Title: Extending the state space of LQG

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Abstract: <p>Instead of formulating the state space of a quantum field theory over a single big Hilbert space, it has been proposed by Jerzy Kijowski to describe quantum states as projective families of density matrices over a collection of smaller, simpler Hilbert spaces. I will discuss the physical motivations for this approach and explain how it can be implemented in the context of LQG. While the resulting state space forms a natural extension of the Ashtekar-Lewandowski Hilbert space, it treats position and momentum variables on equal footing. This paves the way for the construction of semi-classical states beyond fixed graph level, and eventually for the derivation of LQC from full LQG.</p>

Why?

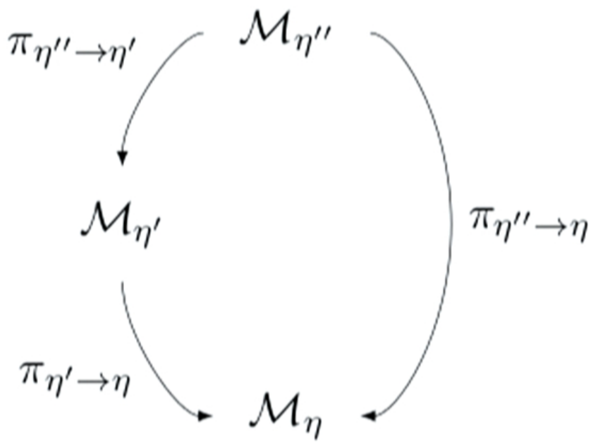
- LQG treatment of holonomies / flux is very unbalanced
  $\rightarrow$ serious issue when looking for well-behaved coherent states
  [see also: Koslowski & Sahlmann '11, Dittrich & Geiller '14]
- working with a stack of small theories is technically comfortable until we try to go beyond fixed graph
  $\rightarrow$ at the end we need to put the pieces together
  [see also: Freidel & Ziprick '11 & '13]
- the LQG way (similar to standard QFT): discrete excitations around a vacuum $\rightsquigarrow \bigoplus$
- alternative: interpret small theories as specialization into specific observables of the continuous theory $\rightsquigarrow \bigotimes$
  [see also: Thiemann & Winkler '01]

How?

- usual construction relies on writing the **configuration** space as a projective limit → let's write the **phase** space as a projective limit... [see also: Thiemann '01]
- transcription at the quantum level → projective families of density matrices, the projections are given by appropriate partial traces [Kijowski '76, Okołów '09 & '13]
- physical insight → a given experiment only measures a finite number of observables

Contents

Projective Systems of Phase Spaces



$$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$$

Collection of partial theories:

- label set $\mathcal{L}$, $\preccurlyeq$
- $\eta \in \mathcal{L}$ = a selection of d.o.f.'s
- 'small' symplectic manifolds $\mathcal{M}_\eta$

Ensuring consistency:

- projections $\pi_{\eta'\to\eta}$ for $\eta \preccurlyeq \eta'$
- compatible with symplectic structures
- 3-spaces-consistency $\to$ projective system

[Projective state spaces: Kijowski '76, Okołów '09 & '13]

Projections & Factorizations

$\pi : \mathcal{M} \to \widetilde{\mathcal{M}}$

$q_1, \ldots\ldots, q_n$
$p_1, \ldots\ldots, p_n;$

$\downarrow$

$\widetilde{q}_1, \ldots, \widetilde{q}_m$
$\widetilde{p}_1, \ldots, \widetilde{p}_m;$

$\mathcal{M} \approx \widetilde{\mathcal{M}} \times \underset{\sim}{\mathcal{M}}$

$q_1, \ldots\ldots, q_n$
$p_1, \ldots\ldots, p_n;$

$\updownarrow$

$\widetilde{q}_1, \ldots, \widetilde{q}_m, \underset{\sim}{q}_{m+1}, \ldots, \underset{\sim}{q}_n$
$\widetilde{p}_1, \ldots, \widetilde{p}_m, \underset{\sim}{p}_{m+1}, \ldots, \underset{\sim}{p}_n;$

$\mathcal{C} \approx \widetilde{\mathcal{C}} \times \underset{\sim}{\mathcal{C}}$

$q_1, \ldots\ldots, q_n$

$\updownarrow$

$\widetilde{q}_1, \ldots, \widetilde{q}_m, \underset{\sim}{q}_{m+1}, \ldots, \underset{\sim}{q}_n$

$\neq$

$\tau : \mathcal{C} \to \widetilde{\mathcal{C}}$

$q_1, \ldots\ldots, q_n$

$\downarrow$

$\widetilde{q}_1, \ldots, \widetilde{q}_m$

$$\mathcal{M}_{\eta'} \xrightarrow{\ \pi\ } \mathcal{M}_{\eta}$$

f, g

$$\{f, g\}_{\eta} \circ \pi = \{f \circ \pi, g \circ \pi\}_{\eta'}$$

Toy Model: Schrödinger Equation

As a classical field theory

$$\mathcal{M}_{\mathcal{I}'} \approx \mathcal{M}_{\mathcal{I}} \times \left(\mathcal{I}^{\perp} \cap \mathcal{I}'\right)$$

$$\psi,\ t,\ E$$

$$\updownarrow$$

$$\left(\Pi_{\mathcal{I}}\,\psi\right),\ t,\ E; \quad \left(\psi - \Pi_{\mathcal{I}}\psi\right)$$

$$\mathcal{I} \subset \mathcal{I}' \subset \mathcal{H}$$

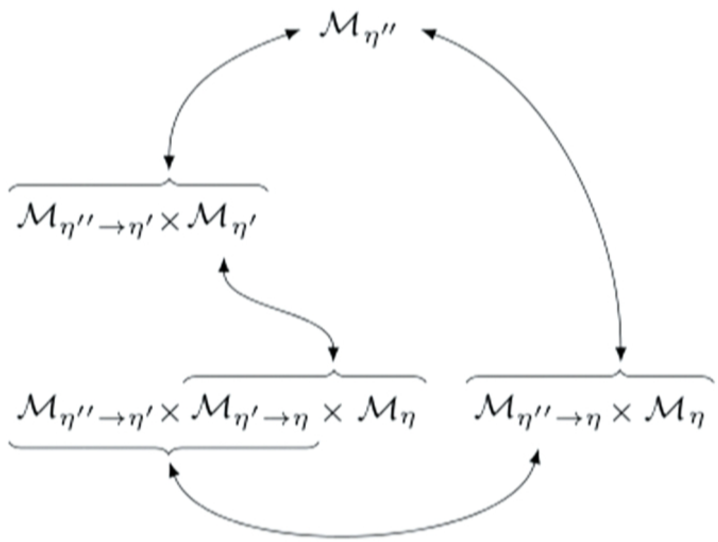
Phase space $\mathcal{H} \times \mathbb{R}^2$:

- Hilbert space $\mathcal{H}$ with $\Omega_{\mathcal{H}} = 2\,\mathrm{Im}\,\langle \cdot, \cdot \rangle$
- $\mathbb{R}^2$ = time & energy

Projective description:

- labels: finite dimensional vector subspaces $\mathcal{I} \subset \mathcal{H}$
- $\mathcal{M}_{\mathcal{I}} = \mathcal{I} \times \mathbb{R}^2$
- $\pi_{\mathcal{I}' \to \mathcal{I}} = \Pi_{\mathcal{I}}|_{\mathcal{I}'} \times \mathrm{id}_{\mathbb{R}^2}$

Projective Systems of Quantum State Spaces



$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$

Modeled on special case:

- classical factorizations
  $\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_{\eta}$
- 3-spaces consistency
  $\mathcal{M}_{\eta'' \to \eta} \approx \mathcal{M}_{\eta'' \to \eta'} \times \mathcal{M}_{\eta' \to \eta}$
- quantum equivalent
  $\to \otimes$-factorizations

Projective families $(\rho_\eta)_{\eta \in \mathcal{L}}$:

- ρ_η density matrix on $\mathcal{H}_\eta$
- $\mathrm{Tr}_{\mathcal{H}_{\eta' \to \eta}} \rho_{\eta'} = \rho_\eta$

[Projective state spaces: Kijowski '76, Okołów '09 & '13]

Toy Model: Schrödinger Equation

Second quantization

$$\widehat{\mathcal{M}}_{\mathcal{I}'} \approx \widehat{\mathcal{M}}_{\mathcal{I}} \otimes (\widehat{\mathcal{I}^{\perp} \cap \mathcal{I}'})$$

$$\left| (n_i)_{i \in I'} \right\rangle \otimes |\psi\rangle_{\mathcal{T}}$$

$$\updownarrow$$

$$\left| (n_i)_{i \in I} \right\rangle \otimes |\psi\rangle_{\mathcal{T}} \otimes \left| (n_i)_{i \in I' \setminus I} \right\rangle$$

$\mathcal{I} \subset \mathcal{I}' \subset \mathcal{H}$

$(e_i)_{i \in I}$ ONB of $\mathcal{I}$, $(e_i)_{i \in I'}$ of $\mathcal{I}'$

Usual quantization $\rightarrow \widehat{\mathcal{H}} \otimes \mathcal{T}$:

- Fock space $\widehat{\mathcal{H}}$ built from $\mathcal{H}$
- $\mathcal{T} = L_2(\mathbb{R}, d\mu_{\mathbb{R}})$

Alternative $\rightarrow$ projective setup:

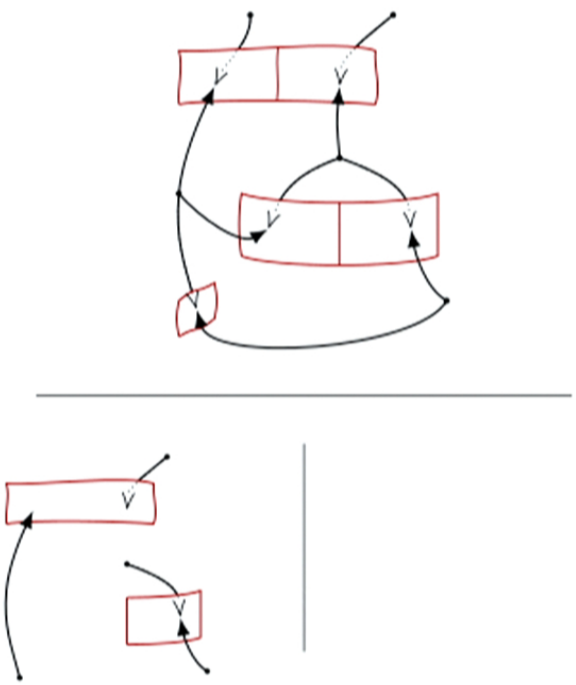
- $\widehat{\mathcal{M}}_{\mathcal{I}} = \widehat{\mathcal{I}} \otimes \mathcal{T}$
- $\widehat{\mathcal{M}}_{\mathcal{I}'} \approx \widehat{\mathcal{M}}_{\mathcal{I}} \otimes (\widehat{\mathcal{I}^{\perp} \cap \mathcal{I}'})$

  from $\widehat{\mathcal{I} \oplus \mathcal{J}} \approx \widehat{\mathcal{I}} \otimes \widehat{\mathcal{J}}$

Contents

Holonomy-Flux Algebra

The factorizations



The state spaces:

- $T^*(G^n)$
- one group variable per edge

The factorizations:

- $G^n \approx G^m \times G^{n-m}$
- selecting specific edges $\rightarrow$ prescribes the factor G^m
- selecting specific flux $\rightarrow$ prescribes the complementary factor G^{n-m}

[Holonomy-flux algebra: Ashtekar, Isham, Rovelli, Smolin, Lewandowski, Pullin, Gambini,...]

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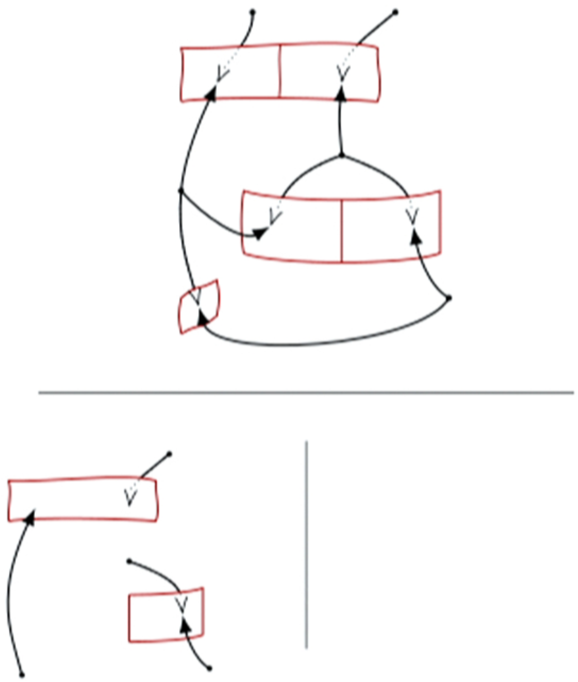

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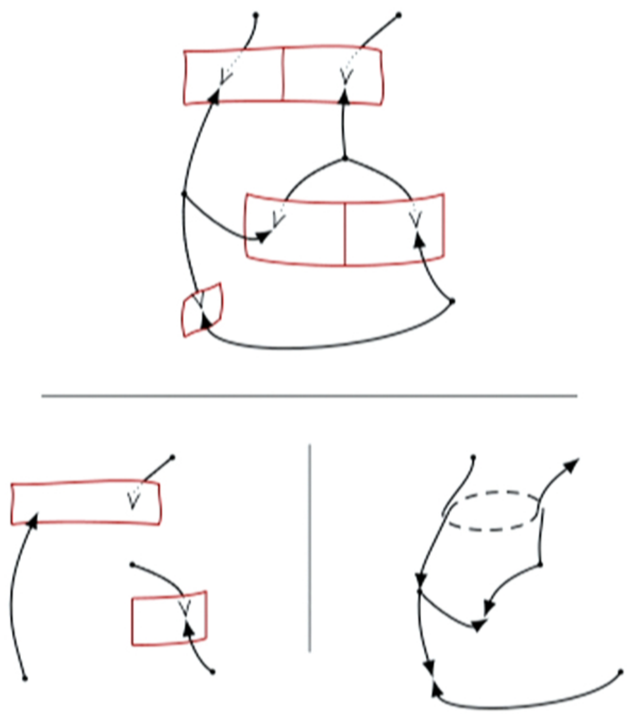

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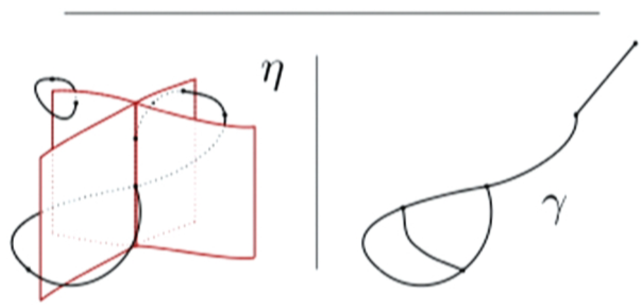
Holonomy-Flux Algebra

Relation to the usual LQG Hilbert space (1)



$\psi \in \mathcal{H}_\gamma \subset \mathcal{H}_{\text{LQG}}$ defines a projective family $(\rho_\eta)_{\eta \in \mathcal{L}}$:

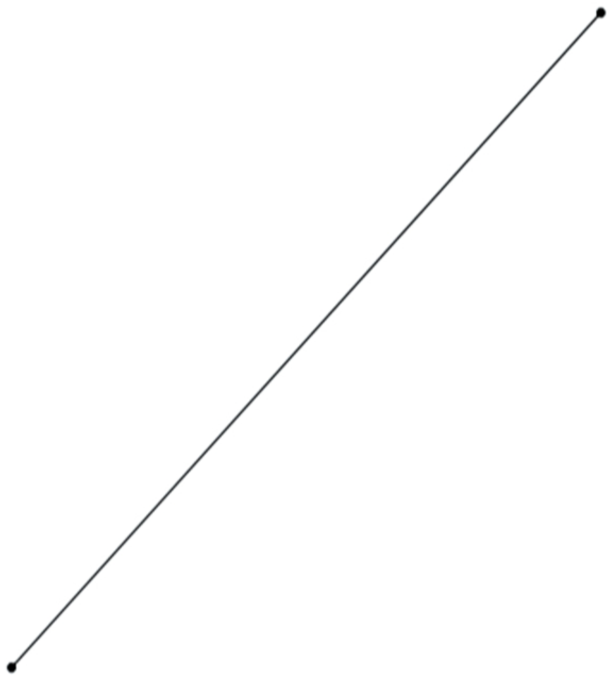
- choose η' with underlying graph γ', such that $\eta \preccurlyeq \eta'$ and $\gamma \preccurlyeq \gamma'$
- $\psi \in \mathcal{H}_\gamma \subset \mathcal{H}_{\gamma'} \approx \mathcal{H}_{\eta'}$
- $\rho_\eta := \text{Tr}_{\eta' \to \eta} |\psi\rangle\langle\psi|$

There is an **injective** map from the space of density matrices on $\mathcal{H}_{\text{LQG}}$ into the projective state space.

[LQG Hilbert space: Isham, Ashtekar, Lewandowski,...]

Holonomy-Flux Algebra

Relation to the usual LQG Hilbert space (2)



The map embedding the LQG state space in the projective one is **not surjective**.

We have states with narrow distribution for infinitely many holonomies:

- first step toward satisfactory coherent states
- but more work needed (restrict the label set...)

[LQG Hilbert space: Isham, Ashtekar, Lewandowski,...]

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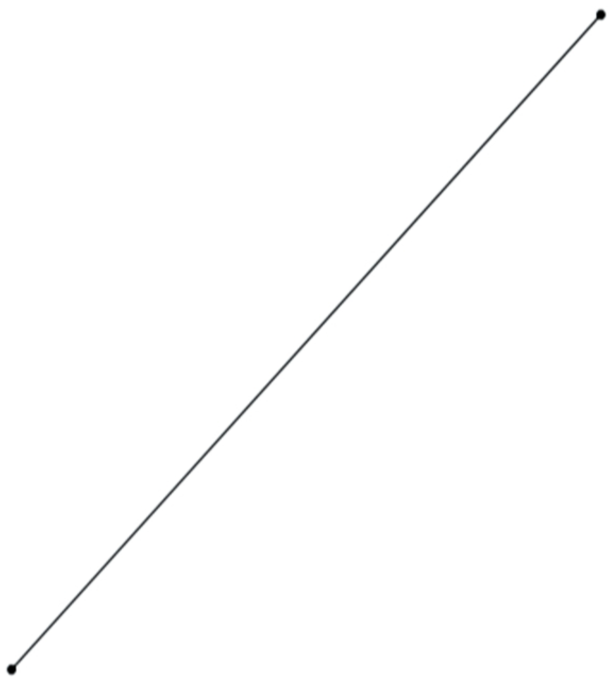

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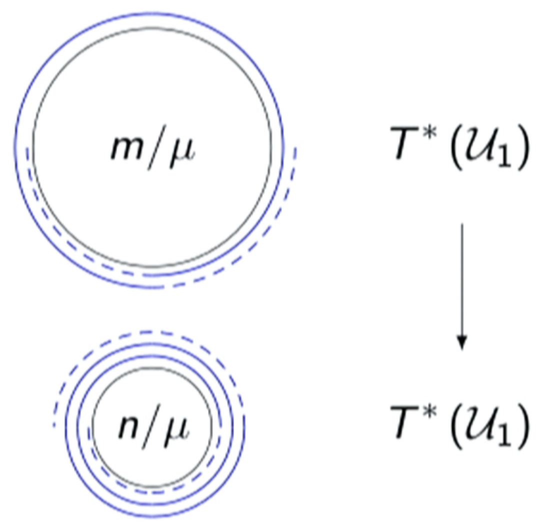
Loop Quantum Cosmology



$n = m/k$

$m,\ n,\ k \in \mathbb{N}$

Label set $\{n \in \mathbb{N}\}$:

- with order $n \mid m$
- less observables than on $\mathcal{H}_{\text{LQC}}$

The classical projections are covering maps:

- no factorization as Cartesian product of symplectic manifolds
- but a $\otimes$-projective structure still exists

[LQC: Bojowald, Ashtekar, Pawlowski, Singh, Lewandowski,...]

Loop Quantum Cosmology

$$L_2(\mathcal{U}_1) \approx L_2(\mathcal{U}_1) \otimes \mathbb{C}^k$$

$$\left| p = k\,q + r \right\rangle_m$$

$$\updownarrow$$

$$\left| q \right\rangle_n \otimes \left| r \right\rangle_{m \to n}$$

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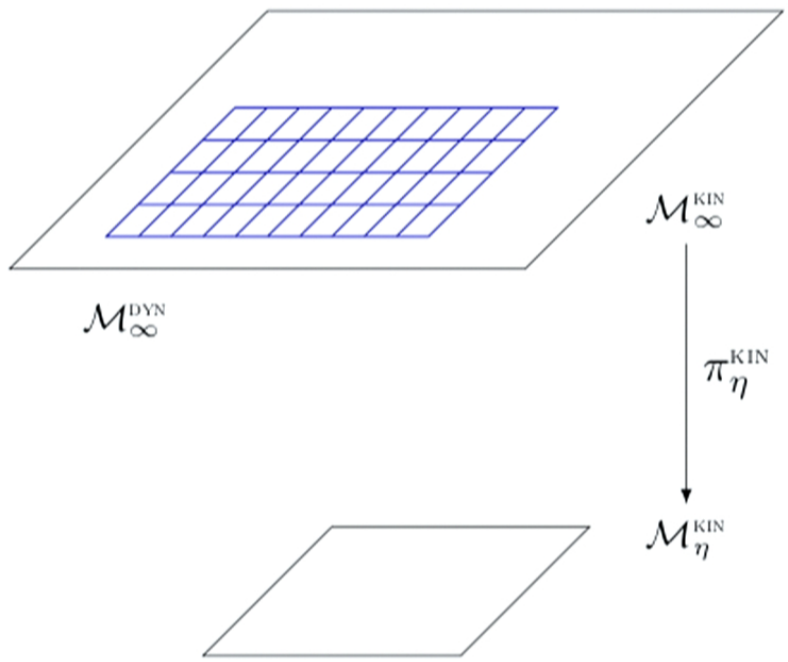
Nice Constraints



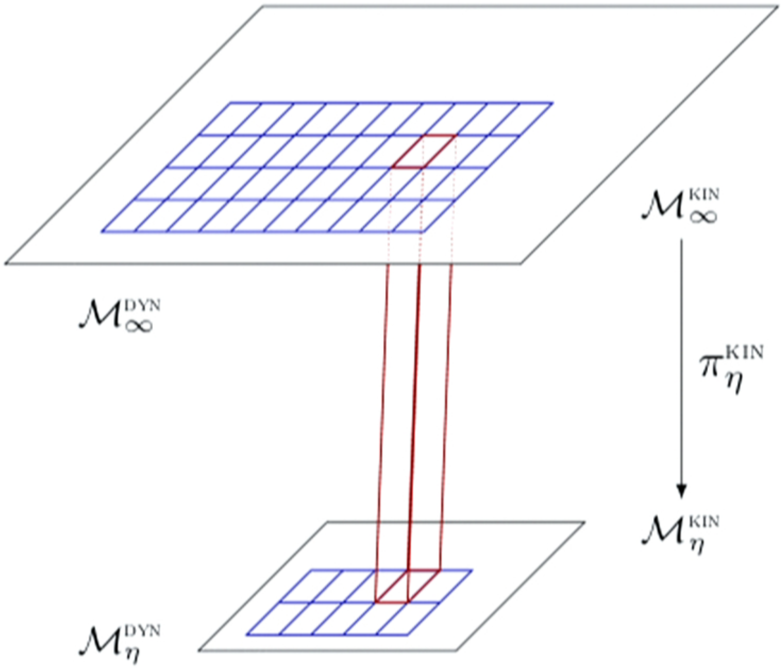
Restrictive requirements:

- orbits are projected on orbits $\rightarrow \pi^{\text{DYN}}_{\eta}$ between reduced phase spaces
- compatible with symplect. structures

Dynamical projective system & transport maps:

- states to projective families of orbits
- observables

Nice Constraints



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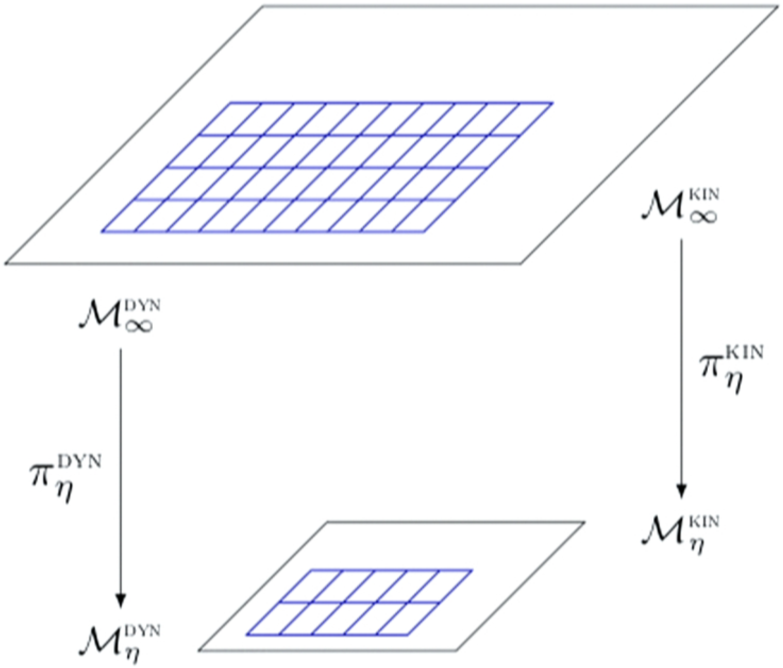

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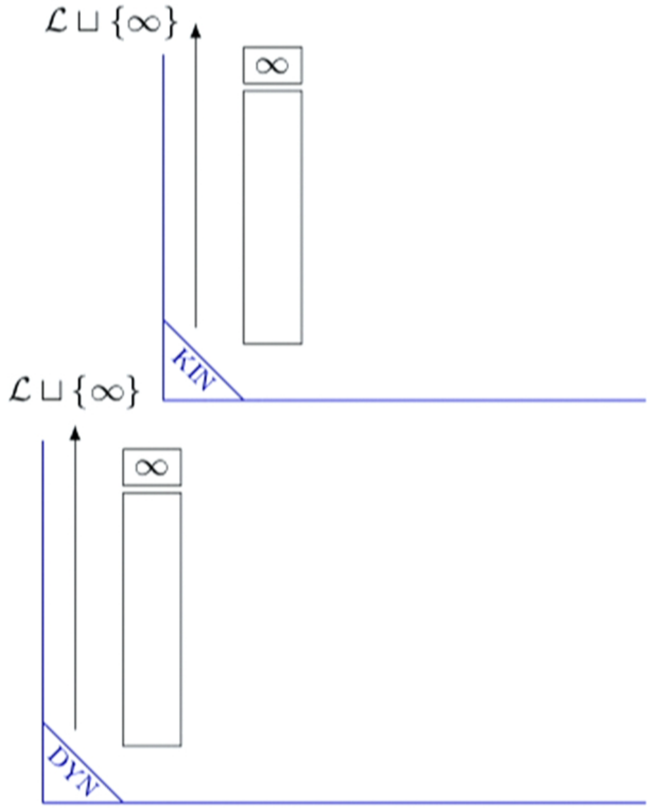
Unfitting Constraints



Successive approximations:

- labeled by $\varepsilon \in \mathcal{E}$
- nice on smaller and smaller cofinal parts of $\mathcal{L}$

Projections between approximated theories:

- dynamical projective system on a subset of $\mathcal{E} \times \mathcal{L}$
- notion of convergence

Toy Model: Schrödinger Equation

Implementation of the Hamiltonian constraint

∞

$$E - \langle \psi, H\psi \rangle = 0$$

Approximations:

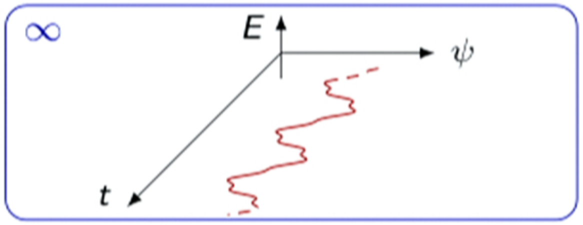
- $\epsilon > 0$ deformation $\to$ compact orbits
- truncation on finite dim. subspace $\mathcal{J}$

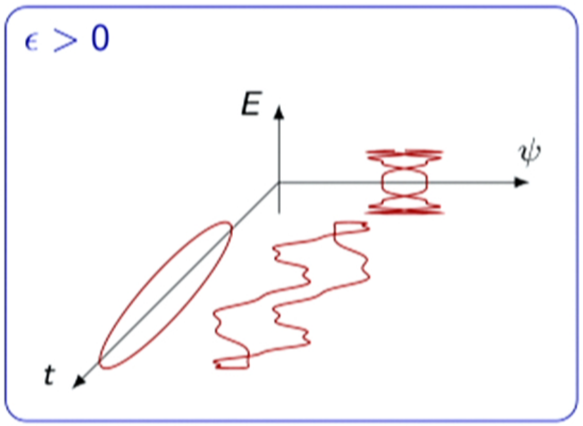
Proof of principle for previous strategy:

- classical $\to$ convergence for normed dynamical states
- quantum $\to$ convergence for Fock dynamical states

Toy Model: Schrödinger Equation

Implementation of the Hamiltonian constraint





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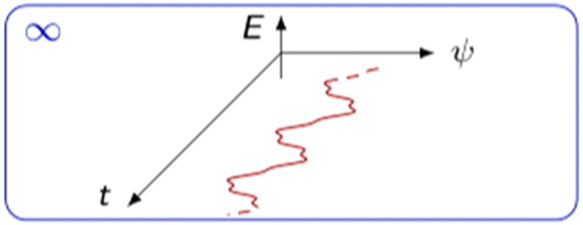

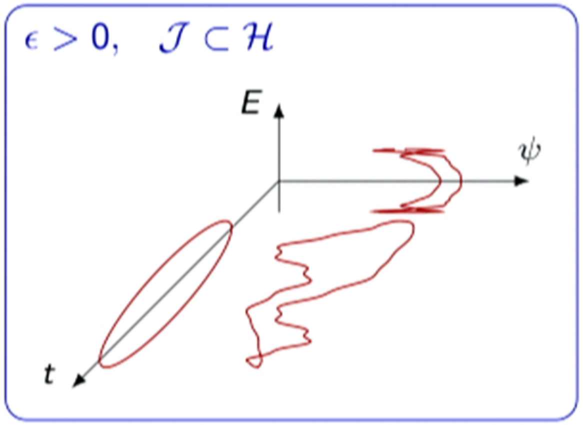

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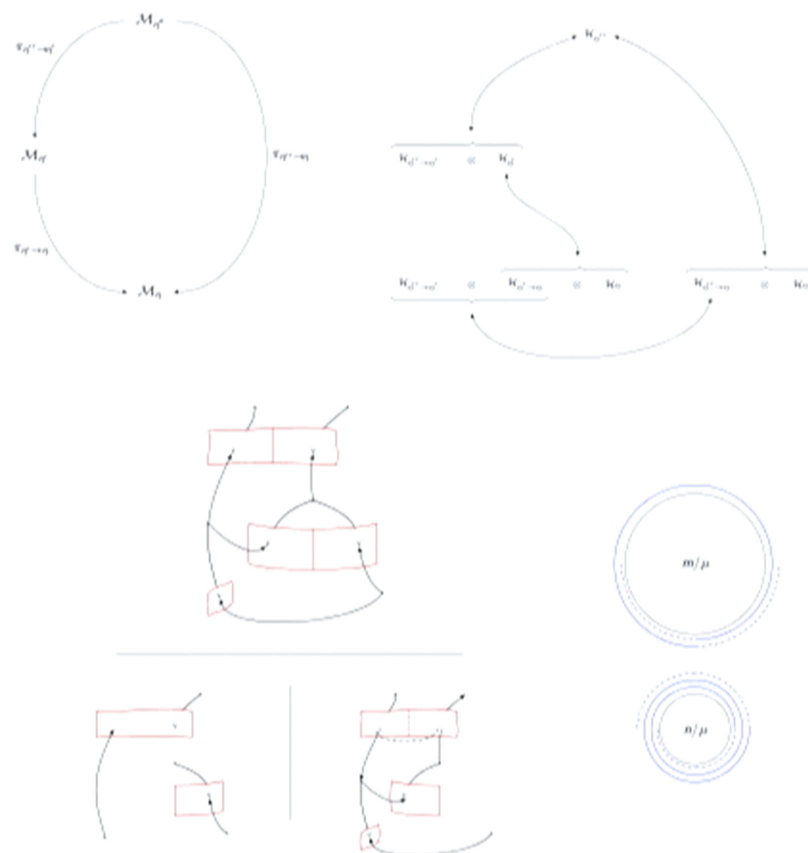
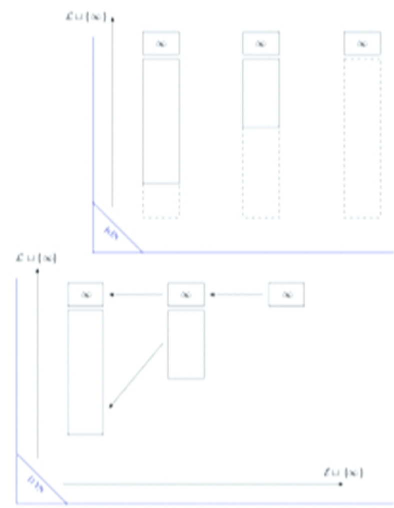
Summary

- we can construct projective state spaces for LQG and LQC
- results obtained in fixed graph can be directly imported
- assembling is done with a different interpretation $\rightarrow$ η selects **observables**, not **states**
- enlarged state space $\rightarrow$ states that were not constructible on $\mathcal{H}_{\text{LQG}}$ can be designed
- needed input for dealing with constraints $\rightarrow$ regularizing scheme $+$ projections between the approximated theories

What next?

- good coherent states: more work needed (because of obstructions in the algebra itself) → cut down the label set...
  [see also: Giesel & Thiemann '06]
- link between LQG and LQC → partly depends on progress in the previous point [see also: Engle '07]
- solving Gauss and diffeo constraints, ultimately even Hamiltonian constraint → by gluing together finer and finer discretizations?
- application to QFT → relation between regularization schemes considered here and renormalization techniques?
  [see also: Dittrich '12]





Thank you!