Title: Exact Results in Quiver Quantum Mechanics and BPS Bound State Counting

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Abstract: <p>We exactly evaluate the partition function (index) of N=4 supersymmetric quiver quantum mechanics in the Higgs phase by using the localization techniques. We show that the path integral is localized at the fixed points, which are obtained by solving the BRST equations, and D-term and F-term conditions. We turn on background gauge fields of R-symmetries for the chiral multiplets corresponding to the arrows between quiver nodes, but the partition function does not depend on these R-charges. We give explicit examples of the quiver theory including a non-coprime dimension vector. The partition functions completely agree with the mathematical formulae of the Poincare polynomials and the wall crossing for the quiver moduli spaces. We also discuss exact computation of the expectation values of supersymmetric (Q-closed) Wilson loops in the quiver theory.</p>

Exact Results in Quiver Quantum Mechanics and BPS State Counting

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based on arXiv:1408.0582 with K. Ohta

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1. Introduction

Motivation

1d N=4 quiver quantum mechanics describes the low energy dynamics of multi-centered BPS particles (black holes) in 4d N=2 SUGRA. Denef (2002)

$U(1) \times U(1)$ gauge theory with k chiral multiplets in bi-fundamental rep.

FI parameter: ζ

2 charged BPS particles with $\langle \gamma_1, \gamma_2 \rangle = k$

Phases of the central charges α_1, α_2

$$\zeta = m \sin(\alpha_1 - \alpha_2)$$

The distance of two particles is given by

$$r = \frac{\langle \gamma_1, \gamma_2 \rangle}{m \sin(\alpha_1 - \alpha_2)} = \frac{k}{\zeta}$$

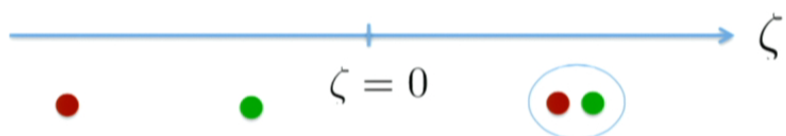
Assume $k > 0$

Two BPS particles are bounded if $\zeta > 0$

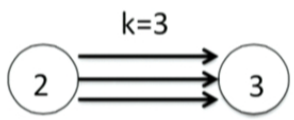
Otherwise, these particles are not bounded.

$\Rightarrow$ Wall crossing phenomenon

In this case, the marginal stability wall is located at $\zeta = 0$



For example,



Our result

$$\mathcal{Z}^3_{2,3} = \frac{1}{t^3}(1 + t + 3t^2 + 3t^3 + 3t^4 + t^5 + t^6) \qquad t = e^{i\beta\epsilon}$$

Agree with Reineke's formula

Wall crossing formula with $\Omega^+(\gamma_1) = \Omega^+(\gamma_2) = 1$, the others = 0

$$\Delta\bar{\Omega}_{\rm ref}(2\gamma_1 + 3\gamma_2, y)|_{k=3} = \frac{1}{y^6}(1 + y^2 + 3y^4 + 3y^6 + 3y^8 + y^{10} + y^{12})$$

Our result agrees with the wall crossing formula if we set $t^{1/2} = -y$

2. N=4 U(N) supersymmetric quantum mechanics (SQM)

4D N=1 U(N) SYM → 1D N=4 U(N) SYM

Dimensional reduction

R-symmetry: $SU(2)_J \times U(1)_R$

(i) Vector multiplet

	A_0	X_i	λ_α	D
$SU(2)_J$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{1}$
$U(1)_R$	0	0	1	0

$(i = 1, 2, 3)$

$(\alpha = 1, 2)$

We introduce the "BRST charge" as

$$Q = \frac{i}{\sqrt{2}}(Q^1 - \bar{Q}^1).$$

And also, after the Wick rotation, we redefine the fields by

$$Z = X_1 - iX_2, \qquad \bar{Z} = X_1 + iX_2, \quad \sigma = X_3, \quad A = A_\tau,$$
$$Y_{\mathbb{R}} = D - \tfrac{1}{2}[Z, \bar{Z}],$$

(for bosons)

$$\lambda_z = \sqrt{2}i\bar{\lambda}_2, \qquad \lambda_{\bar{z}} = -\sqrt{2}i\lambda_2, \quad \eta = -\tfrac{1}{\sqrt{2}}(\lambda_1 + \bar{\lambda}_1),$$
$$\chi_{\mathbb{R}} = \tfrac{i}{\sqrt{2}}(\lambda_1 - \bar{\lambda}_1),$$

(for fermions)

BRST transformations for vector multiplet

$$\begin{aligned}
QZ &= i\lambda_z, & Q\lambda_z &= i(\mathcal{D}_\tau Z + [\sigma, Z]),\\
Q\bar{Z} &= -i\lambda_{\bar{z}}, & Q\lambda_{\bar{z}} &= -i(\mathcal{D}_\tau \bar{Z} + [\sigma, \bar{Z}]),\\
QA &= i\eta, & &\\
Q\sigma &= \eta, & Q\eta &= -\mathcal{D}_\tau\sigma,\\
QY_{\mathbb{R}} &= i(\mathcal{D}_\tau\chi_{\mathbb{R}} + [\sigma, \chi_{\mathbb{R}}]), & Q\chi_{\mathbb{R}} &= iY_{\mathbb{R}}.
\end{aligned}$$

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$$QA = i\eta,$$
$$Q\sigma = \eta, \qquad Q\eta = -\mathcal{D}_\tau \sigma,$$
$$QY_{\mathbb{R}} = i(\mathcal{D}_\tau \chi_{\mathbb{R}} + [\sigma, \chi_{\mathbb{R}}]), \qquad Q\chi_{\mathbb{R}} = iY_{\mathbb{R}}.$$

(ii) Chiral multiplet

	q	ψ_α	F
$SU(2)_J$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$
$U(1)_R$	r	$r-1$	$r-2$

We redefine the fields by

$$Y_{\mathbb{C}} = F + Zq, \qquad \bar{Y}_{\mathbb{C}} = \bar{F} + \bar{q}\bar{Z},$$

(for bosons)

$$\psi = \psi_2, \qquad \bar{\psi} = \bar{\psi}_2,$$
$$\chi_{\mathbb{C}} = -\psi_1, \qquad \bar{\chi}_{\mathbb{C}} = -\bar{\psi}_1.$$

(for fermions)

BRST transformations for chiral multiplet

$$Qq = i\psi, \qquad Q\psi = i(\mathcal{D}_\tau q + \sigma q),$$
$$Q\bar{q} = -i\bar{\psi}, \qquad Q\bar{\psi} = -i(\mathcal{D}_\tau \bar{q} - \bar{q}\sigma),$$
$$QY_{\mathbb{C}} = i(\mathcal{D}_\tau \chi_{\mathbb{C}} + \sigma\chi_{\mathbb{C}}), \qquad Q\chi_{\mathbb{C},} = iY_{\mathbb{C}},$$
$$Q\bar{Y}_{\mathbb{C}} = i(\mathcal{D}_\tau \bar{\chi}_{\mathbb{C}} - \bar{\chi}_{\mathbb{C}}\sigma), \qquad Q\bar{\chi}_{\mathbb{C}} = i\bar{Y}_{\mathbb{C}}.$$

Including the chiral multiplet, this theory has the following twisted R-transformations $U(1)'_J \times U(1)'_R$:

$$q \to e^{ir\theta_R} q, \qquad \psi \to e^{ir\theta_R}\psi,$$
$$Y_{\mathbb{C}} \to e^{i(\theta_J + r\theta_R)} Y_{\mathbb{C}}, \qquad \chi_{\mathbb{C}} \to e^{i(\theta_J + r\theta_R)} \chi_{\mathbb{C}},$$

After gauging the R-symmetries, the BRST transformations become

$$\begin{aligned}
&Q_\epsilon q = i\psi, && Q_\epsilon \psi = i(\mathcal{D}_\tau q + \sigma q + ir\tilde{\epsilon} q),\\
&Q_\epsilon \bar{q} = -i\bar{\psi}, && Q_\epsilon \bar{\psi} = -i(\mathcal{D}_\tau \bar{q} - \bar{q}\sigma - ir\tilde{\epsilon}\bar{q}),\\
&Q_\epsilon Y_{\mathbb{C}} = i(\mathcal{D}_\tau \chi_{\mathbb{C}} + \sigma\chi_{\mathbb{C}} + i(\epsilon + r\tilde{\epsilon})\chi_{\mathbb{C}}), && Q_\epsilon \chi_{\mathbb{C},} = iY_{\mathbb{C}},\\
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\end{aligned}$$

ϵ : constant background gauge field for $U(1)'_J$

$\tilde{\epsilon}$: constant background gauge field for $U(1)'_R$

Including the chiral multiplet, this theory has the following twisted R-transformations $U(1)'_J \times U(1)'_R$:

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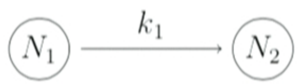
Q_\epsilon Y_{\mathbb{C}} &= i(\mathcal{D}_\tau \chi_{\mathbb{C}} + \sigma\chi_{\mathbb{C}} + i(\epsilon + r\tilde{\epsilon})\chi_{\mathbb{C}}), & Q_\epsilon \chi_{\mathbb{C},} &= iY_{\mathbb{C}}, \\
Q_\epsilon \bar{Y}_{\mathbb{C}} &= i(\mathcal{D}_\tau \bar{\chi}_{\mathbb{C}} - \bar{\chi}_{\mathbb{C}}\sigma - i(\epsilon + r\tilde{\epsilon})\bar{\chi}_{\mathbb{C}}), & Q_\epsilon \bar{\chi}_{\mathbb{C}} &= i\bar{Y}_{\mathbb{C}}.
\end{aligned}
$$

ϵ : constant background gauge field for $U(1)'_J$

$\tilde{\epsilon}$: constant background gauge field for $U(1)'_R$

3. Quiver quantum mechanics (QQM)

A quiver diagram



$U(N_1) \times U(N_2)$ gauge theory

with k_1 chiral multiplets

in the bi-fundamental representation $(\Box, \bar{\Box})$

BRST transformations for vector multiplet at node v

$$
\begin{aligned}
&Q_\epsilon Z_v = i\lambda_{z,v}, && Q_\epsilon \lambda_{z,v} = i(\partial_\tau Z_v + [\Phi_v, Z_v] + i\epsilon Z_v),\\
&Q_\epsilon \bar{Z}_v = -i\lambda_{\bar{z},v}, && Q_\epsilon \lambda_{\bar{z},v} = -i(\partial_\tau \bar{Z}_v + [\Phi_v, \bar{Z}_v] - i\epsilon \bar{Z}_v),\\
&Q_\epsilon A_v = i\eta_v, && \\
&Q_\epsilon \sigma_v = \eta_v, && Q_\epsilon \eta_v = -(\partial_\tau \sigma_v + [\Phi_v, \sigma_v]),\\
&Q_\epsilon Y_{\mathbb{R},v} = i(\partial_\tau \chi_{\mathbb{R},v} + [\Phi_v, \chi_{\mathbb{R},v}]), && Q_\epsilon \chi_{\mathbb{R},v} = iY_{\mathbb{R},v},
\end{aligned}
$$

where $\Phi_v \equiv \sigma_v + iA_v$ is Q-closed: $Q_\epsilon \Phi_v = 0$

BRST transformations for the chiral multiplet along the arrow $a : v \to w$

$$Q_\epsilon q_a = i\psi_a, \qquad Q_\epsilon \psi_a = i(\partial_\tau q_a + \Phi_v q_a - q_a \Phi_w + i\epsilon_a q_a),$$

$$Q_\epsilon \bar{q}_a = -i\bar{\psi}_a, \qquad Q_\epsilon \bar{\psi}_a = -i(\partial_\tau \bar{q}_a - \bar{q}_a \Phi_v + \Phi_w \bar{q}_a - i\epsilon_a \bar{q}_a),$$

$$Q_\epsilon Y_{\mathbb{C},a} = i(\partial_\tau \chi_{\mathbb{C},a} + \Phi_v \chi_{\mathbb{C},a} - \chi_{\mathbb{C},a}\Phi_w + i(\epsilon + \epsilon_a)\chi_{\mathbb{C},a}), \qquad Q_\epsilon \chi_{\mathbb{C},a} = iY_{\mathbb{C},a},$$

$$Q_\epsilon \bar{Y}_{\mathbb{C},a} = i(\partial_\tau \bar{\chi}_{\mathbb{C},a} - \bar{\chi}_{\mathbb{C},a}\Phi_v + \Phi_w \bar{\chi}_{\mathbb{C},a} - i(\epsilon + \epsilon_a)\bar{\chi}_{\mathbb{C},a}), \qquad Q_\epsilon \bar{\chi}_{\mathbb{C},a} = i\bar{Y}_{\mathbb{C},a},$$

where $\epsilon_a \equiv r_a \tilde{\epsilon}$.

r_a : R-charge for q_a

The action can be written by the following Q-exact form:

$$S = S_V + S_C,$$

$$S_V = \frac{1}{2g^2} Q_\epsilon \int d\tau \sum_v \mathrm{Tr}\left[\vec{\mathcal{F}}_v \cdot \overline{Q_\epsilon \vec{\mathcal{F}}_v} - 2i\chi_{\mathbb{R},v}\mu_{\mathbb{R},v}\right]$$

$$S_C = \frac{1}{2} Q_\epsilon \int d\tau \sum_a \mathrm{Tr}\left[\vec{\mathcal{F}}_a \cdot \overline{Q_\epsilon \vec{\mathcal{F}}_a} - 2i\bar{\chi}_{\mathbb{C},a}\mu_{\mathbb{C},a} - 2i\chi_{\mathbb{C},a}\bar{\mu}_{\mathbb{C},a}\right].$$

where $\vec{\mathcal{F}}_v \equiv (\lambda_{z,v}, \lambda_{\bar{z},v}, \eta, \chi_{\mathbb{R},v})$ $\quad \vec{\mathcal{F}}_a \equiv (\psi_{a,v}, \bar{\psi}_{a,v}, \chi_{\mathbb{C},a}, \bar{\chi}_{\mathbb{C},a})$

$$\mu_{\mathbb{R},v} = \frac{1}{2}[Z_v, \bar{Z}_v] + g^2\left(\sum_{a: v\to\bullet} q_a\bar{q}_a - \sum_{a:\bullet\to v} \bar{q}_a q_a - \zeta_v\right)$$: D-term constraint

$$\mu_{\mathbb{C},a} = Z_v q_a - q_a Z_w \qquad \bar{\mu}_{\mathbb{C},a} = \bar{q}_a \bar{Z}_v - \bar{Z}_w \bar{q}_a$$: F-term constraint

To decouple the center-of-mass part, we impose

$$\theta(\mathbf{N}) \equiv \sum_v N_v \zeta_v = 0$$

King (1994), Denef (2002)

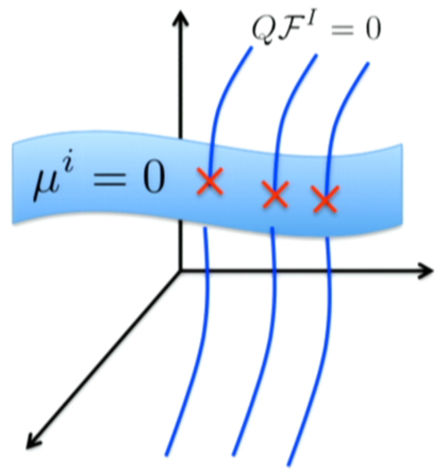
4. Localization and Exact partition function of QQM

We compactify the time direction as $\tau \sim \tau + \beta$

Let us denote

$\mathcal{F}^I$: fermionic fields

$\mathcal{B}^I$: bosonic fields except for Φ



Generically, the Q-exact action takes the following form:

$$S = tQ \int d\tau \, \mathrm{Tr}\left[g_{IJ}\mathcal{F}^I \overline{Q\mathcal{F}^J}\right] - 2it'Q \int d\tau \, \mathrm{Tr}\chi_i \mu^i,$$

t, t' : coupling constants

μ^i : D, F-term constraints

Partition function (and vev of Q-closed operator) are independent of the couplings.

Localization

<u>4. Localization and Exact partition function of QQM</u>

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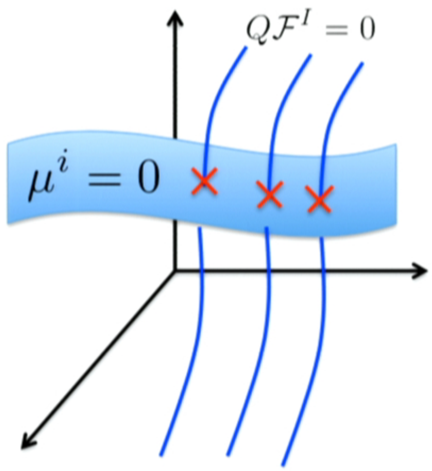

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Localization

Bosonic part of the action

$$S_B = t\int d\tau \mathrm{Tr}|Q\mathcal{F}_I|^2 - t'\int d\tau \mathrm{Tr}(Y^2 + 2iY\mu)$$
$$= t\int d\tau \mathrm{Tr}|Q\mathcal{F}_I|^2 - t'\int d\tau \mathrm{Tr}Y'^2 - t'\int d\tau \mathrm{Tr}\mu^2$$

In the partition function, we have

$$e^{t'\int d\tau \mathrm{Tr}\mu^2}$$

$$\xrightarrow{t' \to -\infty} \begin{cases} 0 & \text{if} \quad \mu \neq 0 \\ 1 & \text{if} \quad \mu = 0 \end{cases}$$

Constraint

By taking $t \to \infty$, the field configuration is localized at the BRST fixed points

$Q\mathcal{F}_I = 0$.

4. Localization and Exact partition function of QQM

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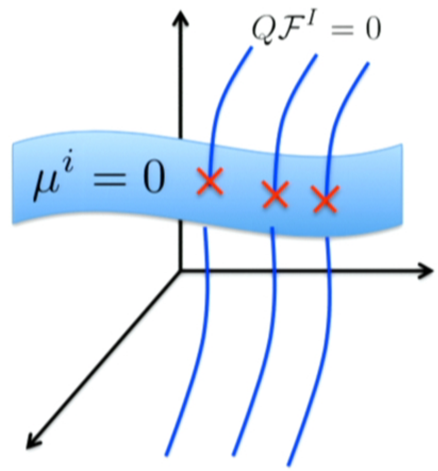

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t, t' : coupling constants

μ^i : D, F-term constraints

Partition function (and vev of Q-closed operator) are independent of the couplings.

Localization

1-loop determinant

$$\Delta(\Phi) = \sqrt{\frac{\det \frac{\delta Q\mathcal{B}^I}{\delta \mathcal{F}^J}}{\det \frac{\delta Q\mathcal{F}^I}{\delta \mathcal{B}^J}}}\Bigg|_{Q\mathcal{F}^I = Q\mathcal{B}^I = 0}$$

↑ BRST fixed points

But we have to take into account the D, F-term constraints $\mu^i = 0$

Let us denote Φ^* as a finite set of the fixed points on the D, F-term constraints.

Localization formula for partition function

$$Z = \sum_{\Phi^* \in \text{fixed points}} \text{Res}_{\Phi=\Phi^*} [\Delta_{gh}(\Phi)\Delta(\Phi)]$$

$\Delta_{gh}(\Phi^*)$: 1-loop det. for ghost

<u>Gauge fixing</u>

After the redefinitions, the theory seems to have $GL(N,\mathbb{C})$ gauge symmetry

with a complexified gauge field $\Phi = \sigma + iA$

Vacuum moduli space

$$\mathcal{M} \equiv \frac{\mu_{\mathbb{R}}^{-1}(0)\cap\mu_{\mathbb{C}}^{-1}(0)\cap\bar{\mu}_{\mathbb{C}}^{-1}(0)}{U(N)} \simeq \frac{\mu_{\mathbb{C}}^{-1}(0)\cap\bar{\mu}_{\mathbb{C}}^{-1}(0)}{GL(N,\mathbb{C})}$$

Using the GL(N,C) rotation, we take the diagonal gauge condition

$$\Phi|_{\text{off-diag}} = 0 \quad \Rightarrow \quad \Phi = \text{diag}(\phi_1,\cdots,\phi_N)$$

We still have some gauge degrees of freedom. ⇒ We impose the D-term constraints after the gauge fixing.

Action for FP ghosts for each node

$$S_{\rm gh} = i \int d\tau \, {\rm Tr}[\bar{c}_v(\partial_\tau c_v + [\Phi_v, c_v])]$$

1-loop det for ghosts

$$\Delta_{{\rm gh},v}(\phi) = \prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{2N_v} \prod_{i\neq j} \prod_{n=-\infty}^{\infty} \left(\frac{2\pi i n}{\beta} + \phi_{v,i} - \phi_{v,j}\right)$$

$$= \beta^{N_v} \prod_{i\neq j} 2\sinh\frac{\beta}{2}(\phi_{v,i} - \phi_{v,j})$$

Partition function of QQM

After the gauge fixing, BRST eq. says

$$Q_\epsilon \eta_v = -\partial_\tau \Phi_v = 0$$

This means the constant modes of Φ_v survive in the path integral.

1-loop det. for vector multiplet

$$\Delta_v^V(\phi) = \Delta_{\mathrm{gh},v}(\phi)\sqrt{\frac{\det \frac{\delta Q_\epsilon \mathcal{B}_v^I}{\delta \mathcal{F}_v^J}}{\det \frac{\delta Q_\epsilon \mathcal{F}_v^I}{\delta \mathcal{B}_v^J}}} = \left(\frac{\beta}{2i\sin\frac{\beta\epsilon}{2}}\right)^{N_v} \prod_{i\neq j} \frac{\sinh\frac{\beta}{2}(\phi_i^v - \phi_j^v)}{\sinh\frac{\beta}{2}(\phi_i^v - \phi_j^v + i\epsilon)}$$

1-loop det. for chiral multiplet

$$\Delta_a^C(\phi) = \sqrt{\frac{\det \frac{\delta Q_\epsilon \mathcal{B}_a^I}{\delta \mathcal{F}_a^J}}{\det \frac{\delta Q_\epsilon \mathcal{F}_a^I}{\delta \mathcal{B}_a^J}}} = \prod_{i=1}^{N_v}\prod_{i'=1}^{N_w} \frac{\sinh\frac{\beta}{2}(\phi_i^v - \phi_{i'}^w + i(\epsilon+\epsilon_a))}{\sinh\frac{\beta}{2}(\phi_i^v - \phi_{i'}^w + i\epsilon_a)}$$

Total partition function

$$\mathcal{Z} = \int \prod_v \frac{1}{N_v!}\prod_{i=1}^{N_v}\frac{d\phi_i^v}{2\pi i}\Delta_v^V(\phi)\prod_a \Delta_a^C(\phi)$$

Comments

- Since the integrand of partition function depends only on the relative variables $\phi_i^v - \phi_j^v$ or $\phi_i^v - \phi_{j'}^w$, one trivial integration is left, which leads to the divergence. This comes from the c.o.m motion. So, we will decouple it.

- We should not choose all poles in the denominators of partition function.

  These poles correspond to BRST fixed points, but we have to choose poles which satisfy the F-term and D-term constraints

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- We should not choose all poles in the denominators of partition function.

  These poles correspond to BRST fixed points, but we have to choose poles which satisfy the F-term and D-term constraints

- The integrand of partition function has infinitely many poles. These poles are related to the large gauge transformation

$$\phi_i^v \to \phi_i^v + \frac{2\pi i w_i^v}{\beta}, \qquad (w_i^v \in \mathbb{Z})$$

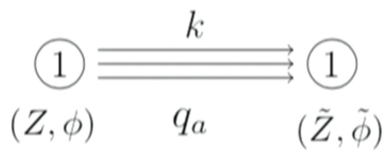
Cordova, Shao (2014)

But since the integrand of partition function is invariant under this large gauge transformation, the partition function trivially diverge if we take into account all of the poles.

To avoid this, we only pick up one of the poles in the sinh function.

5. Examples

Abelian 2 nodes with k arrows



$(a = 1, \cdots, k)$

BRST equations say

$$\begin{cases} Z = \tilde{Z} = 0 \\ (\phi - \tilde{\phi} + i\epsilon_a) q_a = 0 \end{cases}$$

No F-term constraint

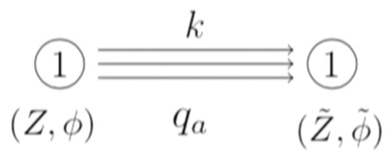
$$q_a = (0, \cdots, 0, q_l, 0, \cdots, 0)$$

$$\phi - \tilde{\phi} + i\epsilon_l = 0$$

Totally, k fixed points.

5. Examples

Abelian 2 nodes with k arrows



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⇒

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Totally, k fixed points.

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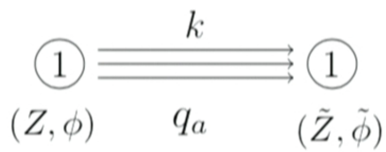

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$Z = \tilde{Z} = 0$ → No F-term constraint

→

$$q_a = (0, \cdots, 0, q_l, 0, \cdots, 0)$$

$$\phi - \tilde{\phi} + i\epsilon_l = 0$$

Totally, k fixed points.

At the fixed point, the D-term constraints become

$$|q_l|^2 = \zeta_1 \qquad -|q_l|^2 = \zeta_2$$

with $\zeta_1 + \zeta_2 = 0$

If $\zeta_1 > 0$, we choose a pole at $\phi - \tilde{\phi} + i\epsilon_l = 0$

But if $\zeta_1 < 0$, the D-term constraint cannot be satisfied!

⇒ Wall crossing phenomenon

After the residue integrations, the partition function becomes

$$\mathcal{Z} = \frac{t^{k/2} - t^{-k/2}}{t^{1/2} - t^{-1/2}} \qquad \text{for } \zeta_1 > 0$$

where $t \equiv e^{i\beta\epsilon}$ $\qquad$ ϵ_a dependence has disappeared!

Hori, Kim, Yi (2014)

This agrees with the Poincare polynomial of $\mathbb{C}P^{k-1}$ and primitive wall crossing formula by setting $t^{1/2} = -y$, $\Omega^+(\gamma_1) = \Omega^+(\gamma_2) = 1$.

$U(1) \times U(N)$ with k arrows

(Z, ϕ) $(\tilde{Z}, \tilde{\phi}_i)$ $(i = 1, \cdots, N)$

At first, we assume $\tilde{Z}$ is diagonal. ➡ No F-term constraint

BRST fixed point for q_a^i (for example)

$$q_a^i = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \quad \text{for} \quad N = 3, k = 5$$

If $a_i = a_j$, we find $\tilde{\phi}_i = \tilde{\phi}_j$. But this case does not contribute to the partition function because of the numerator of 1-loop det of vector multiplet.

Using the Weyl permutation, we have ${}_kC_N$ fixed points.

At the fixed point, the D-term constraints become

$$\sum_{i=1}^{N} |q^i_{a_i}|^2 = \zeta_1$$

$$-|q^1_{a_1}|^2 = -|q^2_{a_2}|^2 = \cdots = -|q^N_{a_N}|^2 = \zeta_2 \qquad (\zeta_1 + N\zeta_2 = 0)$$

The solution is

$$|q^1_{a_1}| = |q^2_{a_2}| = \cdots = |q^N_{a_N}| = \sqrt{\zeta_1/N} \qquad \text{if} \qquad \zeta_1 > 0$$

After the residue integral, we obtain the following partition function:

$$\mathcal{Z}^k_{1,N} = t^{-\frac{1}{2}N(k-N)} \frac{\prod_{j=1}^{k}(1-t^j)}{\prod_{j=1}^{N}(1-t^j)\prod_{j=1}^{k-N}(1-t^j)}$$

This result agrees with the Poincare polynomial of Grassmannian $Gr(N,k)$

※ In this calculation, we could choose the poles from vector multiplets. But these fixed points do not satisfy the F-term and D-term constraints.

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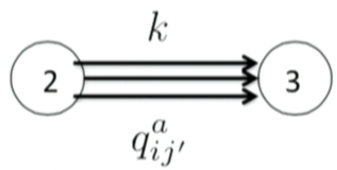
$U(2) \times U(3)$ with k arrows



$i = 1, 2$

$j' = 1, 2, 3$

$$\Phi = \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix} \qquad \tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_1 & & \\ & \tilde{\phi}_2 & \\ & & \tilde{\phi}_3 \end{pmatrix}$$

We cannot choose poles from vector multiplet because of F-term and D-term constraints. So, we only consider the fixed points for chiral multiplets.

BRST eq. for chiral multiplet is

$$(\phi_i - \tilde{\phi}_{j'} + i\epsilon_a) q^a_{ij'} = 0$$

There are 5 integration variables, but one of those is c.o.m. So, we choose 4 poles in the partition function.

A possible set

$$\begin{array}{c} \xrightarrow{\quad\quad\quad} j' \\ \downarrow i \begin{pmatrix} a & b & \\ c & & d \end{pmatrix} \end{array} \longrightarrow \quad q^a_{11}, q^b_{12}, q^c_{21}, q^d_{23} \neq 0$$

From the numerator of 1-loop det. for vector multiplet, we find

$a \neq b$, $a \neq c$ and $c \neq d$.

This type satisfies the D-term constraints if $2\zeta_1 = -3\zeta_2 > 0$.

There are other 5 similar contributions

$$\begin{pmatrix} a & b & \\ & c & d \end{pmatrix}, \begin{pmatrix} a & & b \\ c & d & \end{pmatrix}, \quad \cdots\cdots$$

Total number of fixed points of this type is

$$\frac{1}{3! \cdot 2!} 6k(k-1)^3 = \frac{1}{2} k(k-1)^3$$

Note

$$\begin{pmatrix} a & b & c \\ d & & \end{pmatrix}$$

$a \neq b \neq c, \ a \neq d$

This type does not satisfy the D-term constraints.

$$\begin{pmatrix} a & b & \\ c & d & \end{pmatrix}$$

$$\phi_1 - \tilde{\phi}_1 = -i\epsilon_a$$
$$\phi_1 - \tilde{\phi}_2 = -i\epsilon_b$$
$$\phi_2 - \tilde{\phi}_1 = -i\epsilon_c$$
$$\phi_2 - \tilde{\phi}_2 = -i\epsilon_d$$

Inconsistent

But according to the Reineke's formula, the Euler number of the Higgs branch moduli space of this theory is

$$\chi^k_{2,3} = \frac{1}{6}k(k-1)(3k^2-5k+1)$$

We need another $\frac{1}{6}k(k-1)(k-2)$ fixed points.

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→

$$\phi_1 - \tilde{\phi}_1 = -i\epsilon_a$$
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$$\chi^k_{2,3} = \frac{1}{6}k(k-1)(3k^2-5k+1)$$

We need another $\frac{1}{6}k(k-1)(k-2)$ fixed points.

To find them, we go back to the original expression of partition function.

$$\mathcal{Z}^k_{2,3} = \frac{1}{2!3!}\int \prod_{i=1}^{2}\frac{d\phi_i}{2\pi i}\prod_{i'=1}^{3}\frac{d\tilde{\phi}_{i'}}{2\pi i}\prod_{i\neq j}^{2}\frac{\sinh\frac{\beta}{2}(\phi_i-\phi_j)}{\sinh\frac{\beta}{2}(\phi_i-\phi_j+i\epsilon)}\prod_{i'\neq j'}^{3}\frac{\sinh\frac{\beta}{2}(\tilde{\phi}_{i'}-\tilde{\phi}_{j'})}{\sinh\frac{\beta}{2}(\tilde{\phi}_{i'}-\tilde{\phi}_{j'}+i\epsilon)}$$
$$\cdot\prod_{a=1}^{k}\prod_{i=1}^{2}\prod_{i'=1}^{3}\frac{\sinh\frac{\beta}{2}(\phi_i-\tilde{\phi}_{i'}+i(\epsilon+\epsilon_a))}{\sinh\frac{\beta}{2}(\phi_i-\tilde{\phi}_{i'}+i\epsilon_a)}$$

At first, we pick up the following three poles

$$\phi_1-\tilde{\phi}_1+i\epsilon_a=0 \quad \phi_1-\tilde{\phi}_2+i\epsilon_b=0 \quad \phi_1-\tilde{\phi}_3+i\epsilon_c=0 \qquad \begin{pmatrix} a & b & c \\ & & \end{pmatrix}$$

or

$$\phi_2-\tilde{\phi}_1+i\epsilon_a=0 \quad \phi_2-\tilde{\phi}_2+i\epsilon_b=0 \quad \phi_2-\tilde{\phi}_3+i\epsilon_c=0 \qquad \begin{pmatrix} & & \\ a & b & c \end{pmatrix}$$

where $a\neq b\neq c$.

Then, after integrating over $\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3$, we find the following factor in the integrand:

$$\frac{1}{\sinh\frac{\beta}{2}(\phi_1-\phi_2)}$$

This gives a new pole $\phi_1=\phi_2$

The total number of fixed points of this type is

$$\frac{1}{2!3!}2k(k-1)(k-2) = \frac{1}{6}k(k-1)(k-2)$$

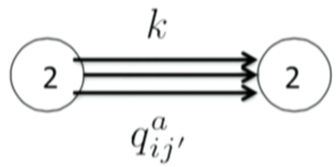
This completely agrees with the number of the missing fixed points!

Our partition function for k=1,…,5

$$\mathcal{Z}^1_{2,3} = 0,$$
$$\mathcal{Z}^2_{2,3} = 1,$$
$$\mathcal{Z}^3_{2,3} = \frac{1}{t^3}(1 + t + 3t^2 + 3t^3 + 3t^4 + t^5 + t^6),$$
$$\mathcal{Z}^4_{2,3} = \frac{1}{t^6}(1 + t + 3t^2 + 4t^3 + 7t^4 + 8t^5 + 10t^6 + 8t^7 + 7t^8 + 4t^9 + 3t^{10} + t^{11} + t^{12}),$$
$$\mathcal{Z}^5_{2,3} = \frac{1}{t^9}(1 + t + 3t^2 + 4t^3 + 7t^4 + 9t^5 + 14t^6 + 16t^7 + 20t^8 + 20t^9 + 20t^{10} + 16t^{11} + 14t^{12} + 9t^{13} + 7t^{14} + 4t^{15} + 3t^{16} + t^{17} + t^{18}).$$

These results agree with the Poincare polynomial of the Higgs branch moduli space (Reineke's formula).

$U(2) \times U(2)$ with k arrows



$i = 1, 2$

$j' = 1, 2$

$$\Phi = \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix} \qquad \tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_1 & \\ & \tilde{\phi}_2 \end{pmatrix}$$

Z $\qquad$ $\tilde{Z}$

In this case, there is a pole of vector multiplet which we can pick up because the corresponding fixed point satisfies the F-term and D-term constraints.

At first, we assume $Z = \tilde{Z} = 0$

We should choose 3 poles from chiral multiplets.

Possible fixed points

$$\begin{pmatrix} a & b \\ c & \end{pmatrix}, \quad \begin{pmatrix} b & a \\ & c \end{pmatrix}$$

where $a \neq b \text{ and } a \neq c$

Moreover, we can take the following 3 poles:

$$\phi_1 - \phi_2 + i\epsilon = 0$$ vector

$$\left.\begin{aligned} \phi_1 - \tilde{\phi}_1 + i\epsilon_a = 0 \\ \phi_2 - \tilde{\phi}_2 + i\epsilon_a = 0 \end{aligned}\right\}$$ chiral

$$\Rightarrow \quad Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & \tilde{z} \\ 0 & 0 \end{pmatrix}, \quad q^a_{11}, q^a_{22} \neq 0$$

F-term constraint $Zq_a - q_a\tilde{Z} = 0$ is satisfied if $q^a_{11}z - q^a_{22}\tilde{z} = 0$.

D-term constraints are

$$\frac{1}{2g^2}|z|^2 + |q^a_{11}|^2 = \zeta_1 + \delta \qquad -\frac{1}{2g^2}|\tilde{z}|^2 - |q^a_{22}|^2 = \zeta_2$$

$$-\frac{1}{2g^2}|z|^2 + |q^a_{22}|^2 = \zeta_1 - \delta \qquad \frac{1}{2g^2}|\tilde{z}|^2 - |q^a_{11}|^2 = \zeta_2$$

where we have introduce $\delta > 0$.

$2\zeta_1 + 2\zeta_2 = 0$ does not be modified.

Moreover, we can take the following 3 poles:

$$\phi_1 - \phi_2 + i\epsilon = 0 \qquad \text{vector}$$

$$\left.\begin{aligned} \phi_1 - \tilde{\phi}_1 + i\epsilon_a = 0 \\ \phi_2 - \tilde{\phi}_2 + i\epsilon_a = 0 \end{aligned}\right\} \quad \text{chiral}$$

$$\Rightarrow \quad Z = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & \tilde{z} \\ 0 & 0 \end{pmatrix}, \qquad q^a_{11}, q^a_{22} \neq 0$$

F-term constraint $Zq_a - q_a\tilde{Z} = 0$ is satisfied if $q^a_{11}z - q^a_{22}\tilde{z} = 0$.

D-term constraints are

$$\frac{1}{2g^2}|z|^2 + |q^a_{11}|^2 = \zeta_1 + \delta \qquad -\frac{1}{2g^2}|\tilde{z}|^2 - |q^a_{22}|^2 = \zeta_2$$

$$-\frac{1}{2g^2}|z|^2 + |q^a_{22}|^2 = \zeta_1 - \delta \qquad \frac{1}{2g^2}|\tilde{z}|^2 - |q^a_{11}|^2 = \zeta_2$$

where we have introduce $\delta > 0$.

$2\zeta_1 + 2\zeta_2 = 0$ does not be modified.

Solution of the constraints

$$|z|^2 = 2g^2 \frac{\delta(\zeta+\delta)}{2\zeta+\delta} \qquad |\tilde{z}|^2 = 2g^2 \frac{\zeta\delta}{2\zeta+\delta}$$

$$|q_{11}^a|^2 = \frac{2\zeta(\zeta+\delta)}{2\zeta+\delta} \qquad |q_{22}^a|^2 = \frac{2\zeta^2}{2\zeta+\delta} \qquad \zeta \equiv \zeta_1 = -\zeta_2 > 0$$

After the residue integrations, we find the partition function for $k = 1, \cdots, 5$

$$\mathcal{Z}_{2,2}^1 = -\frac{t^{1/2}}{2(1+t)},$$

$$\mathcal{Z}_{2,2}^2 = \frac{t^{-1/2}}{2(1+t)}(1+t^2),$$

$$\mathcal{Z}_{2,2}^3 = \frac{t^{-5/2}}{2(1+t)}(1+t^2+t^4)(2+3t+2t^2),$$

$$\mathcal{Z}_{2,2}^4 = \frac{t^{-9/2}}{2(1+t)}(1+t^2+t^4+t^6)(2+4t+5t^2+4t^3+2t^4),$$

$$\mathcal{Z}_{2,2}^5 = \frac{t^{-13/2}}{2(1+t)}(1+t^2+t^4+t^6+t^8)(2+4t+6t^2+7t^3+6t^4+4t^5+2t^6).$$

Our results agree with the wall crossing formula.

6. Summary

- We have derived the exact partition functions of N=4 QQM by the localization.

- We have shown several examples of QQM, which include the non-Abelian quivers such as $U(2) \times U(3)$ and $U(2) \times U(2)$

- We found that our results are consistent with the Poincare polynomials of Higgs branch moduli space or wall crossing formulas.