Title: Purity Without Probability

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Abstract: <p>Pure states and pure transformations play a crucial role in most of the recent reconstructions of quantum theory. In the frameworks of general probabilistic theories, purity is defined in terms of probabilistic mixtures and bears an intuitive interpretation of ``maximal knowledge'' of the state of the system or of the evolution undergone by it. On the other hand, many quantum features do not need the probabilistic structure of the theory. For example, Schumacher and Westmoreland formulated a toy theory that only specifies which events are possible (without quantifying they probability) and still reproduces a large number of quantum features. In this talk I will provide a probability-free definition of pure states and pure transformations, which can expressed in the categorical framework of process theories developed by Abramsky and Coecke and coincides with the usual notion under standard assumptions. Building on this definition, I will present a probability-free version of the purification principle, which allows one to retrieve a large number of quantum features even in the lack of probabilistic structure. This work is part of a larger programme that aims at drawing the line between those aspects of quantum theory that can be defined solely in terms of operations in a circuit and those that rely on the subjective expectations of an agent.</p>

<p> </p>

<p>Related works:
-GC, Distinguishability and copiability of programs in general process theories, arXiv:1411.3035;
-Categorical purification, http://www.cs.ox.ac.uk/CQM2014/programme/Giulio.pdf
-GC, G. M. D'Ariano, and P. Perinotti, Probabilistic theories with purification, Phys. Rev. A 81, 062348 (2010)</p>

PURITY WITHOUT PROBABILITY

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Institute for Interdisciplinary Information Sciences
Tsinghua University, Beijing

Perimeter Institute Quantum Foundation Seminar
November 27, 2014

RECRUITMENT
PROGRAM OF GLOBAL EXPERTS

FQXi
FOUNDATIONAL QUESTIONS INSTITUTE

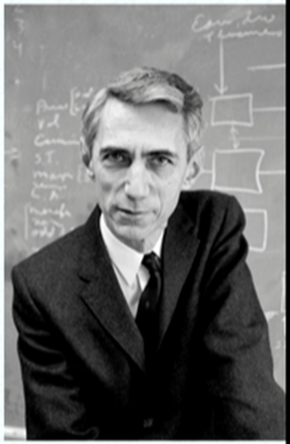
NSFC
National Natural Science Foundation of China

PLAN OF THE TALK

- A simple question
- General probabilistic theories
- A probability-free definition of pure states
- A probability-free definition of faithful states
- Making it work: purification, faithfulness, and the Purity Theorem
- Other de-convexifications

WHAT IS INFORMATION? SHANNON VS TURING

Two traditions at the foundations of information science:

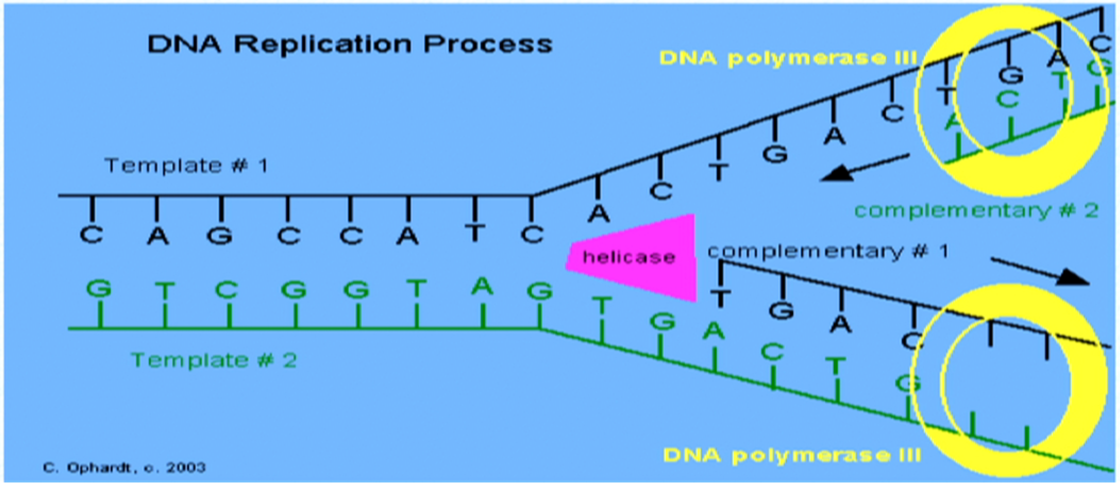


- information as an assignment of probabilities (Shannon 1948), foundation of coding theory

- information as an instruction to execute a process (Turing 1936, Shannon 1938), foundation of computability and complexity theory

TWO EXAMPLES

Example 1:
"Let's make a deal"
The agent's knowledge matters



Example 2:
DNA replication
Instructions matter too

GENERALIZING INFORMATION

Advent of quantum information

$\longrightarrow$ no obvious "natural notion of information"
(we naturally expected information to be classical, but in fact is quantum)

Can we construct an abstract theory of "information" that captures the key features of classical and quantum information?

Find the "rules of the software", independently on the physical theory governing the hardware?

GENERAL PROBABILISTIC THEORIES

namely

yet another example when a mathematical structure
is more important than the problem that motivated it

OPERATIONAL-PROBABILISTIC THEORIES (CDP 2009)

General theories of circuits that can produce random outcomes

Operational-probabilistic theory
=
operational structure
+
probabilistic structure

Clear separation between the probabilistic, Shannon-like component and the algorithmic, Turing-like component.

OPERATIONAL-PROBABILISTIC THEORIES (CDP 2009)

General theories of circuits that can produce random outcomes

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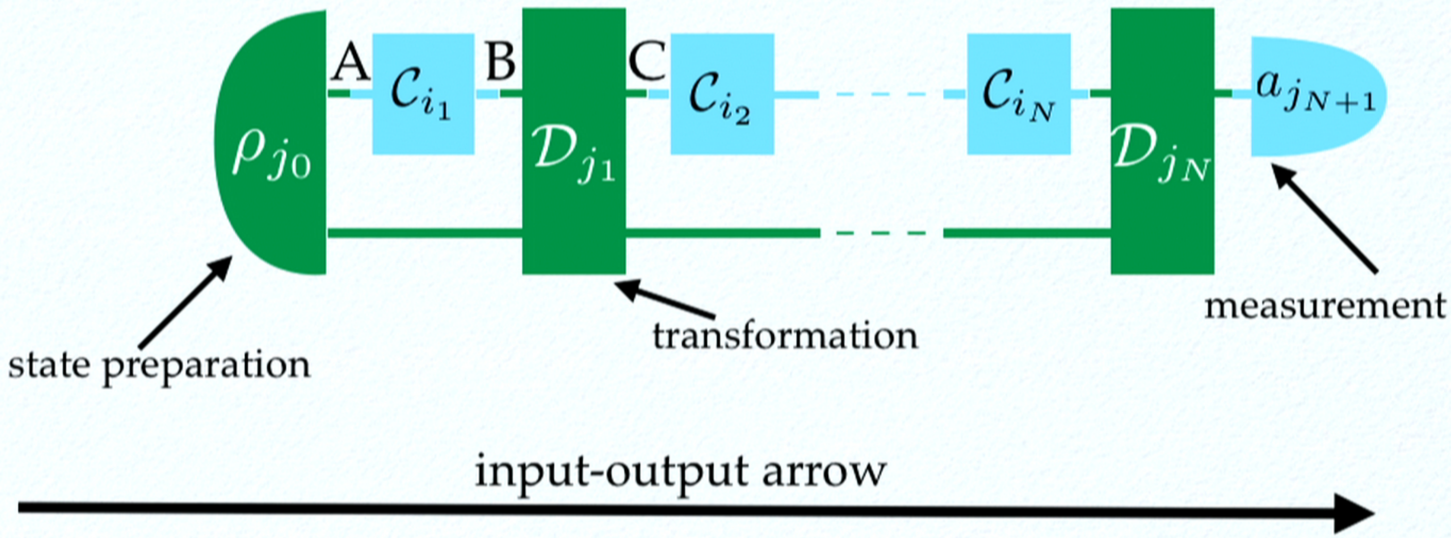
OPERATIONAL STRUCTURE



The operational structure is a **graphical language**,
described by **symmetric strict monoidal categories**

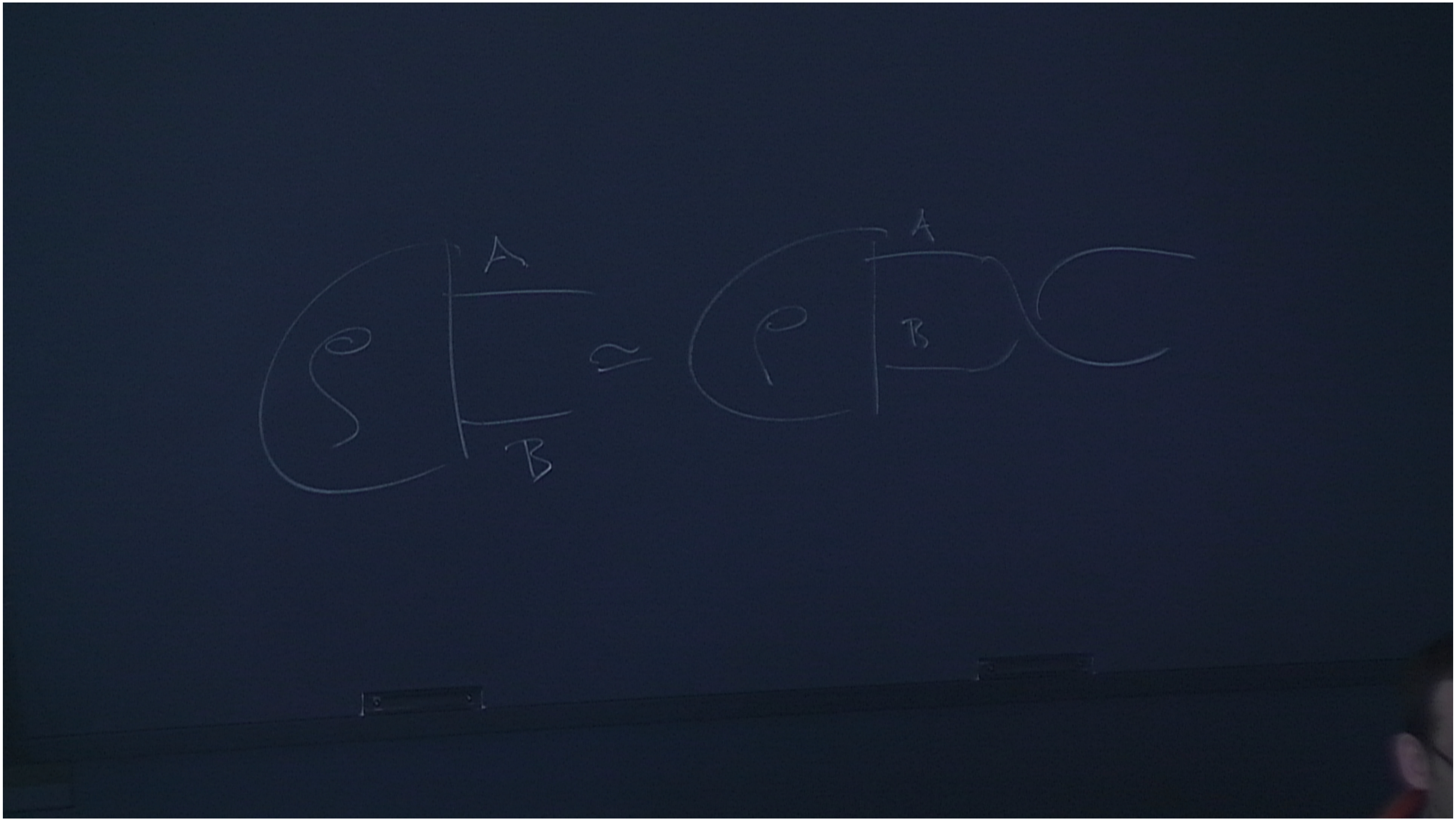
cf. -Abramsky-Coecke 2004
-Coecke's process theories
-Selinger's graphical languages

PROBABILISTIC STRUCTURE

- Preparation + measurement = probability distribution

$$= p(a_j, \rho_i)$$

$$\begin{cases} p(a_j, \rho_i) \geq 0 \\ \sum_{i \in X} \sum_{j \in Y} p(a_j, \rho_i) = 1 \end{cases}$$


A
B
ρ
≃
A
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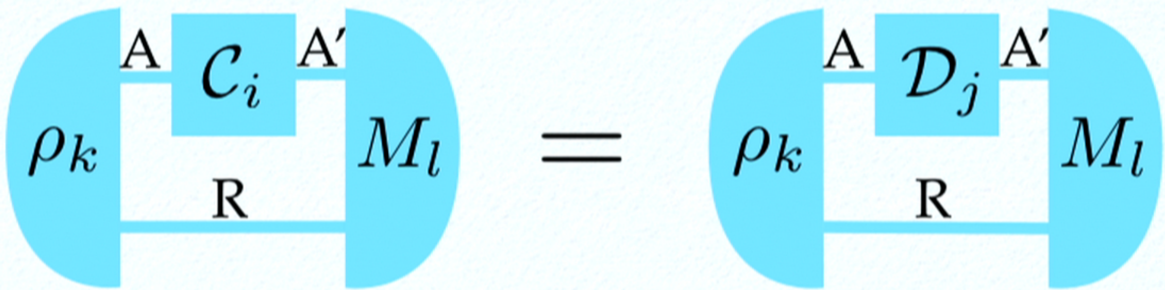
ρ

QUOTIENT THEORIES

and

are statistically equivalent iff



$$\forall R, \forall \rho_k, \forall M_l$$

Taking the quotient one gets a new OPT: the quotient theory

VECTOR SPACE STRUCTURE

Theorem (CDP09):
In the quotient theory

- the processes with input A and output A′ generate a real vector space

- sequential composition is linear in both arguments

$$\left(\sum_i x_i\,\mathcal{C}_i\right)\circ\left(\sum_j y_j\,\mathcal{D}_j\right)=\sum_{i,j} x_i y_j\;(\mathcal{C}_i\circ\mathcal{D}_j)$$

PURE STATES
IN
OPERATIONAL-PROBABILISTIC
THEORIES

PURE STATES IN QUANTUM RECONSTRUCTIONS

Pure states play center stage in most quantum reconstructions:

cf.

- Hardy 2001 "every two pure states of a system connected by a path of reversible transformations"

- Dakic-Brukner, Masanes-Mueller

- CDP "every mixed state is the marginal of pure state" (purification) "the composition of two pure processes is a pure process" (purity preservation)

- Hardy 2011 "for every pure state there exists a unique maximal effect that gives probability one"

AXIOMATIZATION OF QUANTUM PROTOCOLS FROM PURIFICATION (CDP 2009)

- Entanglement
- No Cloning
- No Information Without Disturbance
- Teleportation
- Steering
- Existence of perfectly-correlating states
- Ancilla-assisted process tomography
- Reversible simulation of irreversible processes
- No Bit Commitment
- Principle of Delayed Measurement
- No Programming Theorem
- Error correction balance
- Structure of no-signalling channels

...

COOL, BUT...

- do we really need the probabilistic structure?

cf. Schumacher's and Westmoreland's quantum theory on finite fields

NOT JUST QUANTUM FOUNDATIONS

The foundation of the notion of pure state / pure process is related to two rather fundamental questions:

- What is "maximal knowledge"?
- How can one acquire an integral piece of information?

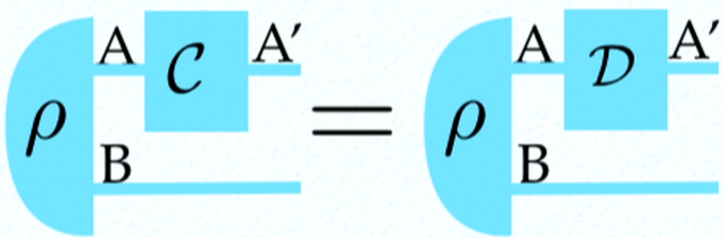
DE-CONVEXIFICATION
OF
PURE STATES

THE FRAMEWORK: CAUSAL DETERMINISTIC CATEGORIES (COECKE-LAL 2010)

Consider a process category Det with the following features:

-the monoidal unit is terminal (i.e. there exists a "partial trace")

-states separate processes

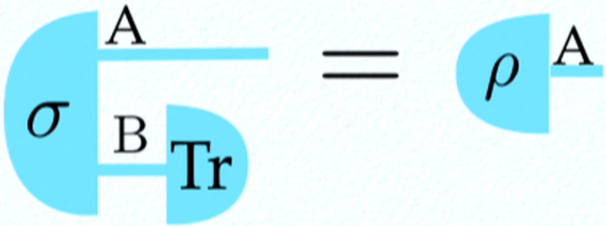


$\forall B, \forall \rho : I \to A \otimes B$

CONTEXTS

Think of a state as a piece of information.
What are the contexts that are compatible with that piece of information?

Definition: σ is an extension of ρ iff



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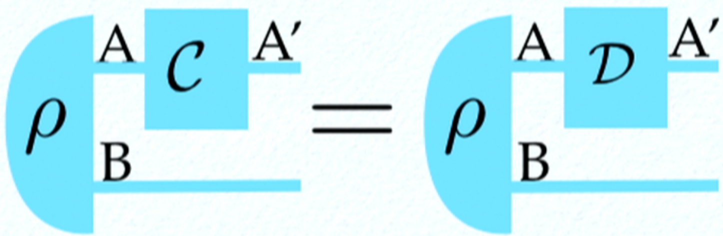
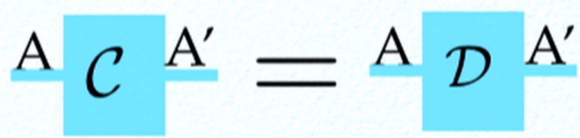


$$\forall B, \forall \rho : I \to A \otimes B$$

$\Longrightarrow$



CONTEXTS

Think of a state as a piece of information.
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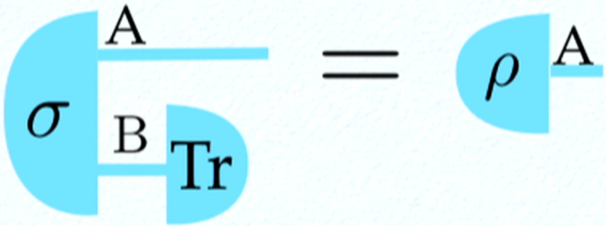
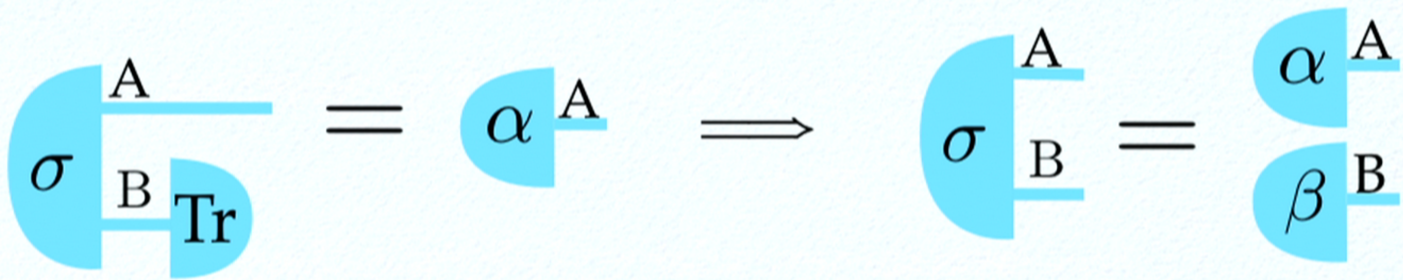


PROBABILITY-FREE DEFINITION OF PURE STATE

Definition: a state is pure iff it only has trivial extensions:

α is pure iff



Informally,
pure state = piece of information that is by definition independent of the context.

PROPERTIES

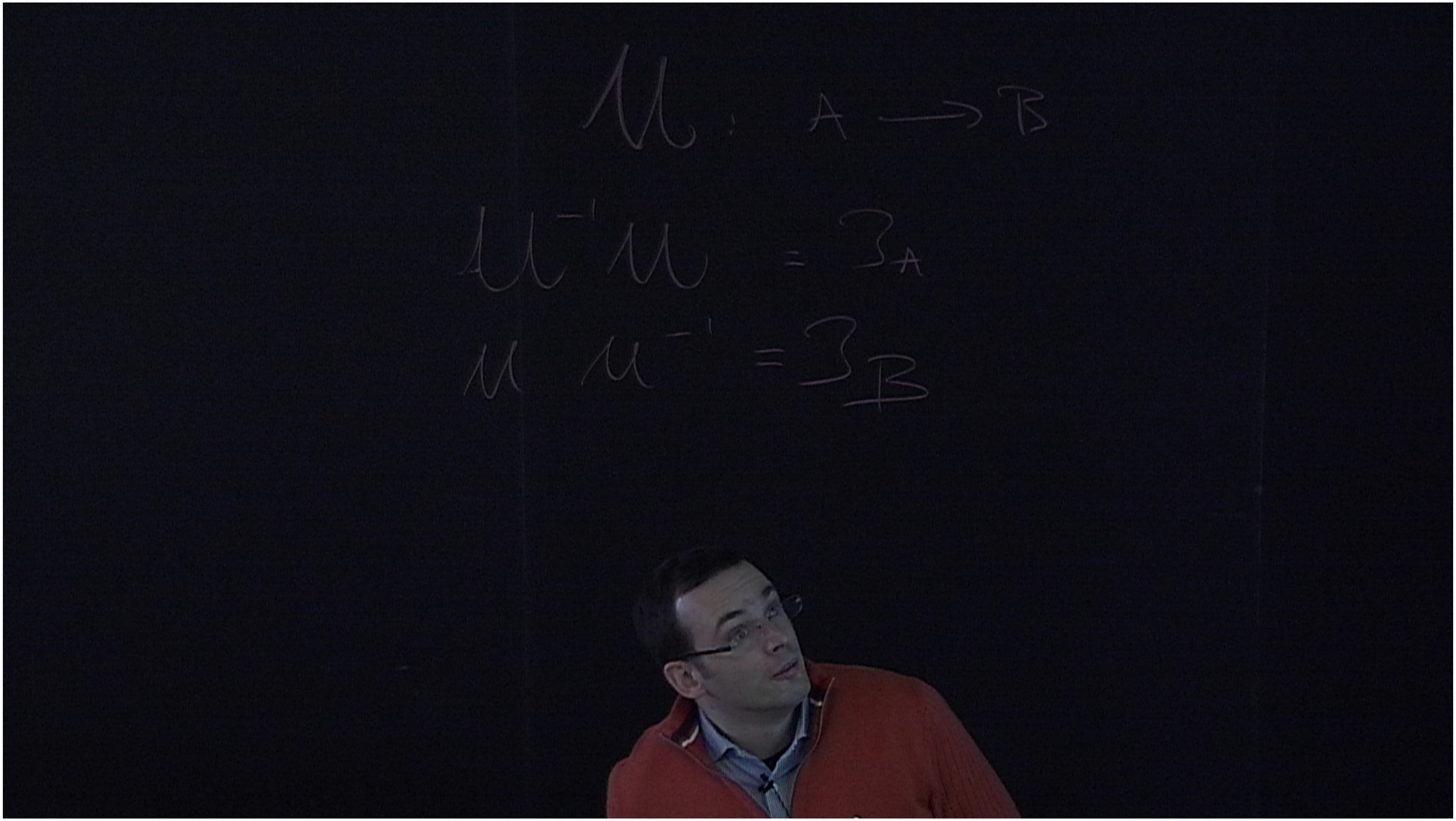
- Pure states form a monoid:

$$\alpha : I \to A \text{ pure}\,, \beta : I \to B \text{ pure}$$

$$\implies \alpha \otimes \beta : I \to A \otimes B \text{ pure}$$

- Reversible transformations (i.e. isomorphisms) preserve pure states

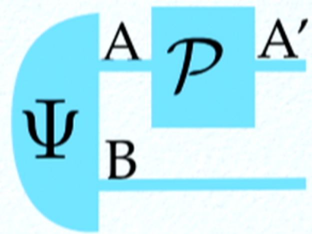
$$\alpha : I \to A \text{ pure}\,,\ \mathcal{U} : A \to B \text{ iso} \quad \implies \quad \mathcal{U} \circ \alpha \text{ pure}$$


$\mathcal{U} : A \longrightarrow B$
$\mathcal{U}^{-1}\mathcal{U} = 1_A$
$\mathcal{U}\,\mathcal{U}^{-1} = 1_B$

THE CATEGORY OF PURITY-PRESERVING PROCESSES

Definition: a process $\mathcal{P}$ is purity-preserving iff

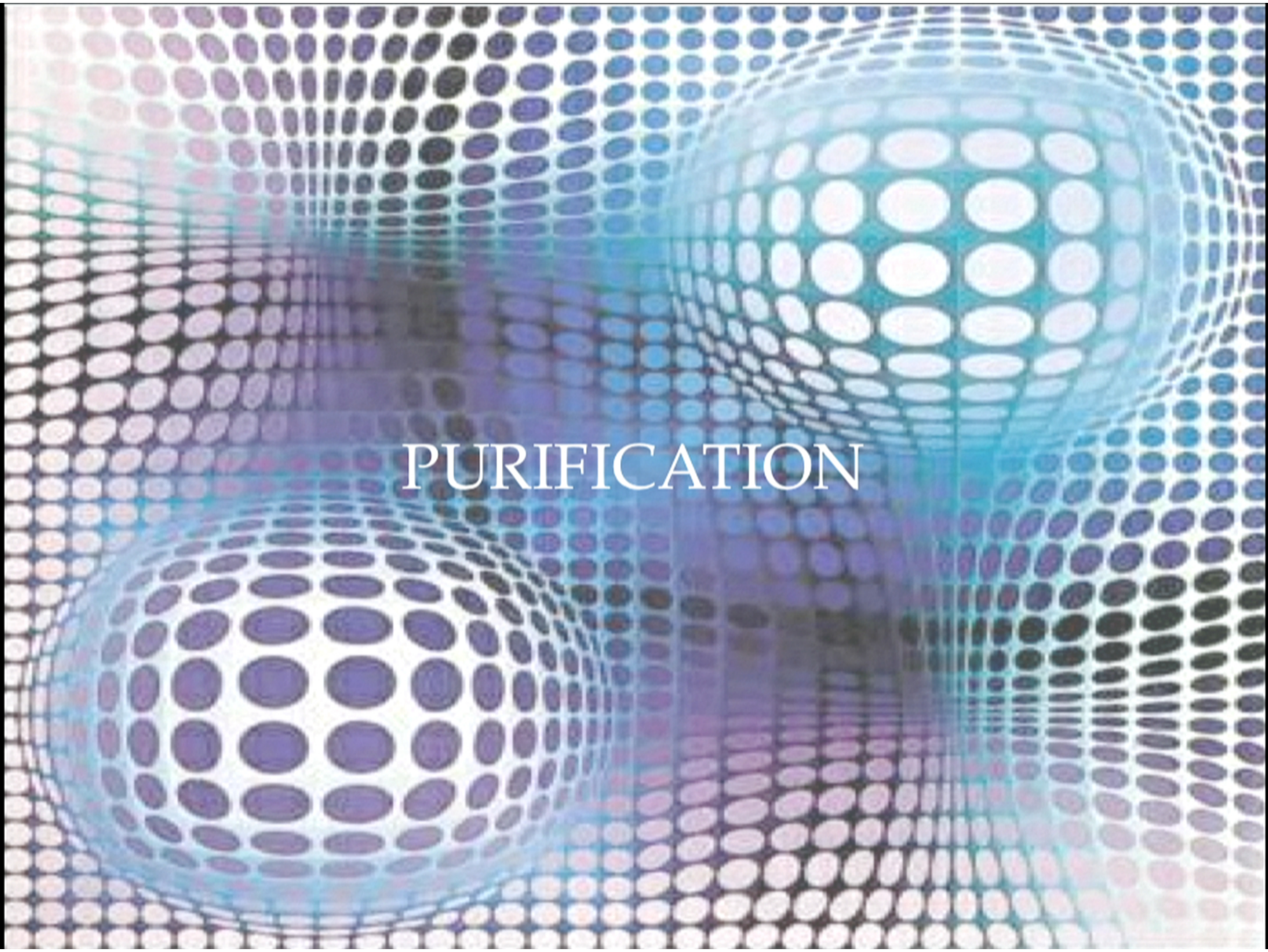
$\in \mathsf{PurSt}(A \otimes B) \implies$



$\in \mathsf{PurSt}(A' \otimes B)$

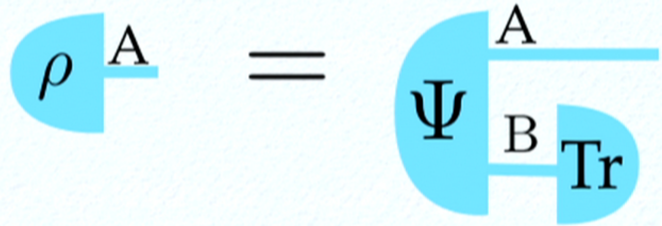
Property:

- purity-preserving processes form a symmetric monoidal subcategory of Det , containing the monoid of pure states

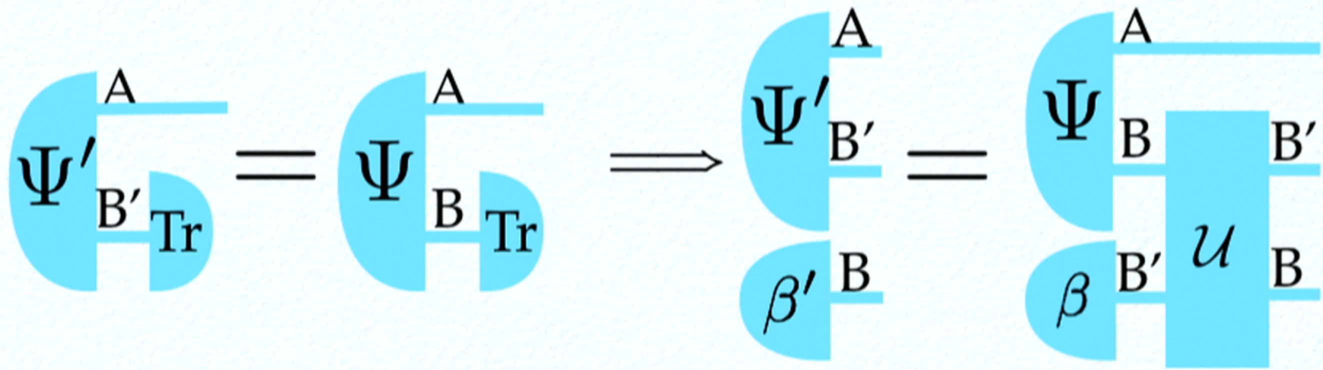

PURIFICATION

CATEGORICAL PURIFICATION

- **Existence:** For every state ρ of A
  there is a system B and a pure state Ψ of $A \otimes B$
  such that



- **Uniqueness:** all purifications of the same state are equivalent
  up to isos on the context



GOOD, BUT...

Does the Categorical Purification Axiom give all the features it gave in the convex world?

like, e. g. entanglement? or no-cloning?

mhm... wait!

We don't know yet if our category contains mixed states!

In fact, classical deterministic computation satisfies Purification, and has no entanglement nor a no-cloning theorem

MIXED STATES, MIXED STATES EVERYWHERE, NOR ANY PROBABILITY TO MAKE MIXTURES...

Definition: a state is mixed if it is not pure

Good, but when is a state "more mixed" than another?

In the convex world, one can say that ρ is "more mixed" than σ

iff

$$\rho = p\,\sigma + (1-p)\,\tau$$

for some p>0 and some state τ

However, the above expression is not "legal" in our language...

EXTENSIONS OF MIXED STATES

In the convex world, if ρ is "more mixed" than σ

then, for every extension of σ, say $\sigma' \in \mathsf{St}(A \otimes B)$

there exists an extension of ρ, say ρ'

that is "more mixed" than σ'

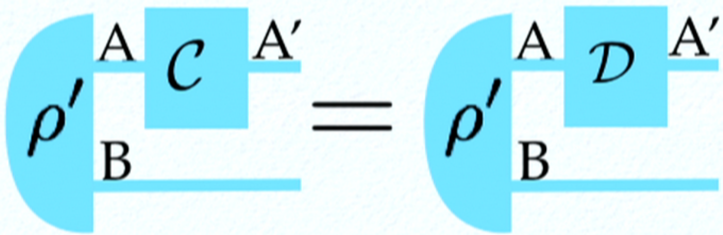
e. g. take $\rho' = p\sigma' + (1-p)\,\tau \otimes \beta$ for abitrary β

Idea: leverage on this property at the categorical level

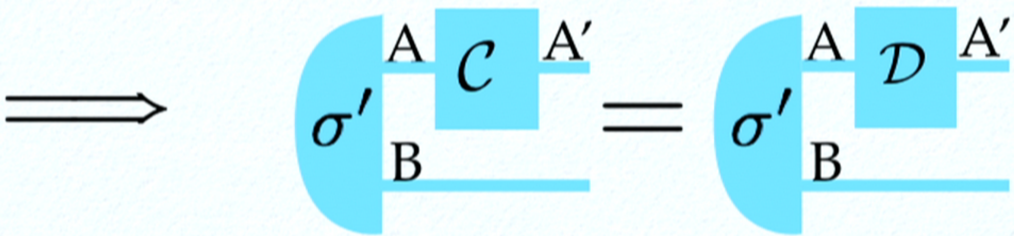
CATEGORICAL DEFINITION OF "MORE MIXED"

Definition: ρ is sufficient for σ, denoted $\rho \succeq \sigma$

iff



for every extension of ρ

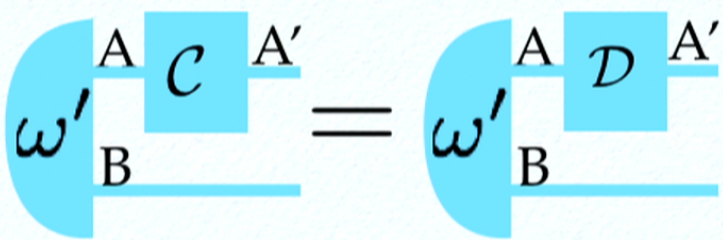


for every extension of σ

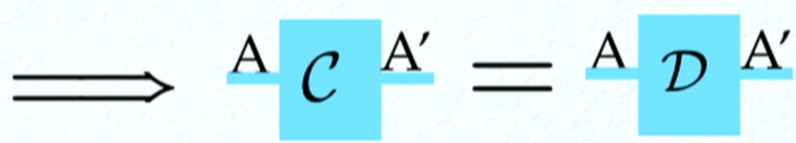
FAITHFUL STATES

Definition: $\omega \in \mathsf{St}(A)$ is faithful iff $\omega \succeq \rho\,, \quad \forall \rho \in \mathsf{St}(A)$

In other words:



for every extension of ω



FAITHFULNESS AXIOM

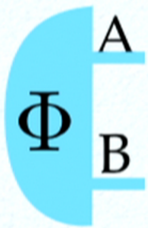
Axiom: for every system type A,
the set of states contains at least one faithful state ω .

Satisfied by all convex operational-probabilistic theories:
-quantum theory on complex and real fields,
-classical probability theory
-...

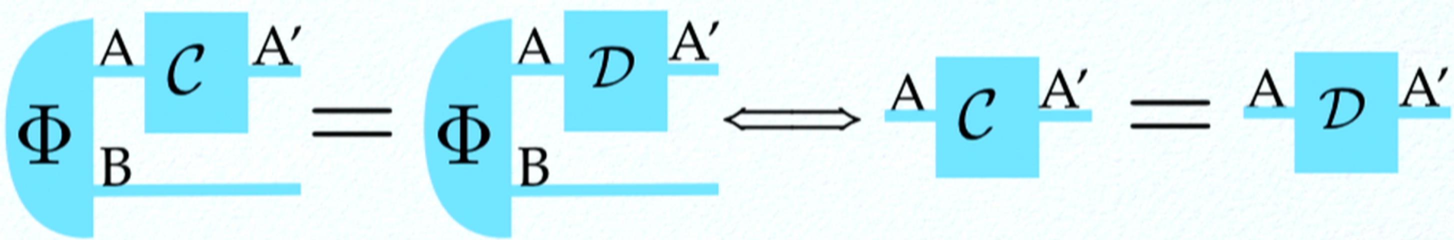
but also by non-probabilistic theories
-Schumacher-Westmoreland quantum theory on finite fields
-...

THE PURE STATE-TRANSFORMATION ISOMORPHISM

PURE STATE-TRANSFORMATION ISOMORPHISM

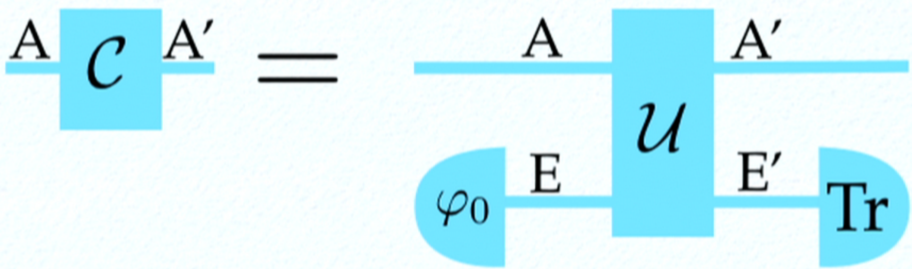


:= purification of the faithful state



PURE AND REVERSIBLE PROCESS SIMULATION

Theorem: For every process
there exist environments E and E′
a pure state of E ,
and a reversible process from AE to BE′ such that



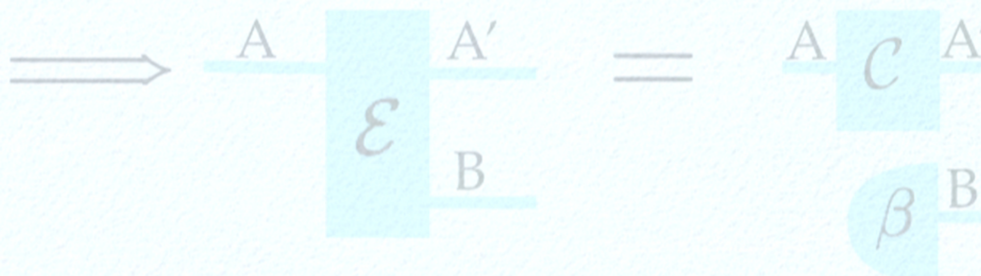
This simulation is unique up to isos on the context.

cf. Stinespring-Kraus' dilation theorem

Theorem: under the validity of Purification and Faithfulness
a process is purity-preserving
if and only if it is pure.

Definition: $\mathcal{E}$ is pure iff

THE PURITY THEOREM



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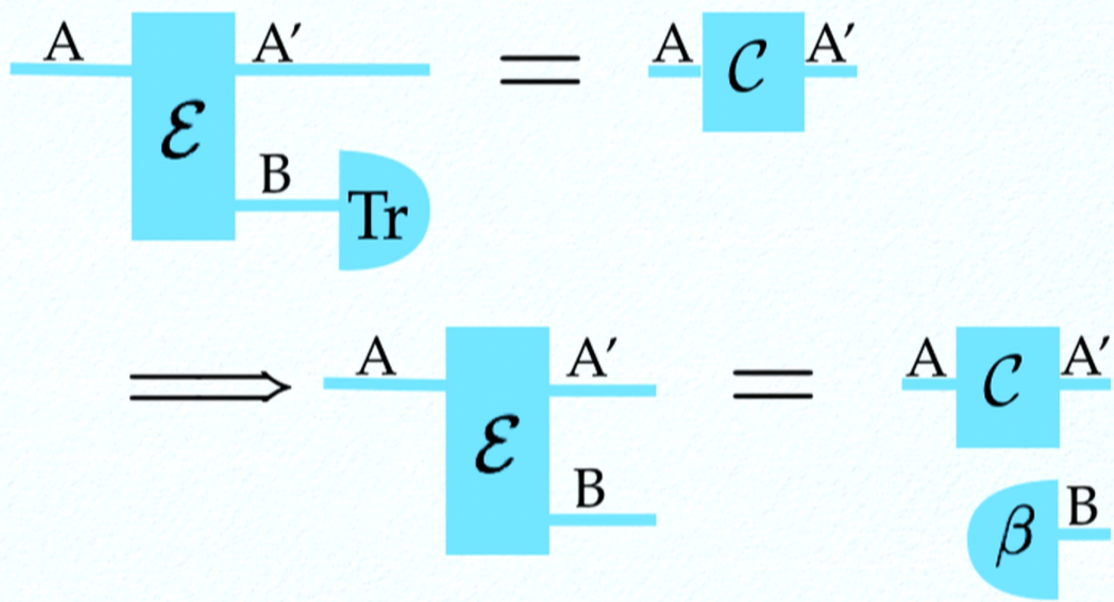
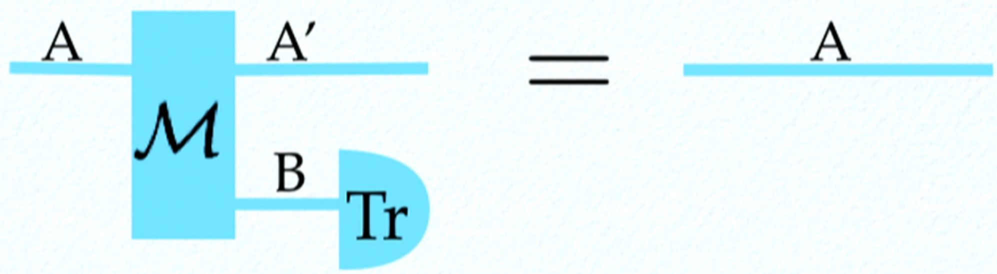
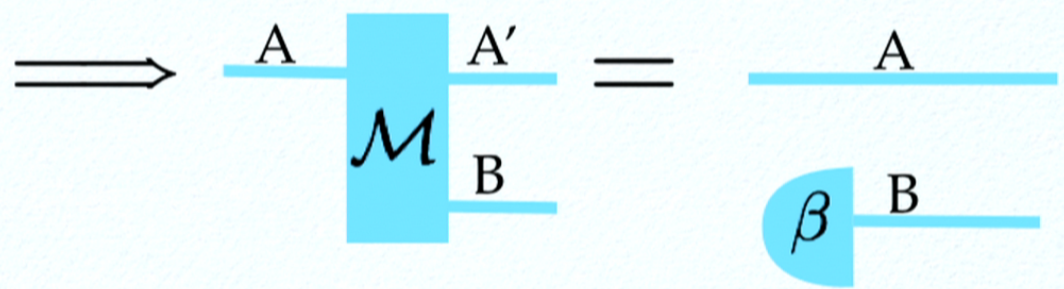


COROLLARIES OF THE PURITY THEOREM

- No Information Without Disturbance



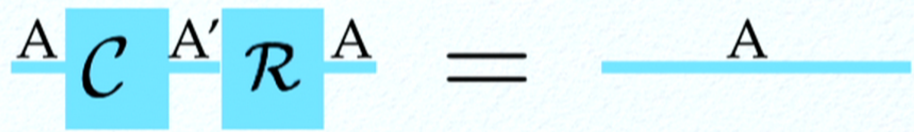


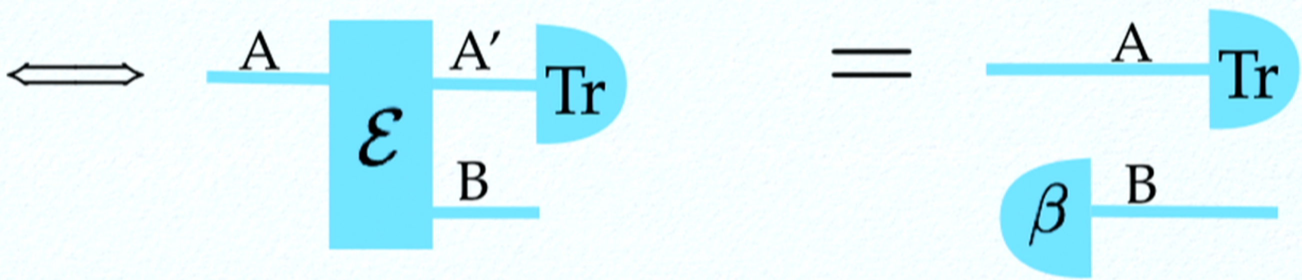
- No Cloning

MORE COROLLARIES OF THE PURITY THEOREM

- Error correction balance

$\exists \mathcal{R} : A' \to A$ s. t.





for every extension $\mathcal{E}$.

CONTRIBUTIONS TO THE AXIOMATIZATION PROGRAMME

- No restriction to finite dimensions
  e.g. everything holds also for (some) infinite dimensional systems

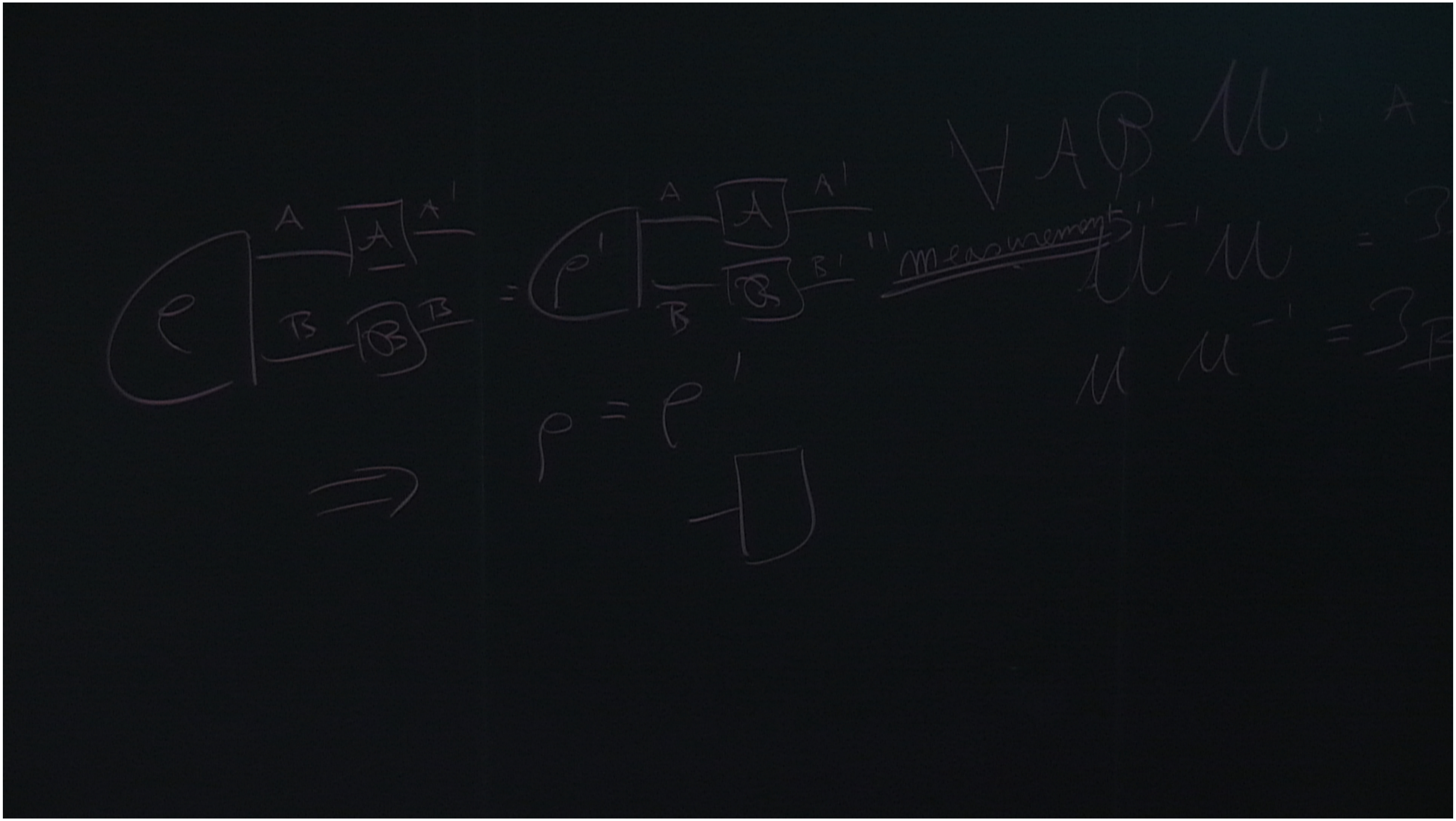
- No no need of "Local Tomography"
  all results hold also for QT on real Hilbert spaces

- No need of "Purity Preservation"
  one of the axioms of CDP2010 was that the product of two pure processes yield a pure process
  In the new framework, this is subsumed by the Purity Theorem

WHAT IS HERE AND WHAT IS NOT (YET)

- Entanglement (Y)
- No Cloning (Y)
- No Information Without Disturbance (Y)
- Teleportation (N)
- Steering (Y/N)
- Existence of perfectly-correlating states (N)
- Ancilla-assisted process tomography (Y)
- Reversible simulation of irreversible processes (Y)
- No Bit Commitment (Y)
- Principle of Delayed Measurement (Y/N)
- No Programming Theorem (Y)
- Error correction balance (Y)
- Structure of no-signalling channels (Y)

...


A
A'
B
B'
ρ = ρ'