Title: Gauge theories and quantum hydrodynamics

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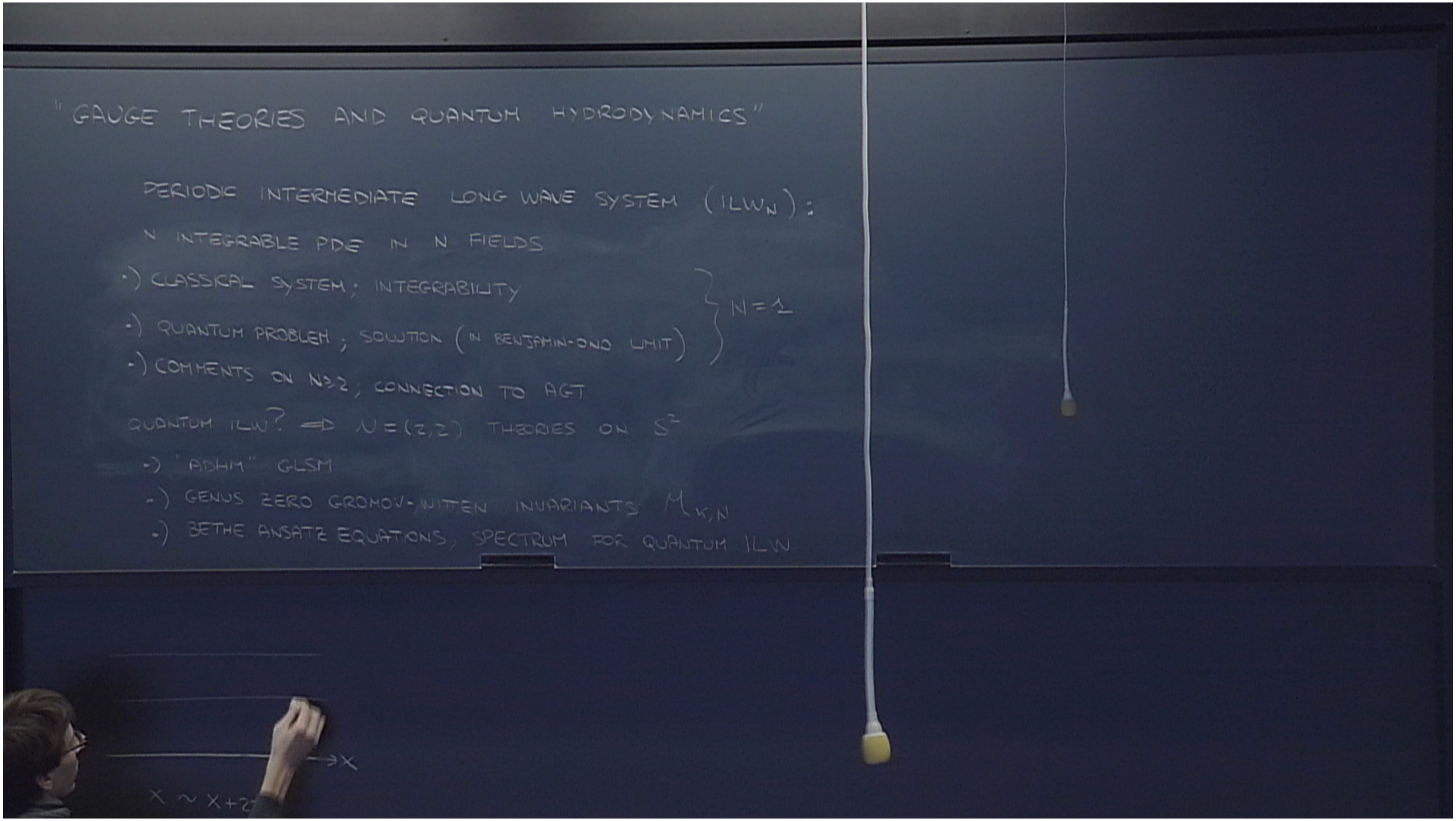
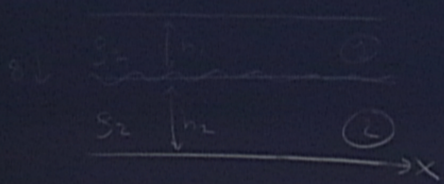
Abstract: Hydrodynamic integrable systems are described in terms of integrable partial
 differential equations.
 I will focus on the periodic Intermediate Long Wave (ILW) system, both at
 the classical and quantum level. The quantum problem has not been solved
 yet, if not in a particular limit (the Benjamin-Ono limit) which is related
 to the AGT correspondence. I will show how a particular two dimensional
 N=(2,2) gauge theory on S^2 can be used to determine the spectrum of the ILW
 system via Bethe Ansatz equations. Moreover the partition function of this
 theory (which represents the instanton partition function on C^2*S^2)
 computes genus zero Gromov-Witten invariants for the instanton moduli space,
 thus relating quantum cohomology to quantum hydrodynamics.


"GAUGE THEORIES AND QUANTUM HYDRODYNAMICS"
PERIODIC INTERMEDIATE LONG WAVE SYSTEM (ILW_N):
N INTEGRABLE PDE IN N FIELDS
•) CLASSICAL SYSTEM; INTEGRABILITY
•) QUANTUM PROBLEM; SOLUTION (IN BENJAMIN-ONO LIMIT)
N = 1
•) COMMENTS ON N≥2; CONNECTION TO AGT
QUANTUM ILW? ⟸ $\mathcal{N}=(2,2)$ THEORIES ON S^2
•) "ADHM" GLSM
•) GENUS ZERO GROMOV-WITTEN INVARIANTS $\mathcal{M}_{K,N}$
•) BETHE ANSATZ EQUATIONS, SPECTRUM FOR QUANTUM ILW
X
X ~ X+2

-) "ADHM" GLSM

-) GENUS ZERO GROMOV-WITTEN INVARIANTS $\mathcal{M}_{k,N}$

-) BETHE ANSATZ EQUATIONS, SPECTRUM FOR QUANTUM ILW



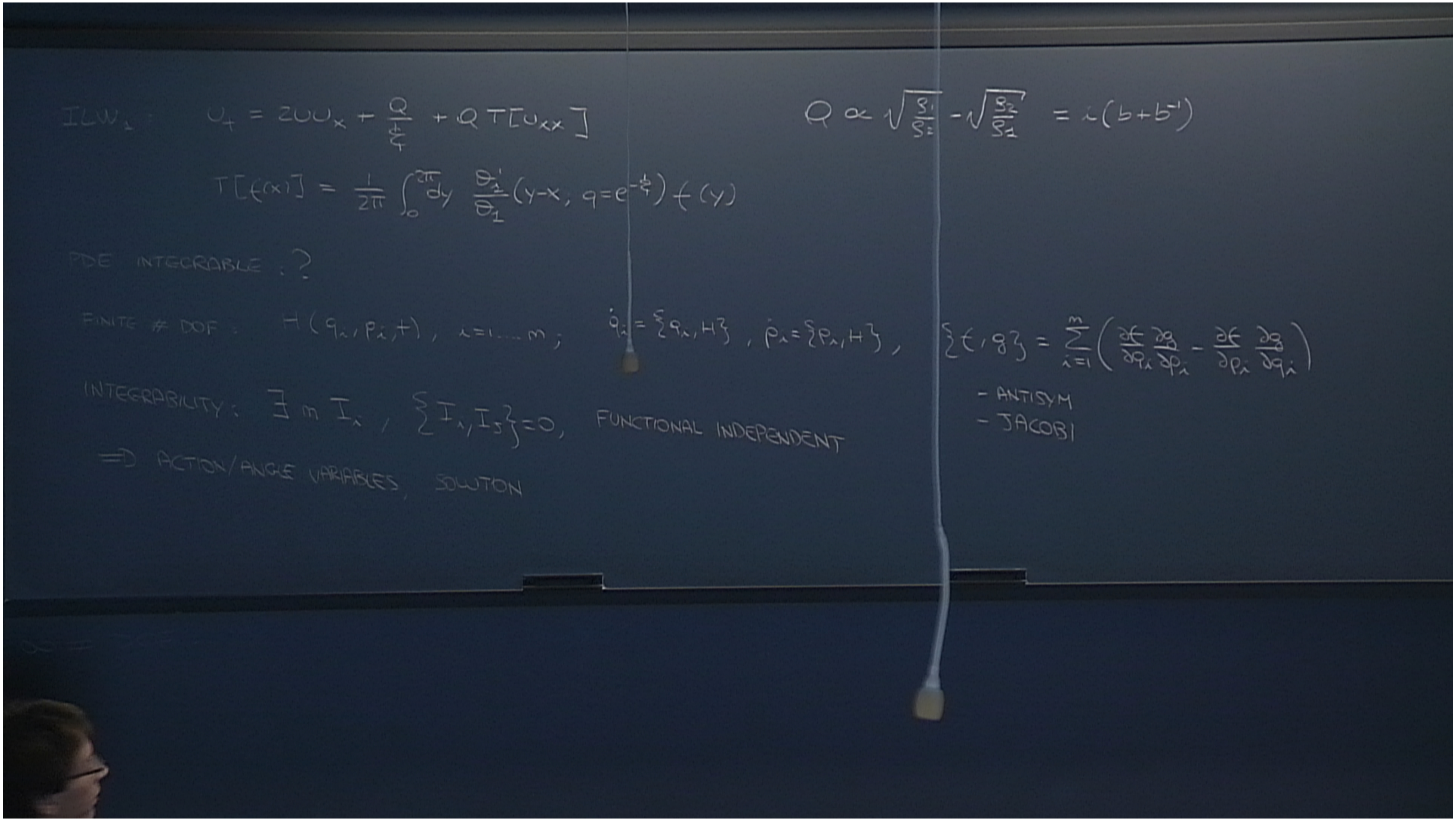
$x \sim x + 2\pi$

$h = h_1 + h_2$ (LONG WAVE)

-) $h \sim \lambda$, $\xi = \frac{h}{\lambda} \sim O(1)$: ILW REGIME

-) $h \gg \lambda$, $\xi \to \infty$: BENJAMIN-ONO REGIME

-) $h \ll \lambda$, $\xi \to 0$: KdV REGIME


ILW1: ut = 2uux + Q/ξ + Q T[uxx]
Q ∝ √(g1/g2) − √(g2/g1) = i(b+b⁻¹)
T[f(x)] = 1/2π ∫0^2π dy θ1'/θ1 (y−x, q=e^−ξ) f(y)
PDE INTEGRABLE ?
FINITE # DOF: H(qi, pi, t), i=1...m; q̇i = {qi, H}, ṗi = {pi, H}, {f, g} = Σi=1^m (∂f/∂qi ∂g/∂pi − ∂f/∂pi ∂g/∂qi)
- ANTISYM
- JACOBI
INTEGRABILITY: ∃ m Ii, {Ii, Ij} = 0, FUNCTIONAL INDEPENDENT
⟹ ACTION/ANGLE VARIABLES, SOLUTION

∞ DOF : INTEGRABILITY :

·) $\{\ ,\ \}$, ANTISYM, JACOBI

·) $\exists\ \infty\ \#\ I_n\ /\ \{I_n, I_m\} = 0$

·) $\exists\ \infty\ \#$ EXACT SOLUTIONS (SOLITON)

ILW$_1$ ... INTEGRABLE

$$\{F[u], G[u]\} = \int_0^{2\pi} \frac{\delta F}{\delta u} \partial_z \frac{\delta G}{\delta u} \, dz$$

$$\{u(x), u(y)\} = \delta'(x-y)$$

$$I_n = \frac{1}{2\pi} \int_0^{2\pi} dx \, G_n[u] \qquad u_t = \{u, I_2\}$$

$$G_1 = \frac{u^2}{2}, \quad G_2 = \frac{u^3}{3} + \frac{Q}{2} u T[u_x], \ldots$$

$$u(x) = \sum_{m \in \mathbb{Z}} a_m e^{imx} \quad , \quad \{a_m, a_n\} = -im\,\delta_{m+n,0} \xrightarrow[\{\,,\,\} \to \frac{[\,,\,]}{i\hbar}]{} [a_m, a_n] = \hbar m\,\delta_{m+n,0}$$

HEISENBERG ALGEBRA

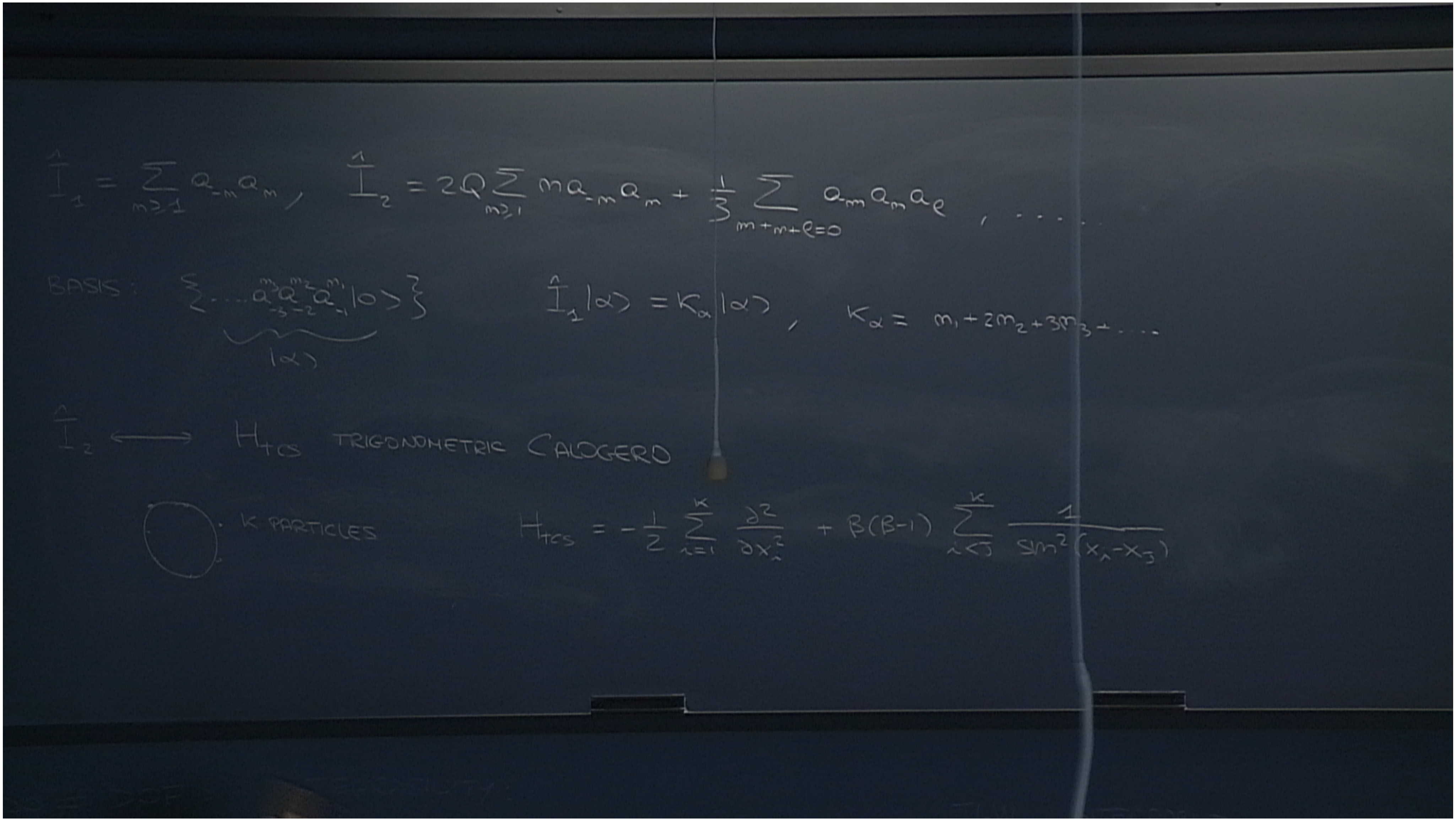
$$I_m \longrightarrow \hat{I}_m \quad , \quad |\alpha\rangle \;/\; \hat{I}_m|\alpha\rangle = E_m^{(\alpha)}|\alpha\rangle$$

$$\hat{I}_m = :\frac{1}{2\pi}\int_0^{2\pi} dx\, G_m[u]: \qquad [\hat{I}_m, \hat{I}_n] = 0 + o(\hbar)$$

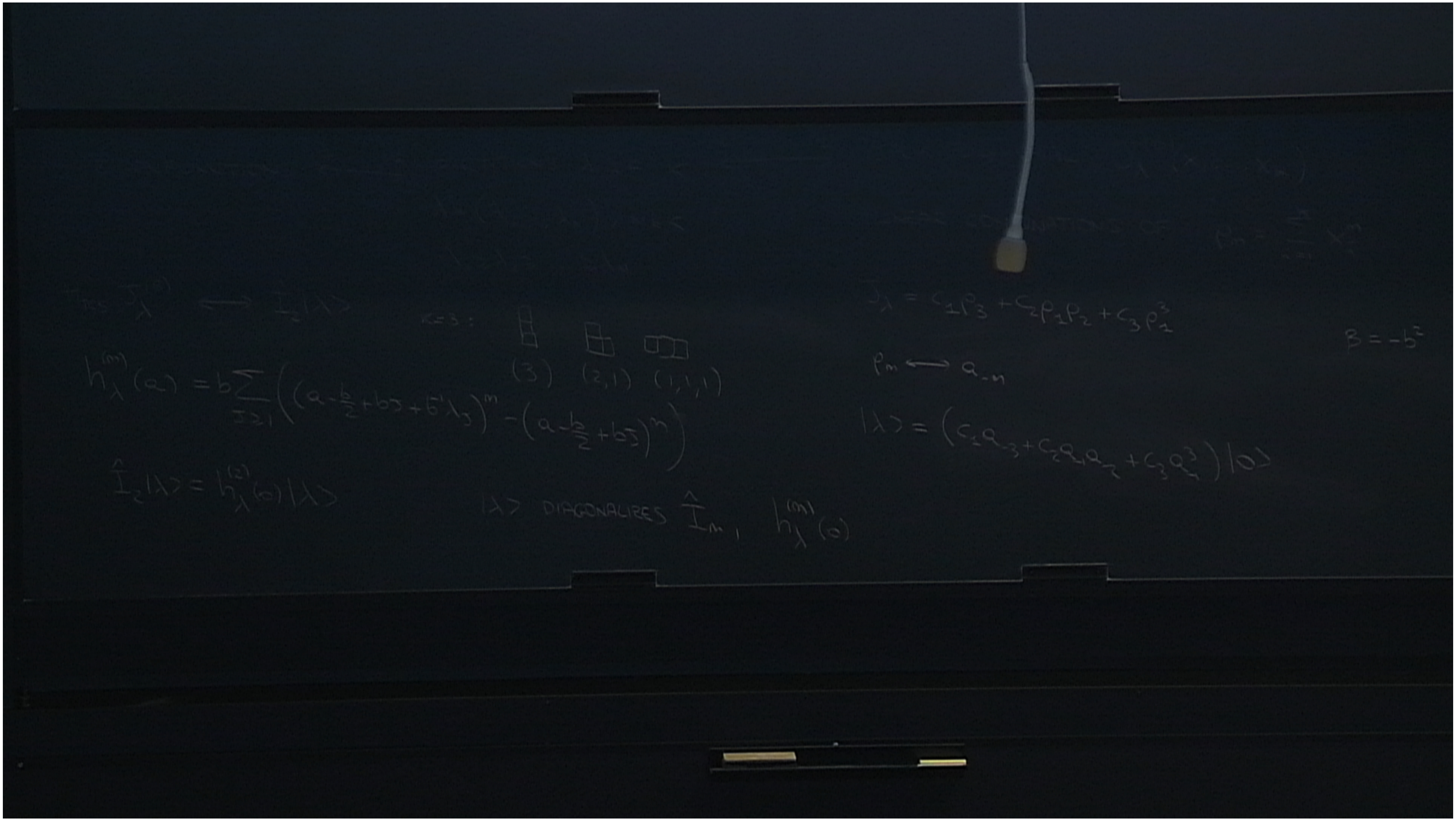
BENJAMIN-ONO LIMIT:

•) $\hat{I}_m$ KNOWN

•) QUANTUM PROBLEM SOLVED



$\hat{I}_1 = \sum_{m\geq 1} a_{-m} a_m$, $\hat{I}_2 = 2Q \sum_{m\geq 1} m\, a_{-m} a_m + \frac{1}{3} \sum_{m+n+l=0} a_m a_n a_l$, ...
BASIS: $\{ \ldots a_{-3}^{m_3} a_{-2}^{m_2} a_{-1}^{m_1} |0\rangle \}$
$|\alpha\rangle$
$\hat{I}_1 |\alpha\rangle = K_\alpha |\alpha\rangle$, $K_\alpha = m_1 + 2m_2 + 3m_3 + \ldots$
$\hat{I}_2 \longleftrightarrow H_{TCS}$ TRIGONOMETRIC CALOGERO
K PARTICLES
$H_{TCS} = -\frac{1}{2} \sum_{i=1}^{K} \frac{\partial^2}{\partial x_i^2} + \beta(\beta-1) \sum_{i<j}^{K} \frac{1}{\sin^2(x_i - x_j)}$



$\hat{I}_2 |\lambda\rangle$
$k=3:$
(3) $(2,1)$ $(1,1,1)$
$h^{(m)}_\lambda(a) = b \sum_{j\geq 1} \left(\left(a - \frac{b}{2} + bj + b^{-1}\lambda_j\right)^m - \left(a - \frac{b}{2} + bj\right)^m \right)$
$\hat{I}_2 |\lambda\rangle = h^{(2)}_\lambda(0) |\lambda\rangle$
$|\lambda\rangle$ DIAGONALIZES $\hat{I}_m$, $h^{(m)}_\lambda(0)$
$J_\lambda = c_1 p_3 + c_2 p_1 p_2 + c_3 p_1^3$
$\beta = -b^2$
$p_n \longleftrightarrow a_{-n}$
$|\lambda\rangle = (c_1 a_{-3} + c_2 a_{-1} a_{-2} + c_3 a_{-1}^3) |0\rangle$

$$\mathrm{ILW}_2 : \quad \begin{cases} u_t = \{u, I_2\} \\ v_t = \{v, I_2\} \end{cases}$$

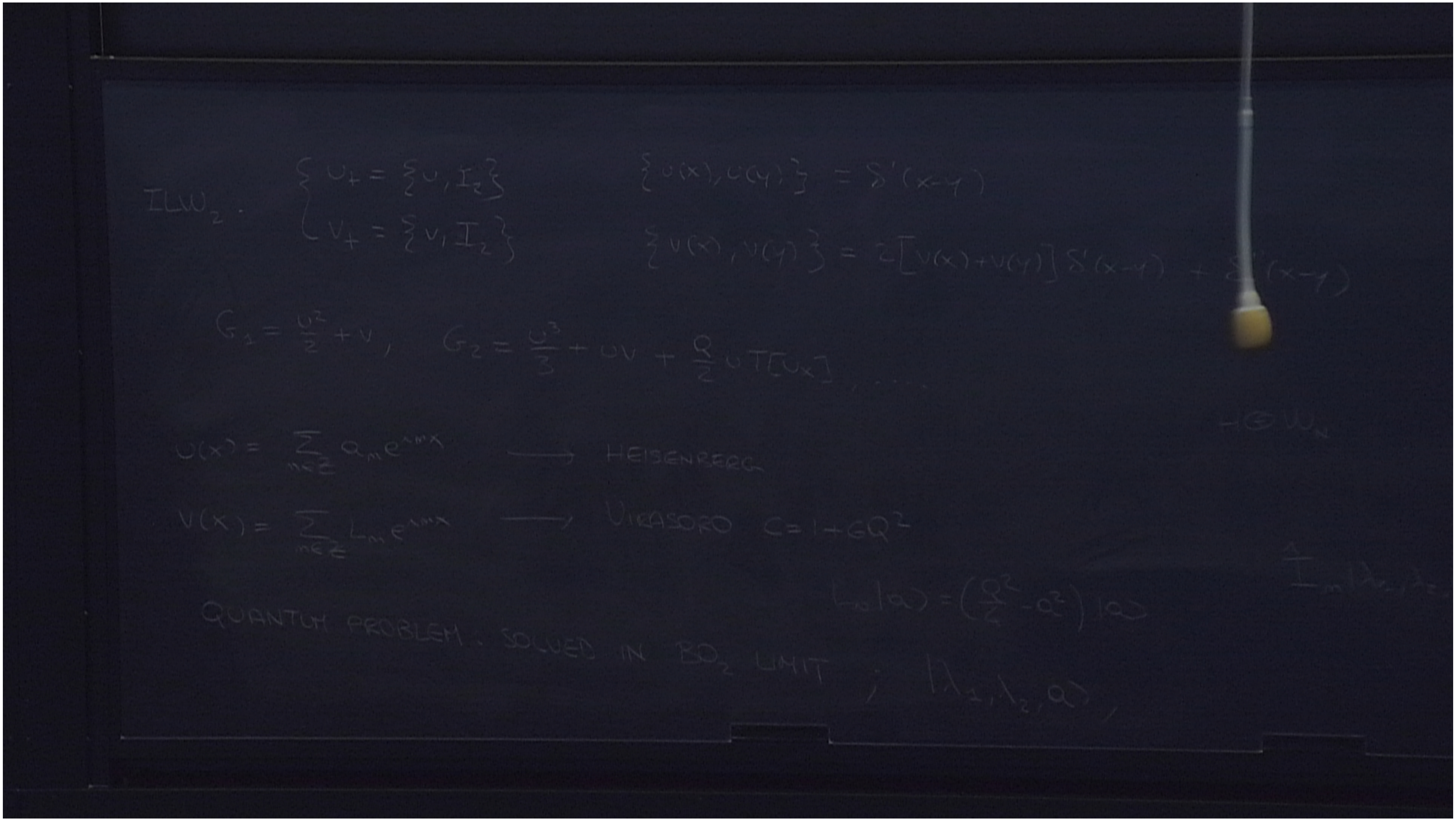
$$\{u(x), u(y)\} = \delta'(x-y)$$

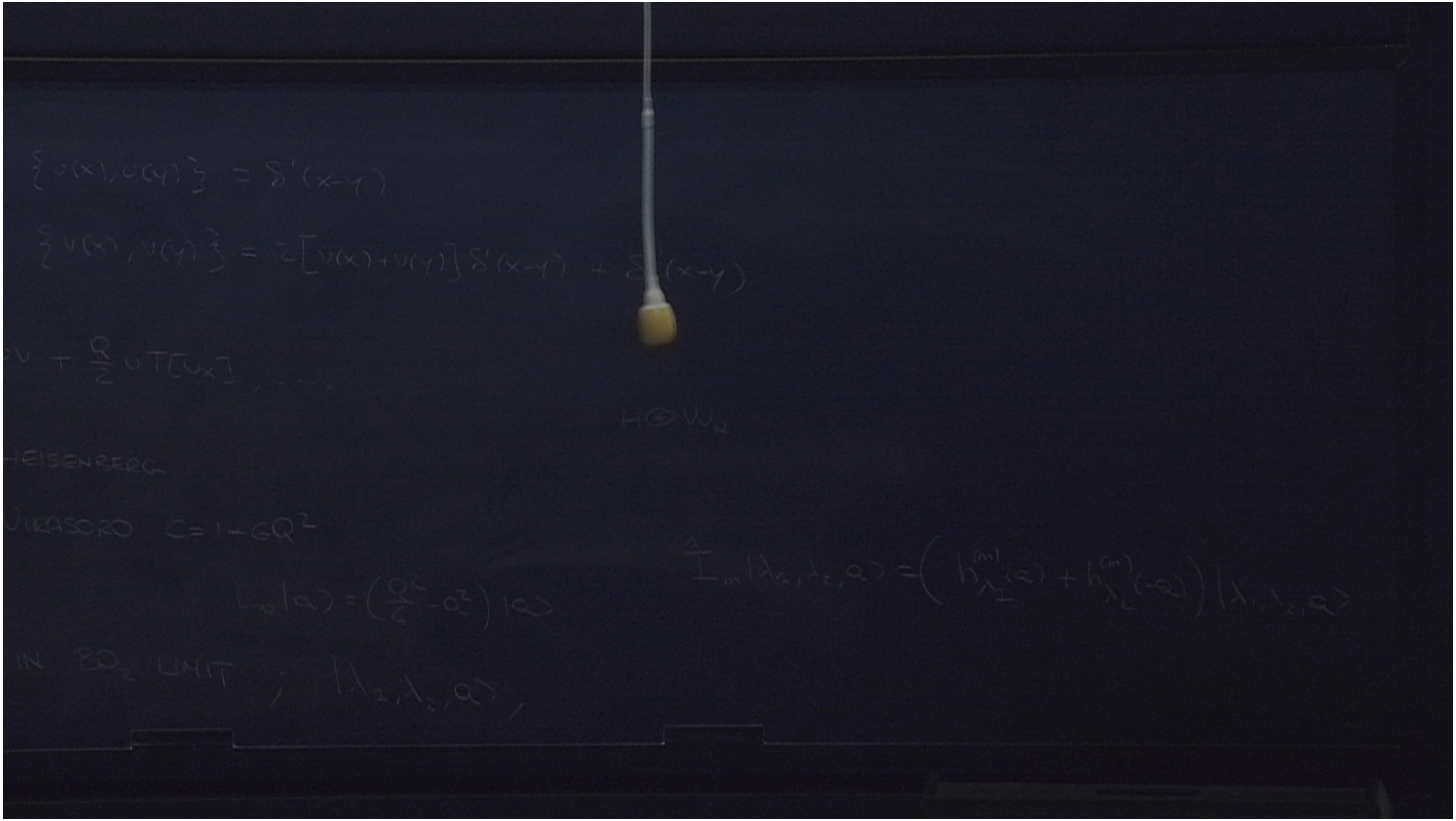
$$\{v(x), v(y)\} = 2\left[v(x)+v(y)\right]\delta'(x-y) + \delta'''(x-y)$$

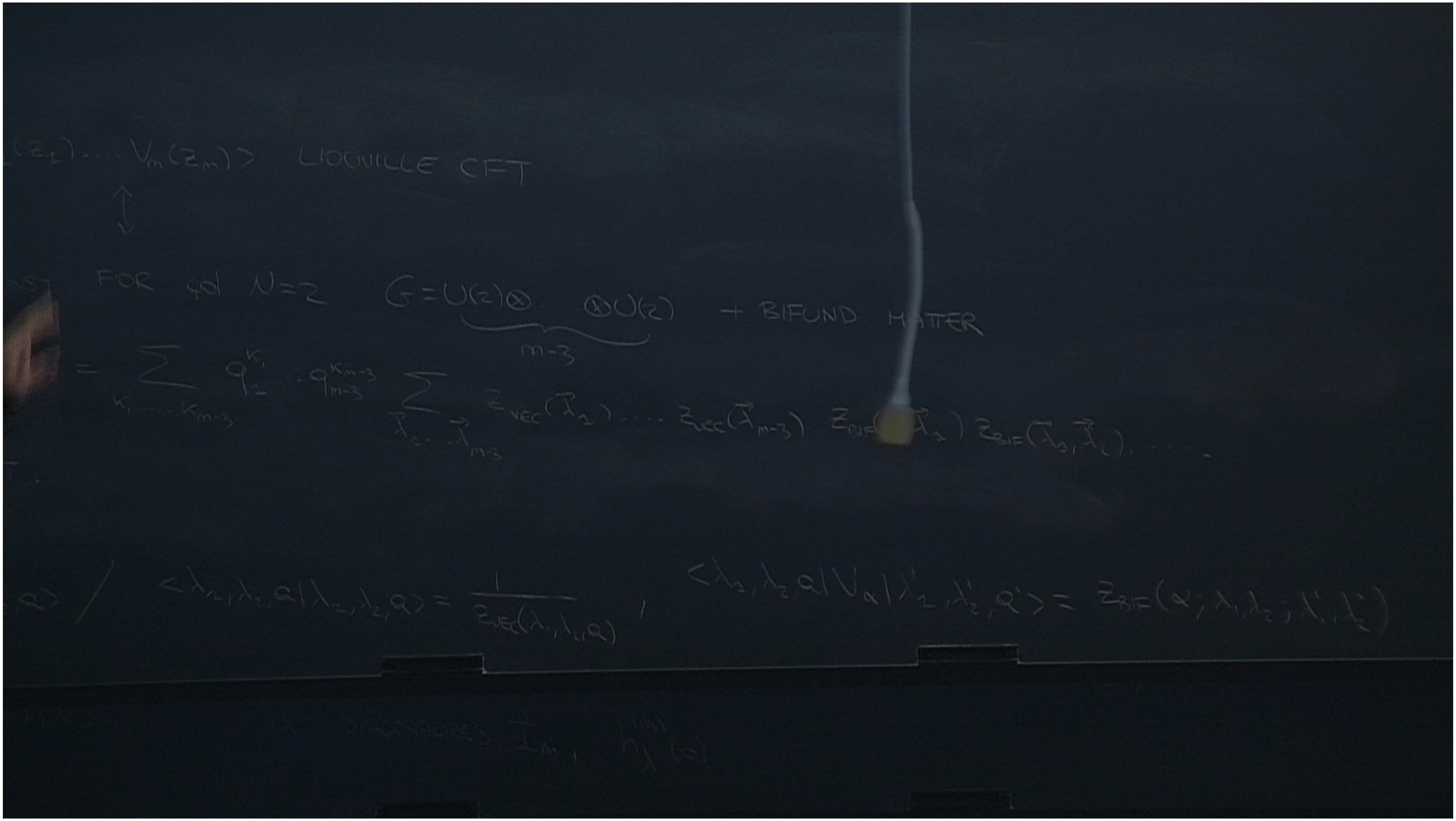
$$G_1 = \frac{u^2}{2} + v, \quad G_2 = \frac{u^3}{3} + uv + \frac{Q}{2} u T[u_x], \; \ldots$$

$$u(x) = \sum_{m \in \mathbb{Z}} a_m e^{imx} \longrightarrow \text{HEISENBERG}$$

$$v(x) = \sum_{m \in \mathbb{Z}} L_m e^{imx} \longrightarrow \text{VIRASORO} \quad c = 1 + 6Q^2$$


ILW₂ : { u_t = {u, I₂} ; v_t = {v, I₂} }
{u(x), u(y)} = δ'(x−y)
{v(x), v(y)} = 2[v(x)+v(y)] δ'(x−y) + δ'''(x−y)
G₁ = u²/2 + v , G₂ = u³/3 + uv + Q/2 u T[u_x] , . . .
u(x) = Σ_{m∈Z} a_m e^{imx} ⟶ HEISENBERG
v(x) = Σ_{m∈Z} L_m e^{imx} ⟶ VIRASORO c = 1 + 6Q²
H ⊕ W_N
L₀|a⟩ = (Q²/4 − a²)|a⟩
QUANTUM PROBLEM : SOLVED IN BO₂ LIMIT ; |λ₁, λ₂, a⟩ ,


HEISENBERG
VIRASORO C=1+6Q^2
IN BO_2 LIMIT


(z1) ... Vm(zm)> LIOUVILLE CFT
FOR 4d N=2 G=U(2)⊗ ... ⊗U(2) + BIFUND MATTER
m-3
= Σ q1^k1 ... qm-3^km-3 Σ Zvec(λ2) ... Zvec(λm-3) ZBIF(λ1) ZBIF(λ3, λ2) ...
<λ1, λ2, a | λ1, λ2, a> = 1 / Zvec(λ1, λ2, a)
<λ1, λ2, a | Vα | λ'1, λ'2, a'> = ZBIF(α; λ1, λ2; λ'1, λ'2)

AGT : $\langle V_1(z_1)\dots V_m(z_m)\rangle$ LIOUVILLE CFT

$\updownarrow$

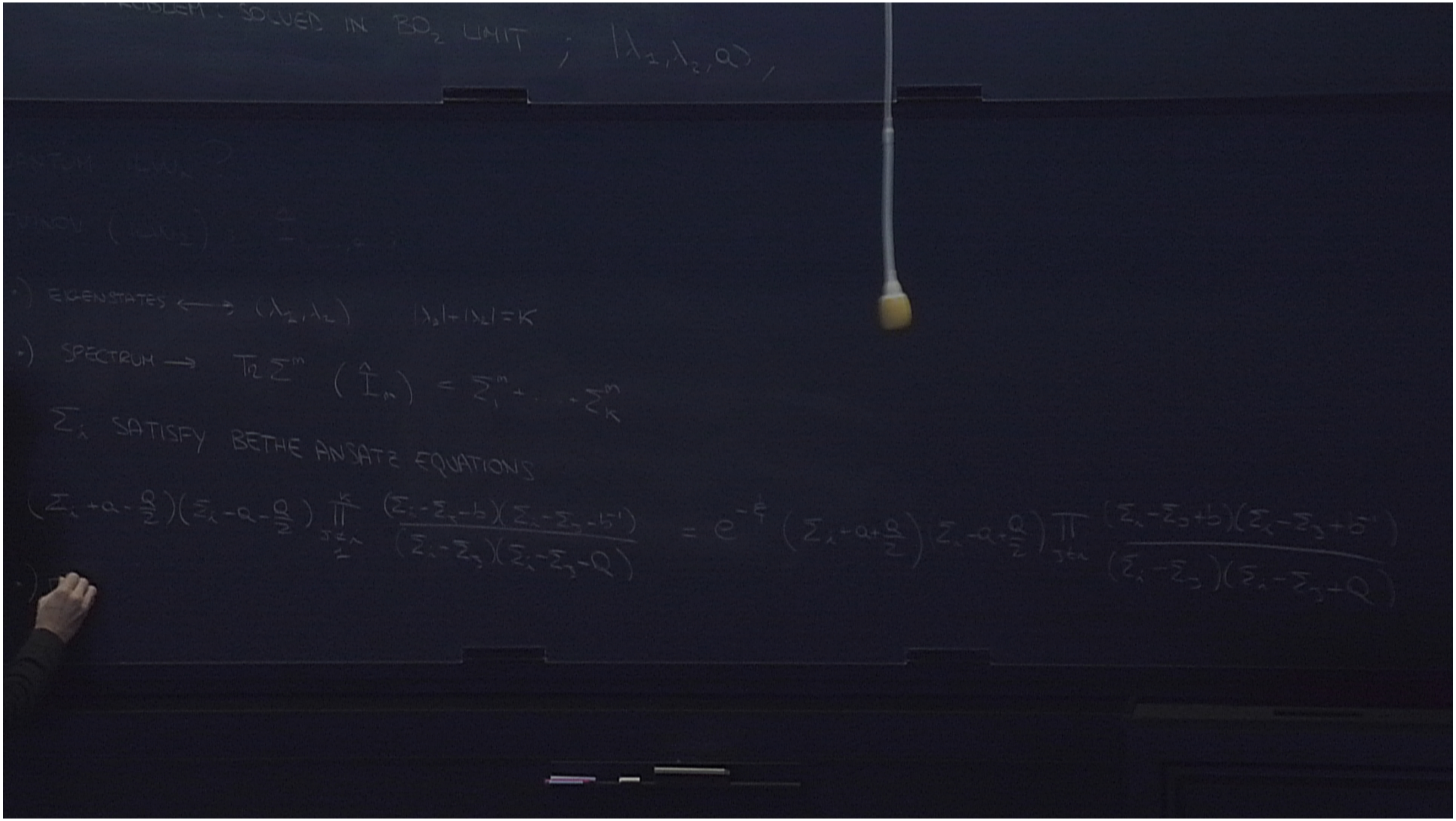
Z_{INST} FOR 4d $\mathcal{N}=2$ $\quad G = \underbrace{U(2)\otimes\dots\otimes U(2)}_{m-3}$ + BIFUND MATTER

$$Z_{INST} = \sum_{k_1\dots k_{m-3}} q_1^{k_1}\dots q_{m-3}^{k_{m-3}} \sum_{\vec\lambda_1\dots\vec\lambda_{m-3}} Z_{VEC}(\vec\lambda_1)\dots Z_{VEC}(\vec\lambda_{m-3})\, Z_{BIF}(\emptyset,\vec\lambda_1)\, Z_{BIF}(\vec\lambda_1,\vec\lambda_2)\dots$$

PROOF IN CFT.

•) $H \oplus Vir$;

•) $\exists!\ |\lambda_1,\lambda_2,a\rangle$ / $\langle\lambda_1,\lambda_2,a|\lambda_1,\lambda_2,a\rangle = \frac{1}{Z_{VEC}(\lambda_1,\lambda_2,a)}$, $\langle\lambda_1,\lambda_2,a|V_\alpha|\lambda'_1,\lambda'_2,a'\rangle = Z_{BIF}$



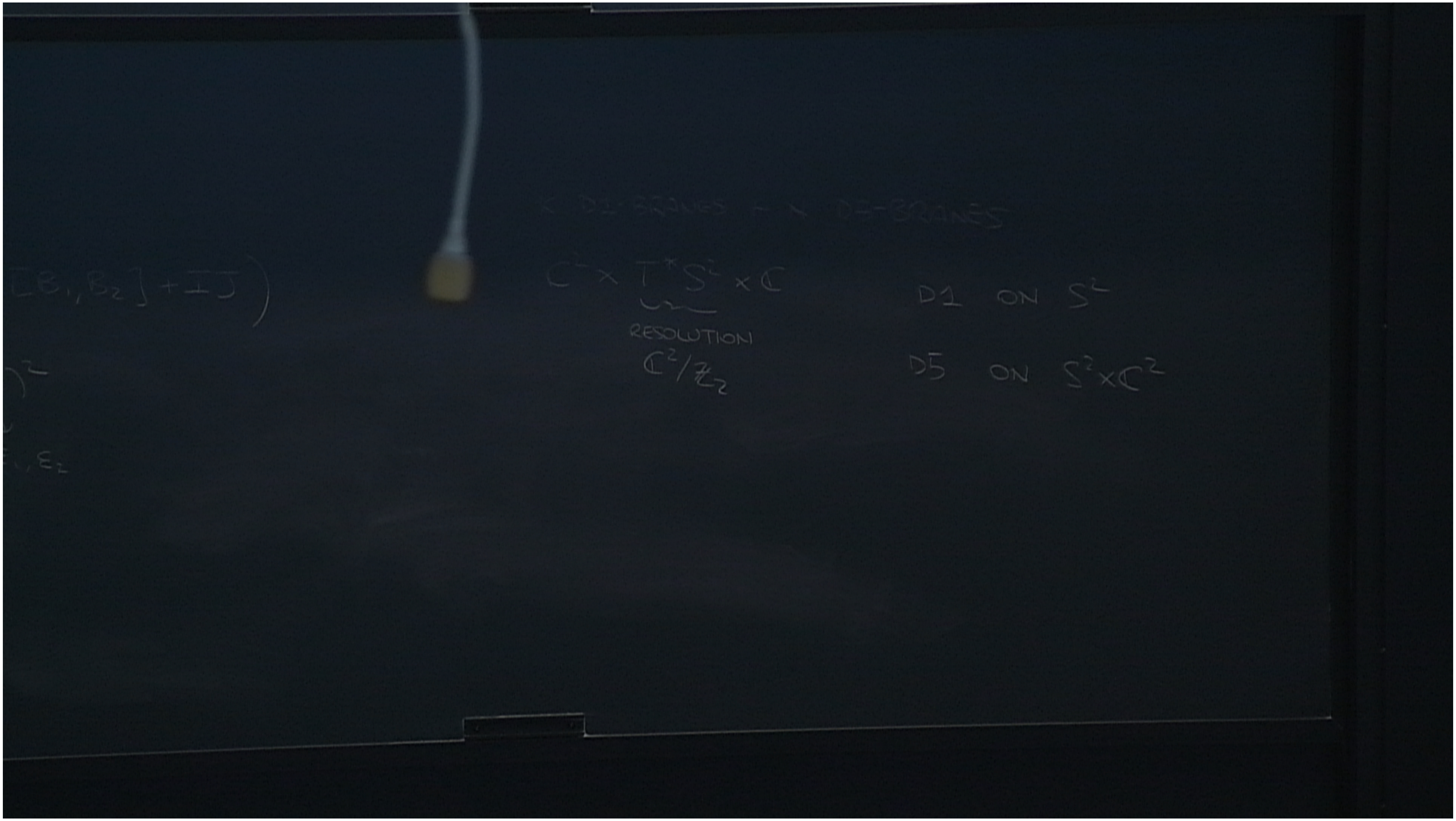
PROBLEM : SOLVED IN BO2 LIMIT ; |λ1, λ2, a⟩ ,
•) EIGENSTATES ⟷ (λ1, λ2) |λ1| + |λ2| = K
•) SPECTRUM → Tr Σ^m (Î_m) = Σ_1^m + ... + Σ_K^m
Σ_i SATISFY BETHE ANSATZ EQUATIONS
(Σ_i + a − Q/2)(Σ_i − a − Q/2) ∏_{j≠i}^{K} (Σ_i − Σ_j − b)(Σ_i − Σ_j − b^{-1}) / (Σ_i − Σ_j)(Σ_i − Σ_j − Q) = e^{−ξ} (Σ_i − a + Q/2)(Σ_i − a + Q/2) ∏_{j≠i} (Σ_i − Σ_j + b)(Σ_i − Σ_j + b^{-1}) / (Σ_i − Σ_j)(Σ_i − Σ_j + Q)

- EIGENSTATES $\longleftrightarrow$ (λ_1, λ_2) $\quad |\lambda_1| + |\lambda_2| = K$
- SPECTRUM $\longrightarrow$ $\mathrm{Tr}\, \Sigma^m \left(\hat{I}_m \right) = \Sigma_1^m + \dots + \Sigma_K^m$

Σ_i SATISFY BETHE ANSATZ EQUATIONS

$$\left(\Sigma_i + a - \frac{Q}{2}\right)\left(\Sigma_i - a - \frac{Q}{2}\right) \prod_{\substack{j \neq i \\ 1}}^{K} \frac{(\Sigma_i - \Sigma_j - b)(\Sigma_i - \Sigma_j - b^{-1})}{(\Sigma_i - \Sigma_j)(\Sigma_i - \Sigma_j - Q)} = e^{-\xi} \left(\Sigma_i + \right.$$

- FORMULA $\langle \, I \, \rangle = \dots$


C²×T*S²×C
RESOLUTION
C²/Z₂
D1 ON S²
D5 ON S²×C²

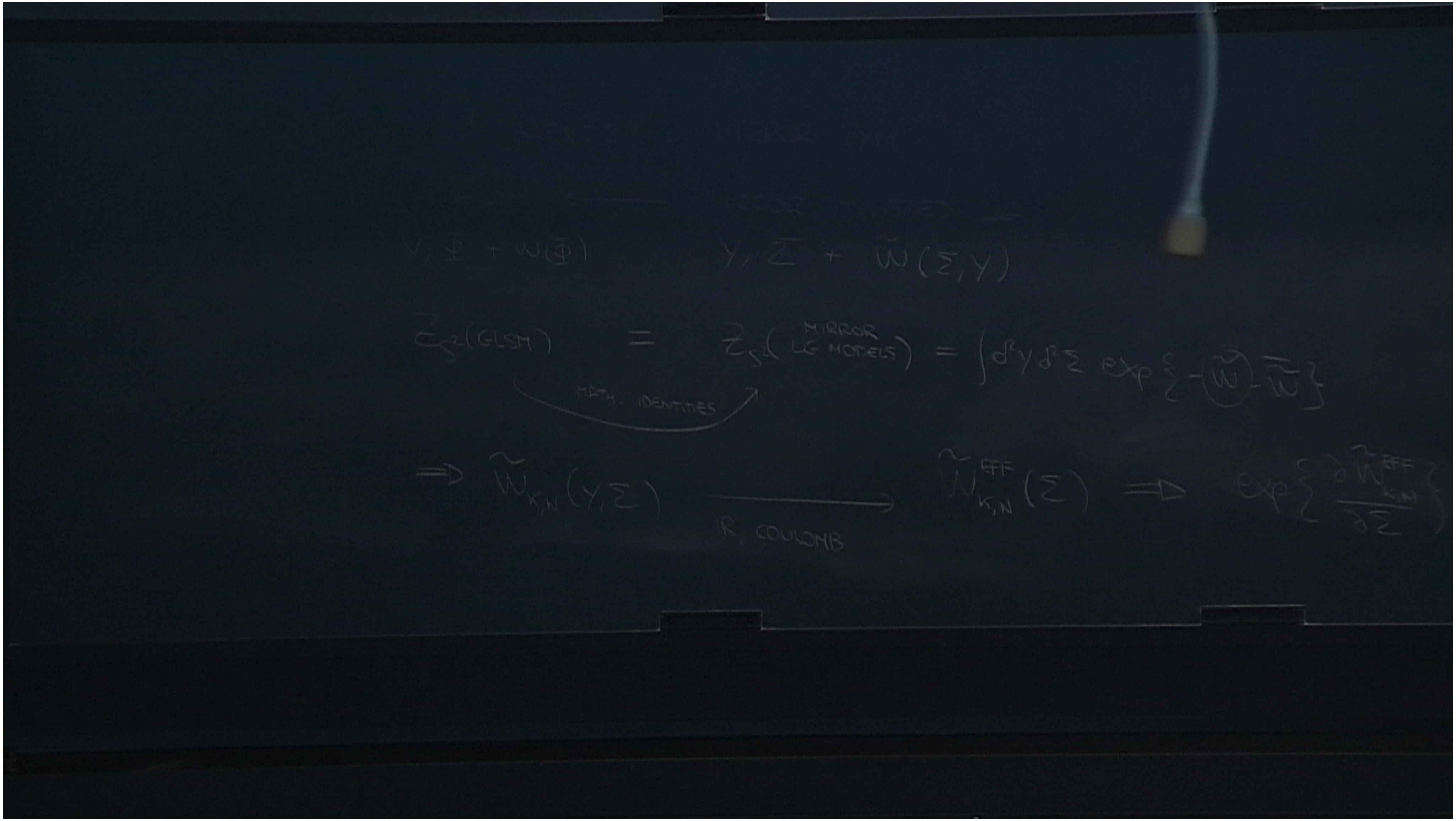
GLSM $\longrightarrow$ TARGET MANIFOLD M ;

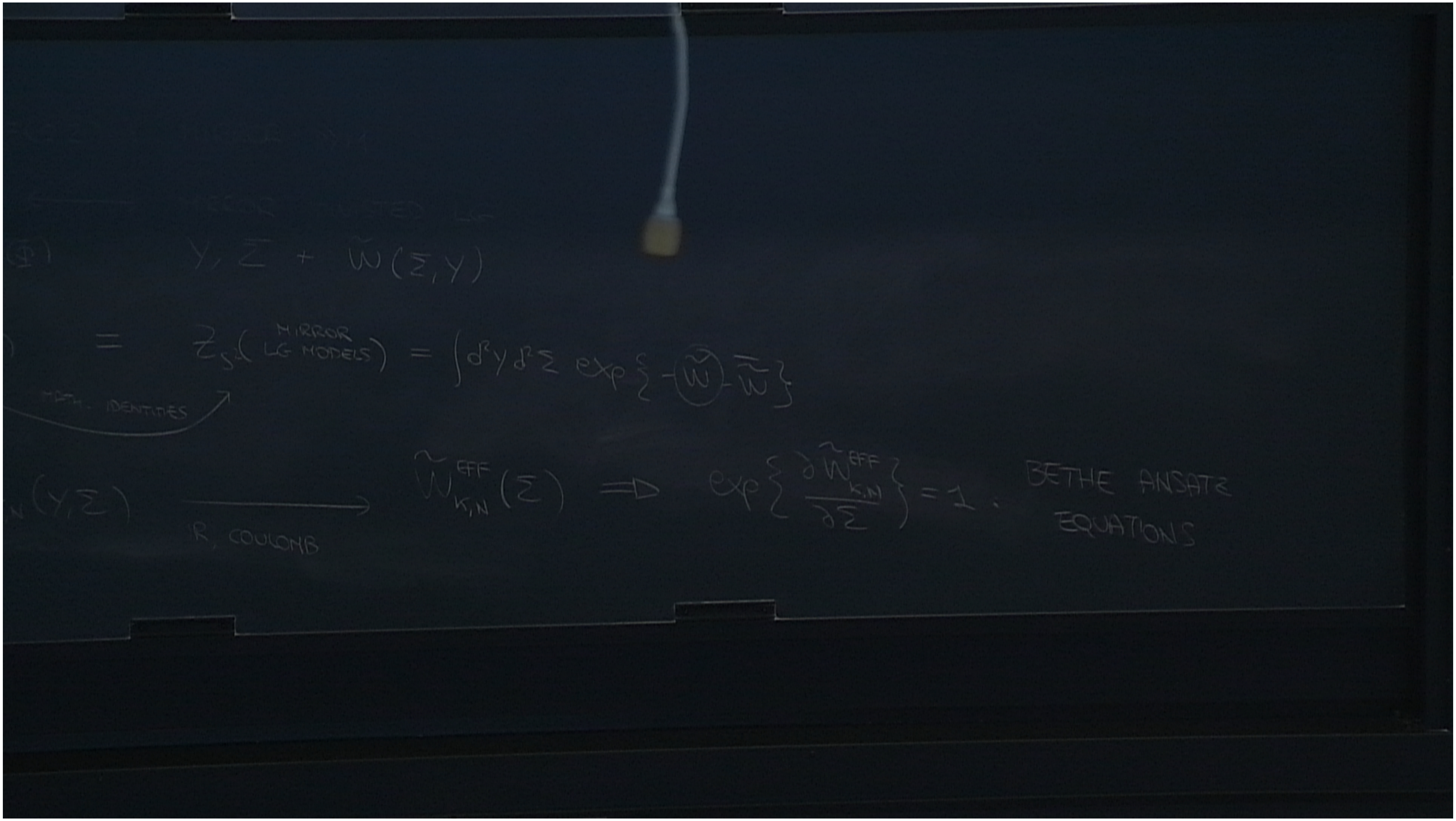
$M = \mathcal{M}_{K,N} = \{$ MODULI SPACE K INSTANTONS U(N) GAUGE THEORY $\}$

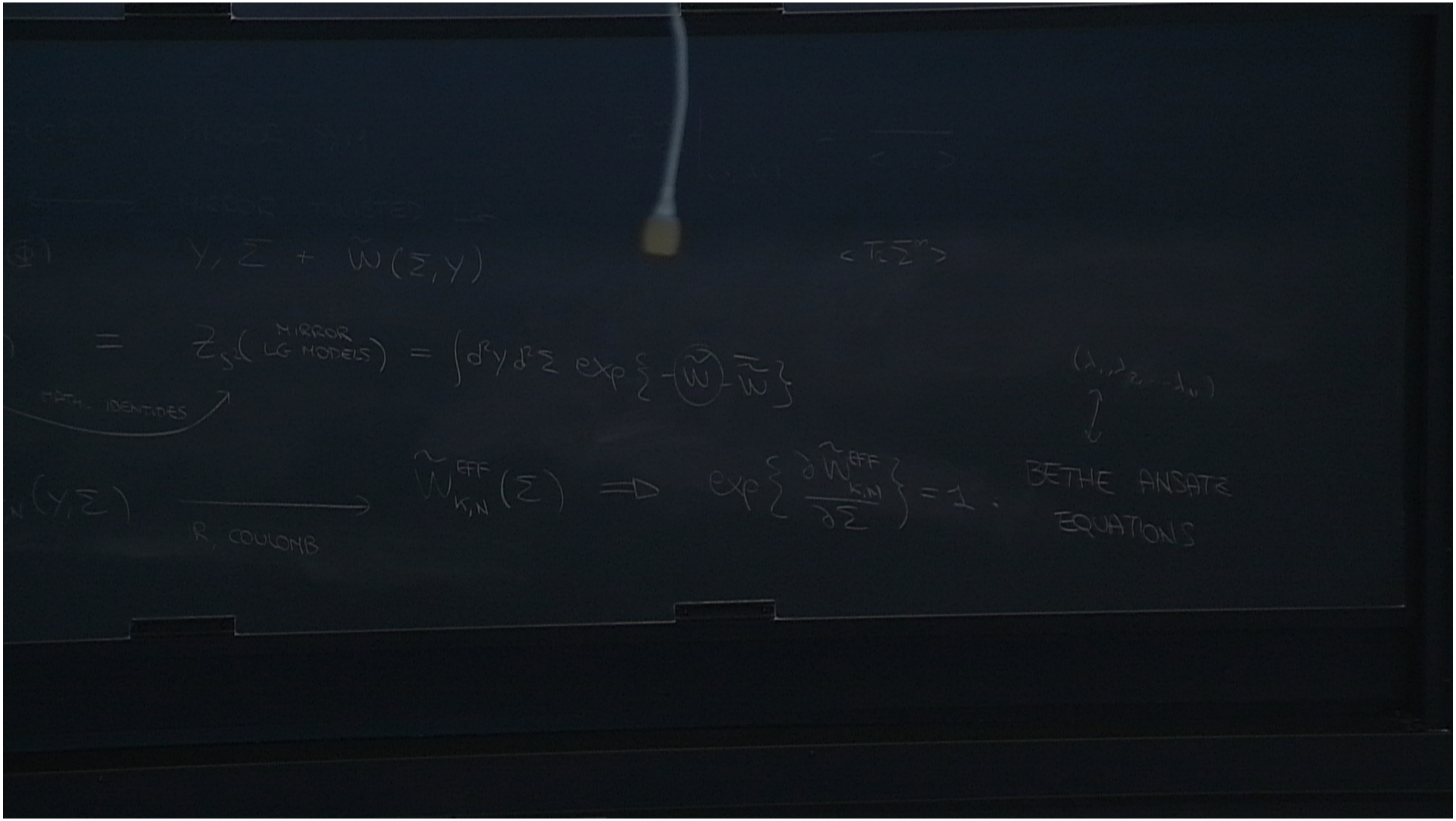
$Z_{S^2}(\text{GLSM}, M) = \exp\{-K_K(M)\}$ $\longrightarrow$ GENUS ZERO GW INVARIANTS M

•) EQUIVARIANT GW $\mathcal{M}_{K,N}$ $\nearrow$ $K_K(M) \longrightarrow F^{(0)}_{GW}$

$\searrow$ $Z_{S^2} = Z_{1L} Z_{V} Z_{AV}$, $Z_V \longleftrightarrow I$ GIVENTAL FUNCTION


V, Φ + W(Φ)
Y, Σ + W̃(Σ, Y)
Z_{S^2}(GLSM) = Z_{S^2}(MIRROR LG MODELS) = ∫ d²Y d²Σ exp{ −W̃ − W̃̄ }
MATH. IDENTITIES
⟹ W̃_{K,N}(Y, Σ)
R, COULOMB
W̃^{EFF}_{K,N}(Σ) ⟹ exp{ ∂W̃^{EFF}_{K,N} / ∂Σ }


MIRROR
LG MODELS
MATH. IDENTITIES
R, COULOMB
BETHE ANSATZ
EQUATIONS


MIRROR
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