Title: Quantum Gravity in 2 dimension and Liouville Theory

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Abstract: Gravity in 1+1 dimension is classically trivial but, as shown by A. Polyakov in 1981, it is a non-trivial quantum theory, in fact a conformal field theory (the Liouville theory), and also a string theory. In the last decades many important results and connexions with various areas of mathematics and theoretical physics have been established, but some important issues remain to be understood. In this colloquium I shall focus on some recent developments and new questions on the relation between discrete and continuous 2 dimensional gravity, probabilities and stochastic processes, random fractal geometries and SLE curves.

Quantum Gravity in 2 dimension and Liouville Theory

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Gravitation is classically trivial in 1+1 dimension

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0$$

The Einstein-Hilbert action is a topological invariant: the Euler characteristic

$$\int_{\mathcal{M}} \sqrt{|g|}\, R = 4\pi\, \chi = 8\pi(1 - h)$$

However, it is a non-trivial quantum theory, thanks to the Weyl anomaly !

2

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Path integral over 2D (Riemannian) metrics

$$\int_{\mathfrak{M}} \mathcal{D}[g] \; \exp\left(-\frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{|g|}(\mu + \gamma R)\right)$$

Invariance under the Diffeomorphisms of M.

This requires gauge fixing.

Polyakov 1981: use conformal gauge (uniformization theory of Riemann surfaces)

$$g_{\mu\nu}(x) = \Lambda(x)\,\hat{g}_{\mu\nu}(x)$$

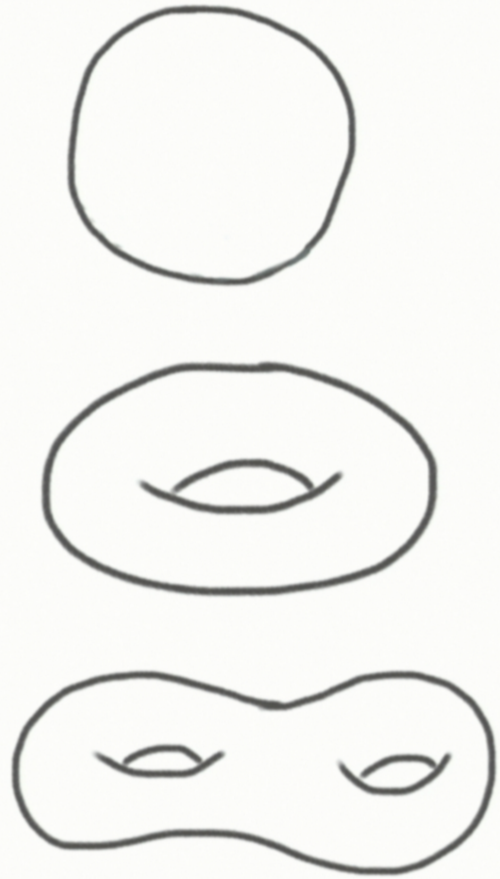
With $\hat{g}_{\mu\nu}(x)$ a fixed reference metric (up to moduli parameters).

Examples:

Sphere (g=0) $\hat{g}$ metric with constant curvature $\hat{R} = 2$

Torus (g=1) $\hat{g}_\tau$ flat metric
Depends on the aspect ratio τ
2 moduli

General Riemann curve (g>1) metric with constant curvature $\hat{R} = -2$
Depends on 3g-3 moduli



Gauge fixing involves a Faddev-Popov determinant

$$\int \mathcal{D}[g] \;=\; \int \mathcal{D}_g[\phi]\;\det[\nabla_g]\det[\overline{\nabla}_g] \;=\; \int \mathcal{D}_g[\phi]\,\mathrm{e}^{-S_{\mathrm{eff}}[\phi]}$$

The effective action depends of the metric, hence of the conformal factor ϕ through the Weyl anomaly and the trace of the E-M tensor for the **bc** ghost system.

$$\frac{\delta S_{\mathrm{eff}}[\phi]}{\delta\phi} = \langle T_{\mathrm{bc}}{}^{\mu}_{\mu}\rangle_g = -\frac{(-26)}{24\pi}R = \frac{26}{24\pi}\mathrm{e}^{-\phi}(-\Delta\phi + \hat{R})$$

We obtain a quantum theory for the ghost fields **bc**, the conformal factor ϕ (now treated as quantum field), plus the other quantum fields coupled to the metric (matter fields).

This effective theory is the Liouville theory

Its action is

$$S_{\mathrm{Liouville}}[\varphi] = \frac{1}{4\pi}\int_M \sqrt{\hat{g}}\,\left((\partial\varphi)^2 + Q\hat{R}\,\varphi + \mu\,\mathrm{e}^{\gamma\varphi}\right)$$

the “quantum” metric is $ds^2 = \mathrm{e}^{\gamma\varphi(z,\bar{z})}\,dz d\bar{z}$

with (conformal invariance) $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$

In the classical limit $\gamma \ll 1$ the equations of motion are the “Liouville equations”

$$R + \mu = 0$$

with *R* the curvature $R = \mathrm{e}^{-\gamma\varphi}\left(-\gamma\Delta\varphi + \hat{R}\right)$

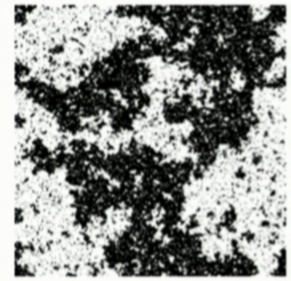
and μ the “cosmological constant”

The Liouville theory is a Conformal Theory (CFT)

$$z \to w = \frac{az + b}{cz + d}$$

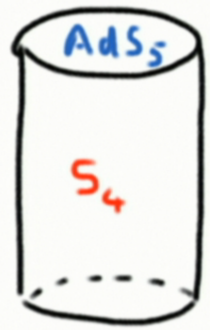
Critical systems 2D (Ising)

Scale+Rotation invariance (+ Unitarity)



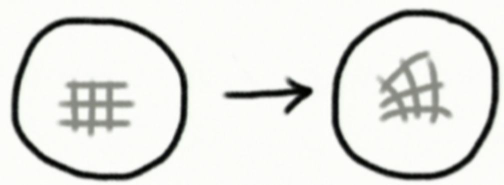
AdS/CFT

Symmetries of AdS space



Liouville

Remnant of diffeomorphims invariance
after conformal gauge fixing



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Liouville is a CFT

central charge $c_{\text{Liouville}} = 1 + 6Q^2$

Weyl anomaly consistency condition (Diff. invariance) $c_{\text{Liouville}} + c_{\text{ghost}} + c_{\text{matter}} = 0$

$$Q^2 = \frac{25 - c_{\text{matter}}}{6}$$

Transformation law of the field

$$z \to w(z) \qquad \varphi(z) \to \tilde{\varphi}(w) = \varphi(z) + Q \log \left| \frac{\partial z}{\partial w} \right|$$

but it is a non-rational CFT, it is much more difficult to study than “ordinary” minimal or rational (and probably the logarithmic) CFT

Vertex operators

the vertex operator $V_\alpha(x) = \exp(\alpha\,\varphi(x))$

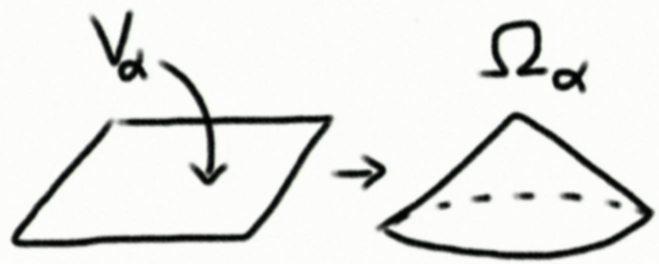
has conformal dimension $\Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$

in particular the interaction term has dimension $\Delta_\gamma = 1$

but $\mu \int \exp(\gamma\varphi)$ is not a "mass term"

Insertion of a vertex operator amounts to insert a curvature singularity (conical point) on the surface with

$$\Omega_\alpha = 2\pi(1 - \alpha/Q)$$



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N-point functions and Seiberg bounds

The sphere is not a solution of Liouville equation !

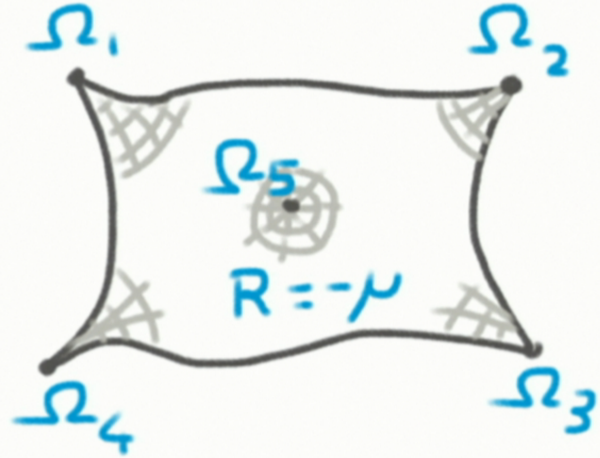
$$Z_{\text{sphere}} = 0$$

But correlation functions $\langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) \cdots V_{\alpha_N}(x_N) \rangle$

make sense, provided they satisfy the Seiberg bounds

$$\alpha_i < Q$$

$$\sum_i \alpha_i > 2Q$$



Liouville theory describes (non-critical) strings

2D gravity = string in a linear Dilaton background

$$\frac{1}{4\pi}\int \sqrt{\hat{g}}\, \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}(X) + T(X) + \hat{R}\,\Phi(X)$$

view Liouville field φ as X^0 coordinate ("time")

$\Phi(X) = QX^0$ linear dilaton background

$T(X) = \mu \exp(X_0)$ exponential tachyon background

Many developments since 30 years…

For instance, the 2004 review by Nakayama contains approx. 500 references. Many more now…

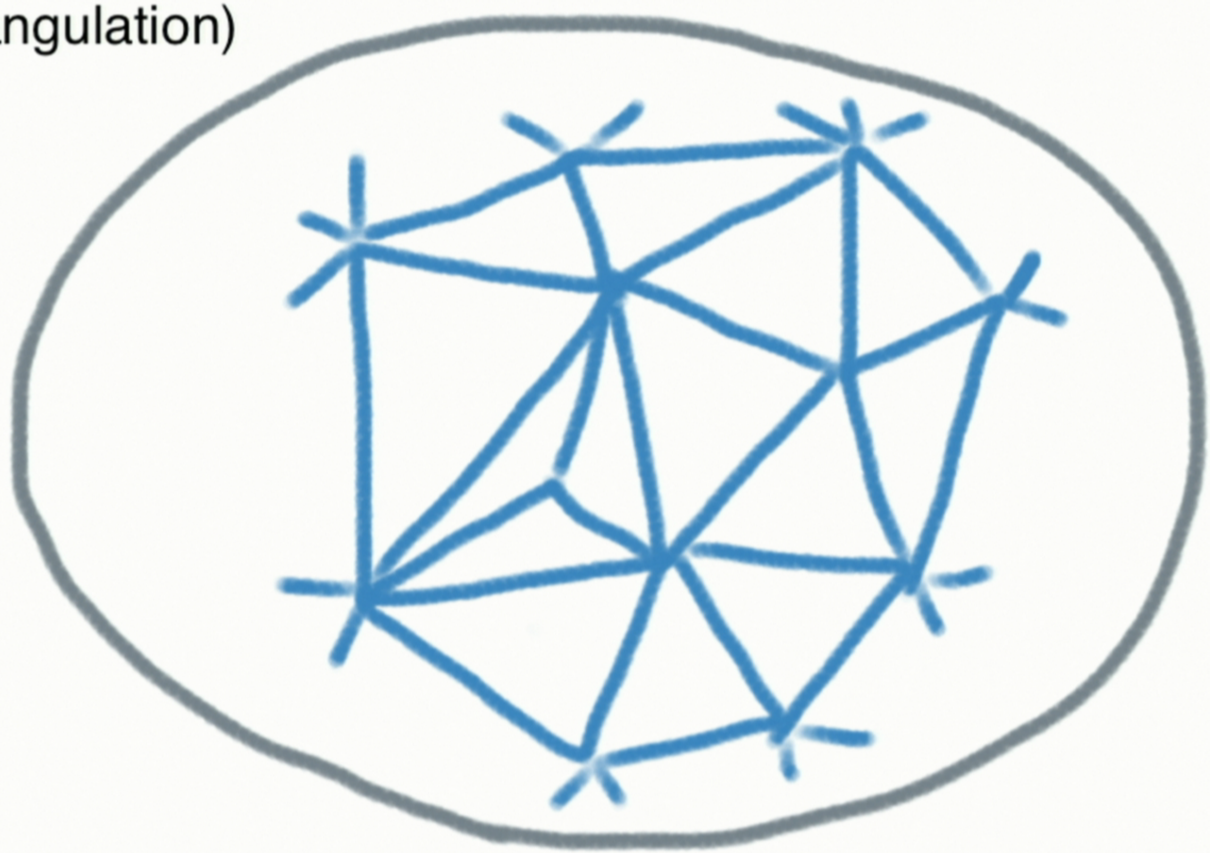
Especially in view of:

Liouville theory in the disk and branes in string theory (FZZT-branes and ZZ-branes)

AGT conjecture (relation between Liouville in 2D and supersymmetric gauge theories in 4D)

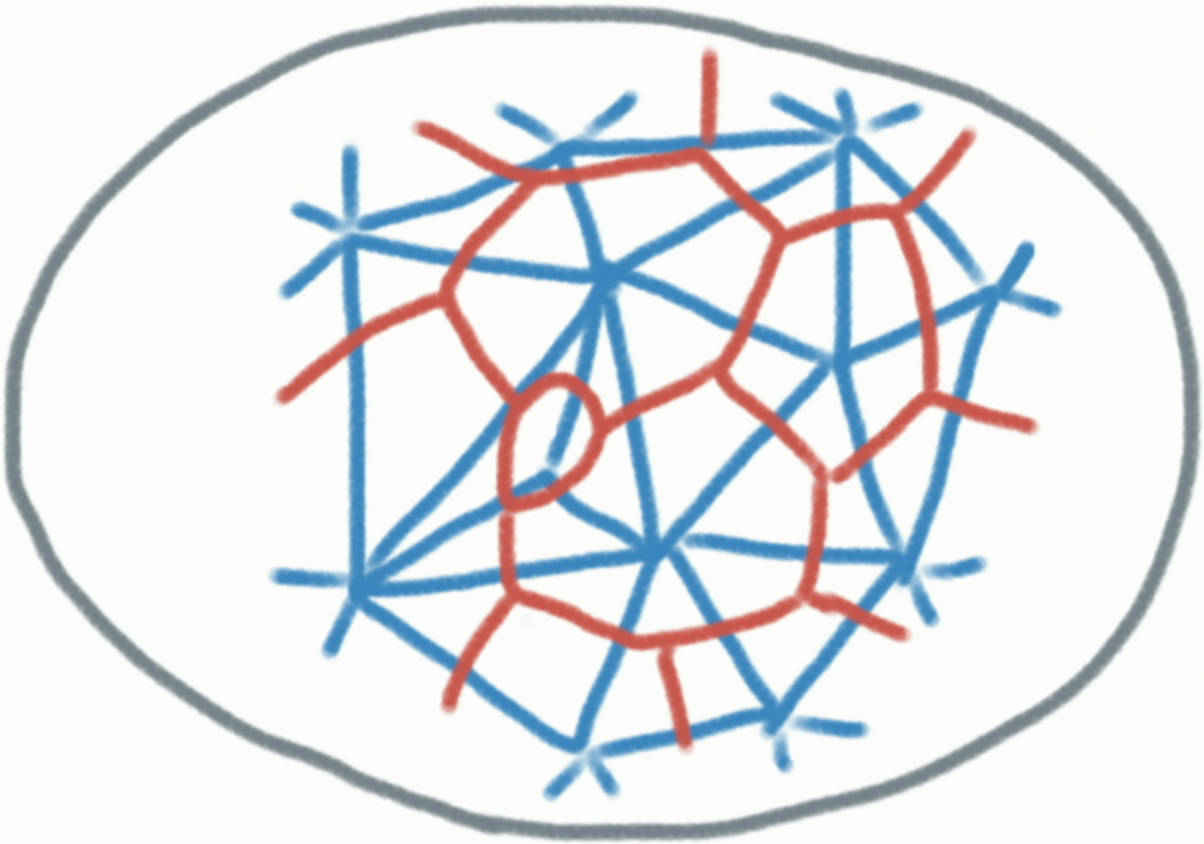
Discrete 2D gravity

Discretized 2D geometry = random maps (here a planar triangulation)



Discrete 2D gravity

The dual is a planar graph (here trivalent)



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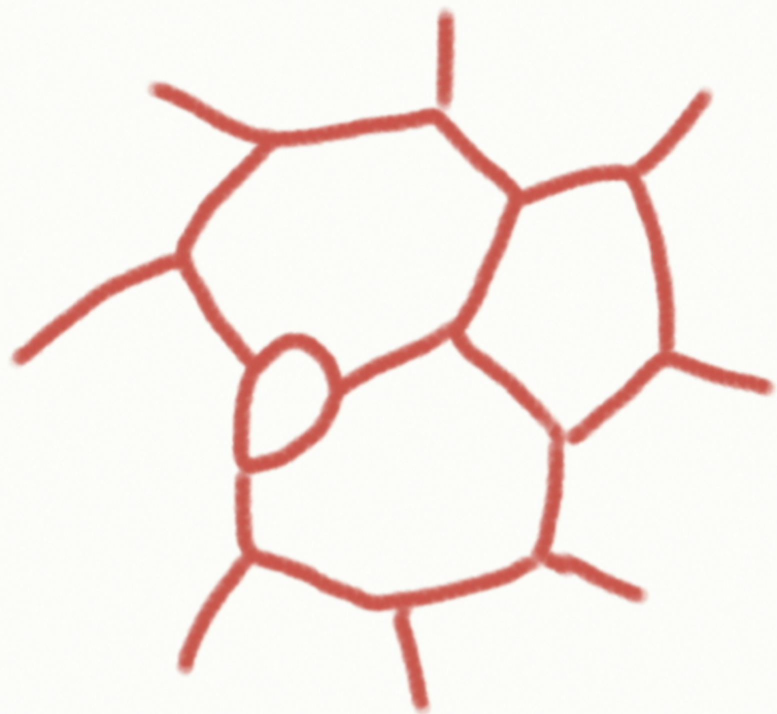
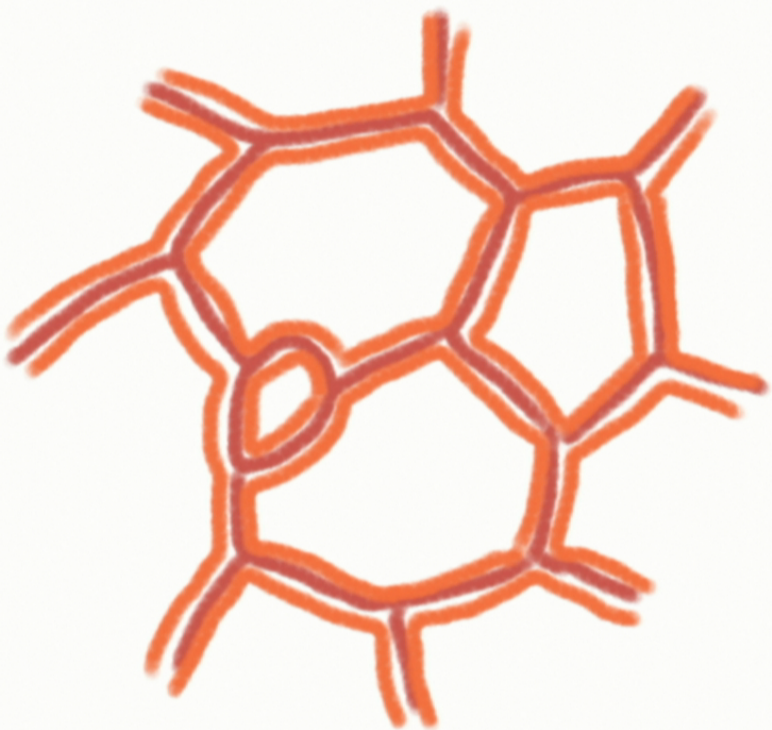


Discrete 2D gravity

The dual is in fact a planar fat graph



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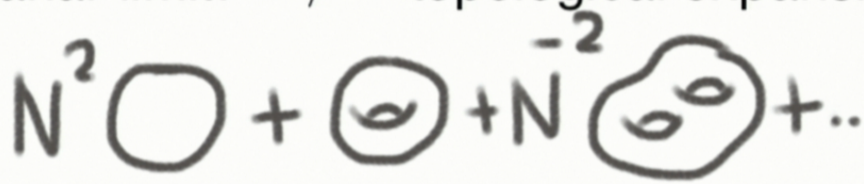
Combinatorics of planar maps

By combinatorics: Tutte recurrence equations

By matrix models: generating functions $N \times N$ matrix

$$\int_{M=M^\dagger} dM \ \exp(-N \ \mathrm{Tr}(M^2 - gM^3))$$

’t Hooft planar limit: $1/N^2$ topological expansion



critical point: $g \to g_c$ large maps and continuum limit

double scaling limit: the topological expansion is an expansion in $z = N^{-2}(g_c - g)^{-2+\gamma_s}$ and it corresponds to non-perturbative solution of string theory

Discrete 2D gravity

The dual is in fact a planar fat graph

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Discrete 2D gravity

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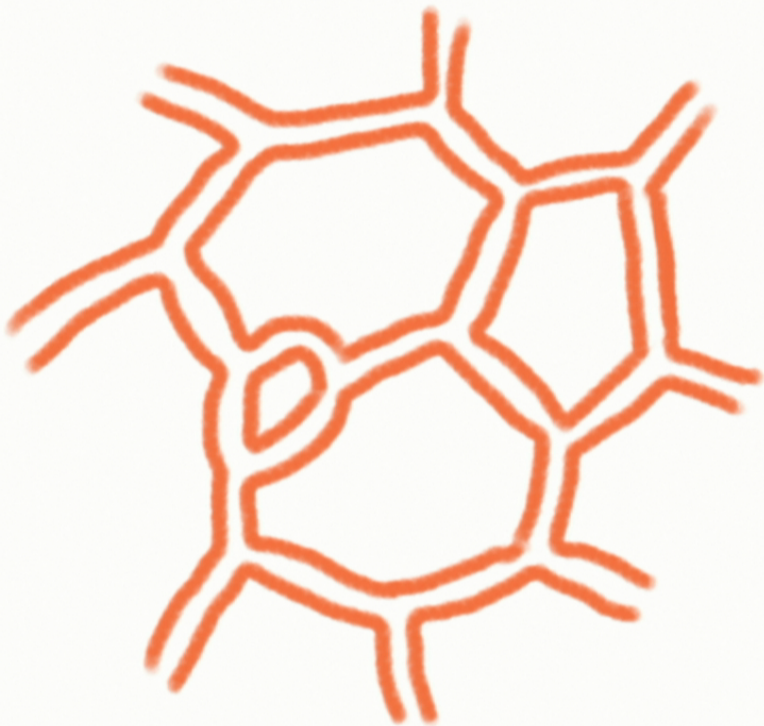

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W. Tutte

G. 't Hooft

É. Brézin, C. Itzykson, G. Parisi, J.-B. Zuber

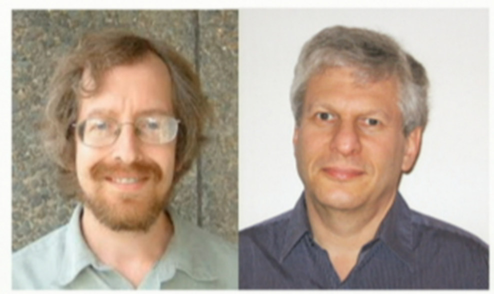
V. Kazakov

F. David

J. Ambjørn, B. Durhuus, J. Fröhlich

E. Brézin, V. Kazakov

D. Gross, A. Migdal



M. Douglas, S. Shenker

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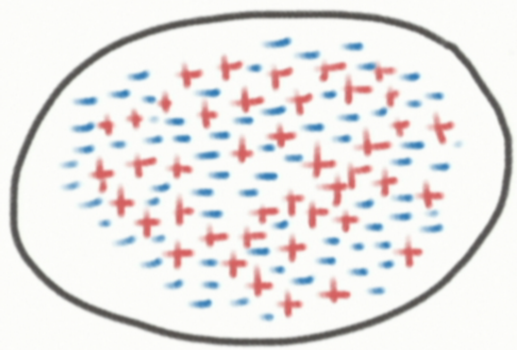
KPZ scaling laws

Consider some matter CFT (Ising, O(n) model) could to 2D gravity.

If some local operator (spin, energy) has dimension x_0 in the CFT, it has dimension x in gravity+CFT theory

$$x_0 = x + \frac{\gamma^2}{4} x(x-1)$$



The “string exponent” is

$$\gamma_s = 1 - \frac{4}{\gamma^2}$$

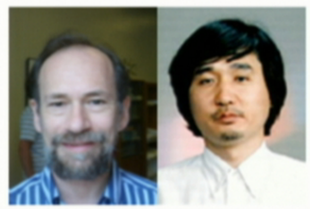
Proven by CFT & Liouville - *basically* $\Delta_\alpha = \frac{\alpha}{2}\left(Q - \frac{\alpha}{2}\right)$

Knizhnik-Polyakov-Zamolodchikov ‘88

FD ’88 , Distler-Kawai ‘89

V. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov

F. David



J. Distler, H. Kawai

some guys to be associated to KPZ, later on....

NOT THIS KPZ !

Many evidences for equivalence Liouville - Matrix models

Explicit calculations for many models

- Same scaling dimensions
- Same recursion relations
- Also relation with Kontsevich model and topological gravity (geometry of moduli space of Riemann surfaces)

However…

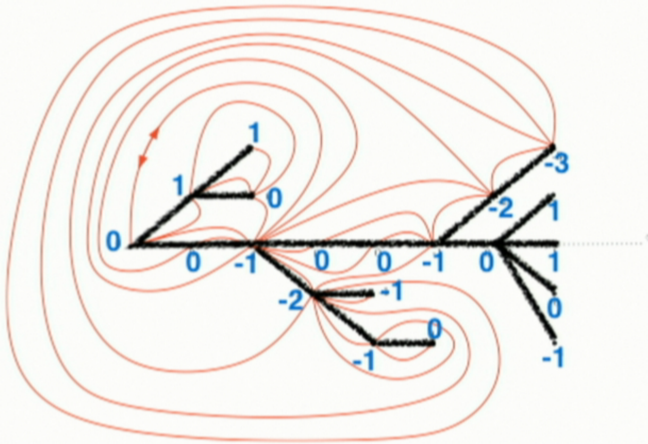
No direct construction and complete understanding of:

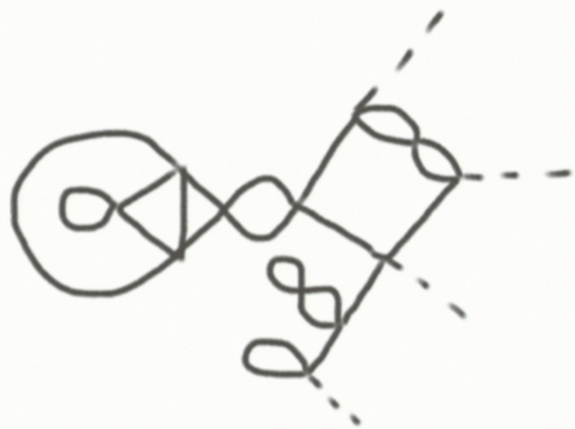
- discrete conformal map + continuum limit
- conformal structure (measure) versus Riemannian structure (distance metric and topology) except for pure gravity $c_{\mathrm{matter}} = 0\ ,\ \gamma = \sqrt{8/3}$

Distance geometry: random maps & random trees

There are bijections between planar maps and labelled trees *Cory-Vauquelin, Schaeffer, …. Ambjørn-Kawai, …*

This gives access to some metric information (distances)





$d_{\text{spectral}} = 2$ random walk on a quantum metric not that different from the classical one

$d_{\text{Hausdorff}} = 4$ geometry of geodesics on a quantum metric is very different

Local limit of Liouville: the Gaussian Free Field (GFF)

"Locally", Liouville CFT "looks like" the Gaussian Free Field

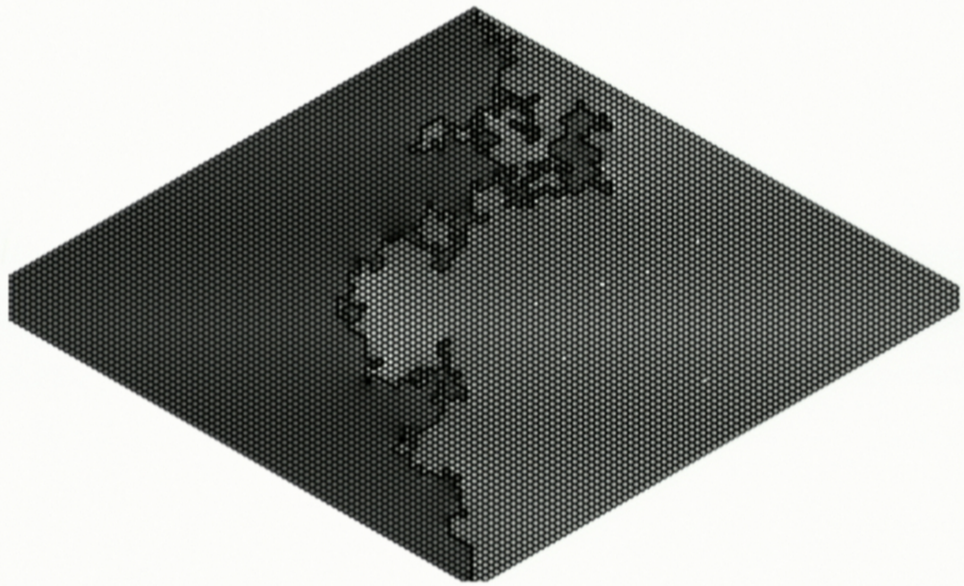
$$S = \frac{1}{4\pi}\int d^2 z\,(\partial\varphi)^2$$

The GFF is already an interesting object

example:

its level-lines are fractal and related to curves created by a SLE process, here SLE_4

Schramm-Sheffield



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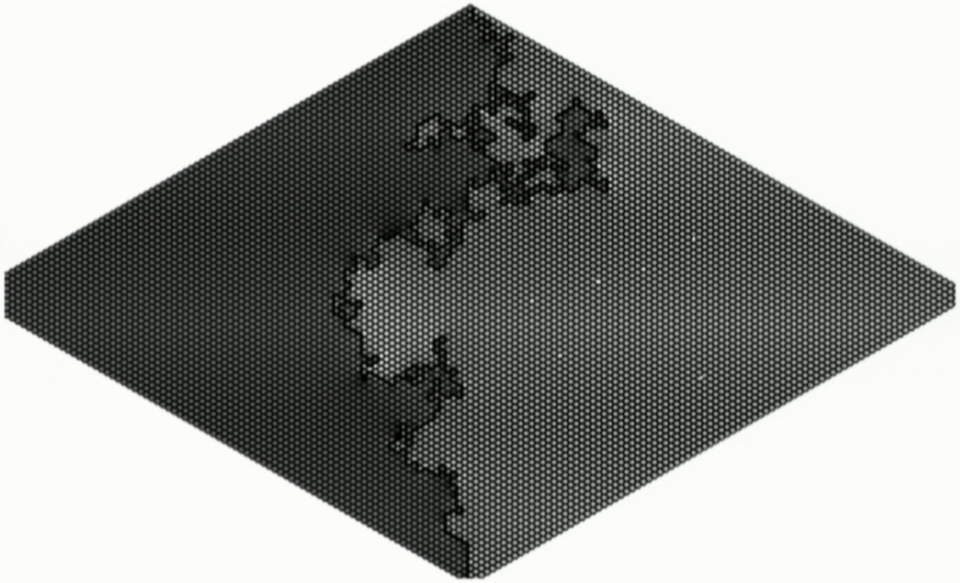


Schramm-Sheffield

Stochastic/Schramm Loewner Evolution (SLE) process

SLE is a curve growth process which has two properties:

1. Conformal invariance
2. Domain Markov property

In the upper half plane (chordal SLE) it is defined from the stochastic equation

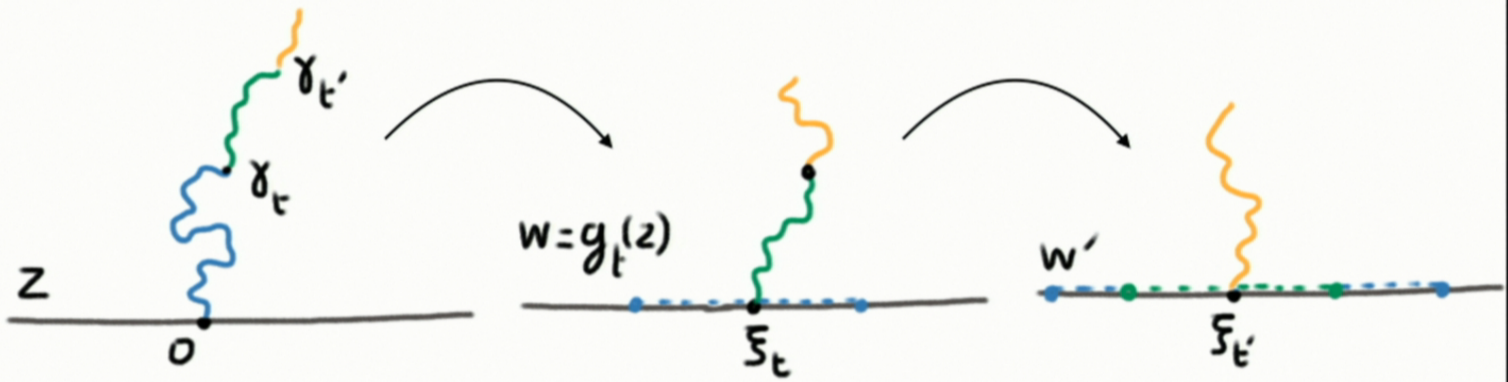
$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t} \qquad \xi_t = \sqrt{\kappa}\, B_t \text{ Brownian process}$$

This defines a conformal mapping from the upper-half-plane minus a curve onto the upper-half-plane

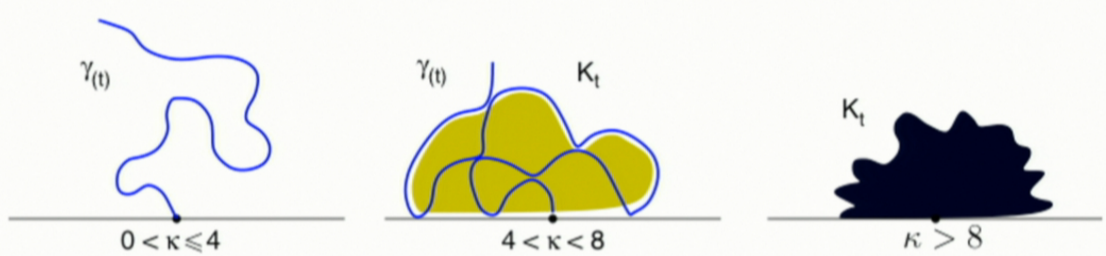
This curve is the SLE curve

O. Schramm '99, ... ,Lawler, Schramm & Werner '01

Some properties of SLE



This seemingly simple process is highly non-trivial



$d_H = 1 + \kappa/8$

Hausdorff dimension

$d_H = 2$

space-filling continuous curve

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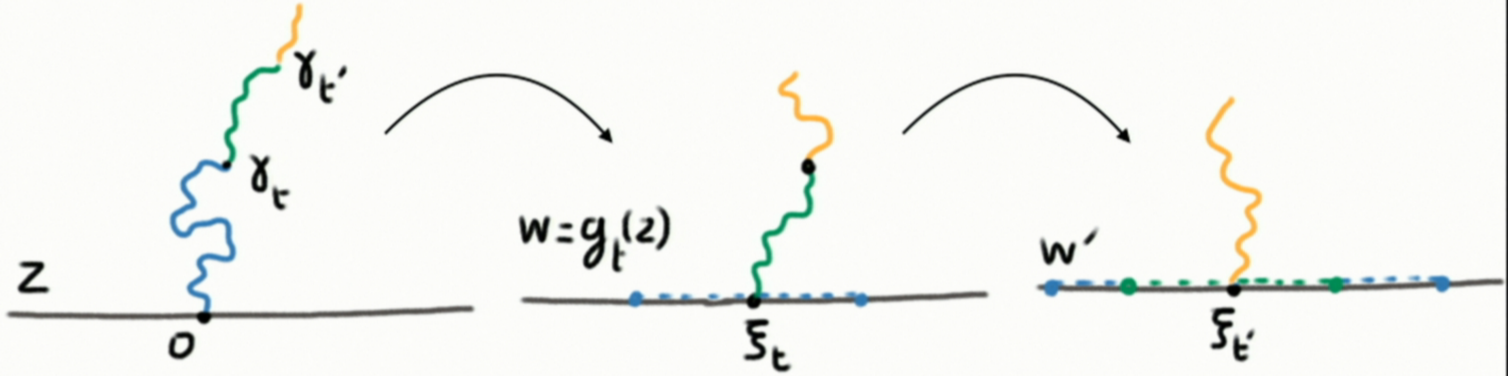
Some properties of SLE



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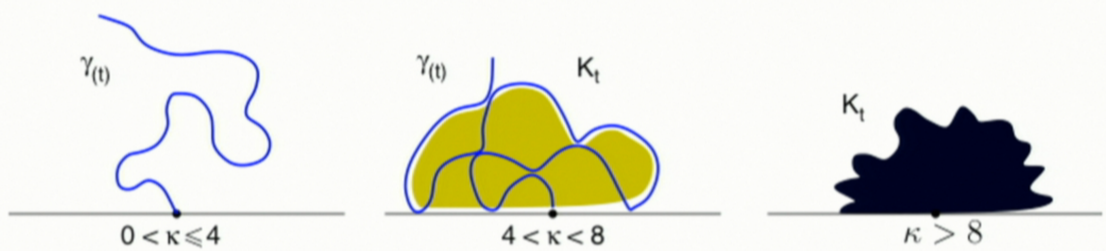

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SLE, Interfaces in CFT and KPZ

SLE has revolutionised our understanding of the multifractal properties of interfaces and clusters in 2D critical systems and CFT.

The SLE_κ curve describes the interface of a critical model in the plane whose continuum limit is a CFT with central charge c with a simple relation between c and κ.

This relation is nothing but a KPZ relation for Liouville+CFT

$$Q^2 = \left(\frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}\right)^2 = \frac{25-c}{6}$$

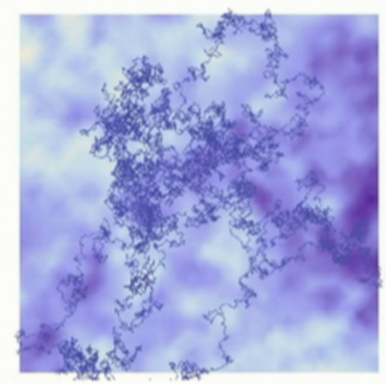
This suggest a deep relation between SLE and Liouville

Geometric KPZ & the GFF

A geometric version of the KPZ relation holds: it relates the fractal dimension of an object in the classical flat measure $\mu_0(dz) = d^2z$ and its fractal dimension in the "quantum" measure $\mu(dz) = d^2z\ \exp(\gamma\varphi)$ with φ a GFF

This requires tools of probability theory (*Kahane multiplicative chaos, etc.*), or studying random walks in a quantum metric.

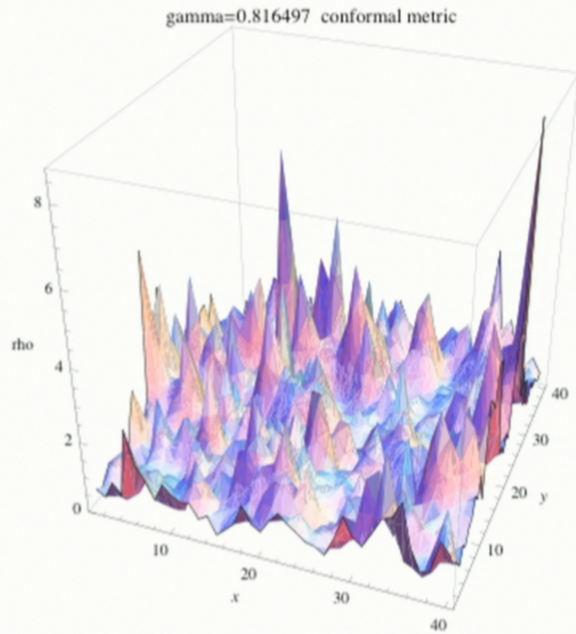
$$d_0 = d + \frac{\gamma^2}{8} d(2-d)$$

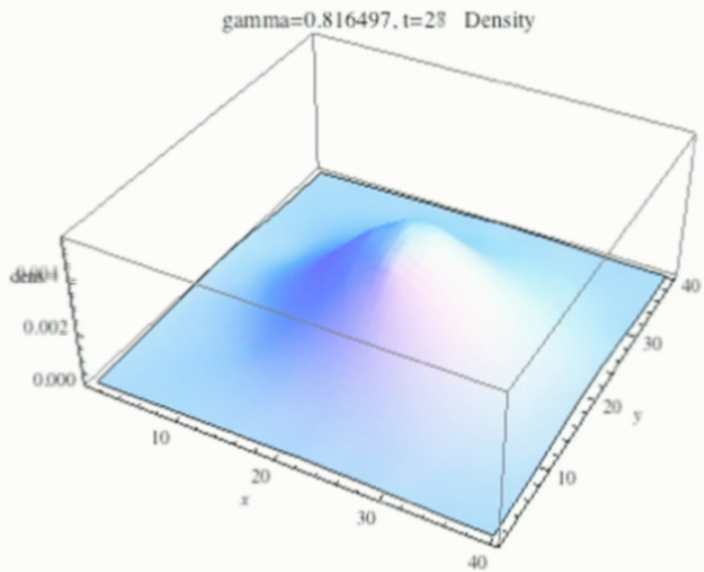


Duplantier-Sheffield, Benjamini-Schramm, Rhodes-Vargas (see also Bauer-David)

Diffusion in random metric

c=−24, κ=2/3 *(Liouville at weak coupling)*





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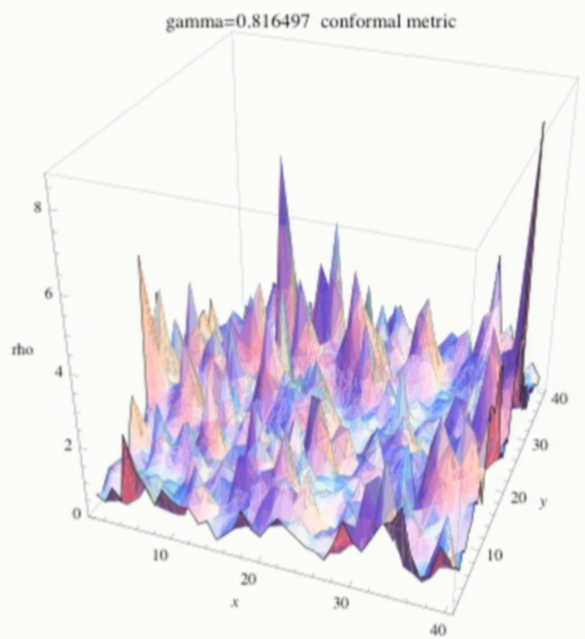

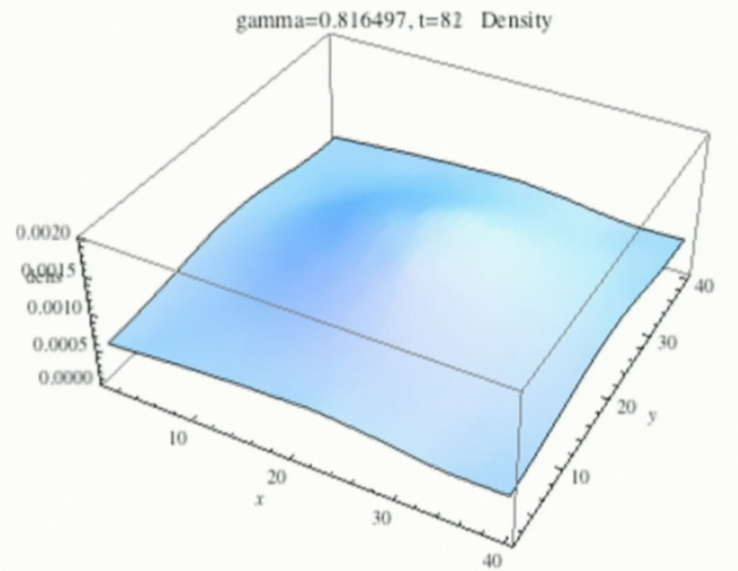

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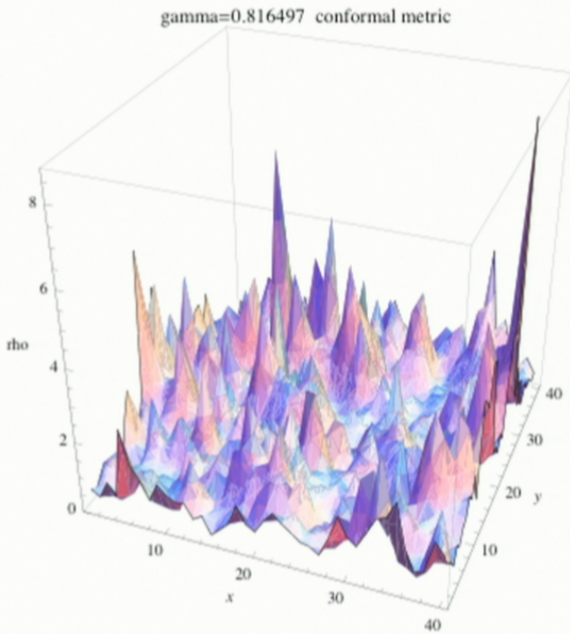
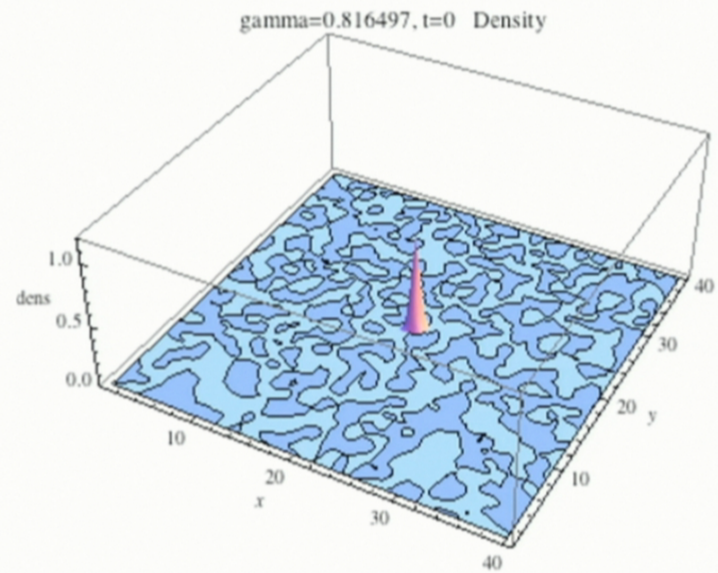
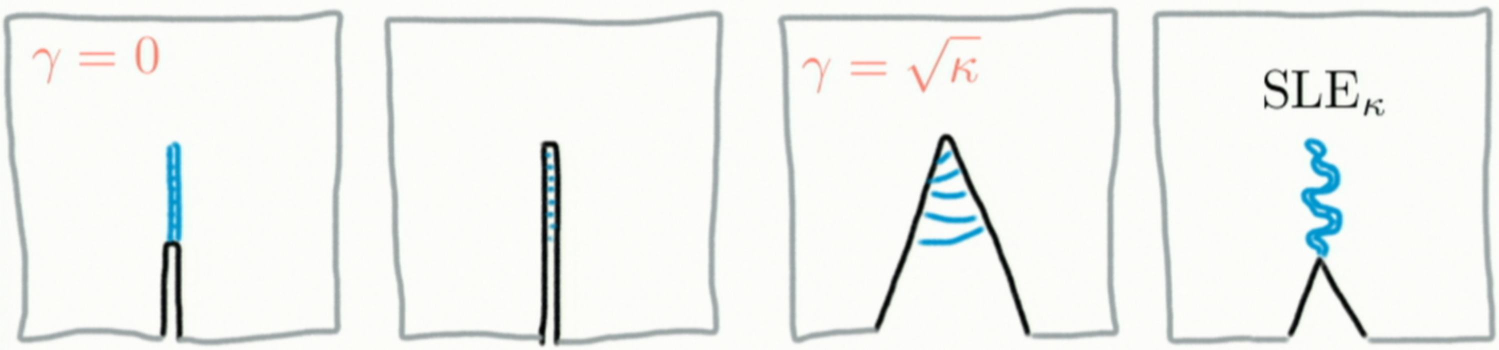




SLE/Liouville: Conformal Welding

One can construct SLE out of Liouville (in its GFF avatar) by "conformally gluing" pieces of quantum surfaces with the quantum metric $ds^2 = e^{\gamma\varphi(z,\bar{z})}\, dz d\bar{z}$ with φ the GFF with free boundary conditions



"classical welding"
flat metric

"quantum welding"
quantum metric

Sheffield, Duplantier-Sheffield (see also Dubedat, Kupiainen et al.)

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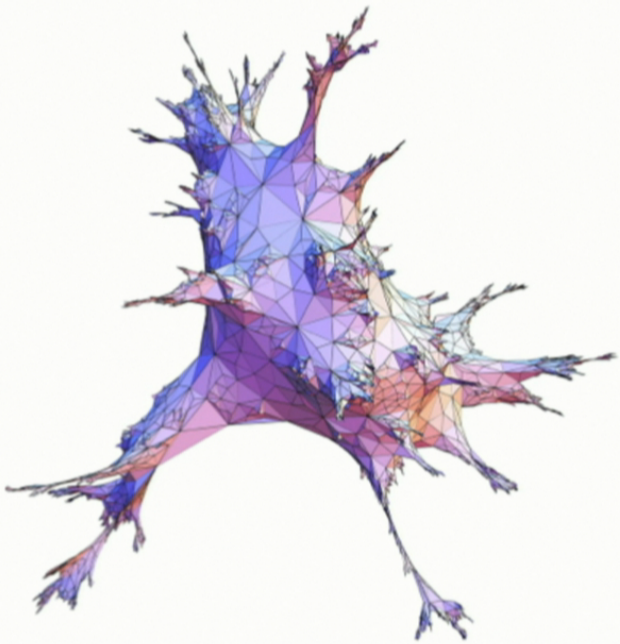
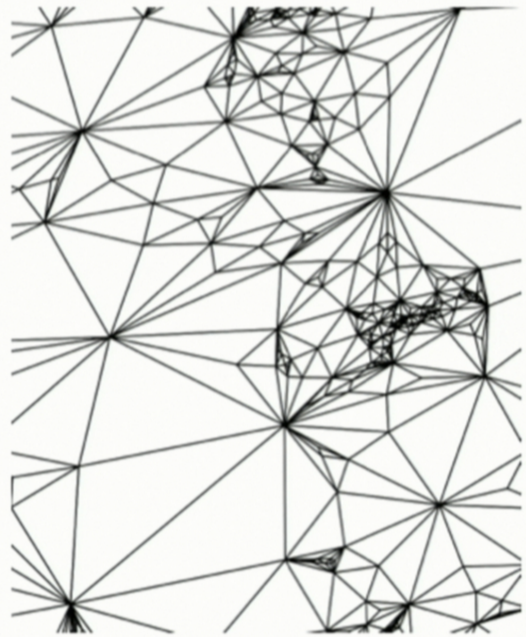
Discrete Conformal & Quasiconformal Maps

How do we embed a random map onto the plane/sphere, preserving its conformal structure?

Exact uniformization *(Curien)*





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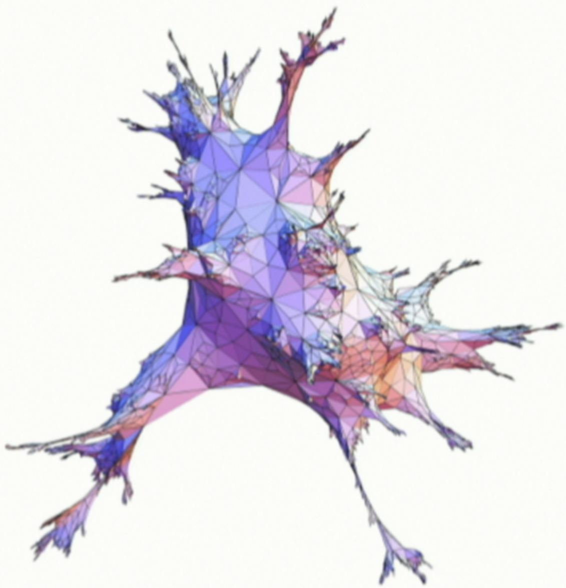
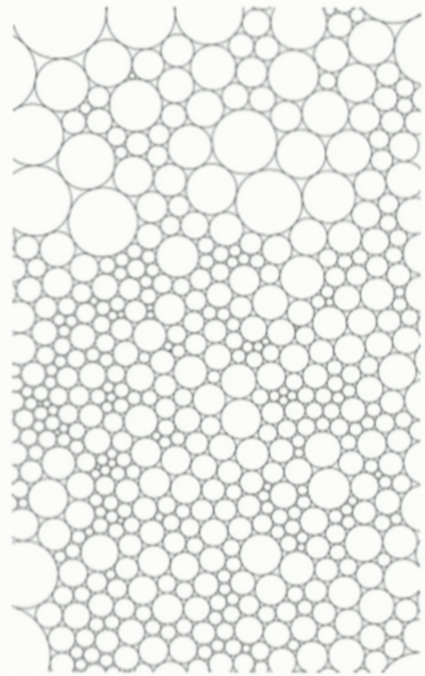
Circle packings? *(Folklore, but few results...)*





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Discrete Conformal & Quasiconformal Maps

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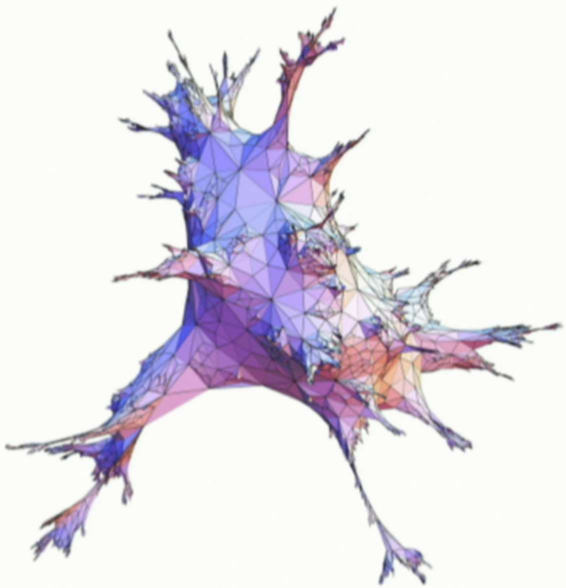
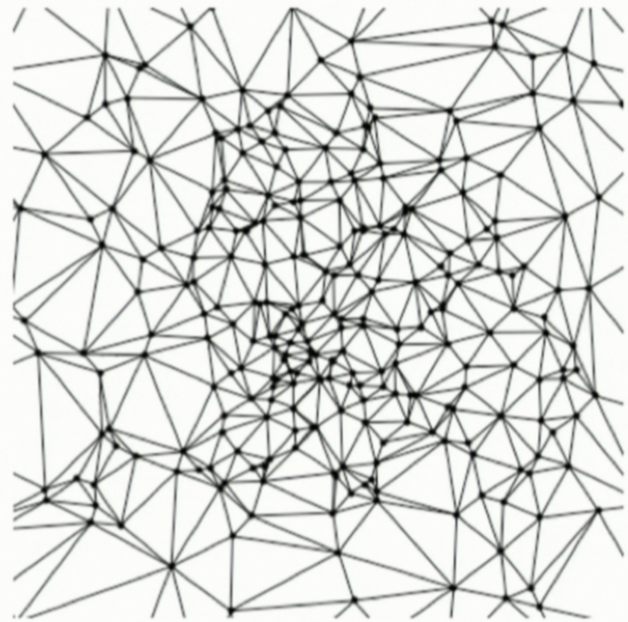
Delaunay circle patterns? *(David-Eynard)*

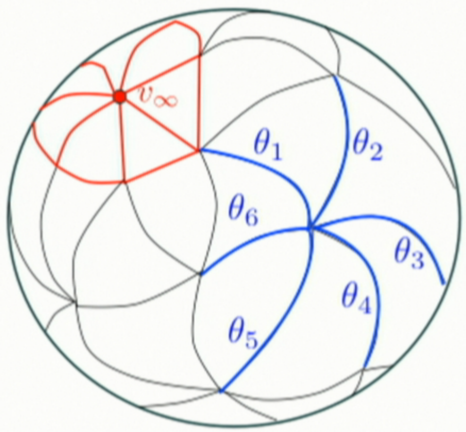




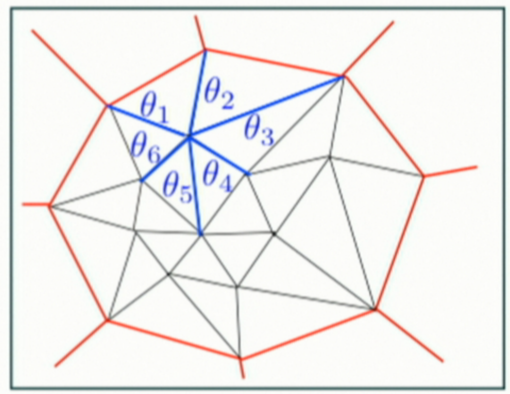
There is a bijection between

triangulations+edge angles
(some truncation of the moduli space of the punctured sphere
$\mathcal{M}_{0,N}$

Delaunay triangulations on the plane
Random distribution on N points on the plane, with non-trivial measure



Sphere



Plane

There is a bijection between

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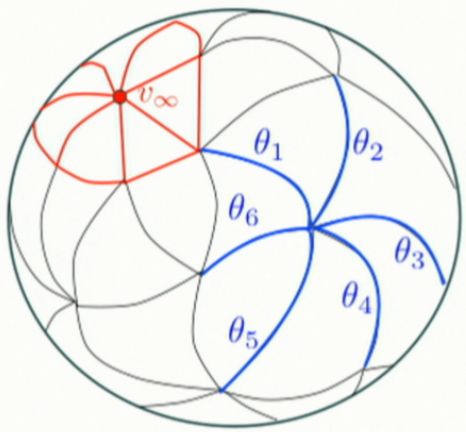
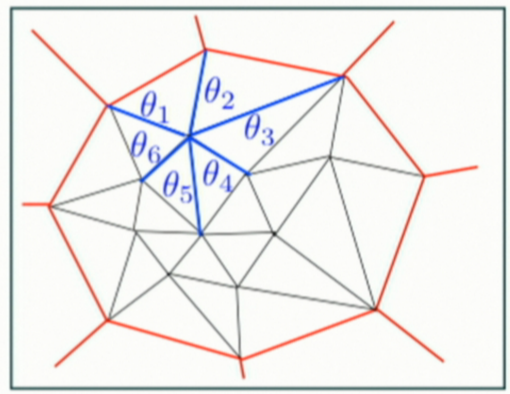
(some truncation of the moduli space of the punctured sphere $\mathcal{M}_{0,N}$

Delaunay triangulations on the plane
Random distribution on N points on the plane, with non-trivial measure



Sphere



Plane

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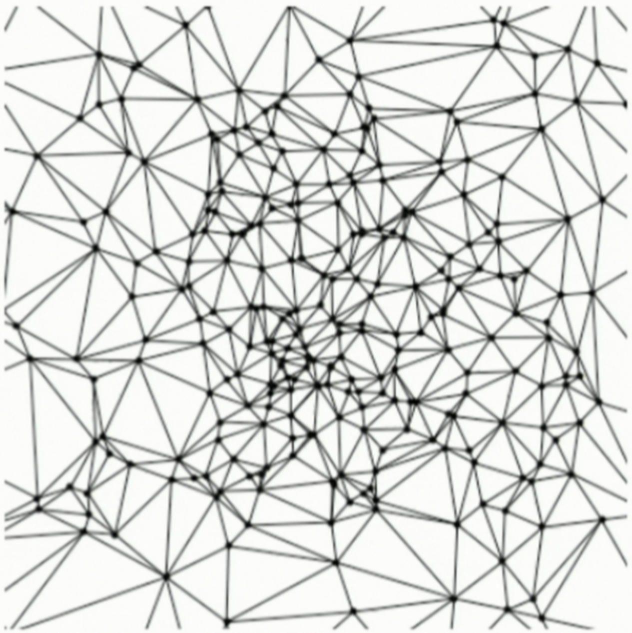
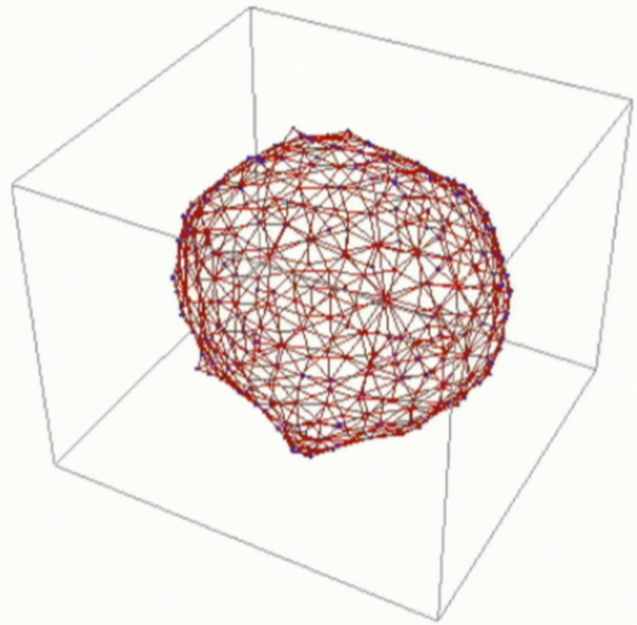
The measure over points is very singular but integrable !
One expects large fluctuations of the density of points at all scales, consequence of conformal invariance





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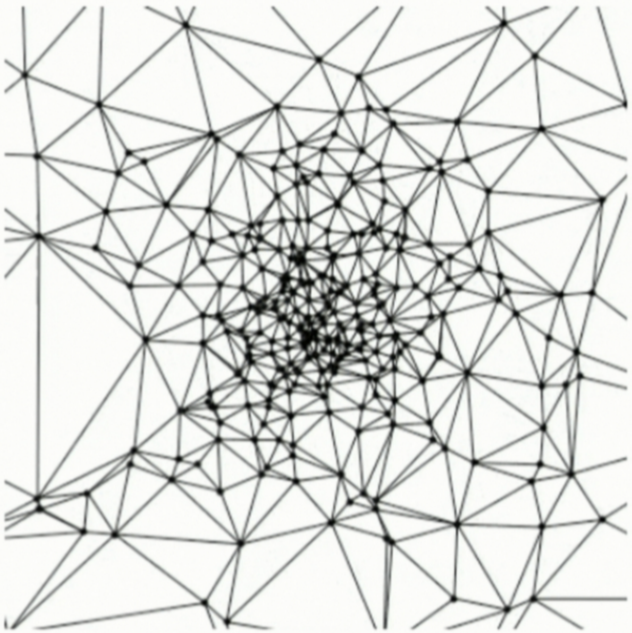

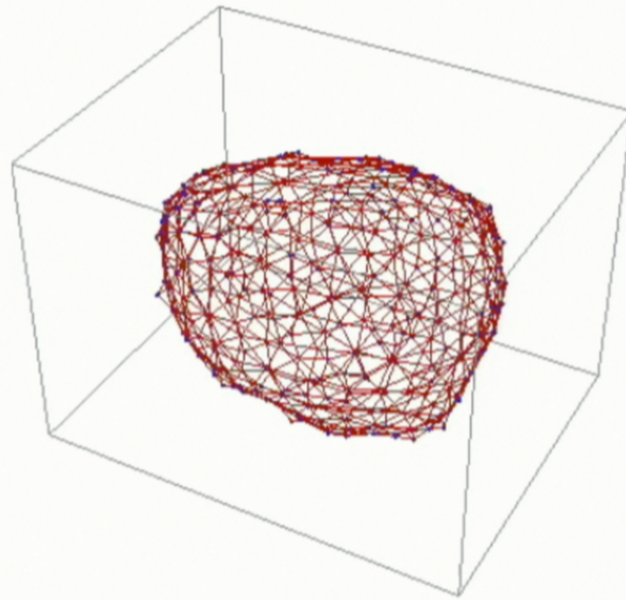

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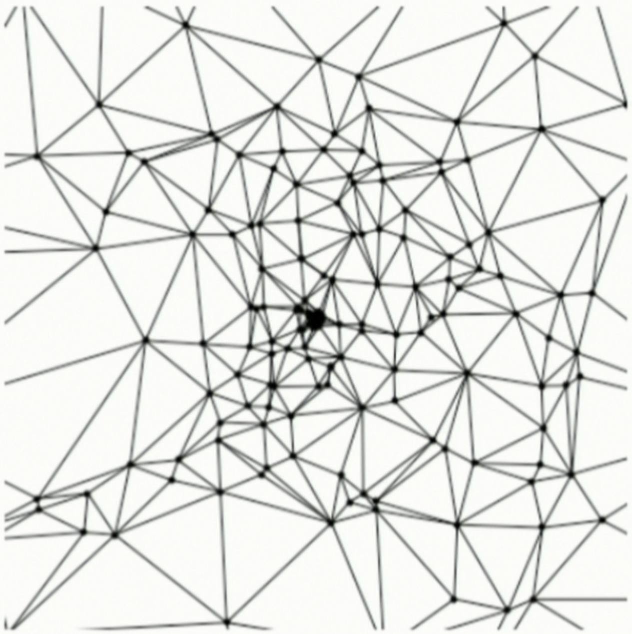

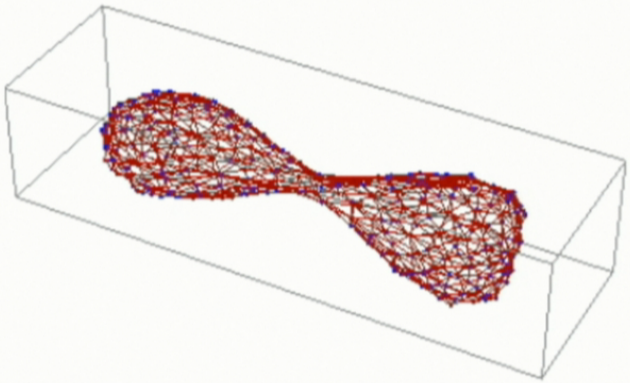

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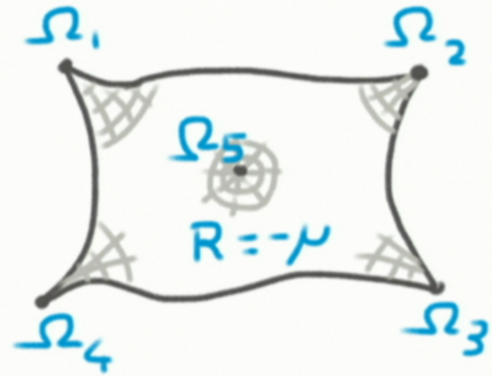
From the GFF to Liouville theory: the probabilistic track

Recent result by *F.D., A. Kupiainen, R. Rhodes & V. Vargas*

A rigorous probabilistic construction of the functional integral and of the correlation functions of the full Liouville theory on the sphere (as a first step…)

$$\langle V \cdots V \rangle_{\mathrm{Liouville}} = \mathbb{E}\left[\exp\left(-\frac{1}{4\pi}\int Q\hat{R}\varphi + \mu\ \exp(\gamma\varphi)\right) V \cdots V\right]_{\mathrm{GFF}}$$

and check of its conformal invariance properties of the KPZ relations for the full fledged Liouville theory.



Liouville Quantum Gravity on the Riemann sphere

François David *, Antti Kupiainen †, Rémi Rhodes ‡, Vincent Vargas §

Tuesday 28^{th} October, 2014

Abstract

In this paper, we rigorously construct $2d$ Liouville Quantum Field Theory on the Riemann sphere introduced in the 1981 seminal work by Polyakov **Quantum Geometry of bosonic strings**. We also establish some of its fundamental properties like conformal covariance under $PSL_2(\mathbb{C})$-action, Seiberg bounds, KPZ scaling laws, KPZ formula and the Weyl anomaly (Polyakov-Ray-Singer) formula for Liouville Quantum Gravity.

Key words or phrases: Liouville Quantum Gravity, Gaussian multiplicative chaos, KPZ formula, KPZ scaling laws, Polyakov formula.
MSC 2000 subject classifications: 81T40, 81T20, 60D05.

Contents

Thank you!