\alpha) \longrightarrow \mathbb{L} = \mathrm{SU}(N-1) \times \mathrm{U}(1)$: simple
  - all α_i are distinct $\longrightarrow \mathbb{L} = \mathrm{U}(1)^N$: full

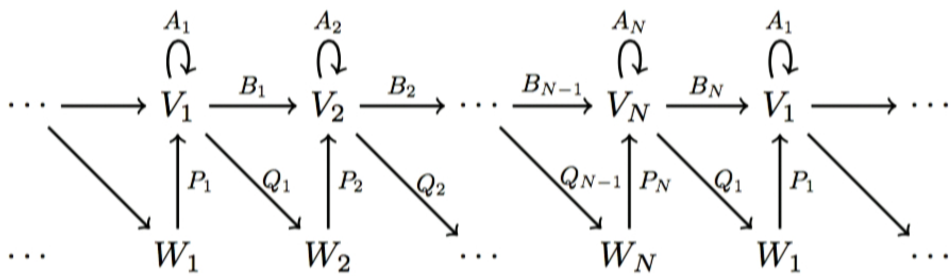
- In addiiton, we insert the phase factor $\exp(i\eta_I \mathfrak{m}^I)$ where η_I are "electric parameters" or "2d theta angles", and $\mathfrak{m}^I$ are magnetic fluxes on $\mathcal{C}$

$$\mathfrak{m}^I = \frac{1}{2\pi} \int_{\mathcal{C}} F^I \qquad (I = 1, \cdots, M) ,$$

where $\sum_I \mathfrak{m}^I = 0$.

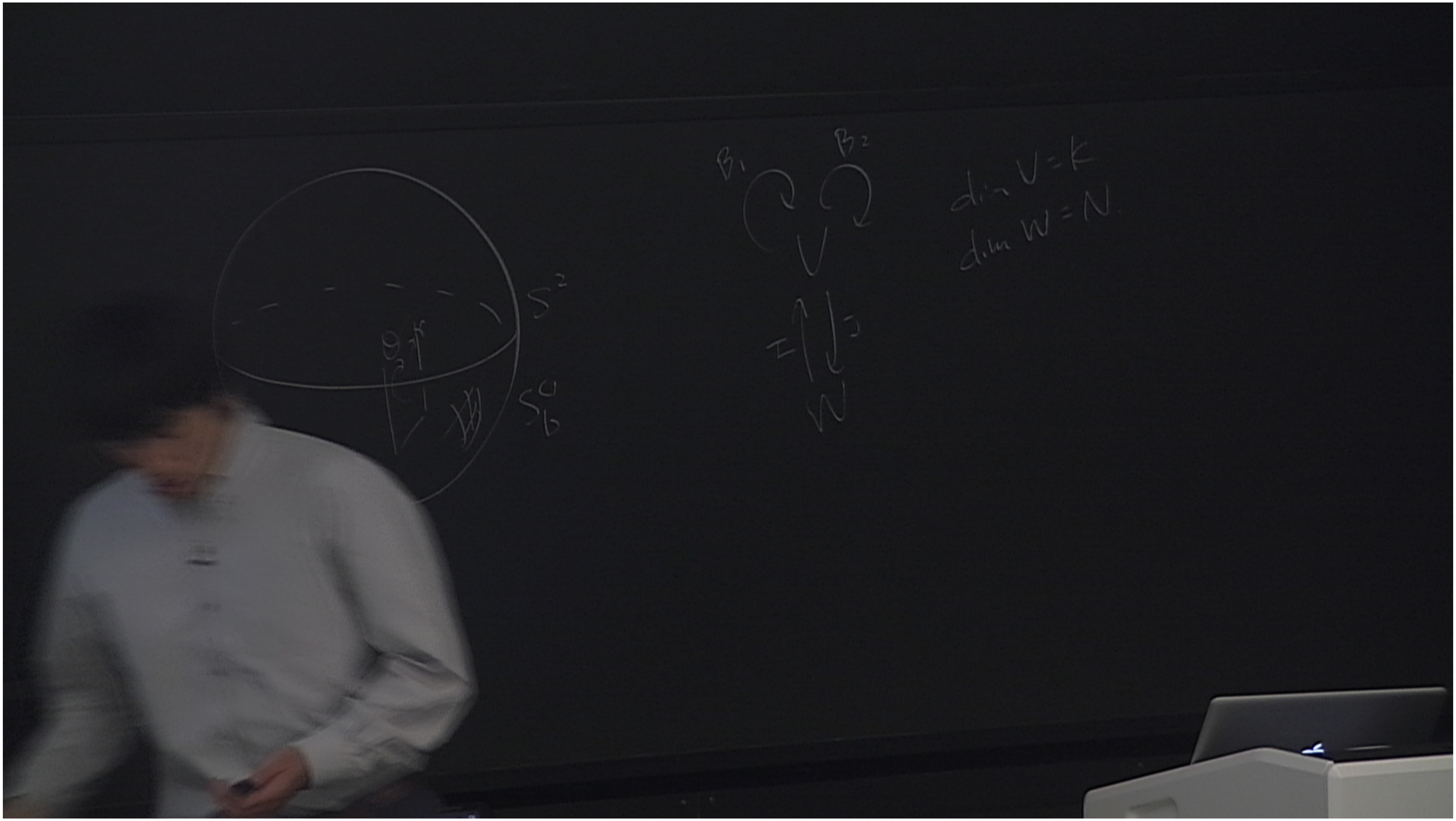
Instanton moduli space

- the moduli space $\mathcal{M}_{\vec{N},k,\vec{\mathfrak{m}}}$ of instanton with the boundary condition of the gauge field on the surface is called affine Laumon space [Feigin-Finkelberg-Negut-Rybnikov]
- affine Laumon space is equivalent to instanton moduli space on an orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$
- it admits quiver representation called chain-saw quiver [Finkelberg-Rybnikov]



where $\dim W_l = N_l$, $\dim V_l = k_l$, with $k_M = k$, $k_{l+1} = k_l + \mathfrak{m}^{l+1}$.

$$\mathcal{M}_{\vec{N},\vec{k}} = \{(A_l, B_l, P_l, Q_l) | \mathcal{E}^{(l)}_{\mathbb{C}} = 0, \text{ stability condition}\}/\mathrm{GL}(k_1, \mathbb{C}) \otimes \cdots \otimes \mathrm{GL}(k_M, \mathbb{C})$$



S2
B1
B2
V
I
J
W
dim V = K
dim W = N

Instanton moduli space

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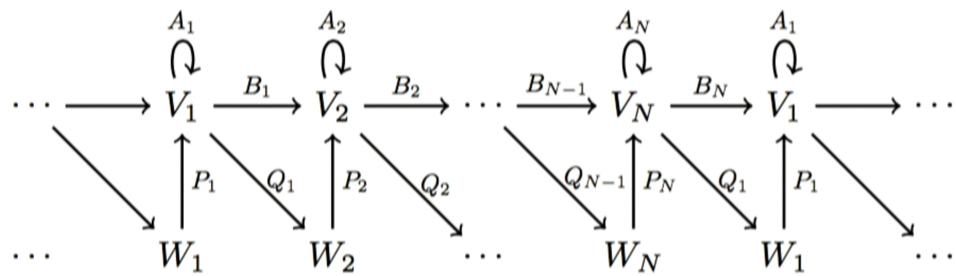

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Instanton partition function

- the action of the Cartan torus $\mathrm{U}(1)^2 \times \mathrm{U}(1)^N$ of the spacetime and the gauge symmetry
- Due to the orbifold spcae, the equivariant parameter

$$(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2/M} z_2) \ .$$

- non-trivial holonomy shifts the equivariant parameters $(a_1, \cdots, a_N)$ of $\mathrm{U}(1)^N$ by

$$a_{s,I} \to a_{s,I} - \tfrac{I-1}{M}\epsilon_2 \ , \qquad (s = 1, \cdots, N_I) \ .$$

- character of the equivariant action at the fixed points yields the Nekrasov insanton partition function

$$\mathscr{Z}_{\rm inst}[\vec{N}] = \sum_{\vec{k}} \prod_{I=1}^{M} z_I^{k_I} \mathscr{Z}_{\vec{N},\vec{k}}(\epsilon_1, \epsilon_2, a, m) \ ,$$

Instanton partition function

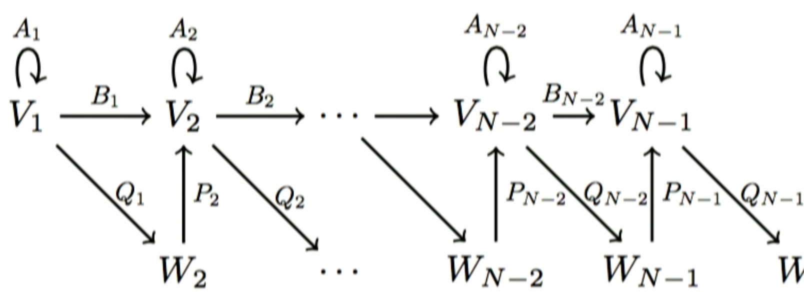
- making change of variables

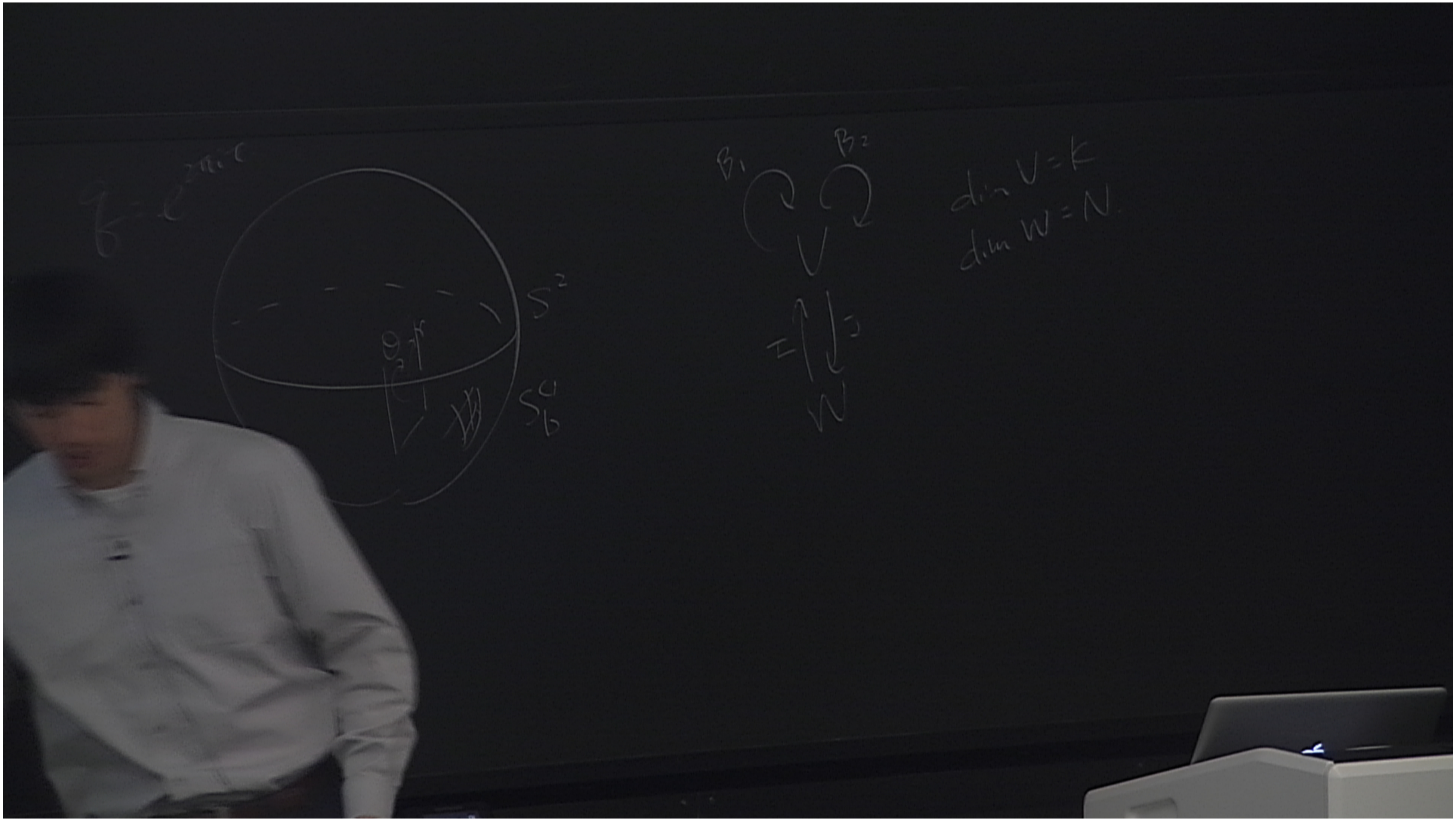
$$\prod_{I=1}^{M} z_I = q \;, \qquad z_I = e^{t_I - t_{I+1}} \;,$$

the instanton partition function can be re-arranged

$$\mathscr{Z}_{\rm inst}[\vec{N}] = \sum_{k=0}^{\infty} \sum_{\mathfrak{m} \in \Lambda_{\mathbb{L}}} q^k e^{t \cdot \mathfrak{m}} \mathscr{Z}_{\vec{N},k,\vec{\mathfrak{m}}}(\epsilon_1, \epsilon_2, a, m) \;,$$

- when $k = 0$, the partition function encodes only 2d dynamics on the support of the surface operator
- chain-saw quiver demotes to hand-saw quiver when $k = 0$ which describes Laumon space




$q = e^{2\pi i \tau}$
S^2
S^4_b
B_1
B_2
V
I
J
W
dim $V = K$
dim $W = N$.


B_1
B_2
dim V=K
dim W=N.
S^2

Dual 2d CFT

- Dual 2d CFT changes since the co-dimension two defect wraps on a Riemann surface $C_{g,n}$ [Alday-Tachikawa, Kozcaz-Pasquetti-Passerini-Wyllard, Wyllard]
- For a surface operator of type $\vec{N} = [N_1, \cdots, N_M]$, the 2d symmetry is the W-algebra $W(\widehat{\mathfrak{sl}}(N), \vec{N})$ obtained by the quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{sl}}(N)$ for the embedding $\rho_{\vec{N}} : \mathfrak{sl}(2) \to \mathfrak{sl}(N)$ corresponding to the partition $\vec{N}$
- For exapmle, when $N = 3$,

Partition	2d symmetry	surface op.
$[3]$	W_3	absent
$[2,1]$	$W_3^{(2)}$	simple
$[1,1,1]$	$\widehat{\mathfrak{sl}}(3)$	full

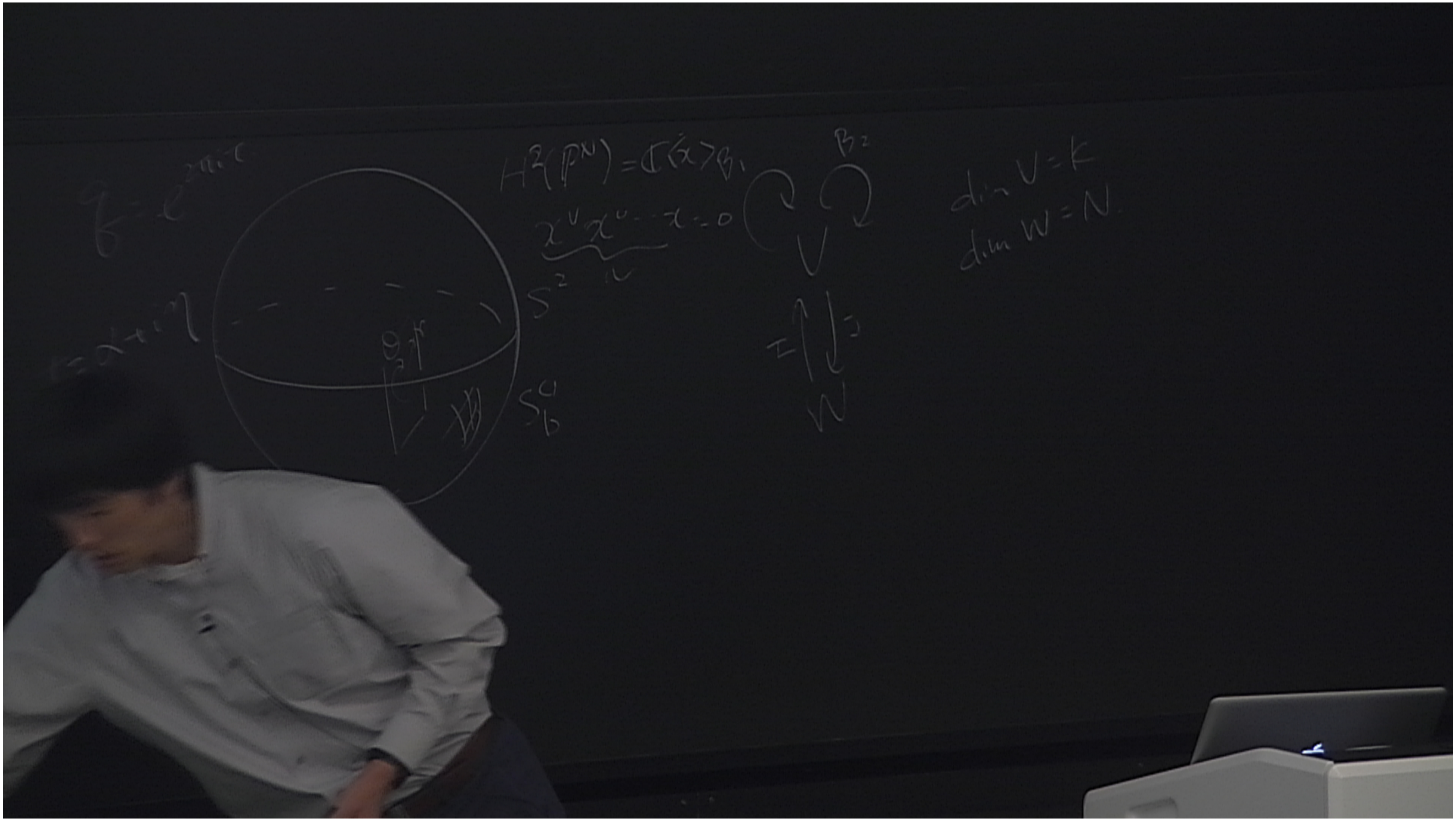
- equivalence between instanton partition function and $W(\widehat{\mathfrak{sl}}(N), \vec{N})$-conformal blocks has been checked [Wyllard, Kanno-Tachikawa]
- we will show the relation between one-loop determinant and structure constant of $\mathrm{SL}(2, \mathbb{R})$ WZNW model

Quantization

The AGT relation with a surface operator is closely related to quantization of cohomology ring

- cohomology ring of $\mathbf{P}^{N-1}$

$$H^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x]/(x^N)\ .$$


dim V = K
dim W = N

Equivariant cohomology ring

let us define the S^1-equivariant action on $\mathbf{P}^{N-1}$ by

$$\lambda[z_0 : \cdots : z_{N-1}] = [\lambda^{r_0} z_0 : \cdots : \lambda^{r_{N-1}} z_{N-1}] \ ,$$

for $\lambda \in S^1$. Then, one can construct bundle

$$(\mathbf{P}^{N-1})_{S^1} = (ES^1 \times \mathbf{P}^{N-1})/S^1 \to BS^1 \ ,$$

the S^1-equivariant cohomology ring of $\mathbf{P}^{N-1}$ is given by

$$H^*_{S^1}(\mathbf{P}^{N-1}) = H^*((\mathbf{P}^{N-1})_{S^1}) \cong \mathbb{C}[x,\hbar]/(\prod_{i=0}^{N-1}(x - r_i\hbar)) \ ,$$

where $\hbar$ represents the hyperplane class of the base manifold of the universal S^1-bundle

$$S^{2\infty+1} = ES^1 \to BS^1 = \mathbf{P}^\infty \ ,$$

Quantization

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- S^1-equivariant cohomology ring of $\mathbf{P}^{N-1}$

$$H^*_{S^1}(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,\hbar]/(\prod_{i=0}^{N-1}(x - r_i\hbar)) \ ,$$

Qunatum cohomology ring

- the cohomology ring is quantized based on Gromov-Witten theory. The quantum cohomology is ordinary cohomology with a quantum product defined by

$$T_i \circ T_j = \sum_{k,\ell} C_{ijk}(t)\eta^{k\ell} T_\ell \ ,$$

for a basis T_i of the cohomology group.

- Here the structure constants $C_{ijk}(t) := \frac{\partial^3 F_0}{\partial T_i \partial T_j \partial T_k}$ is the third derivative of the genus-zero prepotential depending on the complexified Kähler parameter t and $\eta_{ij} := \int T_i \cup T_j$ is the metric on the cohomology group.
- the WDVV equation is equivalent to the associativity of the quantum product, and therefore the quantum product can be thought of as quantum deformation of the cup product of cohomology.
- Writing $q = e^t$, the quantum cohomology ring of $\mathbf{P}^{N-1}$ is isomorphic to

$$QH^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,q]/(x^N - q) \ .$$

Quantization

The AGT relation with a surface operator is closely related to quantization of cohomology ring

- cohomology ring of $\mathbf{P}^{N-1}$

$$H^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x]/(x^N) \ .$$

- S^1-equivariant cohomology ring of $\mathbf{P}^{N-1}$

$$H^*_{S^1}(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,\hbar]/(\prod_{i=0}^{N-1}(x - r_i\hbar)) \ ,$$

- quantum cohomology ring of $\mathbf{P}^{N-1}$

$$QH^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,q]/(x^N - q) \ .$$

S^1-equivariant Floer homology

- the S^1-equivariant Floer homology of the universal covering $\widetilde{L\mathbf{P}^{N-1}}$ of the loop space $L\mathbf{P}^{N-1} := \mathrm{Map}(S^1, \mathbf{P}^{N-1})$ of the projective space

$$HF^*_{S^1}(\widetilde{L\mathbf{P}^{N-1}}) = \bigoplus_{m\in\mathbb{Z}} \bigoplus_{k=0}^{N-1} \mathbb{C}[\hbar] \cdot (x - m\hbar)^k \cdot \prod_{j<m} (x - j\hbar)^N \ .$$

- the S^1-equivariant Floer homology $HF^*_{S^1}(\widetilde{L\mathbf{P}^{N-1}})$ turns out to be endowed with $\mathscr{D}$-module structure [Givental]

$$\mathscr{D}/(p^N - q) \ ,$$

where we define

$$\begin{aligned} p \cdot J(x,\hbar) &= x \cdot J(x,\hbar) \ , \\ q \cdot J(x,\hbar) &= J(x-\hbar,\hbar) \ , \end{aligned}$$

for $J(x,\hbar) \in HF^*_{S^1}(\widetilde{L\mathbf{P}^{N-1}})$. From the definition, it is easy to see $[p,q] = \hbar q$ so that p can be regarded as a differential operator $\hbar q \frac{d}{dq}$ on functions of q.

- the $\mathscr{D}$-module structurecan be rephrased as

$$\left[\left(\hbar q \frac{d}{dq}\right)^N - q\right] J(q) = 0 \ .$$

Quantization

The AGT relation with a surface operator is closely related to quantization of cohomology ring

- cohomology ring of $\mathbf{P}^{N-1}$

$$H^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x]/(x^N) \ .$$

- S^1-equivariant cohomology ring of $\mathbf{P}^{N-1}$

$$H^*_{S^1}(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,\hbar]/(\prod_{i=0}^{N-1}(x - r_i\hbar)) \ ,$$

- quantum cohomology ring of $\mathbf{P}^{N-1}$

$$QH^*(\mathbf{P}^{N-1}) \cong \mathbb{C}[x,q]/(x^N - q) \ .$$

- $\mathscr{D}$-module structure of S^1-equivariant Floer homology of $\mathbf{P}^{N-1}$

$$\mathscr{D}/(p^N - q) \ ,$$

Quantization and Givental J-function

- solution of quantum (Dubrovin) connection

$$\left[\left(\hbar q \frac{d}{dq}\right)^N - q\right] J(q) = 0 \ .$$

is given by the Givental J-function

$$J[\mathbf{P}^{N-1}] = e^{\frac{tx}{\hbar}} \sum_{d=0}^{\infty} \frac{e^{td}}{\prod_{j=1}^{d}(x+j\hbar)^N} \ .$$

- The J-function of X is defined by using the *psi class* $\psi = c_1(\mathcal{L}_1)$

$$J(X) = e^{\delta/\hbar}\left(1 + \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{a=1}^{m} q^\beta \left\langle \frac{T_a}{\hbar-\psi}, 1 \right\rangle_{0,\beta} T^a \right) \ ,$$

where $\delta = \sum_{i=1}^{r} t_i T_i$ and $q^\beta = e^{\int_\beta \delta}$. Thus, it is regarded as a generating function for once-punctured genus zero Gromov-Witten invariants with gravitational descendants.

connection to geometric representation theory

- Braverman and Etingof has constructed invariants equivalent to J-function of complete flag variety Fl_N of degree $\vec{k}$

$$J[\mathrm{Fl}_N] \text{``}=\text{''} \sum_{\vec{k}} z^k \int_{\mathcal{QM}_{\vec{k}}} 1$$

where $\mathcal{QM}_{\vec{k}}$ represents resolution of moduli space of quasi-maps $\mathbf{P}^1 \to \mathrm{Fl}_N$. This is actually the Laumon space

- Nekrasov partition function can be thought of J-function of affine flag variety

$$\mathscr{Z}_{\mathrm{inst}}[1^N] \text{``}=\text{''} \sum_{\vec{k}} z^k \int_{\mathcal{QM}^{\mathrm{aff}}_{\vec{k}}} 1$$

where $\mathcal{QM}^{\mathrm{aff}}_{\vec{k}}$ represents resolution of moduli space of quasi-maps $\mathbf{P}^1 \to \mathrm{Fl}^{\mathrm{aff}}_N$ of degree $\vec{k}$

- the AGT relation of class $\mathcal{S}$ theories with a surface operator generally provides a rich arena for a vast generalization of Givental theory, and quantum connections therefore appear as differential equations of Knizhnik-Zamolodchikov type.

Pure Yang-Mills

Instanton partition function of pure Yang-Mills theory in the presence of a full surface operator

$$\mathscr{Z}^{\rm pure}_{\rm inst}[1^N] = \sum_{\vec{k}} \Big(\prod_{l=1}^{N} z_l^{k_l} \Big) \mathscr{Z}^{\rm pure}_{[1^N],\vec{k}} ,$$

where

$$\mathscr{Z}^{\rm pure}_{[1^N],\vec{k}} = \epsilon_1^{-\sum_{l=1}^N k_l} \oint \prod_{l=1}^{N} \prod_{s=1}^{k_l} \frac{d\phi_s^{(l)}}{(\phi_s^{(l)} + a_l - \frac{(l-1)\epsilon_2}{N})(\phi_s^{(l)} + a_{l+1} + \epsilon - \frac{l\epsilon_2}{N})}$$
$$\prod_{l=1}^{N} \prod_{s=1}^{k_l} \prod_{t \neq s}^{k_l} \frac{\phi_{st}^{(l)}}{\phi_{st}^{(l)} + \epsilon_1} \prod_{l=1}^{N} \prod_{s=1}^{k_l} \prod_{t=1}^{k_{l+1}} \frac{\phi_s^{(l)} - \phi_t^{(l+1)} + \epsilon}{\phi_s^{(l)} - \phi_t^{(l+1)} + \frac{\epsilon_2}{N}} .$$

It satisfies periodic Toda equation

$$\left[\frac{\epsilon_1^2}{2} \sum_{l=1}^{N} (z_l \partial_l - z_{l+1} \partial_{l+1})^2 + \epsilon_1 \sum_{l=1}^{N} u_l z_l \partial_l - \sum_{l=1}^{N} z_l \right] \mathscr{Z}^{\rm pure}_{\rm inst}[1^N] = 0$$

Pure Yang-Mills

In fact, making the change of variables

$$\prod_{I=1}^{N} z_I = \Lambda \ , \qquad z_I = e^{t_I - t_{I+1}} \ \ (I = 1, \cdots, N-1) \ ,$$

where Λ can be interpreted as the dynamical scale of the pure Yang-Mills, one can bring the equation into the more familiar form [Braverman-Etingof]

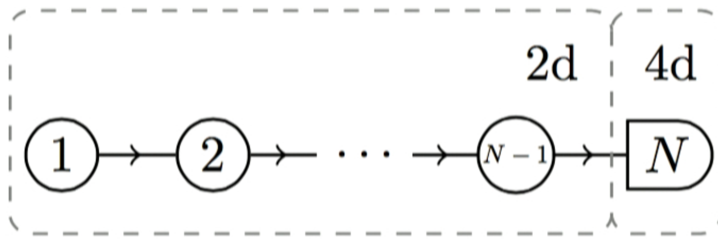
$$\left[2\epsilon_1\epsilon_2\Lambda\frac{\partial}{\partial\Lambda} + \epsilon_1^2\Delta_{\mathfrak{h}} - 2\left(\Lambda e^{t_N - t_1} + \sum_{\alpha\in\Pi} e^{\langle t,\alpha\rangle}\right)\right](e^{-\frac{\langle a,t\rangle}{\epsilon_1}}\mathscr{Z}^{\text{pure}}_{\text{inst}}[1^N]) = \langle a,a\rangle(e^{-\frac{\langle a,t\rangle}{\epsilon_1}}\mathscr{Z}^{\text{pure}}_{\text{inst}}[1^N])$$

where Π represents the set of simple roots of $\mathfrak{sl}(N)$ so that $\sum_{\alpha\in\Pi} e^{\langle t,\alpha\rangle} = \sum_{I=1}^{N} e^{t_I - t_{I+1}}$, and the rest of notations is as follows:

$$\Delta_{\mathfrak{h}} = \sum_{I=0}^{N-1}\frac{\partial^2}{\partial t_I^2} \ , \qquad \langle a,t\rangle = \sum_{I=1}^{N-1} a_I t_I \ , \qquad \langle a,a\rangle = \sum_{I=1}^{N} a_I^2 \ .$$

$\mathcal{N} = (2,2)$ susy gauge theory for full surface operator

- The other way to describe a surface operator is to couple a 4d $\mathcal{N} = 2$ gauge theory to an $\mathcal{N} = (2,2)$ supersymmetric gauge theory on the surface.
- For the surface operator in the pure Yang-Mills, the 2d theory flows at infrared to the $\mathcal{N} = (2,2)$ supersymmetric NLSM with complete flag variety $\mathrm{SL}(N,\mathbb{C})/B$ as a target.
- In this description, the combined parameters $\vec{t} = 2\pi i(\vec{\eta} + i\vec{\alpha})$ are identified with the complexified Kähler parameters of the NLSM.



- The UV description of the 2d theory is given by the quiver diagram where the D-term equation describes complete flag variety
- 2d quiver gauge theory is coupled to the 4d pure Yang-Mills by gauging the flavor symmetry $\mathrm{U}(N)$. Hence, the Coulomb branch parameters a_i in the 4d theory become the twisted masses of the fundamentals in the 2d theory.

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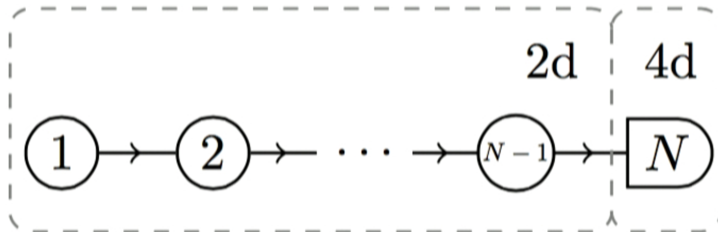

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Coulomb branch formula

Coulomb branch formula of the partition function is given by [Benini-Cremonesi, Doroud-Gomis-Le Floch-Lee]

$$Z[\mathrm{Fl}_N] = \frac{1}{1!\cdots(N-1)!}\sum_{\substack{\vec{B}^{(I)}\\ I=1\cdots N-1}}\int\prod_{I=1}^{N-1}\prod_{s=1}^{I}\frac{d\tau_s^{(I)}}{2\pi i}e^{4\pi\xi^{(I)}\tau_s^{(I)}-i\theta^{(I)}B_s^{(I)}}Z_{\text{vector}}Z_{\text{bifund}}Z_{\text{fund}}\ ,$$

$$Z_{\text{vector}} = \prod_{I=2}^{N-1}\prod_{s<t}^{I}\left(\frac{(B_{st}^{(I)})^2}{4}-(\tau_{st}^{(I)})^2\right)\ ,$$

$$Z_{\text{bifund}} = \prod_{I=1}^{N-2}\prod_{s=1}^{I}\prod_{t=1}^{I+1}\frac{\Gamma\left(\tau_s^{(I)}-\tau_t^{(I+1)}-\frac{B_s^{(I)}}{2}+\frac{B_t^{(I+1)}}{2}\right)}{\Gamma\left(1-\tau_s^{(I)}+\tau_t^{(I+1)}-\frac{B_s^{(I)}}{2}+\frac{B_t^{(I+1)}}{2}\right)}\ ,$$

$$Z_{\text{fund}} = \prod_{s=1}^{N-1}\prod_{t=1}^{N}\frac{\Gamma\left(\tau_s^{(N-1)}-\frac{B_s^{(N-1)}}{2}-\hbar^{-1}a_t\right)}{\Gamma\left(1-\tau_s^{(N-1)}-\frac{B_s^{(N-1)}}{2}+\hbar^{-1}a_t\right)}\ ,$$

J-function of complete flag variety

towers of poles can be dealt by making changes of variables

$$\tau_s^{(l)} = \frac{B_s^{(l)}}{2} - \ell_s^{(l)} - \hbar^{-1} H_s^{(l)} \ ,$$

one can manipulate the partition function into

$$Z[\mathrm{Fl}_N] \quad = \quad \frac{1}{1! \cdots (N-1)!} \oint \prod_{l=1}^{N-1} \prod_{s=1}^{l} \frac{-dH_s^{(l)}}{2\pi\hbar i} (z_l \bar{z}_l)^{\hbar^{-1}|H^{(l)}|} \widetilde{Z}_{\text{1-loop}}(H_s^{(l)}) \widetilde{Z}_{\text{v}}(H_s^{(l)}) \widetilde{Z}_{\text{av}}(H_s^{(l)}) \ ,$$

From $\widetilde{Z}_{\text{v}}(H_s^{(l)})$, one can obtain the expression of hypergeometric type for J-function of Fl_N
[Bonelli-Sciarappa-Tanzini-Vasko,]

$$\begin{aligned} J[\mathrm{Fl}_N] \quad = \quad & \sum_{\vec{k}^{(l)}} \hbar^{-\sum_{l=1}^{N-2}(|k^{(l)}|(l+1)-|k^{(l+1)}|l)-N|k^{(N-1)}|} \prod_{l=1}^{N-1} z_l^{|k^{(l)}|} \prod_{l=2}^{N-1} \prod_{s\neq t}^{l} \frac{1}{(\hbar^{-1}H_{st}^{(l)})_{k_s^{(l)}-k_t^{(l)}}} \\ & \prod_{l=1}^{N-2} \prod_{s=1}^{l} \prod_{t=1}^{l+1} \frac{1}{(1+\hbar^{-1}H_s^{(l)}-\hbar^{-1}H_t^{(l+1)})_{k_s^{(l)}-k_t^{(l+1)}}} \prod_{s=1}^{N-1} \prod_{t=1}^{N} \frac{1}{(1+\hbar^{-1}H_s^{(N-1)}-\hbar^{-1}H_t^{(N)})_{k_s^{(N-1)}}} \ . \end{aligned}$$

Here we identify $H_s^{(l)}$ $(s = 1, \ldots, l)$ with Chern roots to the duals of the universal bundles $\mathcal{S}_l$:

$$0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_{N-1} \subset \mathcal{S}_N = \mathbb{C}^N \otimes \mathcal{O}_{\mathrm{Fl}_N} \ .$$

Higgs branch formula

Higgs branch formula of the partition function is given by

$$Z[\mathrm{Fl}_N] = \frac{1}{1!\cdots(N-1)!}\sum_{\sigma\in S_N}\prod_{I=1}^{N-1}(z_I\bar{z}_I)^{-\hbar^{-1}\sum_{t=1}^{I}a_{\sigma(t)}}Z_{\text{1-loop}}(a_{\sigma(i)})Z_{\text{v}}(a_{\sigma(i)})Z_{\text{av}}(a_{\sigma(i)}) ,$$

$$Z_{\text{1-loop}} = \prod_{s<t}^{N}\gamma\left(\frac{a_s-a_t}{\hbar}\right) ,$$

$$Z_{\text{v}} = \sum_{\vec{k}^{(I)}}\hbar^{-\sum_{I=1}^{N-2}(|k^{(I)}|(I+1)-|k^{(I+1)}|I)-N|k^{(N-1)}|}\prod_{I=1}^{N-1}z_I^{|k^{(I)}|}\prod_{I=2}^{N-1}\prod_{s\neq t}^{I}\frac{1}{(-\hbar^{-1}a_{st})_{k_s^{(I)}-k_t^{(I)}}}$$
$$\prod_{I=1}^{N-2}\prod_{s=1}^{I}\prod_{t=1}^{I+1}\frac{1}{(1-\hbar^{-1}a_{st})_{k_s^{(I)}-k_t^{(I+1)}}}\prod_{s=1}^{N-1}\prod_{t=1}^{N}\frac{1}{(1-\hbar^{-1}a_{st})_{k_s^{(N-1)}}}$$

It turns out that the vortex partition function can be obtained from the instanton partition function by setting the instanton number $k = k_N = 0$

$$Z_{\text{v}}[\mathrm{Fl}_N](z_I, a, \hbar) = \mathscr{Z}_{\text{inst}}^{\text{pure}}[1^N](z_I, z_N = 0, a, \epsilon_1 = \hbar)$$

Toda Hamiltonian

We have seen that instanton partition function obeys the periodic Toda equation

$$\left[2\epsilon_1\epsilon_2\Lambda\frac{\partial}{\partial\Lambda}+\epsilon_1^2\Delta_{\mathfrak{h}}-2\left(\Lambda e^{t_N-t_1}+\sum_{\alpha\in\Pi}e^{\langle t,\alpha\rangle}\right)\right](e^{-\frac{\langle a,t\rangle}{\epsilon_1}}\mathscr{Z}_{\rm inst}^{\rm pure}[1^N])=\langle a,a\rangle(e^{-\frac{\langle a,t\rangle}{\epsilon_1}}\mathscr{Z}_{\rm inst}^{\rm pure}[1^N])$$

Therefore, the vortex partition function of complete flag variety becomes an eigenfunction of the Toda Hamiltonian [Kim]

$$\left(\hbar^2\Delta_{\mathfrak{h}}-2\sum_{\alpha\in\Pi}e^{\langle t,\alpha\rangle}\right)\left[e^{-\frac{\langle a,t\rangle}{\hbar}}Z_v[\mathrm{Fl}_N]\right]=\langle a,a\rangle\left[e^{-\frac{\langle a,t\rangle}{\hbar}}Z_v[\mathrm{Fl}_N]\right],$$

One-loop determinant

For $Q^2 = \mathcal{R}$ and transversally elliptic operator $D : \Gamma(E_0) \to \Gamma(E_1)$, we have

$$\mathscr{Z}_{\text{1-loop}} = \frac{\det_{\text{Coker}D} \mathcal{R}}{\det_{\text{Ker}D} \mathcal{R}}$$

This can be obtained from

$$\text{ind}\, D = \text{tr}_{\text{Ker}D} e^{\mathcal{R}} - \text{tr}_{\text{Coker}D} e^{\mathcal{R}} \ ,$$

where we can convert the index into the determinant via

$$\sum_j c_j e^{w_j(\epsilon_1, \epsilon_2, a, m_f)} \to \prod_j w_j(\epsilon_1, \epsilon_2, a, m_f)^{c_j} \ ,$$

The index can be computed from Atiyah-Singer index formula

$$\text{ind}\, D = \sum_{p \in F} \frac{\text{tr}_{E_0(p)} \mathcal{R} - \text{tr}_{E_1(p)} \mathcal{R}}{\det_{T_pM}(1 - \mathcal{R})}$$

One-loop determinant

One-loop determinant of vector multiplet [Pestun]

$$
\begin{aligned}
\mathscr{Z}^{\text{pure}}_{\text{1-loop}} &= \prod_{\alpha\in\Delta} \left[\Gamma_2\left(\langle a,\alpha\rangle|\epsilon_1,\epsilon_2\right)\Gamma_2\left(\langle a,\alpha\rangle+\epsilon_1+\epsilon_2|\epsilon_1,\epsilon_2\right)\right]^{-1} , \\
&= \prod_{\alpha\in\Delta} \Upsilon\left(\langle a,\alpha\rangle|\epsilon_1,\epsilon_2\right) ,
\end{aligned}
$$

To apply the index theorem on $\mathbb{C}\times(\mathbb{C}/\mathbb{Z}_N)$, we shift the parameters

$$
\epsilon_2 \to \frac{\epsilon_2}{N} , \qquad a_i \to a_i - \frac{i-1}{N}\epsilon_2 ,
$$

due to the orbifold operation. This re-parametrization alters the one-loop determinant

$$
\begin{aligned}
&\prod_{\alpha\in\Delta} \Gamma_2\left(\langle a,\alpha\rangle|\epsilon_1,\epsilon_2\right)\Gamma_2\left(\langle a,\alpha\rangle+\epsilon_1+\epsilon_2|\epsilon_1,\epsilon_2\right) \\
\to &\prod_{i,j=1,i\neq j}^{N} \Gamma_2\left(a_i-a_j+\tfrac{j-i}{N}\epsilon_2|\epsilon_1,\tfrac{\epsilon_2}{N}\right)\Gamma_2\left(a_i-a_j+\epsilon_1+\tfrac{1+j-i}{N}\epsilon_2|\epsilon_1,\tfrac{\epsilon_2}{N}\right) .
\end{aligned}
$$

One-loop determinant

Averaging over the finite group $\mathbb{Z}_N$, we have the one-loop determinant in the existence of the full surface operator

$$\mathscr{Z}^{\text{pure}}_{\text{1-loop}}[1^N] = \prod_{i,j=1,i\neq j}^{N} \left[\Gamma_2(a_i - a_j + \left\lceil \tfrac{j-i}{N} \right\rceil \epsilon_2|\epsilon_1,\epsilon_2)\Gamma_2(a_i - a_j + \epsilon_1 + \left\lceil \tfrac{1+j-i}{N} \right\rceil \epsilon_2|\epsilon_1,\epsilon_2)\right]^{-1}$$
$$= \prod_{i,j=1,i\neq j}^{N} \Upsilon\left(a_i - a_j + \left\lceil \tfrac{j-i}{N} \right\rceil \epsilon_2|\epsilon_1,\epsilon_2\right) ,$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.
if the 4d contribution $\mathscr{Z}^{\text{pure}}_{\text{1-loop}}$ is subtracted from the one-loop determinant $\mathscr{Z}^{\text{pure}}_{\text{1-loop}}[1^N]$, only the 2d effect $Z_{\text{1-loop}}[\mathrm{Fl}_N]$ should be evident

$$\frac{\mathscr{Z}^{\text{pure}}_{\text{1-loop}}[1^N]}{\mathscr{Z}^{\text{pure}}_{\text{1-loop}}}(a,\epsilon_1=\hbar) = \prod_{\alpha\in\Delta^+} \hbar^{\frac{\langle a,\alpha\rangle}{\hbar}-1}\gamma\left(\frac{\langle a,\alpha\rangle}{\hbar}\right) \text{“}=\text{''}\, Z_{\text{1-loop}}[\mathrm{Fl}_N](a,\hbar)$$

$\mathcal{N} = 2^*$ theory

Instanton partition function of $\mathcal{N} = 2^*$ theory in the presence of a full surface operator

$$\mathscr{Z}^{\mathcal{N}=2^*}_{\text{inst}}[1^N] = \sum_{\vec{k}} \Big(\prod_{I=1}^{N} z_I^{k_I} \Big) \mathscr{Z}^{\mathcal{N}=2^*}_{[1^N],\vec{k}} \ ,$$

where

$$\begin{aligned} \mathscr{Z}^{\mathcal{N}=2^*}_{[1^N],\vec{k}} \quad = \quad & \left[\frac{\epsilon_1 - \mu_{\text{adj}}}{\epsilon_1 \mu_{\text{adj}}} \right]^{\sum_{I=1}^N k_I} \\ & \oint \prod_{I=1}^{N} \prod_{s=1}^{k_I} d\phi_s^{(I)} \frac{(\phi_s^{(I)} + a_I - \frac{(I-1)\epsilon_2}{N} + \mu_{\text{adj}})(\phi_s^{(I)} + a_{I+1} + \epsilon - \frac{I\epsilon_2}{N} - \mu_{\text{adj}})}{(\phi_s^{(I)} + a_I - \frac{(I-1)\epsilon_2}{N})(\phi_s^{(I)} + a_{I+1} + \epsilon - \frac{I\epsilon_2}{N})} \\ & \prod_{I=1}^{N} \prod_{s=1}^{k_I} \prod_{t \neq s}^{k_I} \frac{\phi_{st}^{(I)}(\phi_{st}^{(I)} + \epsilon_1 - \mu_{\text{adj}})}{(\phi_{st}^{(I)} + \mu_{\text{adj}})(\phi_{st}^{(I)} + \epsilon_1)} \\ & \prod_{I=1}^{N} \prod_{s=1}^{k_I} \prod_{t=1}^{k_{I+1}} \frac{(\phi_s^{(I)} - \phi_t^{(I+1)} + \epsilon)(\phi_s^{(I)} - \phi_t^{(I+1)} + \frac{\epsilon_2}{N} - \mu_{\text{adj}})}{(\phi_s^{(I)} - \phi_t^{(I+1)} + \frac{\epsilon_2}{N})(\phi_s^{(I)} - \phi_t^{(I+1)} + \epsilon - \mu_{\text{adj}})} \ . \end{aligned}$$

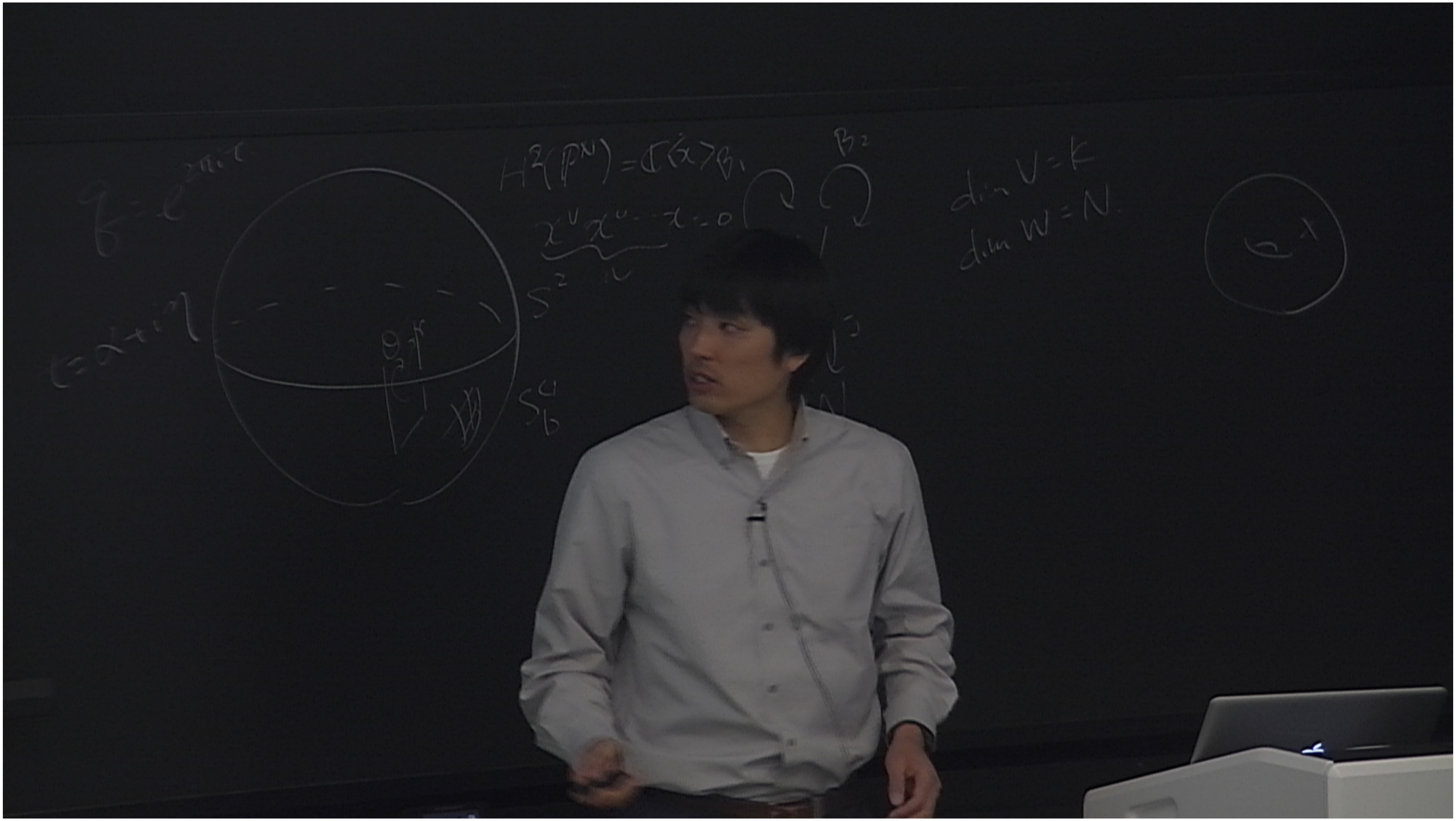
$\mathcal{N} = 2^*$ theory

- Nekrasov partition function is proportional to $SL(N, \mathbb{R})$ WZNW conformal block on the once-punctured torus.
- If we define

$$\mathscr{Y}(t, q, a, \mu_{\rm adj}, \epsilon_1, \epsilon_2) := e^{-\frac{\langle a, t\rangle}{\epsilon_1}} f(t, q)^{-\frac{\mu_{\rm adj}}{\epsilon_1}+1} \prod_{i=1}^{\infty}(1-q^i)^{-\frac{\mu_{\rm adj}(N\epsilon_1+\epsilon_2-N\mu_{\rm adj})}{\epsilon_1\epsilon_2}+1} \mathscr{Z}^{\mathcal{N}=2^*}_{\rm inst}[1^N]$$

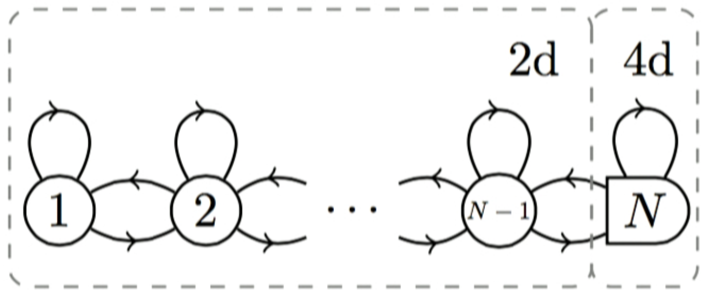
it satisfies the KZB equation

$$\left[2\epsilon_1\epsilon_2 q\frac{\partial}{\partial q} + \epsilon_1^2\Delta_{\mathfrak{h}} + 2\mu_{\rm adj}(\mu_{\rm adj} - \epsilon_1)\sum_{\alpha\in\Delta^+}\left(\frac{1}{4\pi^2}\wp(\langle t, \alpha\rangle; \tau) + \frac{1}{12}\right)\right]\mathscr{Y} = \langle a, a\rangle\mathscr{Y} \ .$$


dim V = K
dim W = N.

$\mathcal{N} = (2,2)$ susy gauge theory for full surface operator

- Since the $\mathcal{N} = 2^*$ theory is a mass deformation of the $\mathcal{N} = 4$ SCFT, the dynamics on the support of surface operator is also described by a deformation of an $\mathcal{N} = (4,4)$ supersymmetric gauge theory
- For the surface operator in the $\mathcal{N} = 2^*$ theory, the 2d theory flows at infrared to the $\mathcal{N} = (2,2)$ supersymmetric NLSM with $T^*\mathrm{Fl}_N$



- superpotential is determined by $\mathcal{N} = (4,4)$ supersymmetry

$$W = \sum_{I=1}^{N-1} \mathrm{Tr}\, \widetilde{Q}^{(I)} \Phi^{(I)} Q^{(I)} + \sum_{I=1}^{N-2} \mathrm{Tr}\, Q^{(I)} \Phi^{(I+1)} \widetilde{Q}^{(I)} ,$$

- it is deformed by turning the twisted mass m of $\widetilde{Q}^{(I)}$ and $\Phi^{(I)}$ $(I = 1, \cdots, N-1)$

Higgs branch formula / Coulomb branch formula

Higgs branch formula of the partition function is given by

Coulomb branch formula of the partition function is given by

$$Z[T^*\mathrm{Fl}_N] \propto \frac{1}{1!\cdots(N-1)!}\sum_{\vec{B}^{(l)}} \int \prod_{l=1}^{N-1}\prod_{s=1}^{l}\frac{d\tau_s^{(l)}}{2\pi i}\sum_{\sigma\in S_N} (z_l)^{\ldots} Z_{\text{vect}} Z_{1\text{-loop}}(a_{\sigma(i)}) Z_{\text{adj}} Z_{\text{bifund}} Z_{\text{v}}(a_{\sigma(i)}) Z_{\text{fund-anti}} Z_{\text{av}}(a_{\sigma(i)}) ,$$

$$Z_{1\text{-loop}} = \prod_{s<t}^{N} \gamma\left(\frac{a_s-a_t}{\hbar}\right)\gamma\left(\frac{a_t-a_s-m}{\hbar}\right) ,$$

$$Z_{\text{vect}} = \prod_{l=1}^{N-1}\prod_{s<t}^{l}\left(\frac{(B_{st}^{(l)})^2}{4}-(\tau_{st}^{(l)})^2\right), \quad Z_{\text{adj}} = \prod_{l=1}^{N-1}\prod_{s\neq t}^{l}\frac{\Gamma\left(1+\tau_{st}^{(l)}-\frac{B_{st}^{(l)}}{2}+\hbar^{-1}m\right)}{\Gamma\left(-\tau_{st}^{(l)}-\frac{B_{st}^{(l)}}{2}-\hbar^{-1}m\right)} .$$

$$Z_{\text{v}} = \sum_{\vec{k}^{(l)}} \prod_{l=1}^{N-1} z_l^{|k^{(l)}|} \prod_{l=2}^{N-1}\prod_{s\neq t} \frac{(\hbar^{-1}a_{st}+\hbar^{-1}m)_{k_s^{(l)}-k_t^{(l)}}}{(-\hbar^{-1}a_{st})_{k_s^{(l)}-k_t^{(l)}}}$$

$$Z_{\text{bifund}} = \prod_{l=1}^{N-2}\prod_{s=1}^{l}\prod_{t=1}^{l+1} \frac{\Gamma\left(\tau_s^{(l)}-\tau_t^{(l+1)}-\frac{B_s^{(l)}}{2}+\frac{B_t^{(l+1)}}{2}\right)}{\Gamma\left(1-\tau_s^{(l)}+\tau_t^{(l+1)}-\frac{B_s^{(l)}}{2}+\frac{B_t^{(l+1)}}{2}\right)} \prod_{s=1}^{N-1}\prod_{t=1}^{N} \frac{\Gamma\left(-\tau_s^{(l)}+\tau_t^{(l+1)}+\frac{B_s^{(l)}}{2}-\frac{B_t^{(l+1)}}{2}-\hbar^{-1}m\right)}{\Gamma\left(1+\tau_s^{(l)}-\tau_t^{(l+1)}+\frac{B_s^{(l)}}{2}-\frac{B_t^{(l+1)}}{2}+\hbar^{-1}m\right)} ,$$

$$\frac{(-\hbar^{-1}a_{st}-\hbar^{-1}m)_{k_s^{(l)}-k_t^{(l+1)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(l)}-k_t^{(l+1)}}} \quad \frac{(\hbar^{-1}a_{st}-\hbar^{-1}m)_{k_s^{(N-1)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(N-1)}}}$$

It turns out that the vortex partition function can be obtained from the instanton partition function by setting the instanton number $k = (k_N = 0, \ldots)$

$$Z_{\text{fund-anti}} = \prod_{s=1}^{N-1}\left[\frac{\Gamma\left(\tau_s^{(N-1)}-\frac{B_s^{(N-1)}}{2}\right)\Gamma\left(-\tau_s^{(N-1)}+\frac{B_s^{(N-1)}}{2}-\hbar^{-1}m\right)}{\Gamma\left(1-\tau_s^{(N-1)}-\frac{B_s^{(N-1)}}{2}\right)\Gamma\left(1+\tau_s^{(N-1)}+\frac{B_s^{(N-1)}}{2}+\hbar^{-1}m\right)}\right]^N$$

$$Z_{\text{v}}[T^*\mathrm{Fl}_N](z_l, a, m, \hbar) = Z_{\text{inst}}(z_1+z_N=0, a, \mu_{\text{adj}}=m+\hbar, \epsilon_1=\hbar)$$

Coulomb branch formula

Coulomb branch formula of the partition function is given by

$$Z[T^*\mathrm{Fl}_N] \quad \propto \quad \sum_{\substack{\vec{B}^{(I)} \\ I=1\cdots N-1}} \int \prod_{I=1}^{N-1} \prod_{s=1}^{I} \frac{d\tau_s^{(I)}}{2\pi i} e^{4\pi\xi^{(I)}\tau_s^{(I)} - i\theta^{(I)} B_s^{(I)}} Z_{\text{vect}} Z_{\text{adj}} Z_{\text{bifund}} Z_{\text{fund-anti}} \ ,$$

$$Z_{\text{vect}} = \prod_{I=2}^{N-1} \prod_{s<t}^{I} \left(\frac{(B_{st}^{(I)})^2}{4} - (\tau_{st}^{(I)})^2 \right) \ , \qquad Z_{\text{adj}} = \prod_{I=2}^{N-1} \prod_{s\neq t}^{I} \frac{\Gamma\left(1+\tau_{st}^{(I)} - \frac{B_{st}^{(I)}}{2} + \hbar^{-1} m\right)}{\Gamma\left(-\tau_{st}^{(I)} - \frac{B_{st}^{(I)}}{2} - \hbar^{-1} m\right)} \ .$$

$$Z_{\text{bifund}} = \prod_{I=1}^{N-2} \prod_{s=1}^{I} \prod_{t=1}^{I+1} \frac{\Gamma\left(\tau_s^{(I)} - \tau_t^{(I+1)} - \frac{B_s^{(I)}}{2} + \frac{B_t^{(I+1)}}{2}\right) \Gamma\left(-\tau_s^{(I)} + \tau_t^{(I+1)} + \frac{B_s^{(I)}}{2} - \frac{B_t^{(I+1)}}{2} - \hbar^{-1} m\right)}{\Gamma\left(1-\tau_s^{(I)} + \tau_t^{(I+1)} - \frac{B_s^{(I)}}{2} + \frac{B_t^{(I+1)}}{2}\right) \Gamma\left(1+\tau_s^{(I)} - \tau_t^{(I+1)} + \frac{B_s^{(I)}}{2} - \frac{B_t^{(I+1)}}{2} + \hbar^{-1} m\right)} \ ,$$

$$Z_{\text{fund-anti}} = \prod_{s=1}^{N-1} \left[\frac{\Gamma\left(\tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2}\right) \Gamma\left(-\tau_s^{(N-1)} + \frac{B_s^{(N-1)}}{2} - \hbar^{-1} m\right)}{\Gamma\left(1-\tau_s^{(N-1)} - \frac{B_s^{(N-1)}}{2}\right) \Gamma\left(1+\tau_s^{(N-1)} + \frac{B_s^{(N-1)}}{2} + \hbar^{-1} m\right)} \right]^N$$

Higgs branch formula

Higgs branch formula of the partition function is given by

$$Z[T^*\mathrm{Fl}_N] = \frac{1}{1!\cdots(N-1)!}\sum_{\sigma\in S_N}\prod_{l=1}^{N-1}(z_l\bar{z}_l)^{-\hbar^{-1}\sum_{t=1}^{l}a_{\sigma(t)}} Z_{\text{1-loop}}(a_{\sigma(i)})Z_{\text{v}}(a_{\sigma(i)})Z_{\text{av}}(a_{\sigma(i)}) ,$$

$$Z_{\text{1-loop}} = \prod_{s<t}^{N}\gamma\left(\frac{a_s-a_t}{\hbar}\right)\gamma\left(\frac{a_t-a_s-m}{\hbar}\right) ,$$

$$Z_{\text{v}} = \sum_{\vec{k}^{(l)}}\prod_{l=1}^{N-1} z_l^{|k^{(l)}|}\prod_{l=2}^{N-1}\prod_{s\neq t}^{l}\frac{(1-\hbar^{-1}a_{st}+\hbar^{-1}m)_{k_s^{(l)}-k_t^{(l)}}}{(-\hbar^{-1}a_{st})_{k_s^{(l)}-k_t^{(l)}}}$$

$$\prod_{l=1}^{N-2}\prod_{s=1}^{l}\prod_{t=1}^{l+1}\frac{(-\hbar^{-1}a_{st}-\hbar^{-1}m)_{k_s^{(l)}-k_t^{(l+1)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(l)}-k_t^{(l+1)}}}\prod_{s=1}^{N-1}\prod_{t=1}^{N}\frac{(-\hbar^{-1}a_{st}-\hbar^{-1}m)_{k_s^{(N-1)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(N-1)}}}$$

It turns out that the vortex partition function can be obtained from the instanton partition function by setting the instanton number $k = k_N = 0$

$$Z_{\text{v}}[T^*\mathrm{Fl}_N](z_l, a, m, \hbar) = \mathscr{Z}_{\text{inst}}^{\mathcal{N}=2^*}[1^N](z_l, z_N = 0, a, \mu_{\text{adj}} = m+\hbar, \epsilon_1 = \hbar)$$

Calogero-Moser Hamiltonian

- Multiplying the following factor to the vortex partition function

$$Y(t,a,m,\hbar) = e^{-\frac{\langle a,t\rangle}{\hbar}} \prod_{\alpha\in\Delta^+} (1-e^{\langle t,\alpha\rangle})^{-\frac{m}{\hbar}} \, Z_v[T^*\mathrm{Fl}_N] \ ,$$

  it becomes an eigenfunction of the trigonometric Calogero-Moser Hamiltonian [Negut, Braverman-Maulik-Okounkov]

$$\left[\hbar^2\Delta_{\mathfrak{h}} - 2m(m+\hbar)\sum_{\alpha\in\Delta^+}\frac{1}{(e^{\langle t,\alpha\rangle/2}-e^{-\langle t,\alpha\rangle/2})^2}\right] Y = \langle a,a\rangle Y \ ,$$

  where Δ^+ represents the set of positive roots of $\mathfrak{sl}(N)$.
- Furthermore, the monodromy matrices of this differential equation satisfy the affine Hecke algebra. This algebra admits a natural physical interpretation as the action of the loop operators on a full surface operator [Gukov-Witten, Okuda-Honda]

Mirror Landau-Ginzburg model

Let us study the twisted chiral ring in the Landau-Ginzburg (LG) model mirror dual to the NLSM with $T^*\mathrm{Fl}_N$. To bring the partition function into the LG description, let us define

$$\Sigma_s^{(I)} = \sigma_s^{(I)} - i\frac{B_s^{(I)}}{2r} ,$$

which become the twisted chiral multiplet corresponding to the I-th vector multiplet for $\mathrm{U}(I)$. In addition, every ratio of Gamma functions can be replaced by [Gomis-Lee]

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} = \int \frac{d^2Y}{2\pi}\exp\Big\{-e^{-Y} + ir\Sigma Y + e^{-\overline{Y}} + ir\overline{\Sigma Y}\Big\} ,$$

where Y, $\overline{Y}$ represent the twisted chiral fields for the matter sector of the LG model. To study the Coulomb branch of this theory in the infrared, we integrate out the twisted chiral fields Y, $\overline{Y}$. Performing a semiclassical approximation

$$Y = -\ln(-ir\Sigma) , \qquad \overline{Y} = -\ln(ir\overline{\Sigma}) ,$$

we are left with

$$\frac{\Gamma(-ir\Sigma)}{\Gamma(1+ir\overline{\Sigma})} \sim \exp\Big\{\varpi(-ir\Sigma) - \frac{1}{2}\ln(-ir\Sigma) - \varpi(ir\overline{\Sigma}) - \frac{1}{2}\ln(ir\overline{\Sigma})\Big\} ,$$

Twisted chiral ring

Using this prescription, the partition function can be written

$$Z[T^*\mathrm{Fl}_N] \sim \frac{1}{1!\cdots(N-1)!}\int\prod_{l=1}^{N-1}\prod_{s=1}^{l}\frac{d^2(r\Sigma_s^{(l)})}{2\pi}\left|Q(\Sigma)^{\frac{1}{2}}e^{-\widetilde{\mathcal{W}}(\Sigma)}\right|^2 ,$$

where $\widetilde{\mathcal{W}}(\Sigma)$ is the effective twisted superpotential of the mirror LG model in the Coulomb branch

$$\begin{aligned}\widetilde{\mathcal{W}}(\Sigma) \;=\; & \sum_{l=1}^{N-1}\sum_{s=1}^{l}(-2\pi\xi^{(l)}+i\theta^{(l)})(ir\Sigma_s^{(l)}) + \sum_{l=2}^{N-1}\sum_{s\neq t}^{l}\varpi(-ir\Sigma_s^{(l)}+ir\Sigma_t^{(l)}+ir\hat{m}) \\ & + \sum_{l=1}^{N-2}\sum_{s=1}^{l}\sum_{u=1}^{l+1}\left[\varpi(-ir\Sigma_s^{(l)}+ir\Sigma_u^{(l+1)})+\varpi(ir\Sigma_s^{(l)}-ir\Sigma_u^{(l+1)}-ir\hat{m})\right] \\ & + \sum_{s=1}^{N-1}\left[\varpi(-ir\Sigma_s^{(N-1)})+\varpi(ir\Sigma_s^{(N-1)}-ir\hat{m})\right] .\end{aligned}$$

the twisted chiral ring is given by the equation of supersymmetric vacua [Nekrasvo-Shatashvili]

$$\exp\left(\frac{\partial\widetilde{\mathcal{W}}}{\partial(ir\Sigma_s^{(l)})}\right) = 1 .$$

Twisted chiral ring

Vacuum equation turns out to be nested Bethe ansatz equations for $\mathfrak{sl}(N)$ spin chain for $I = 1$,

$$\prod_{t=1}^{2} \frac{\Sigma_s^{(1)} - \Sigma_t^{(2)}}{\Sigma_s^{(1)} - \Sigma_t^{(2)} - \hat{m}} = e^{-2\pi\xi^{(1)} + i\theta^{(1)}} ,$$

for $1 < I < N-1$,

$$\prod_{t \neq s}^{I} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I)} - \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I)} + \hat{m}} \prod_{t=1}^{I-1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I-1)} + \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I-1)}} \prod_{t=1}^{I+1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I+1)}}{\Sigma_s^{(I)} - \Sigma_t^{(I+1)} - \hat{m}} = \pm e^{-2\pi\xi^{(I)} + i\theta^{(I)}} ,$$

for $I = N-1$,

$$\prod_{t \neq s}^{N-1} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} + \hat{m}} \prod_{t=1}^{N-2} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)} + \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)}} = \pm e^{-2\pi\xi^{(N-1)} + i\theta^{(N-1)}} \left[\frac{\Sigma_s^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)}} \right]^N .$$

Equivariant quantum cohomology ring

- For the cotangent bundle $T^*\mathrm{Gr}(r,N)$ of a Grassmannian, vacuum equation is BAE for $\mathfrak{sl}(2)$ XXX spin chain [Nekrasvo-Shatashvili]

$$\prod_{a=1}^{r} \frac{\Sigma_i + m_a^{\mathrm{f}}}{\Sigma_i - m_a^{\bar{\mathrm{f}}}} = -e^{2\pi i t} \prod_{j=1}^{N} \frac{\Sigma_i - \Sigma_j - m^{\mathrm{adj}}}{\Sigma_i - \Sigma_j + m^{\mathrm{adj}}}$$

- It is now proven that the Baxter subalgebra of $\mathfrak{sl}(2)$ Yangian is isomorphic to the equivariant quantum cohomology [Maulik-Okounkov]

$$Y(\mathfrak{sl}(2)) \supset B \cong QH_T^*(\coprod_r T^*\mathrm{Gr}(r,N))$$

- Therefore, it is natural to expect

$$Y(\mathfrak{sl}(N)) \supset B \cong QH_T^* T(\coprod_{\vec{d}} T^*\mathrm{Fl}(\vec{d}))$$

Twisted chiral ring

Vacuum equation turns out to be nested Bethe ansatz equations for $\mathfrak{sl}(N)$ spin chain for $I = 1$,

$$\prod_{t=1}^{2} \frac{\Sigma_s^{(1)} - \Sigma_t^{(2)}}{\Sigma_s^{(1)} - \Sigma_t^{(2)} - \hat{m}} = e^{-2\pi\xi^{(1)} + i\theta^{(1)}} ,$$

for $1 < I < N-1$,

$$\prod_{t \neq s}^{I} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I)} - \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I)} + \hat{m}} \prod_{t=1}^{I-1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I-1)} + \hat{m}}{\Sigma_s^{(I)} - \Sigma_t^{(I-1)}} \prod_{t=1}^{I+1} \frac{\Sigma_s^{(I)} - \Sigma_t^{(I+1)}}{\Sigma_s^{(I)} - \Sigma_t^{(I+1)} - \hat{m}} = \pm e^{-2\pi\xi^{(I)} + i\theta^{(I)}} ,$$

for $I = N-1$,

$$\prod_{t \neq s}^{N-1} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-1)} + \hat{m}} \prod_{t=1}^{N-2} \frac{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)} + \hat{m}}{\Sigma_s^{(N-1)} - \Sigma_t^{(N-2)}} = \pm e^{-2\pi\xi^{(N-1)} + i\theta^{(N-1)}} \left[\frac{\Sigma_s^{(N-1)} - \hat{m}}{\Sigma_s^{(N-1)}} \right]^N .$$

Correlation function of $\mathrm{SL}(2)$ WZNW model

- For $\mathrm{SL}(2)$ WZNW model, both two-point and three-point functions are known. [Teschner, Ooguri-Maldacena]
- the two-point function takes the form

$$\langle \mathbb{V}_j(x_1;z_1)\mathbb{V}_j(x_2;z_2)\rangle = B(j)\frac{|x_{12}|^{4j}}{|z_{12}|^{4\Delta(j)}}$$

where $\Delta(j) = \frac{j(j+1)}{2(k+2)}$ is the conformal dimension of the primary field and the reflection amplitude $B(j)$ is given by

$$B(j) = -\frac{k+2}{\pi}\frac{\nu_2^{1+2j}}{\gamma\left(\frac{2j+1}{k+2}\right)}\,, \qquad \nu_2 = \pi\frac{\Gamma\left(1+\frac{1}{k+2}\right)}{\Gamma\left(1-\frac{1}{k+2}\right)}\,.$$

- In addition, the conformal invariance and the affine symmetry determine the three-point function

$$\langle \mathbb{V}_{j_1}(x_1;z_1)\mathbb{V}_{j_2}(x_2;z_2)\mathbb{V}_{j_3}(x_3;z_3)\rangle = D(j_1,j_2,j_3)\frac{|x_{12}|^{2j_{12}^3}|x_{13}|^{2j_{13}^2}|x_{23}|^{2j_{23}^1}}{|z_{12}|^{2\Delta_{12}^3}|z_{13}|^{2\Delta_{13}^2}|z_{23}|^{2\Delta_{23}^1}}$$

where the structure coefficient $D(j_1,j_2,j_3)$ is given by

$$\begin{aligned} &D(j_3,j_2,j_1) \\ = \;& -\frac{\nu_2^{j_1+j_2+j_3+1}\widetilde{\Upsilon}_{k+2}(1)\widetilde{\Upsilon}_{k+2}(-2j_1-1)\widetilde{\Upsilon}_{k+2}(-2j_2-1)\widetilde{\Upsilon}_{k+2}(-2j_3-1)}{2\pi^2\gamma\left(\frac{k+1}{k+2}\right)\widetilde{\Upsilon}_{k+2}(-j_1-j_2-j_3-1)\widetilde{\Upsilon}_{k+2}(j_3-j_1-j_2)\widetilde{\Upsilon}_{k+2}(j_2-j_1-j_3)\widetilde{\Upsilon}_{k+2}(j_1-j_2-j_3)}\,. \end{aligned}$$

Correlation function of $\mathrm{SL}(N)$ WZNW model

- the one-loop determinant of the the $\mathrm{SU}(N)$ SCFT with $N_F = 2N$ in the presence of a full surface operator can be written as

$$\mathscr{Z}^{N_F=2N}_{\text{1-loop}}[1^N] = \frac{\prod_{\alpha\in\Delta^+}\Upsilon(\langle a,\alpha\rangle+\epsilon_2|\epsilon_1,\epsilon_2)\Upsilon(-\langle a,\alpha\rangle|\epsilon_1,\epsilon_2)}{\prod_{p,q}\Upsilon\left(\langle a,h_p\rangle+\mu_q+\left\lceil\frac{N-p-q+1}{N}\right\rceil\epsilon_2|\epsilon_1,\epsilon_2\right)\Upsilon(-\langle a,h_p\rangle-\widetilde{\mu}_q+\left\lceil\frac{p-q}{N}\right\rceil\epsilon_2|\epsilon_1,\epsilon_2)}\,.$$

- the parameters between the 4d gauge theory and $\mathrm{SL}(N,\mathbb{R})$ WZNW model are identified with

$$\frac{a}{\epsilon_1}=j+\rho\,,\qquad -\frac{\epsilon_2}{\epsilon_1}=k+N\,,\qquad \frac{\mu_i}{\epsilon_1}=-\frac{\varkappa}{N}+\langle j_1+\rho,h_i\rangle\,,\qquad \frac{\widetilde{\mu}_i}{\epsilon_1}=\frac{\widetilde{\varkappa}}{N}-\langle\widetilde{j}_1+\rho,h_i\rangle\,.$$

- Using this identification, one can easily deduce the form of three-point function

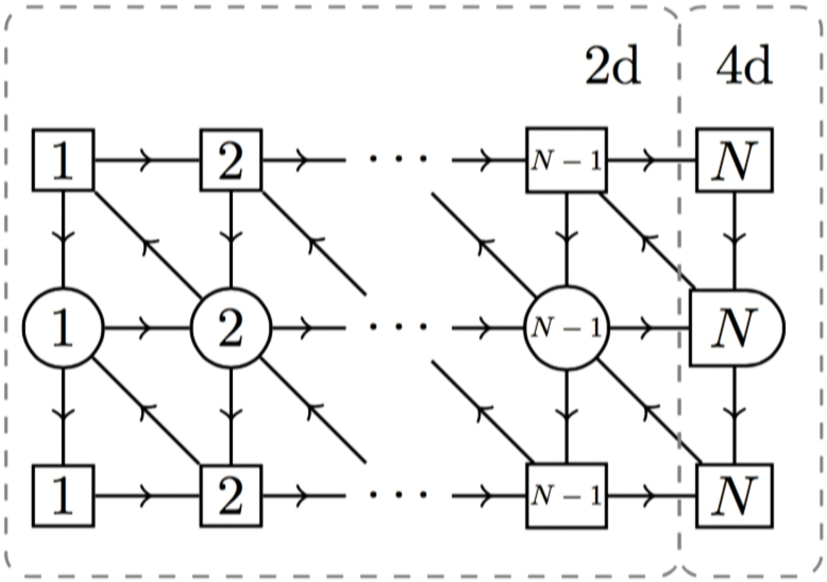
$$D(j_1,\varkappa\omega_{N-1},j_3) = A_1\frac{\left(\widetilde{\Upsilon}_{k+N}(1)\right)^{N-1}\widetilde{\Upsilon}_{k+N}(-\varkappa-1)\prod_{\alpha\in\Delta^+}\widetilde{\Upsilon}_{k+N}(-\langle j_1+\rho,\alpha\rangle)\widetilde{\Upsilon}_{k+N}(-\langle j_3+\rho,\alpha\rangle)}{\prod_{p,q=1}^{N}\widetilde{\Upsilon}_{k+N}\left(-\frac{\varkappa}{N}+\langle j_1+\rho,h_q\rangle+\langle j_3+\rho,h_p\rangle-\left\lceil\frac{N-p-q+1}{N}\right\rceil(k+N)\right)}\,.$$

- Subsequently, the form of reflection coefficient can be obtained from the three-point function

$$B(j)=D(j,0,j^*)=\frac{A_2}{\prod_{\alpha\in\Delta^+}\gamma\left(\frac{\langle 2j+\rho,\alpha\rangle}{k+N}\right)}\,.$$

$N_f = 2N$

Using the equivalence between macroscopic and microscopic description, one can read off (2,2) GLSM description of a full surface operator in $N_f = 2N$



$N_f = 2N$

- Higgs branch of the (2,2) GLSM should be a certain vector bundle of complete flag variety
- J-function of the Higgs branch is

$$Z_v = \sum_{\vec{k}^{(I)}} \hbar^{-\sum_{I=1}^{N-2}(|k^{(I)}|(I-1)-|k^{(I+1)}|I)-N|k^{(N-1)}|} \prod_{I=1}^{N-1} z_I^{|k^{(I)}|} \prod_{I=2}^{N-1}\prod_{s\neq t}^{I} \frac{1}{(-\hbar^{-1}a_{st})_{k_s^{(I)}-k_t^{(I)}}}$$
$$\prod_{I=1}^{N-2}\prod_{s=1}^{I}\prod_{t=1}^{I+1} \frac{(\hbar^{-1}m_{I+1}-\hbar^{-1}a_s)_{k_s^{(I)}}\,(1+\hbar^{-1}\tilde{m}_I-\hbar^{-1}a_s)_{k_s^{(I)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(I)}-k_t^{(I+1)}}}$$
$$\prod_{s=1}^{N-1}\prod_{t=1}^{N} \frac{(\hbar^{-1}m_N-\hbar^{-1}a_s)_{k_s^{(N-1)}}\,(1+\hbar^{-1}\tilde{m}_{N-1}-\hbar^{-1}a_s)_{k_s^{(N-1)}}}{(1-\hbar^{-1}a_{st})_{k_s^{(N-1)}}}$$

- Mutiplying an appropriate factor, this is an eigenfunction of Painlevé VI Hamiltonian

$$H_{VI}(q=0)\; Z_v = \langle a, a\rangle Z_v$$

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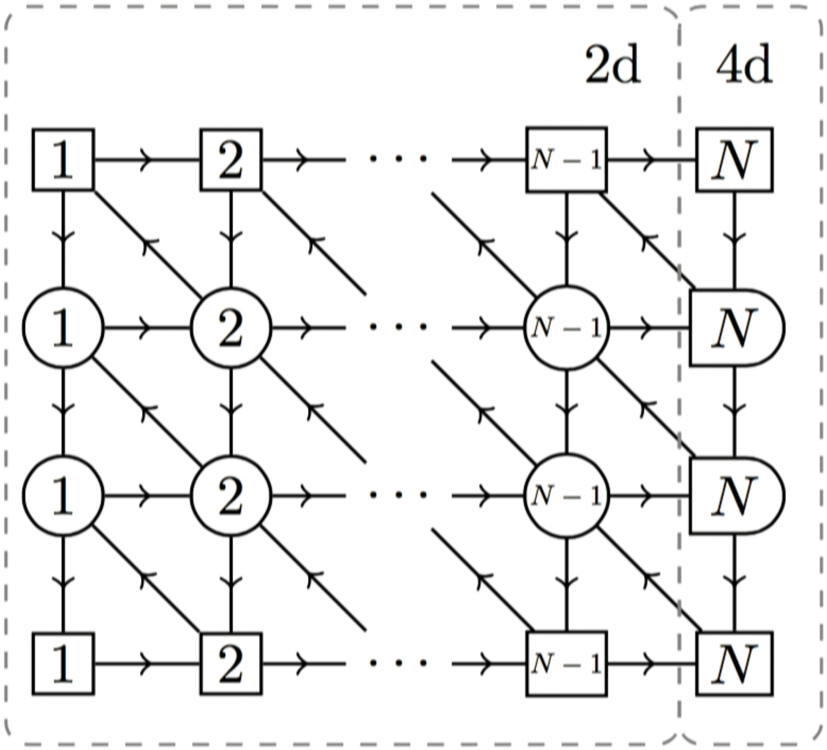
Linear quiver



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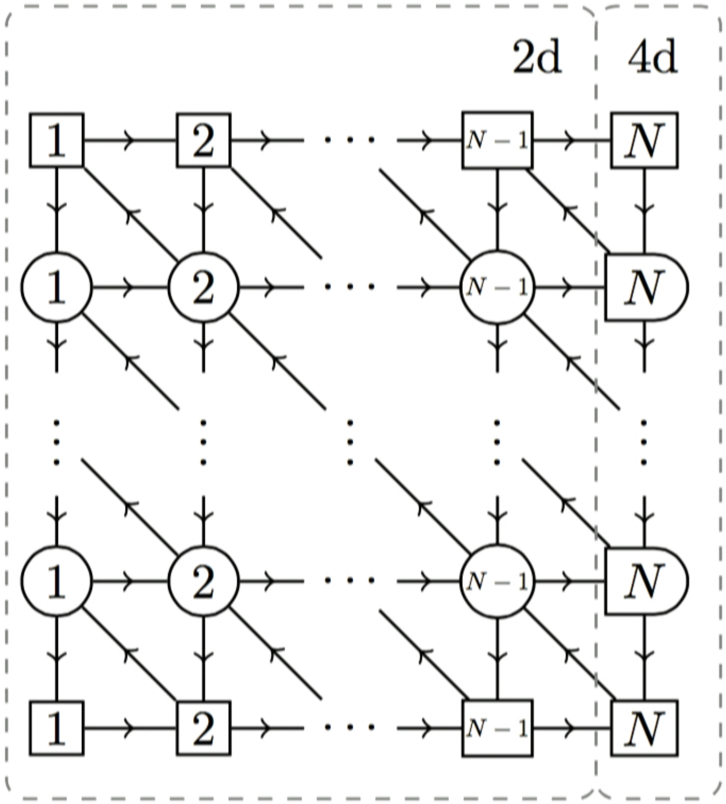
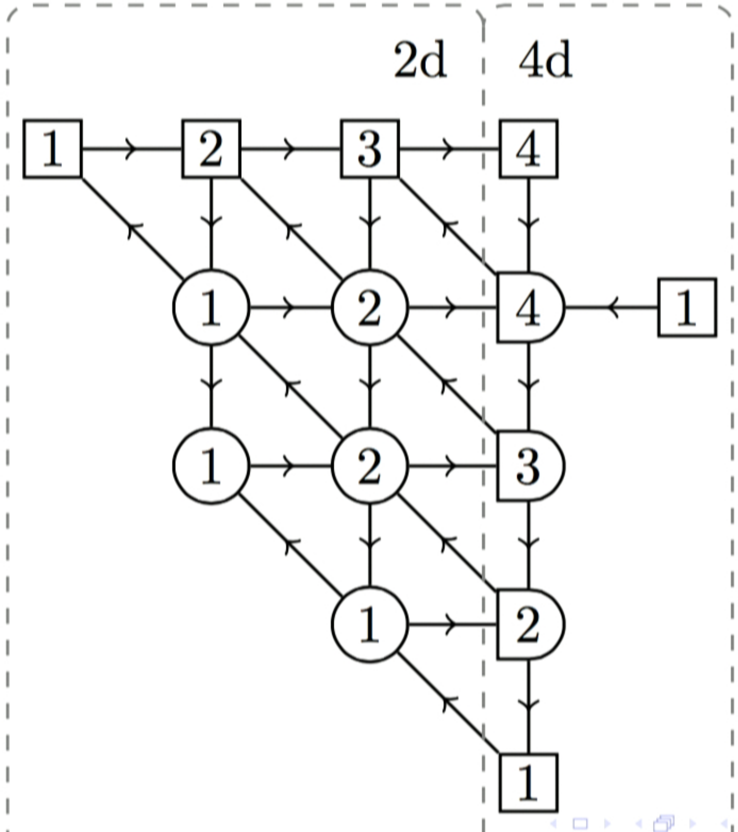


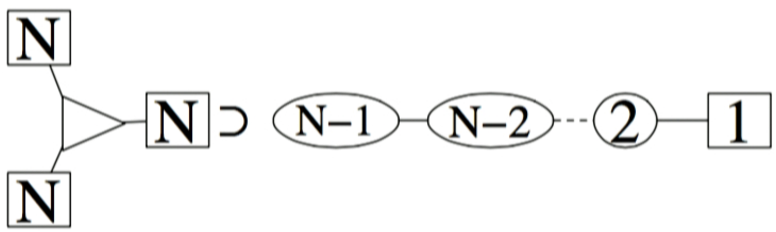
More general quiver

Can you put a surface operator of different Levi types in different gauge nodes?



More general quiver

What are the microscopic descriptions of surface operators in 4d theories without Lagrangian descriptions?



Gaiotto construction of 4d $\mathcal{N} = 2$ theories would lead to a large family of $\mathcal{N} = (2, 2)$ gauge theories without Lagrangian descriptions

Future direction

- Study $\mathrm{SL}(N)$ WZNW model directly
- Why can J-functions be extracted from S^2 partition functions? How does supersymmetric localization provides gravitational descendants?
- Microscopic description of surface operators in more general class $\mathcal{S}$ theory
- K-theoretic J-functions and q-difference equations by using 5d and 3d supersymmetric gauge theory
- gauge groups of different type (BCDEFG type)
- Relation between codim-2 and codim-4 surface operator in the higher rank guage group

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Thank you!