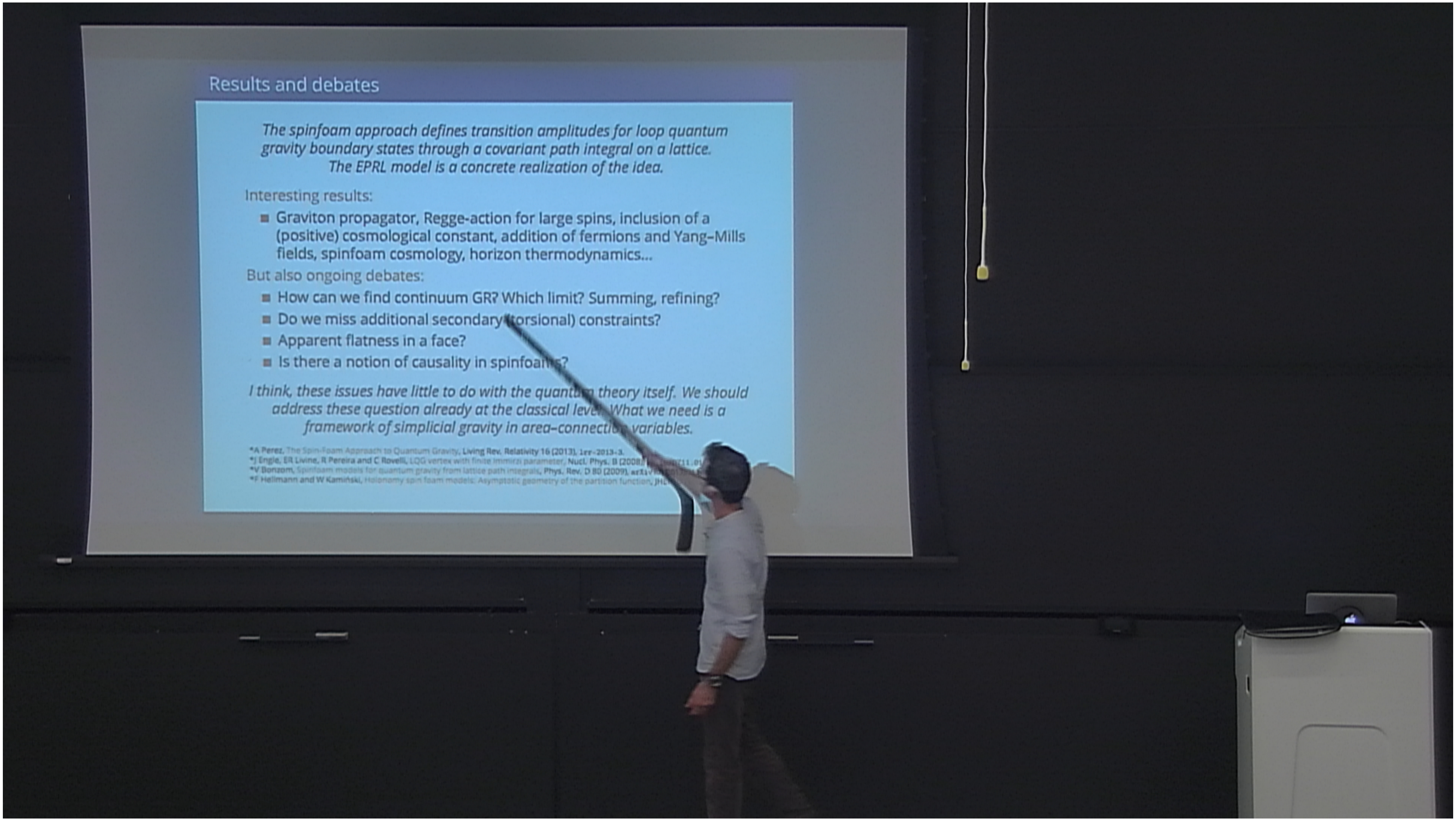
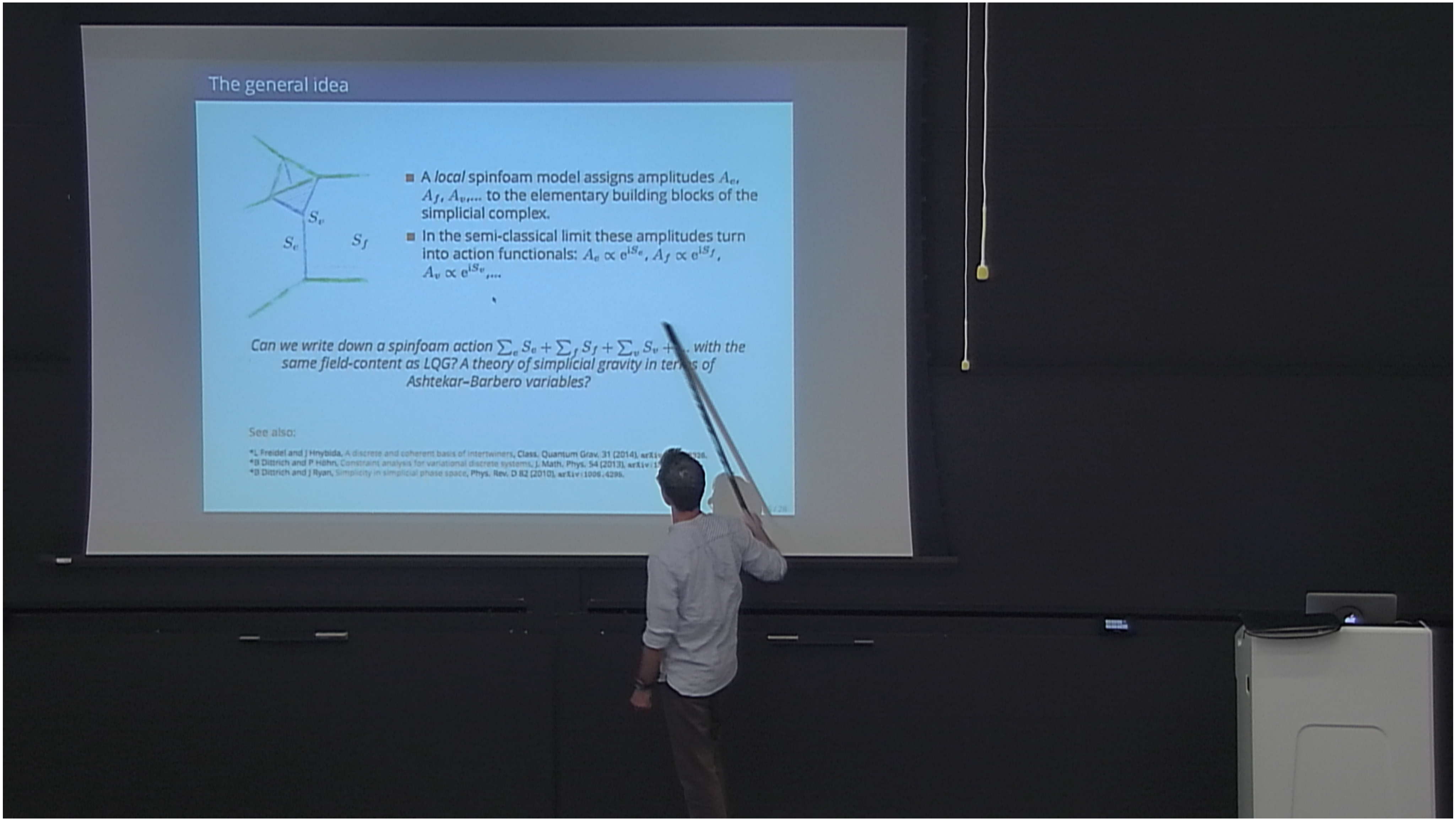
Title: Worldline formalism for covariant loop gravity

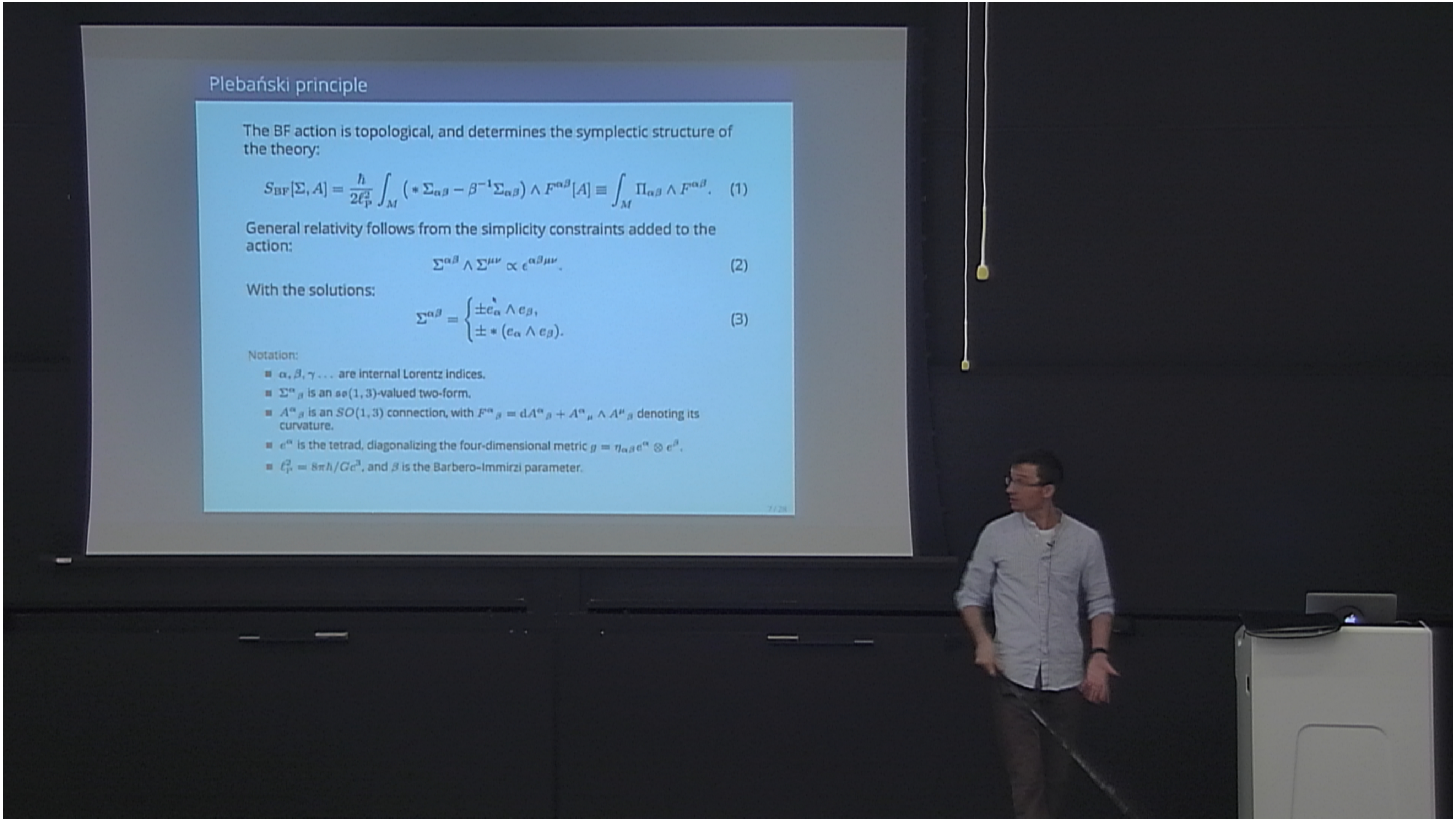
Date: Nov 06, 2014 02:15 PM

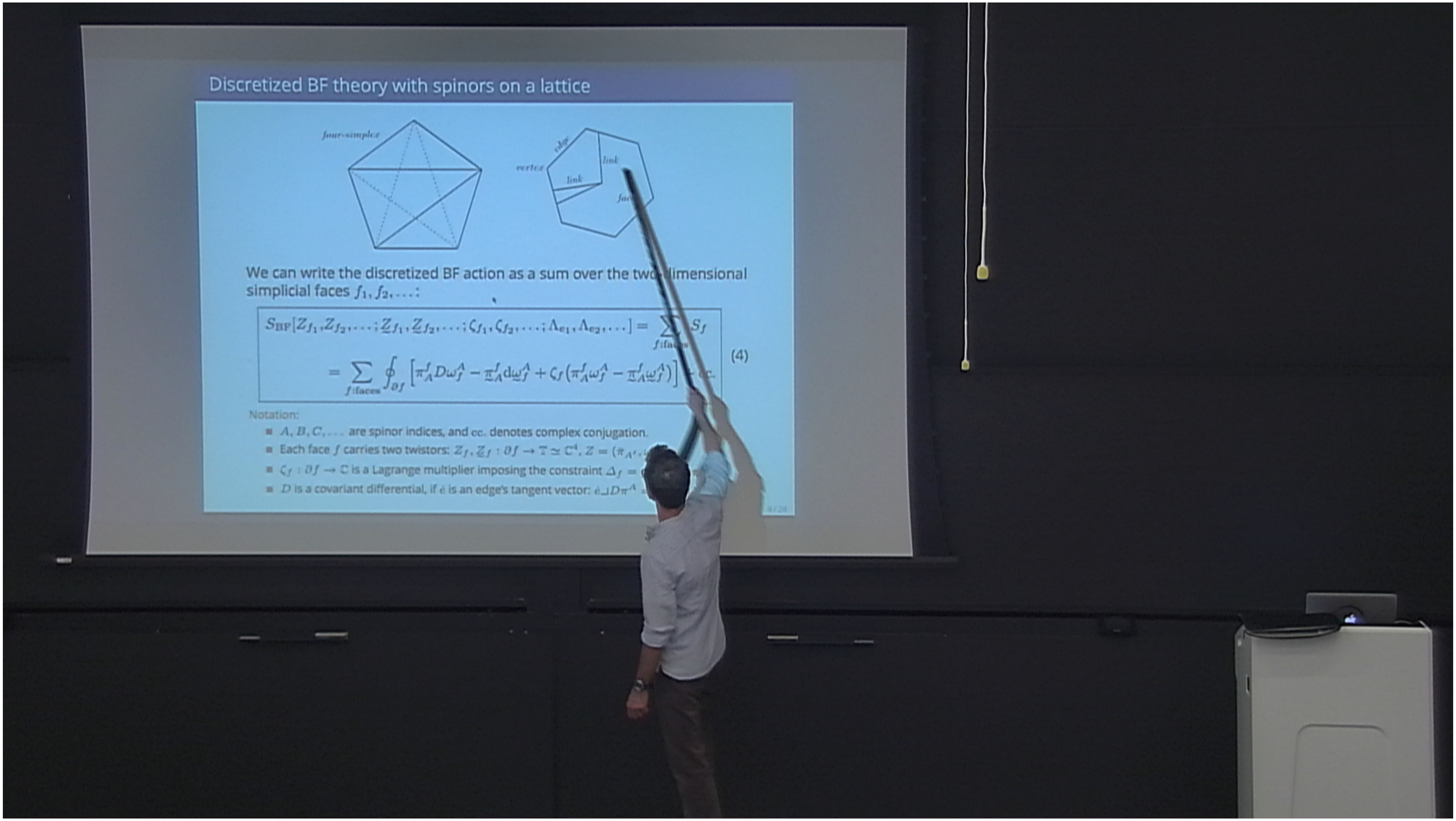
URL: http://pirsa.org/14110116

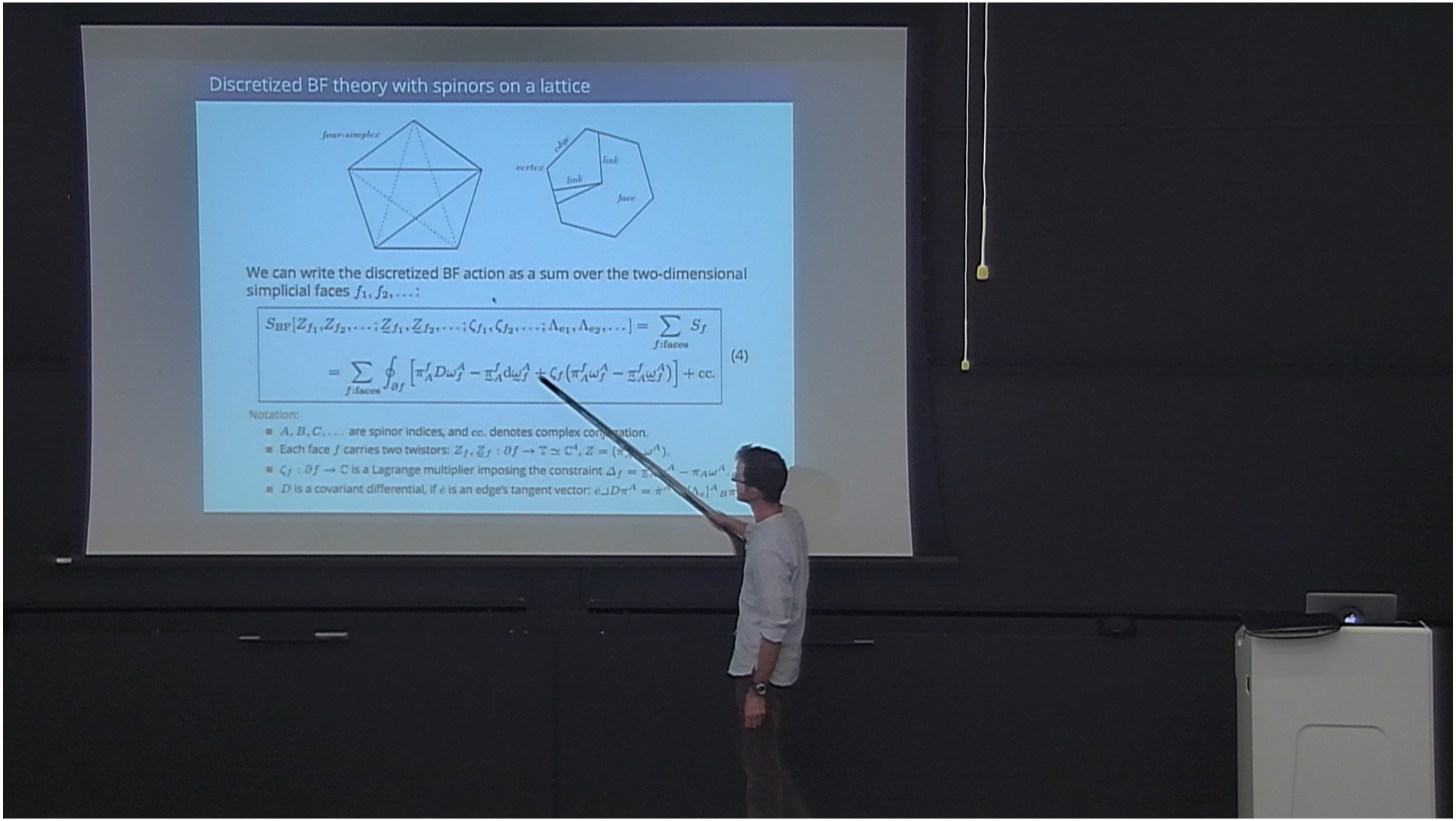
Abstract: I present a proposal for a worldline action for discretized gravity with the same field content as loop quantum gravity. The proposal is defined through its action, which is a one-dimensional integral over the edges of the discretization. Every edge carries a finite-dimensional phase space, and the evolution equations are generated by a Hamiltonian, which is a sum over the constraints of the theory. I will explain the relevance of the model, and close with possible relations to other approaches of quantum gravity, including: relative locality, causal sets and twistor theory.


Results and debates
The spinfoam approach defines transition amplitudes for loop quantum gravity boundary states through a covariant path integral on a lattice.
The EPRL model is a concrete realization of the idea.
Interesting results:
Graviton propagator, Regge-action for large spins, inclusion of a (positive) cosmological constant, addition of fermions and Yang–Mills fields, spinfoam cosmology, horizon thermodynamics...
But also ongoing debates:
How can we find continuum GR? Which limit? Summing, refining?
Do we miss additional secondary (torsional) constraints?
Apparent flatness in a face?
Is there a notion of causality in spinfoams?
I think, these issues have little to do with the quantum theory itself. We should address these question already at the classical level. What we need is a framework of simplicial gravity in area–connection variables.
*A Perez, The Spin-Foam Approach to Quantum Gravity, Living Rev. Relativity 16 (2013), lrr-2013-3.
*J Engle, ER Livine, R Pereira and C Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. B (2008)
*V Bonzom, Spinfoam models for quantum gravity from lattice path integrals, Phys. Rev. D 80 (2009)
*F Hellmann and W Kamiński, Holonomy spin foam models: Asymptotic geometry of the partition function


The general idea
S_v
S_e
S_f
A *local* spinfoam model assigns amplitudes A_e, A_f, A_v,... to the elementary building blocks of the simplicial complex.
In the semi-classical limit these amplitudes turn into action functionals: $A_e \propto e^{iS_e}$, $A_f \propto e^{iS_f}$, $A_v \propto e^{iS_v}$,...
Can we write down a spinfoam action $\sum_e S_e + \sum_f S_f + \sum_v S_v +$ ... with the same field-content as LQG? A theory of simplicial gravity in ter s of Ashtekar–Barbero variables?
See also:
*L Freidel and J Hnybida, A discrete and coherent basis of intertwiners, Class. Quantum Grav. 31 (2014), arXiv
*B Dittrich and P Höhn, Constraint analysis for variational discrete systems, J. Math. Phys. 54 (2013), arXiv:1
*B Dittrich and J Ryan, Simplicity in simplicial phase space, Phys. Rev. D 82 (2010), arXiv:1006.4295.


Plebański principle
The BF action is topological, and determines the symplectic structure of the theory:
$S_{\mathrm{BF}}[\Sigma, A] = \frac{\hbar}{2\ell_{\mathrm{P}}^2} \int_M \left(* \Sigma_{\alpha\beta} - \beta^{-1} \Sigma_{\alpha\beta} \right) \wedge F^{\alpha\beta}[A] \equiv \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta}.$ (1)
General relativity follows from the simplicity constraints added to the action:
$\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}.$ (2)
With the solutions:
$\Sigma^{\alpha\beta} = \begin{cases} \pm e_\alpha \wedge e_\beta, \\ \pm * (e_\alpha \wedge e_\beta). \end{cases}$ (3)
Notation:
$\alpha, \beta, \gamma \ldots$ are internal Lorentz indices.
$\Sigma^\alpha{}_\beta$ is an $\mathfrak{so}(1,3)$-valued two-form.
$A^\alpha{}_\beta$ is an $SO(1,3)$ connection, with $F^\alpha{}_\beta = \mathrm{d}A^\alpha{}_\beta + A^\alpha{}_\mu \wedge A^\mu{}_\beta$ denoting its curvature.
e^α is the tetrad, diagonalizing the four-dimensional metric $g = \eta_{\alpha\beta} e^\alpha \otimes e^\beta$.
$\ell_{\mathrm{P}}^2 = 8\pi\hbar/Gc^3$, and β is the Barbero–Immirzi parameter.


Discretized BF theory with spinors on a lattice
four-simplex
edge
vertex
link
link
We can write the discretized BF action as a sum over the two
imensional simplicial faces $f_1, f_2, \dots$:
$S_{\mathrm{BF}}[Z_{f_1}, Z_{f_2}, \dots; \underline{Z}_{f_1}, \underline{Z}_{f_2}, \dots; \zeta_{f_1}, \zeta_{f_2}, \dots; \Lambda_{e_1}, \Lambda_{e_2}, \dots] =$
S_f
$= \sum_{f:\mathrm{faces}} \oint_{\partial f} \left[\pi^f_A D\omega^A_f - \underline{\pi}^f_A \mathrm{d}\underline{\omega}^A_f + \zeta_f \left(\pi^f_A \omega^A_f - \underline{\pi}^f_A \underline{\omega}^A_f \right) \right]$
(4)
Notation:
$A, B, C, \dots$ are spinor indices, and cc. denotes complex conjugation.
Each face f carries two twistors: $Z_f, \underline{Z}_f : \partial f \to \mathbb{T} \simeq \mathbb{C}^4$, $Z = (\pi_{A'},$
$\zeta_f : \partial f \to \mathbb{C}$ is a Lagrange multiplier imposing the constraint $\Delta_f =$
D is a covariant differential, if $\dot{e}$ is an edge's tangent vector: $\dot{e} \lrcorner D\pi^A$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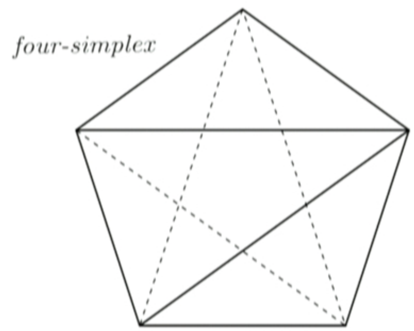
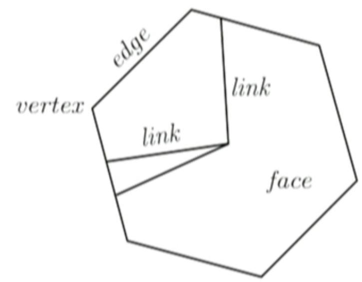
Discretized BF theory with spinors on a lattice





We can write the discretized BF action as a sum over the two-dimensional simplicial faces $f_1, f_2, \ldots$:

$$
\begin{aligned}
S_{\mathrm{BF}}[Z_{f_1}, Z_{f_2}, \ldots; \underset{\sim}{Z}_{f_1}, \underset{\sim}{Z}_{f_2}, \ldots; \zeta_{f_1}, \zeta_{f_2}, \ldots; \Lambda_{e_1}, \Lambda_{e_2}, \ldots] &= \sum_{f:\text{faces}} S_f \\
&= \sum_{f:\text{faces}} \oint_{\partial f} \Big[\pi^f_A D\omega^A_f - \underset{\sim}{\pi}^f_A \mathrm{d}\underset{\sim}{\omega}^A_f + \zeta_f\big(\pi^f_A\omega^A_f - \underset{\sim}{\pi}^f_A\underset{\sim}{\omega}^A_f\big)\Big] + \text{cc.}
\end{aligned}
\tag{4}
$$

Notation:

- $A, B, C, \ldots$ are spinor indices, and cc. denotes complex conjugation.
- Each face f carries two twistors: $Z_f, \underset{\sim}{Z}_f : \partial f \to \mathbb{T} \simeq \mathbb{C}^4$, $Z = (\bar{\pi}_{A'}, \omega^A)$.
- $\zeta_f : \partial f \to \mathbb{C}$ is a Lagrange multiplier imposing the constraint $\Delta_f = \underset{\sim}{\pi}_A \underset{\sim}{\omega}^A - \pi_A \omega^A$.
- D is a covariant differential, if $\dot{e}$ is an edge's tangent vector: $\dot{e} \lrcorner D\pi^A = \dot{\pi}^A + [\Lambda_e]^A{}_B \pi^B$.

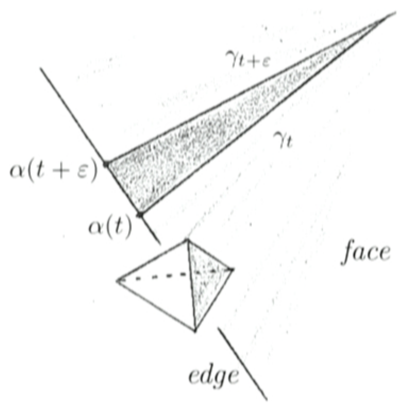
Key ideas of the proof, 1/2

- Step 1: Discretize the action:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \approx \sum_{f:\mathrm{faces}} \int_{\tau_f} \Pi_{\alpha\beta} \int_f F^{\alpha\beta} \equiv \sum_{f:\mathrm{faces}} S_f .$$

- Step 2: Define the smeared flux:

$$\Pi_f^{\alpha\beta}(t) = \int_{\tau_f} \mathrm{d}x\, \mathrm{d}y \, [h_{\gamma(t,x,y)}]^{\alpha}{}_{\mu} [h_{\gamma(t,x,y)}]^{\beta}{}_{\nu} \, [\Pi_{p(x,y)}(\partial_x, \partial_y)]^{\mu\nu} .$$

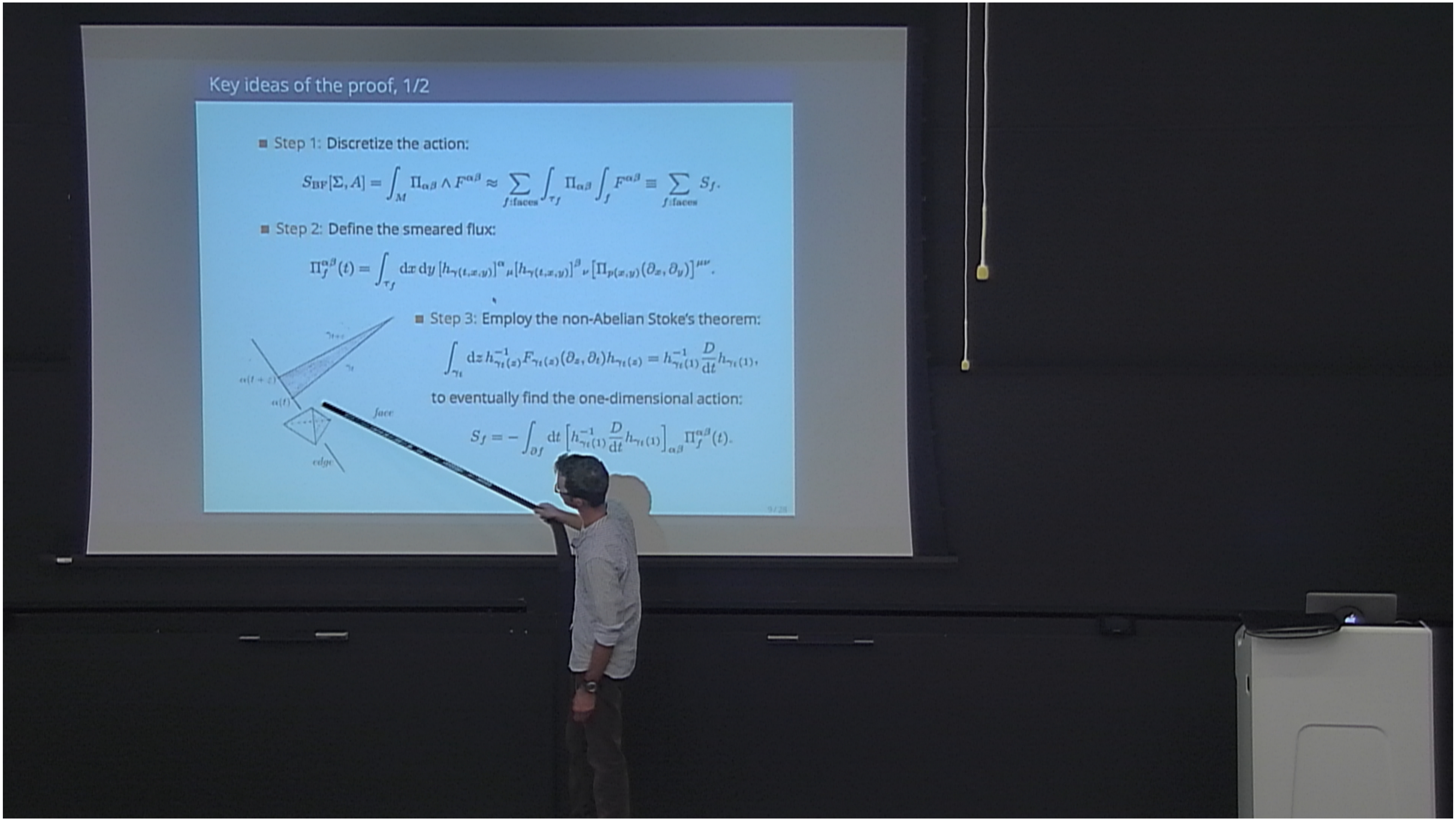


- Step 3: Employ the non-Abelian Stoke's theorem:

$$\int_{\gamma_t} \mathrm{d}z \, h^{-1}_{\gamma_t(z)} F_{\gamma_t(z)}(\partial_z, \partial_t) h_{\gamma_t(z)} = h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)},$$

to eventually find the one-dimensional action:

$$S_f = - \int_{\partial f} \mathrm{d}t \left[h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)} \right]_{\alpha\beta} \Pi_f^{\alpha\beta}(t).$$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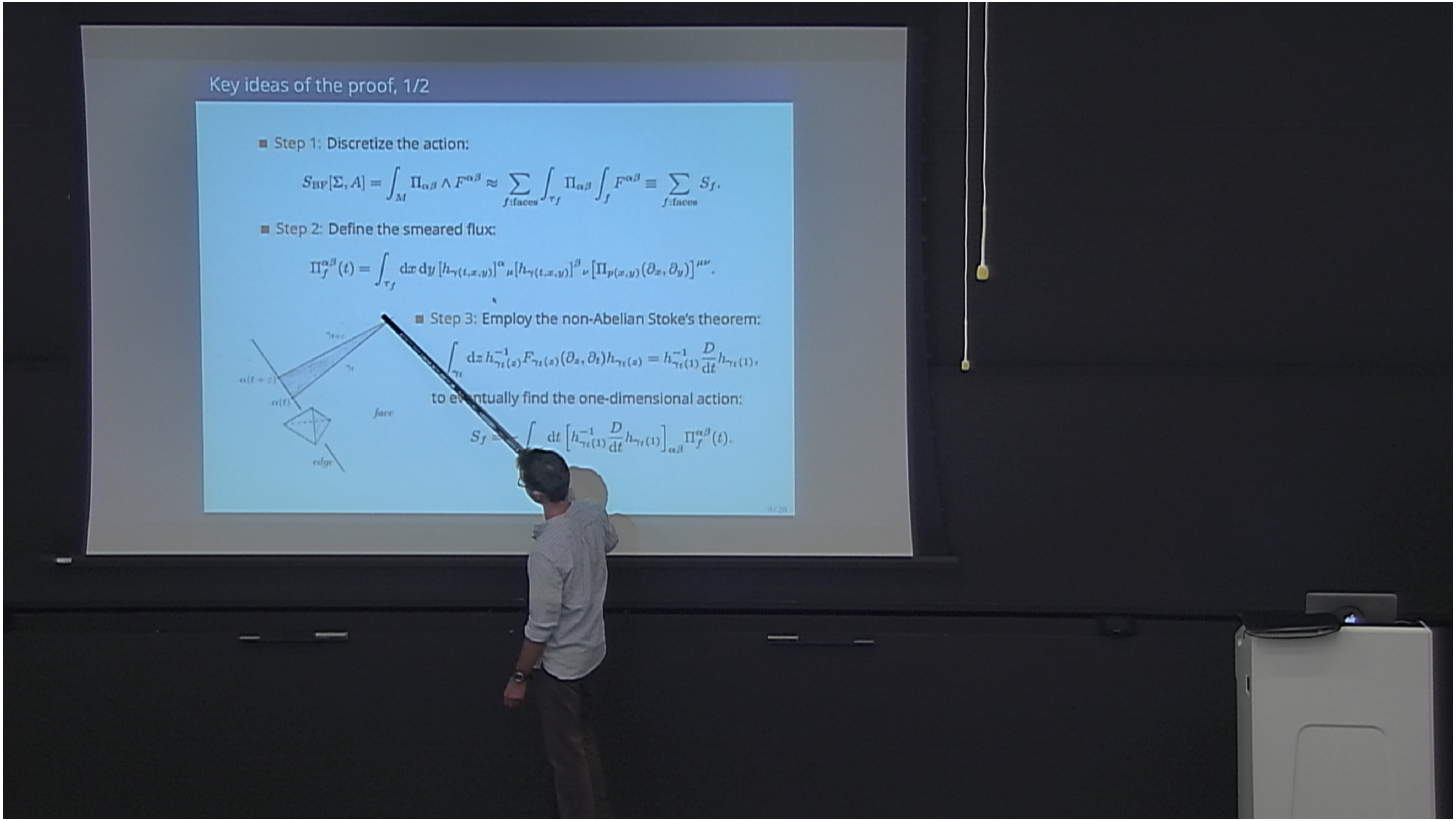
face
edge
9/28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Step 3: Employ the non-Abelian Stoke's theorem:
$\int_{\gamma_t} dz\, h^{-1}_{\gamma_t(z)} F_{\gamma_t(z)}(\partial_z, \partial_t) h_{\gamma_t(z)} = h^{-1}_{\gamma_t(1)} \frac{D}{dt} h_{\gamma_t(1)},$
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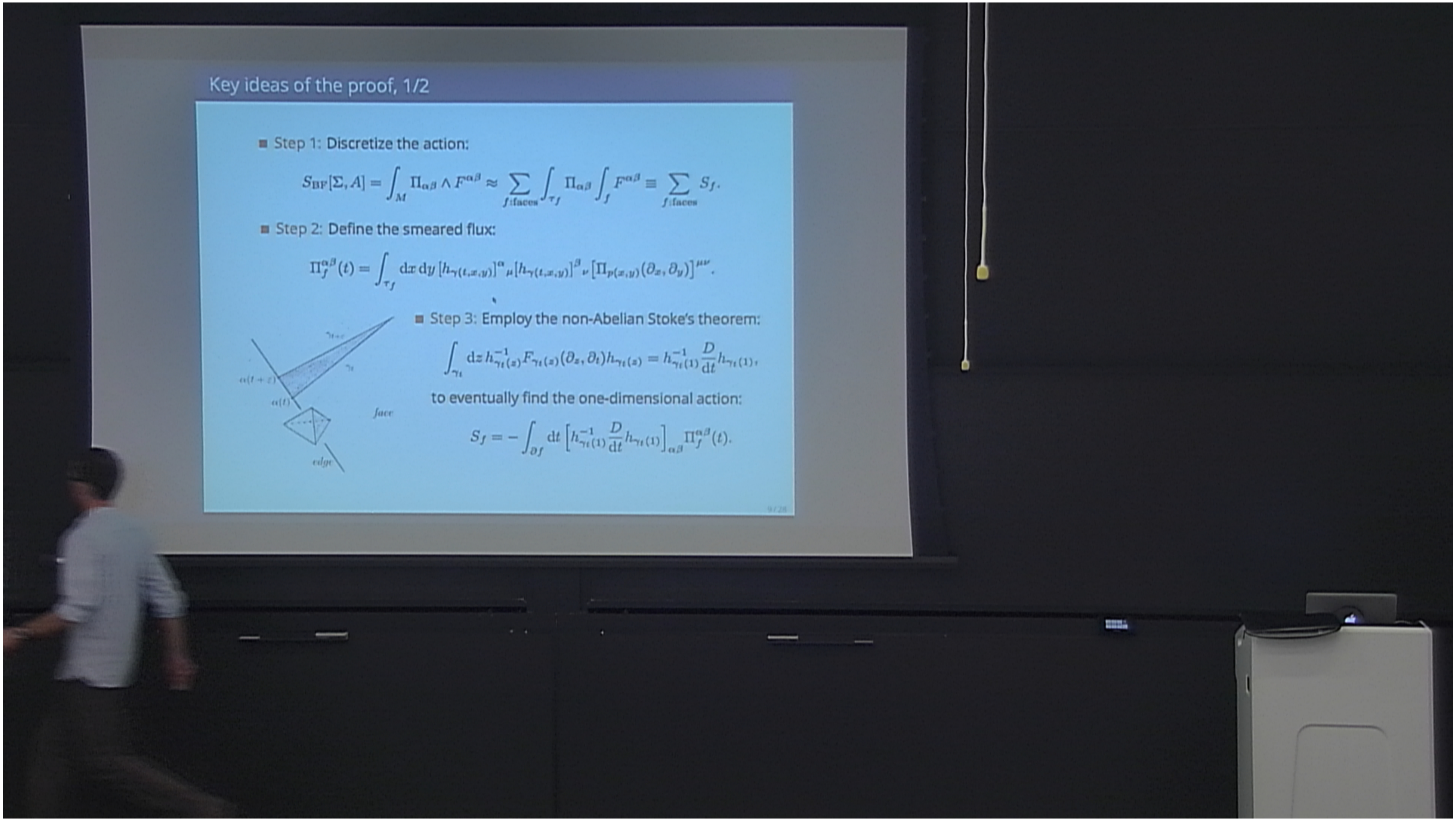
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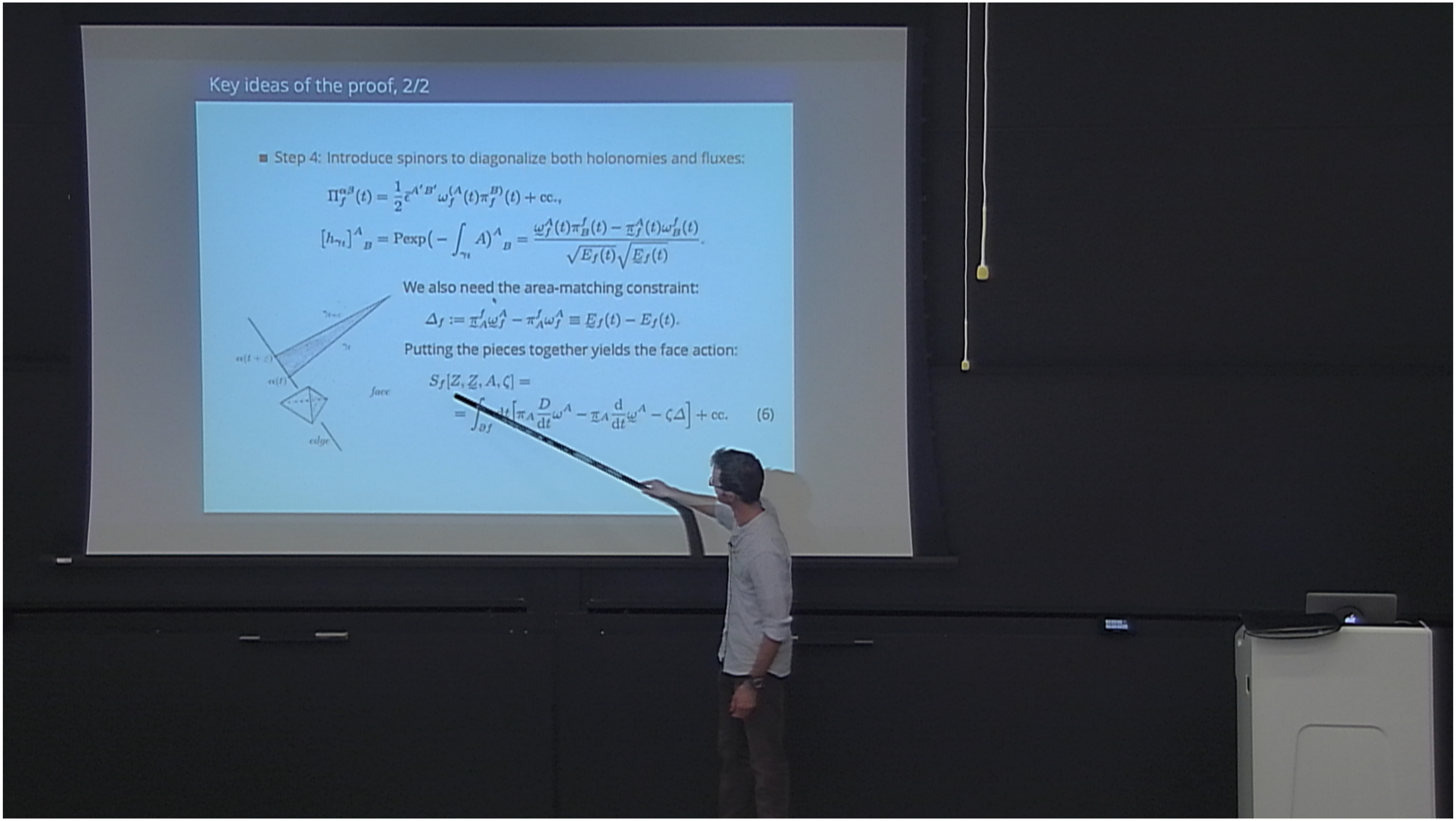
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Step 4: Introduce spinors to diagonalize both holonomies and fluxes:
$\Pi_f^{\alpha\beta}(t) = \frac{1}{2}\bar{\epsilon}^{A'B'}\omega_f^{(A}(t)\pi_f^{B)}(t) + \text{cc.},$
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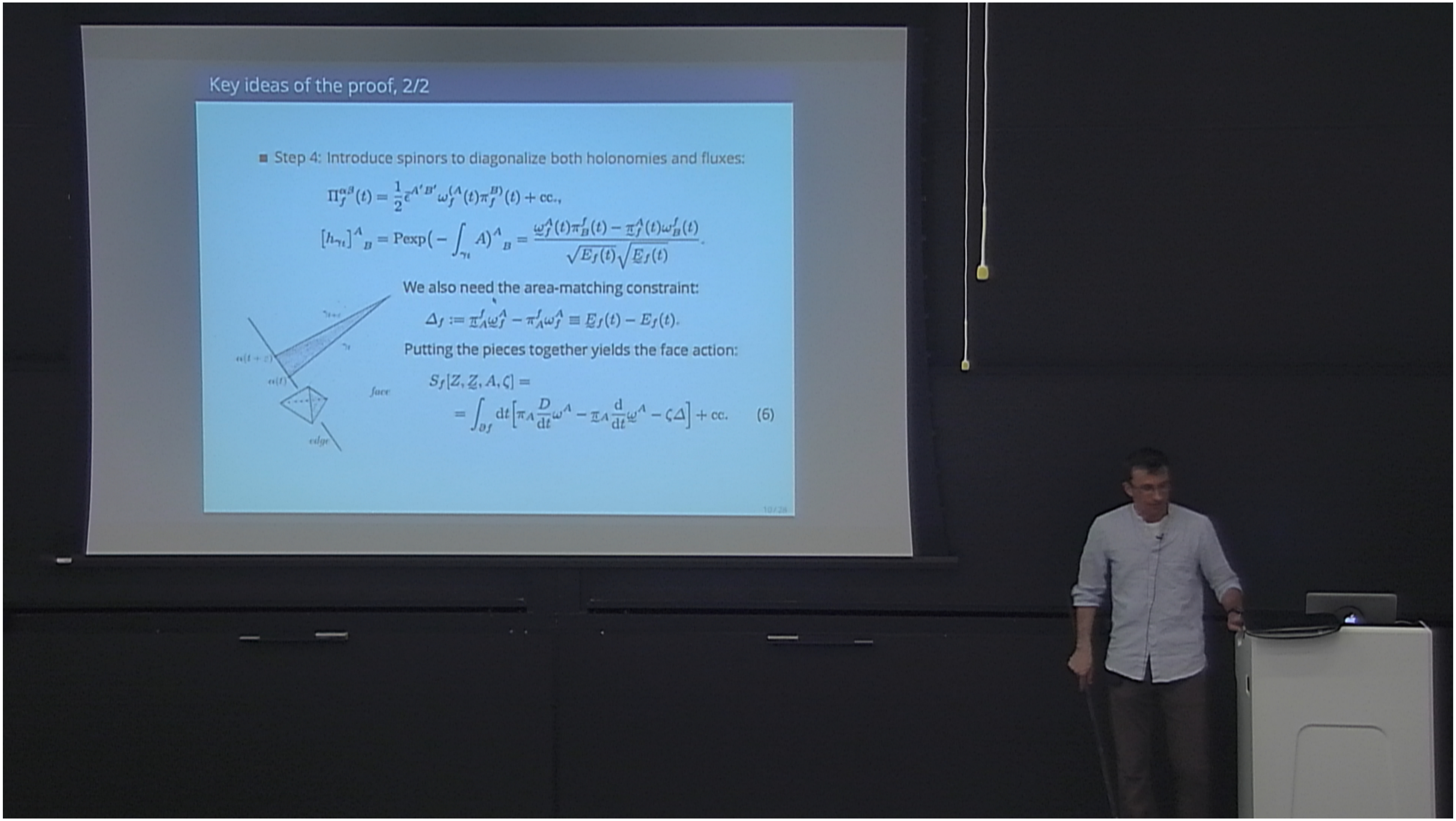
We also need the area-matching constraint:
$\Delta_f := \underline{\pi}_A^f\underline{\omega}_f^A - \pi_A^f\omega_f^A \equiv \underline{E}_f(t) - E_f(t).$
Putting the pieces together yields the face action:
$S_f[Z, \underline{Z}, A, \zeta] =$
$= \int_{\partial f} dt \left[\pi_A \frac{D}{dt}\omega^A - \underline{\pi}_A \frac{d}{dt}\underline{\omega}^A - \zeta\Delta\right] + \text{cc.}$ (6)
face
edge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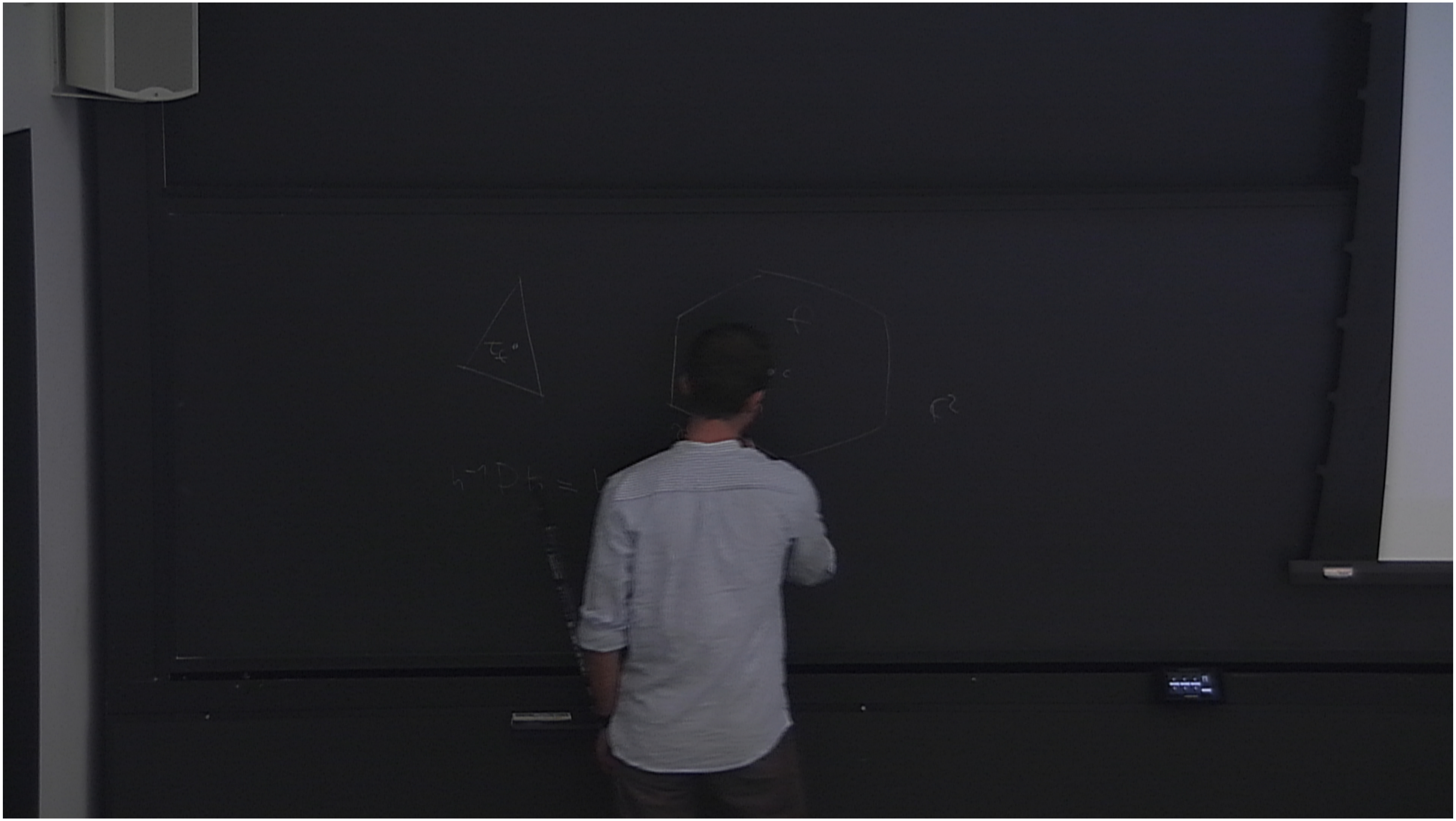
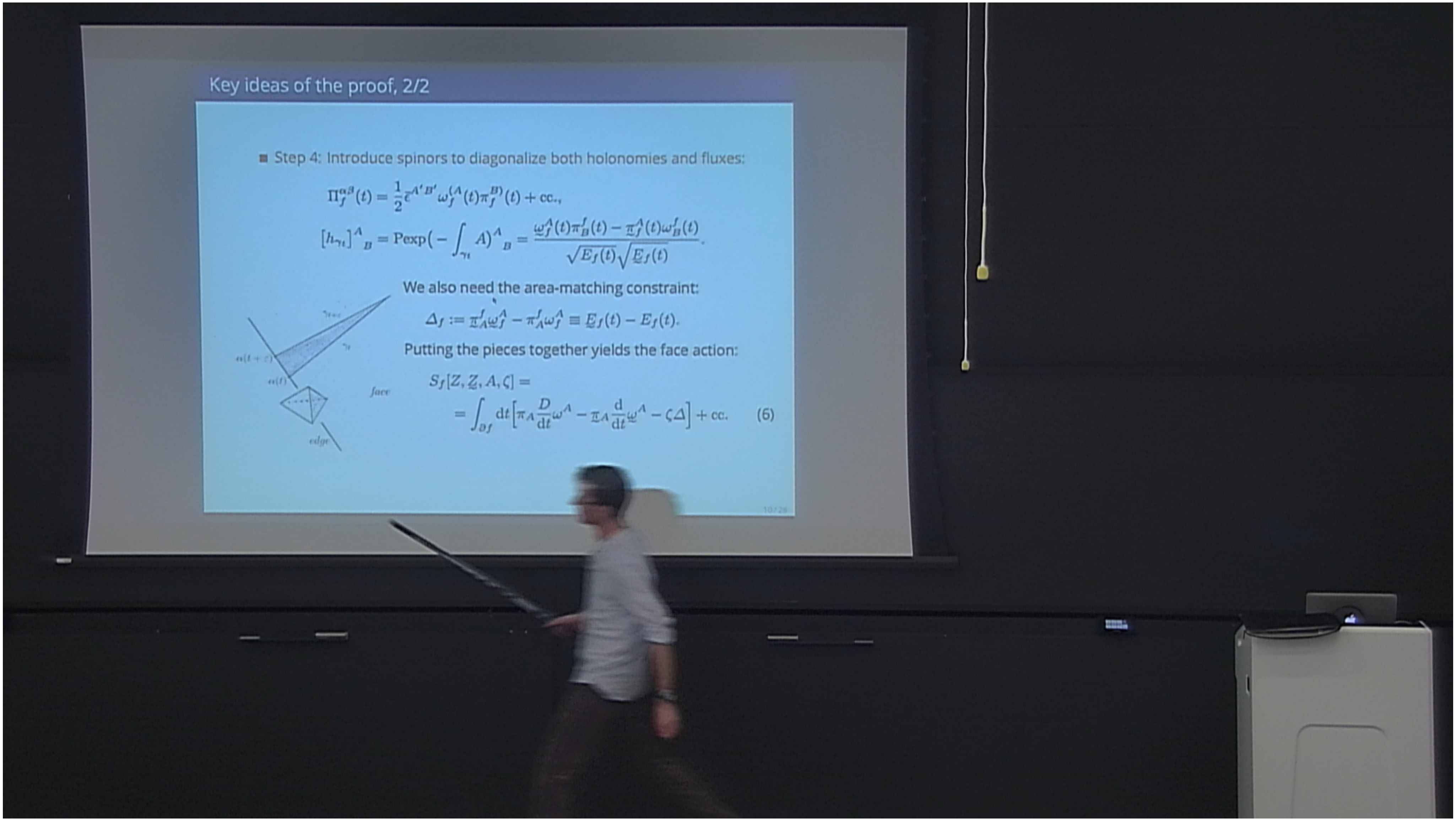


h⁻¹ D h = h⁻¹ F h



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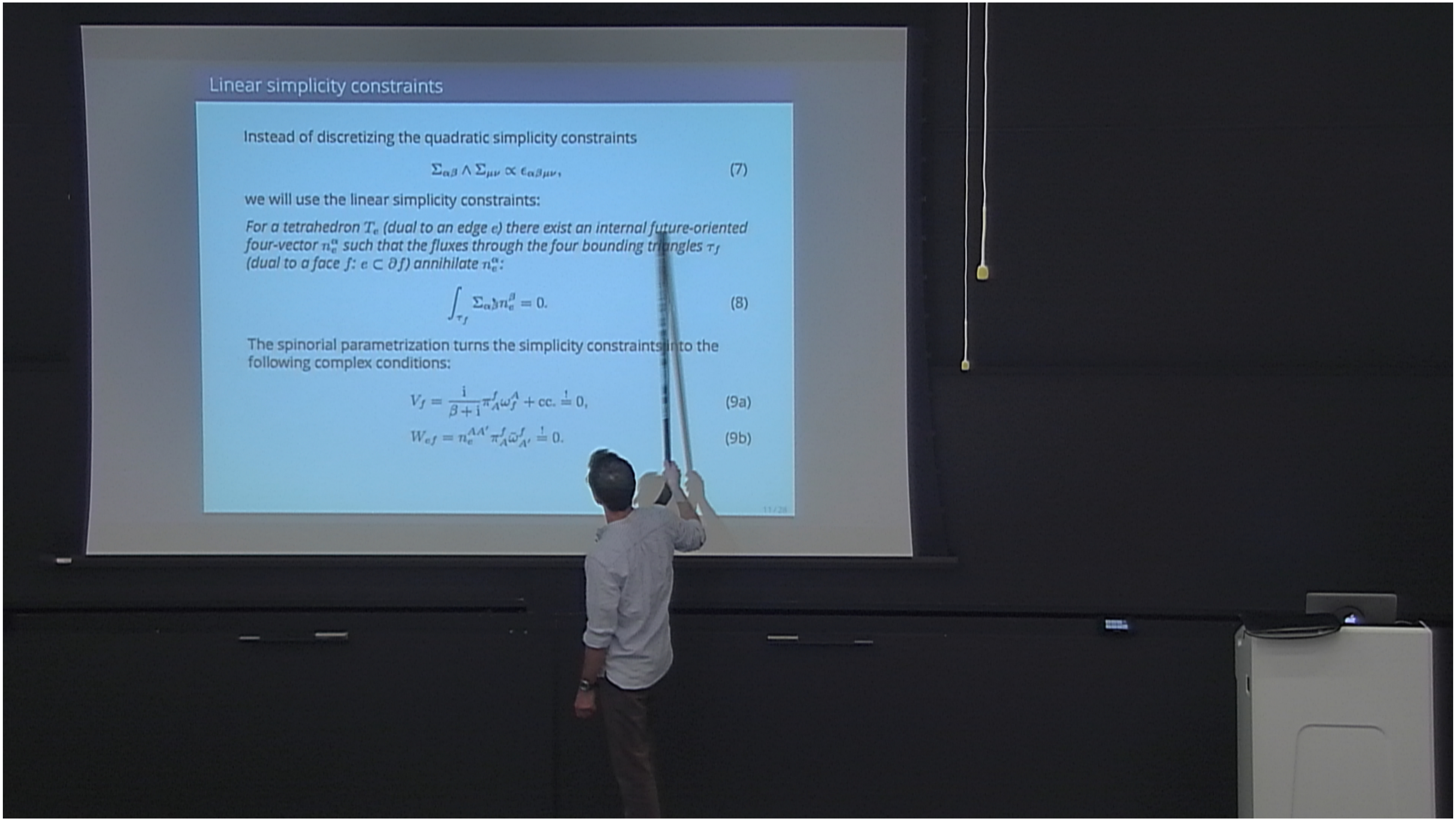
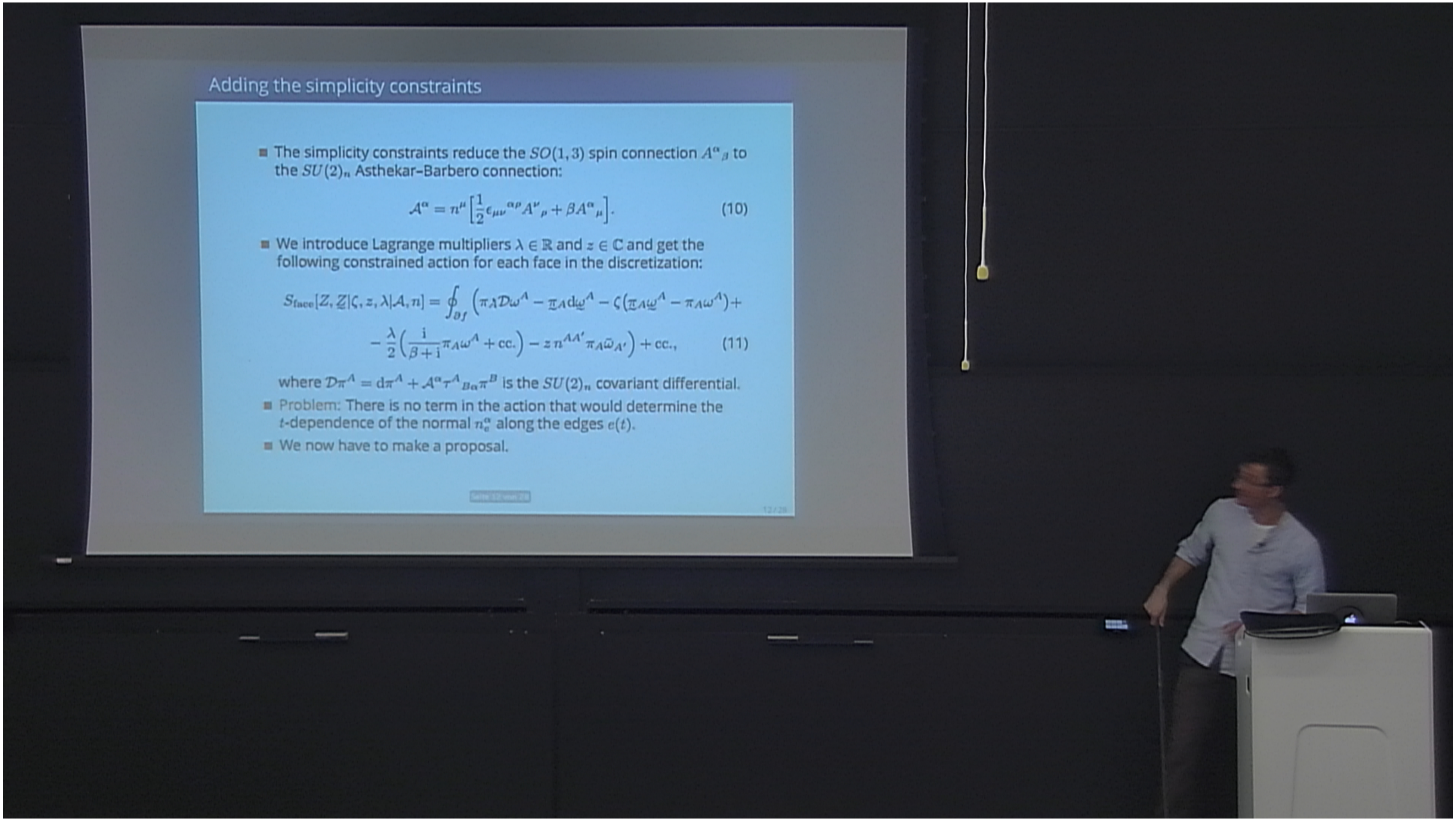
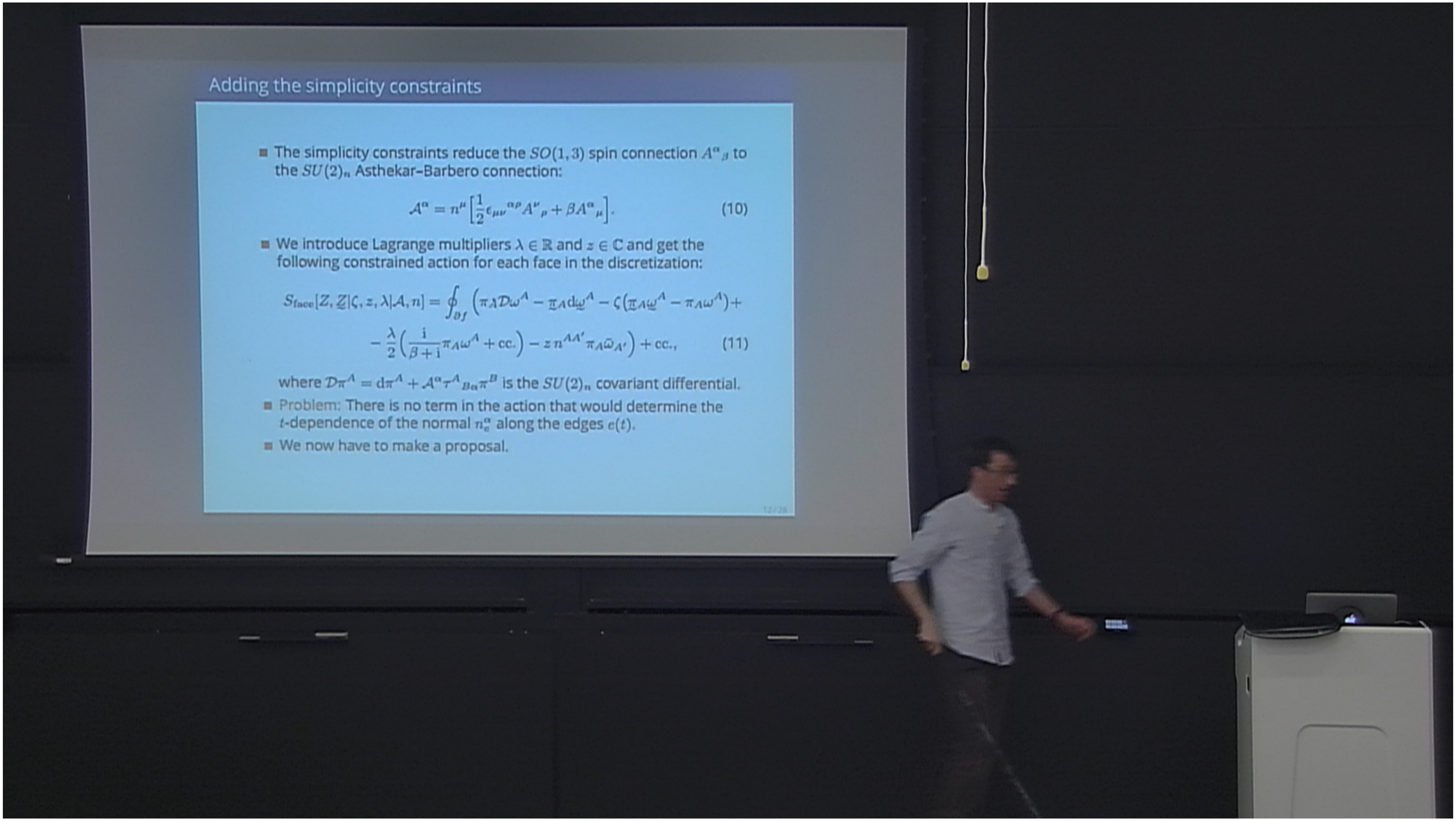

Linear simplicity constraints
Instead of discretizing the quadratic simplicity constraints
$\Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu},$ (7)
we will use the linear simplicity constraints:
For a tetrahedron T_e (dual to an edge e) there exist an internal
four-vector n_e^α such that the fluxes through the four bounding
(dual to a face f: $e \subset \partial f$) annihilate n_e^α:
$\int_{\tau_f} \Sigma_{\alpha\beta} n_e^\beta = 0.$ (8)
The spinorial parametrization turns the simplicity constraints
following complex conditions:
$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0,$ (9a)
$W_{ef} = n_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \stackrel{!}{=} 0.$ (9b)


Adding the simplicity constraints
The simplicity constraints reduce the $SO(1,3)$ spin connection $A^{\alpha}{}_{\beta}$ to the $SU(2)_n$ Asthekar–Barbero connection:
$\mathcal{A}^{\alpha} = n^{\mu}\Big[\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\rho}A^{\nu}{}_{\rho} + \beta A^{\alpha}{}_{\mu}\Big].$ (10)
We introduce Lagrange multipliers $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ and get the following constrained action for each face in the discretization:
$S_{\mathrm{face}}[Z,\underline{Z}|\zeta,z,\lambda|\mathcal{A},n] = \oint_{\partial f}\Big(\pi_A\mathcal{D}\omega^A - \underline{\pi}_A\mathrm{d}\underline{\omega}^A - \zeta\big(\underline{\pi}_A\underline{\omega}^A - \pi_A\omega^A\big) +$
$-\frac{\lambda}{2}\Big(\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi_A\omega^A + \mathrm{cc.}\Big) - z\,n^{AA'}\pi_A\bar{\omega}_{A'}\Big) + \mathrm{cc.},$ (11)
where $\mathcal{D}\pi^A = \mathrm{d}\pi^A + \mathcal{A}^{\alpha}\tau^A{}_{B\alpha}\pi^B$ is the $SU(2)_n$ covariant differential.
Problem: There is no term in the action that would determine the t-dependence of the normal n^{α}_e along the edges $e(t)$.
We now have to make a proposal.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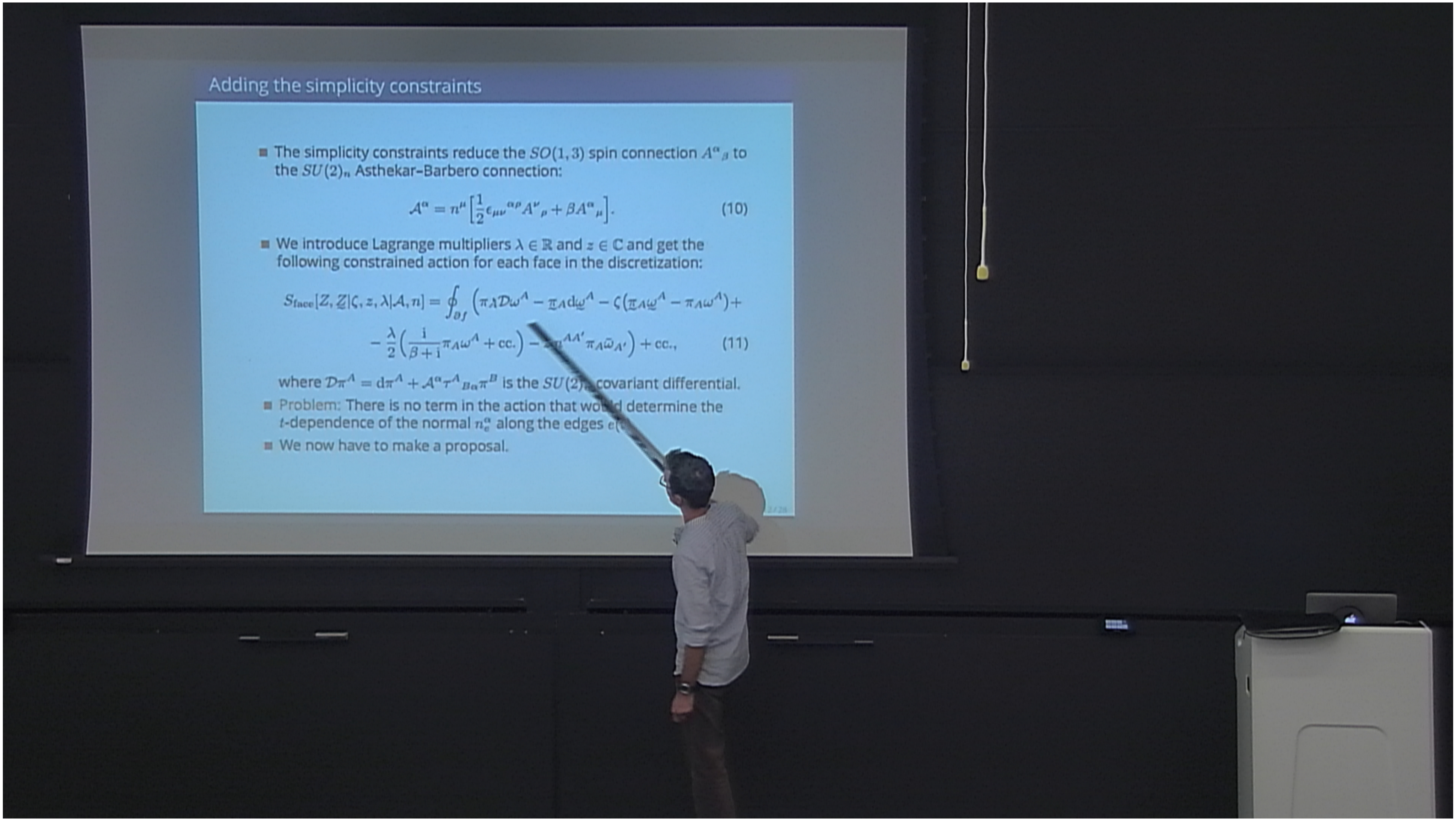
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We introduce Lagrange multipliers $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ and get the following constrained action for each face in the discretization:
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where $\mathcal{D}\pi^{A} = d\pi^{A} + \mathcal{A}^{\alpha}\tau^{A}{}_{B\alpha}\pi^{B}$ is the $SU(2)$ covariant differential.
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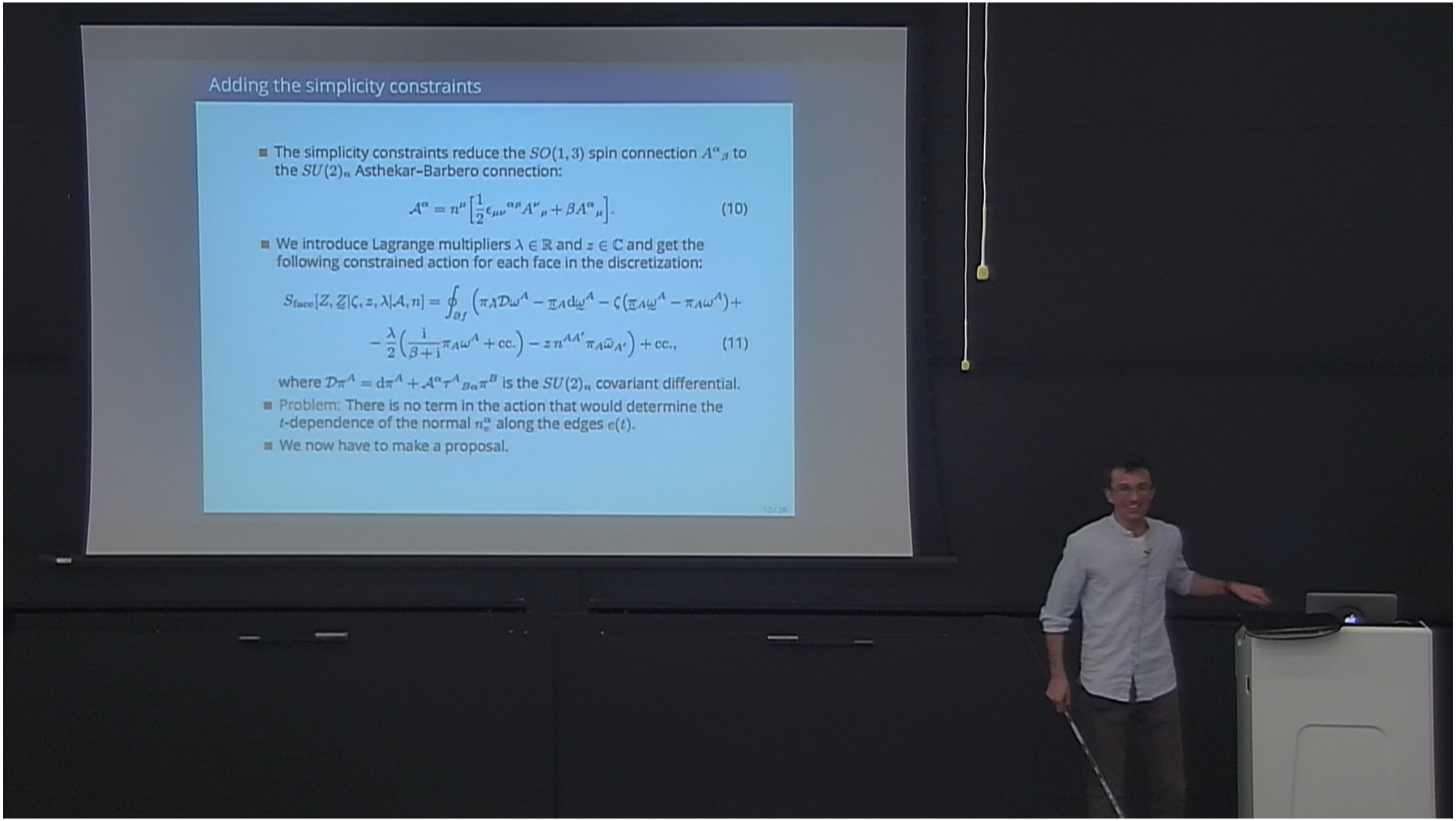
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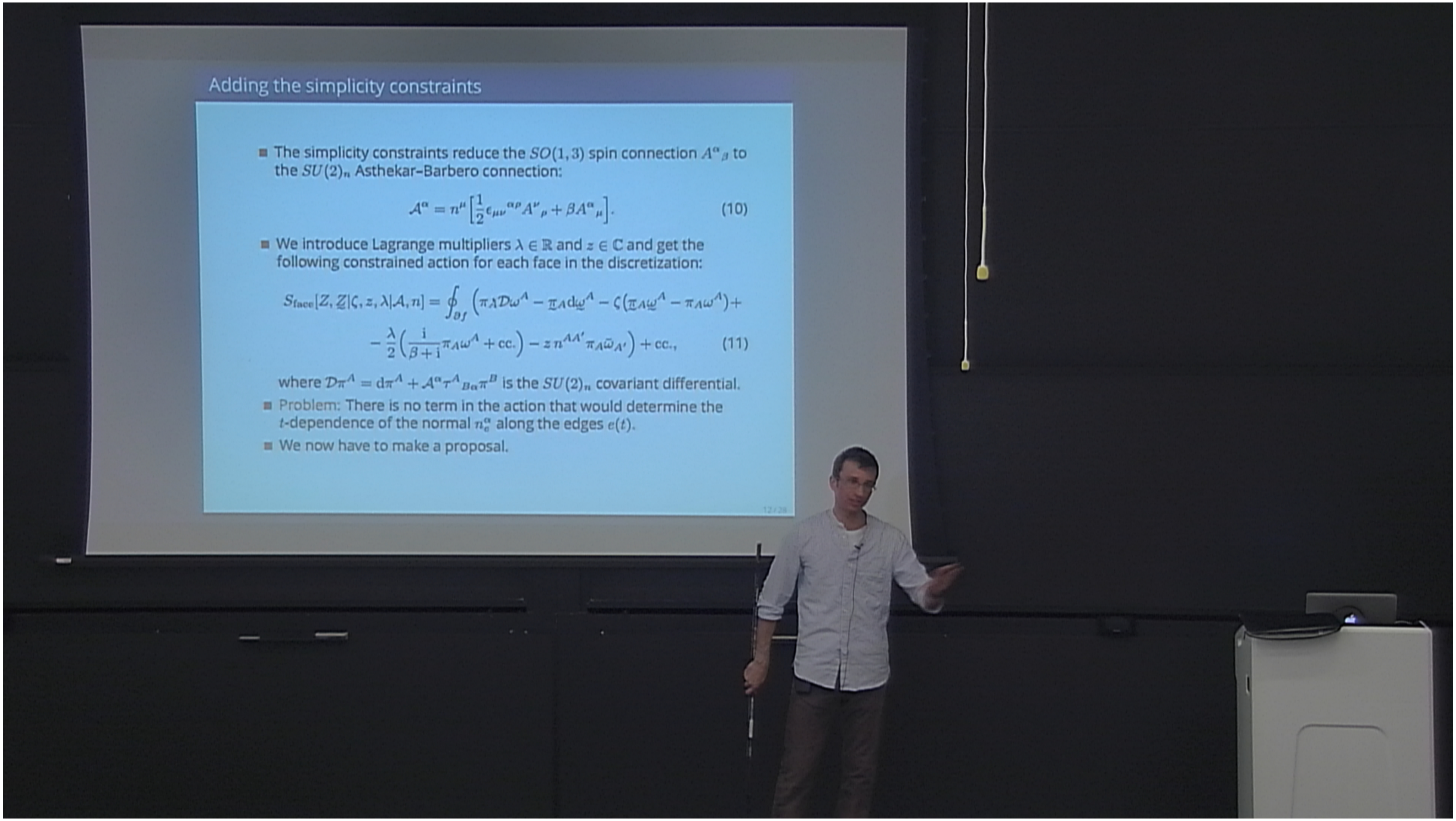

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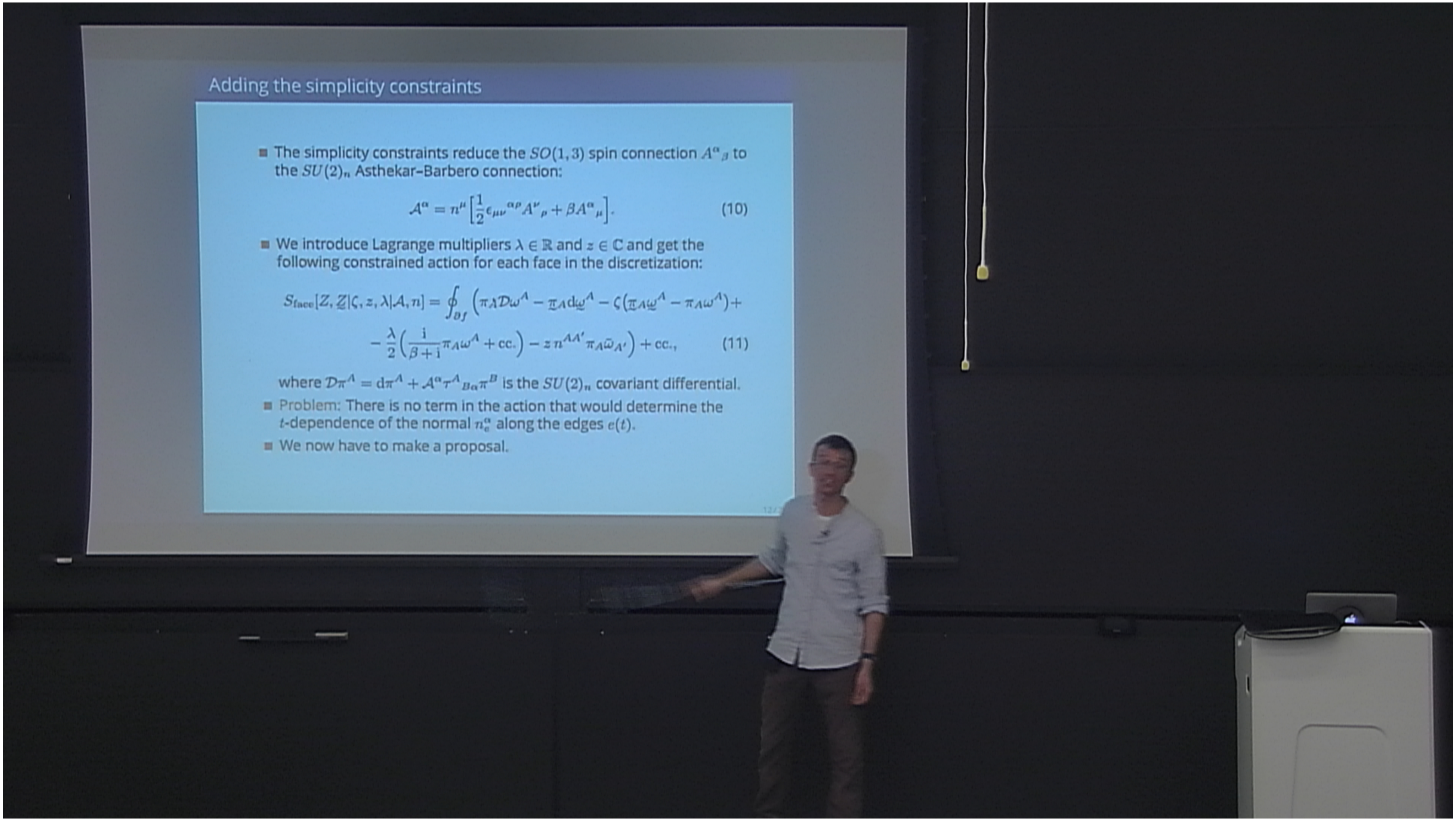
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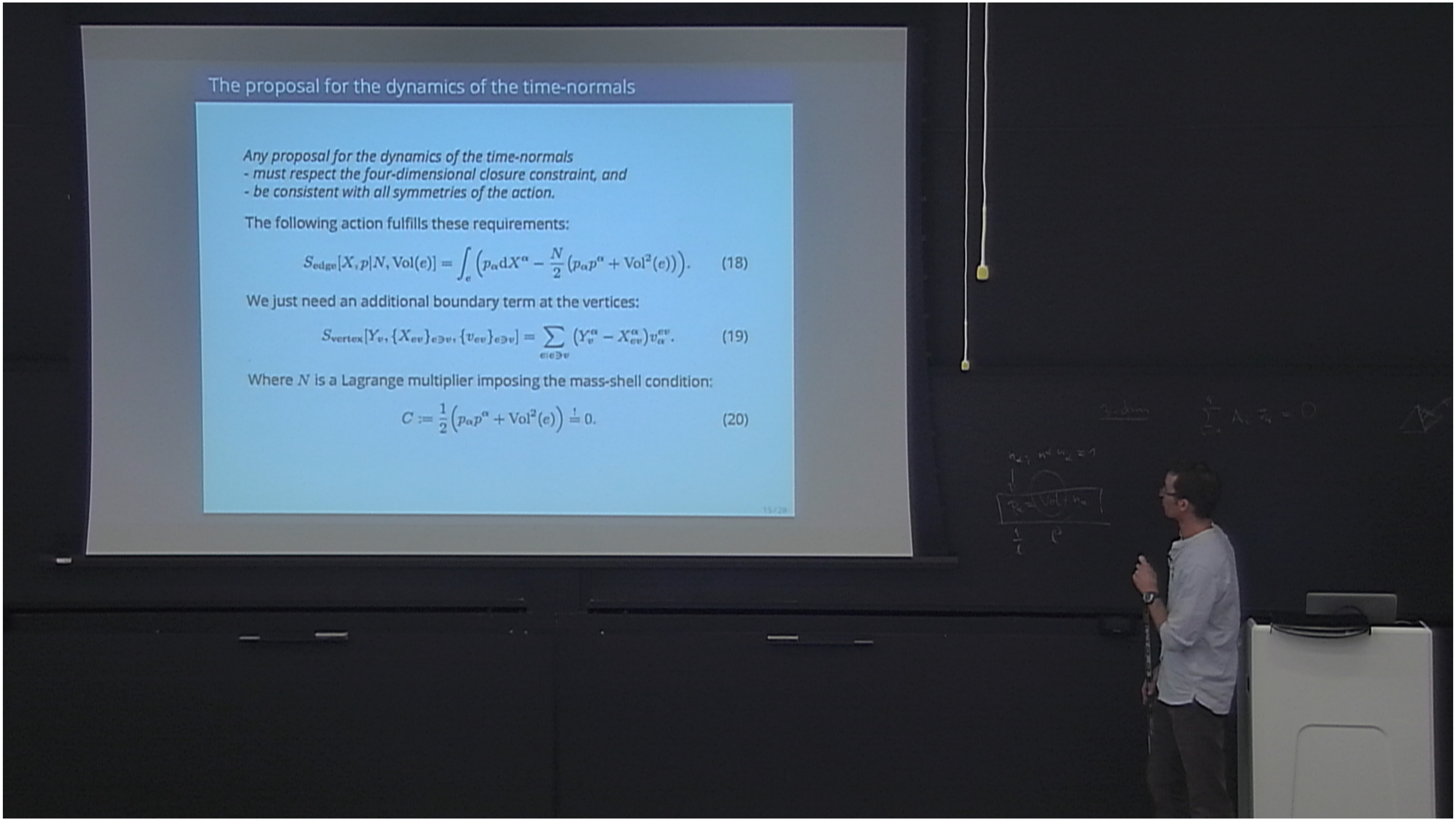
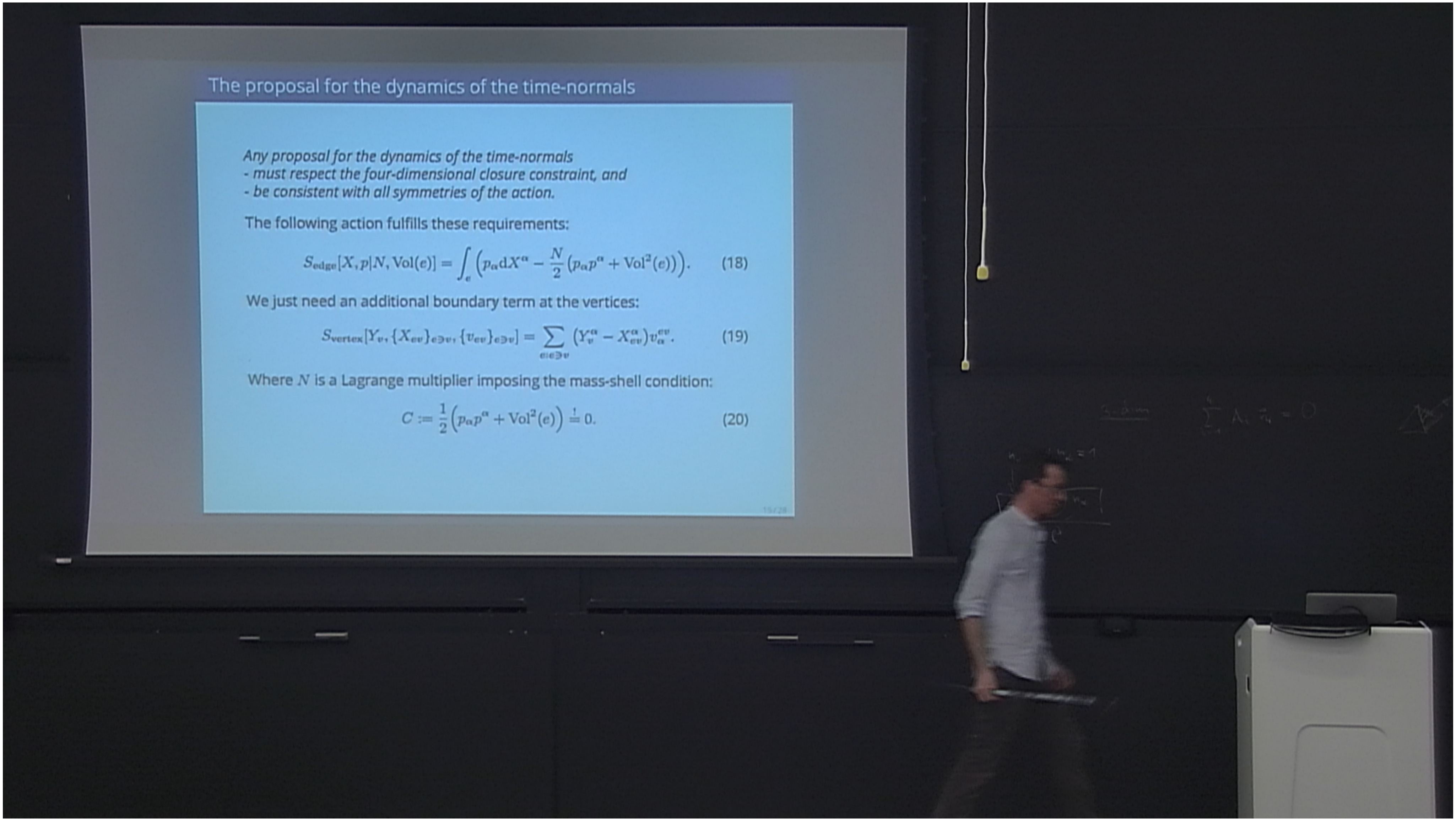

The proposal for the dynamics of the time-normals
Any proposal for the dynamics of the time-normals
- must respect the four-dimensional closure constraint, and
- be consistent with all symmetries of the action.
The following action fulfills these requirements:
$S_{\mathrm{edge}}[X, p|N, \mathrm{Vol}(e)] = \int_e \left(p_\alpha \mathrm{d}X^\alpha - \frac{N}{2}(p_\alpha p^\alpha + \mathrm{Vol}^2(e))\right).$ (18)
We just need an additional boundary term at the vertices:
$S_{\mathrm{vertex}}[Y_v, \{X_{ev}\}_{e \ni v}, \{v_{ev}\}_{e \ni v}] = \sum_{e: e \ni v} (Y_v^\alpha - X_{ev}^\alpha) v_\alpha^{ev}.$ (19)
Where N is a Lagrange multiplier imposing the mass-shell condition:
$C := \frac{1}{2}\left(p_\alpha p^\alpha + \mathrm{Vol}^2(e)\right) \stackrel{!}{=} 0.$ (20)



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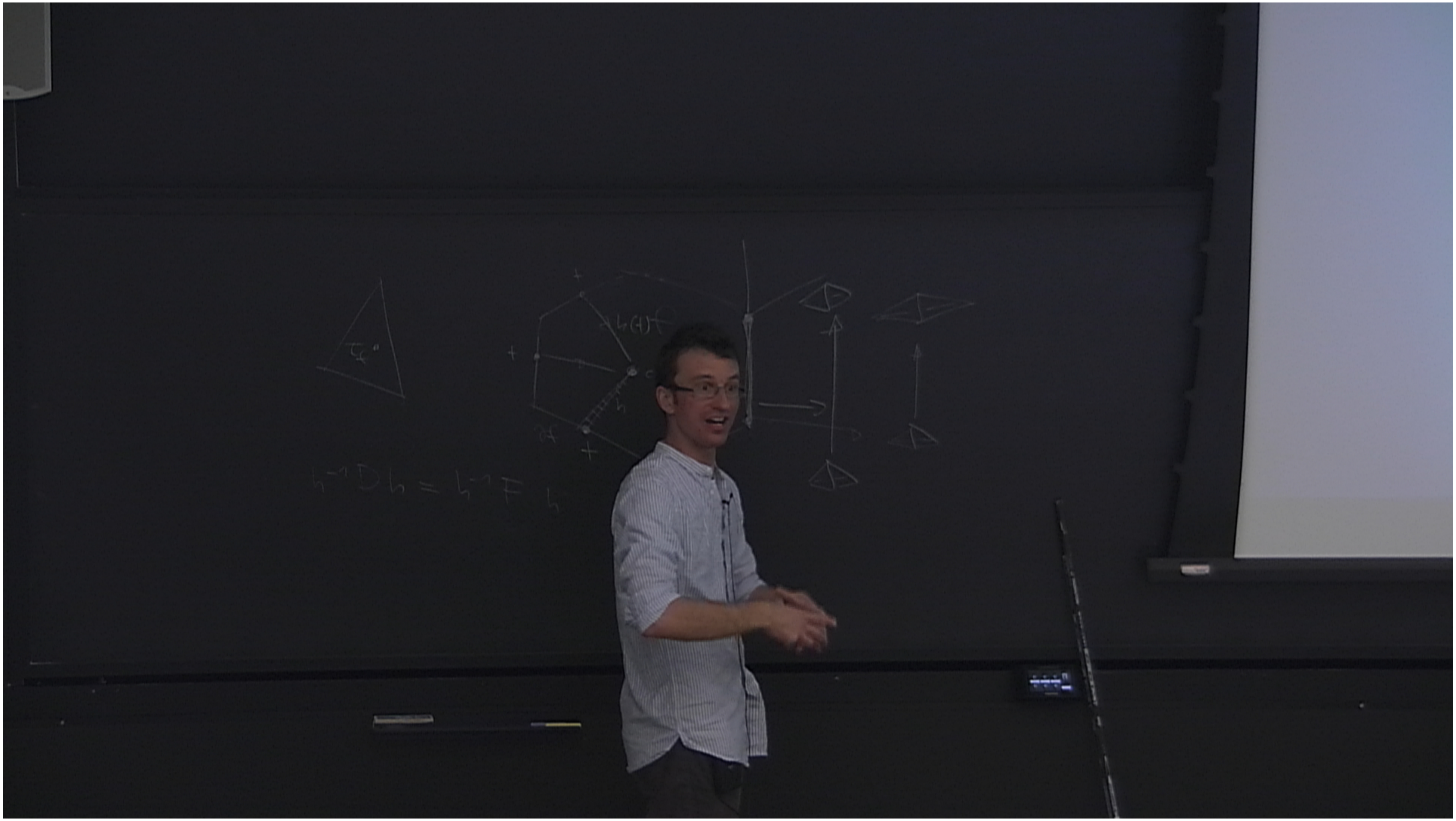

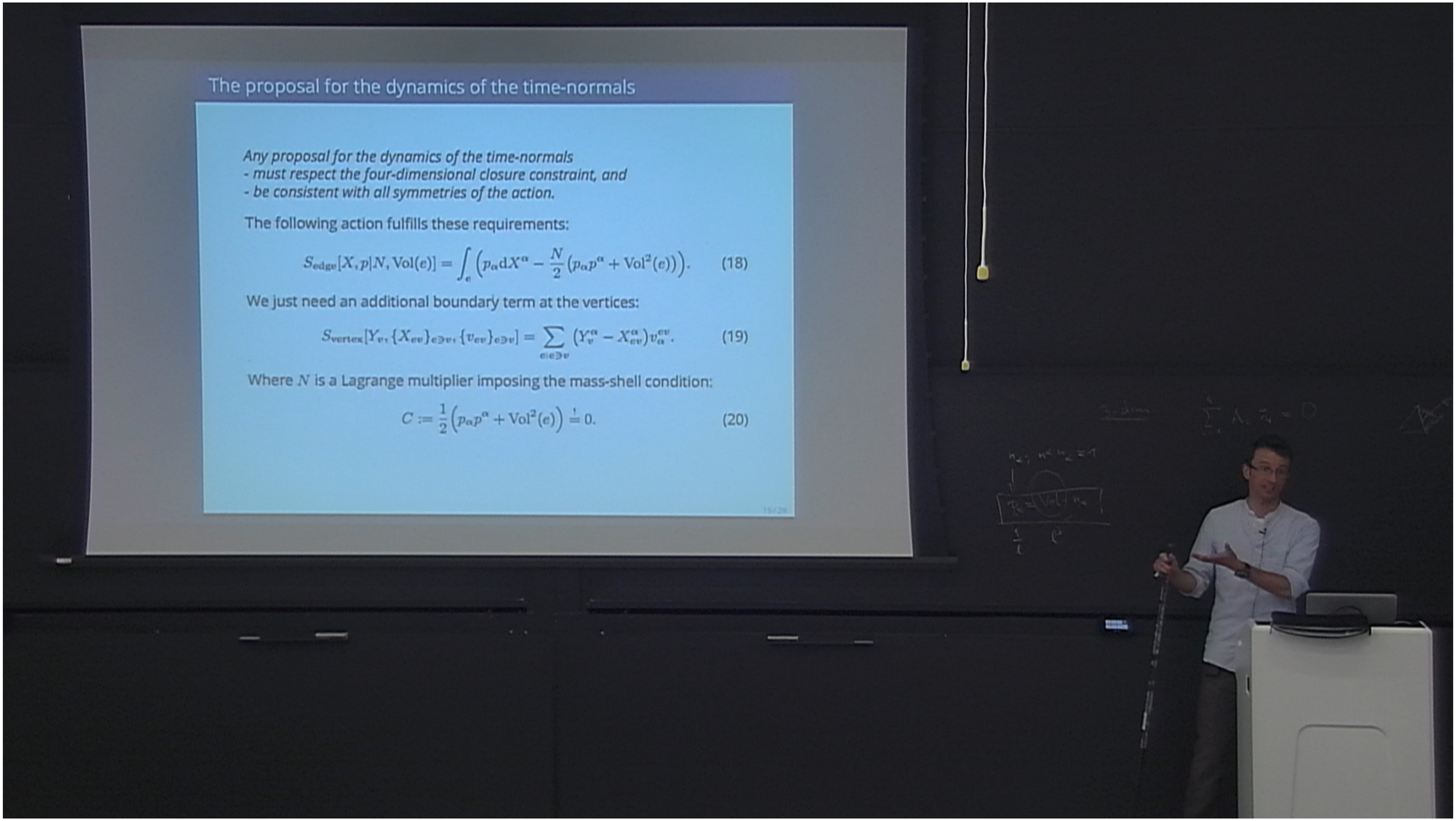
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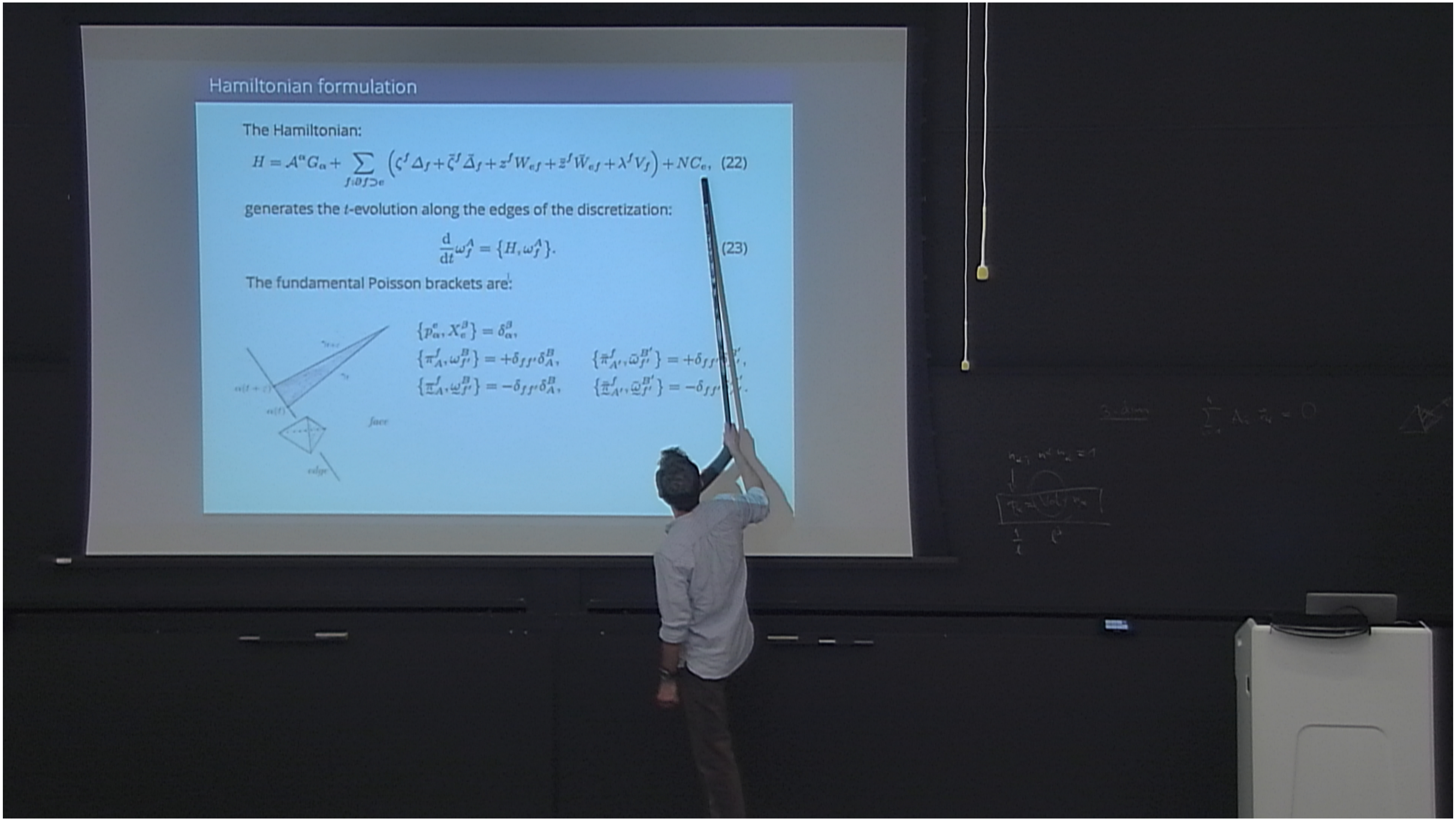
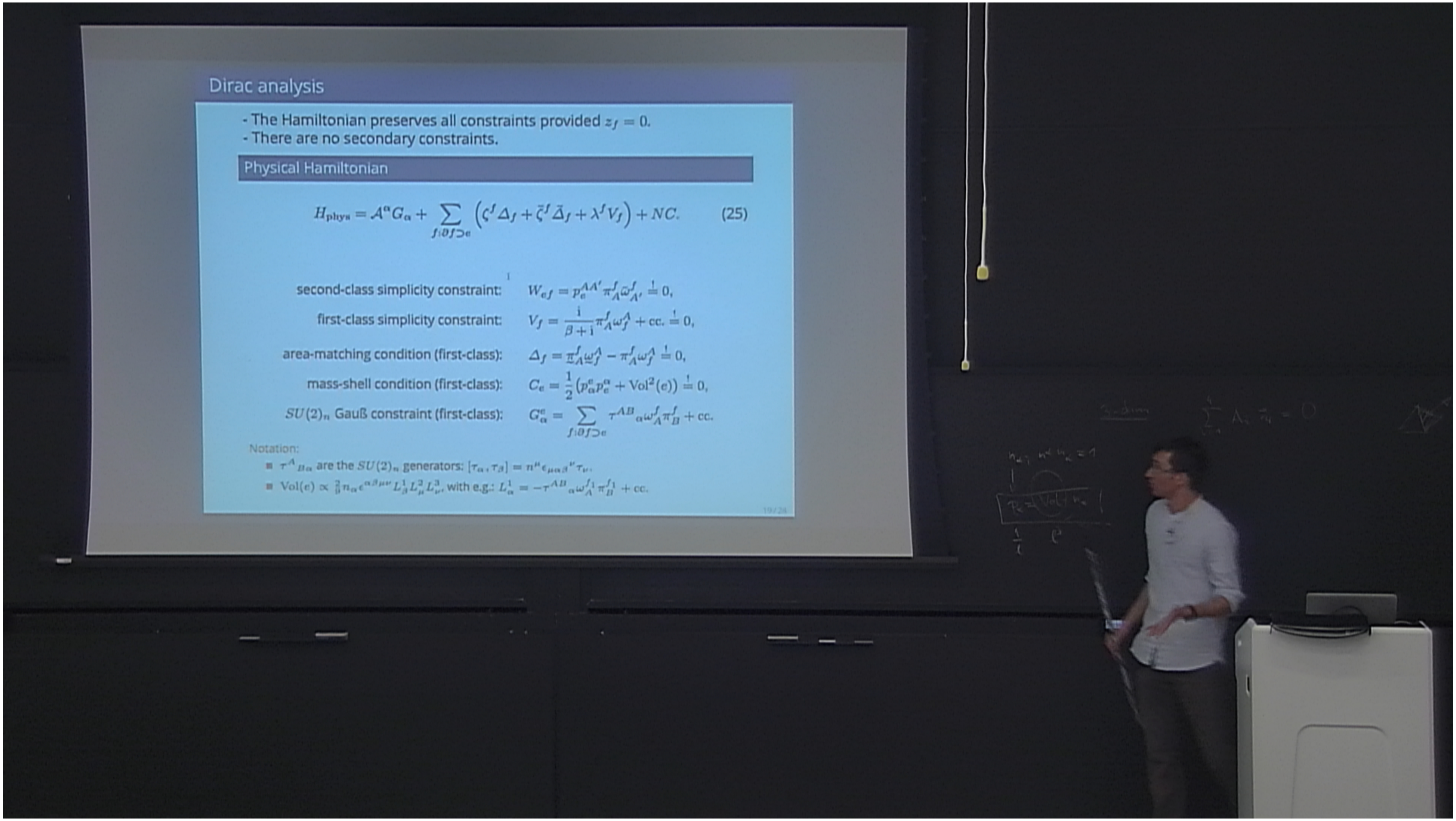
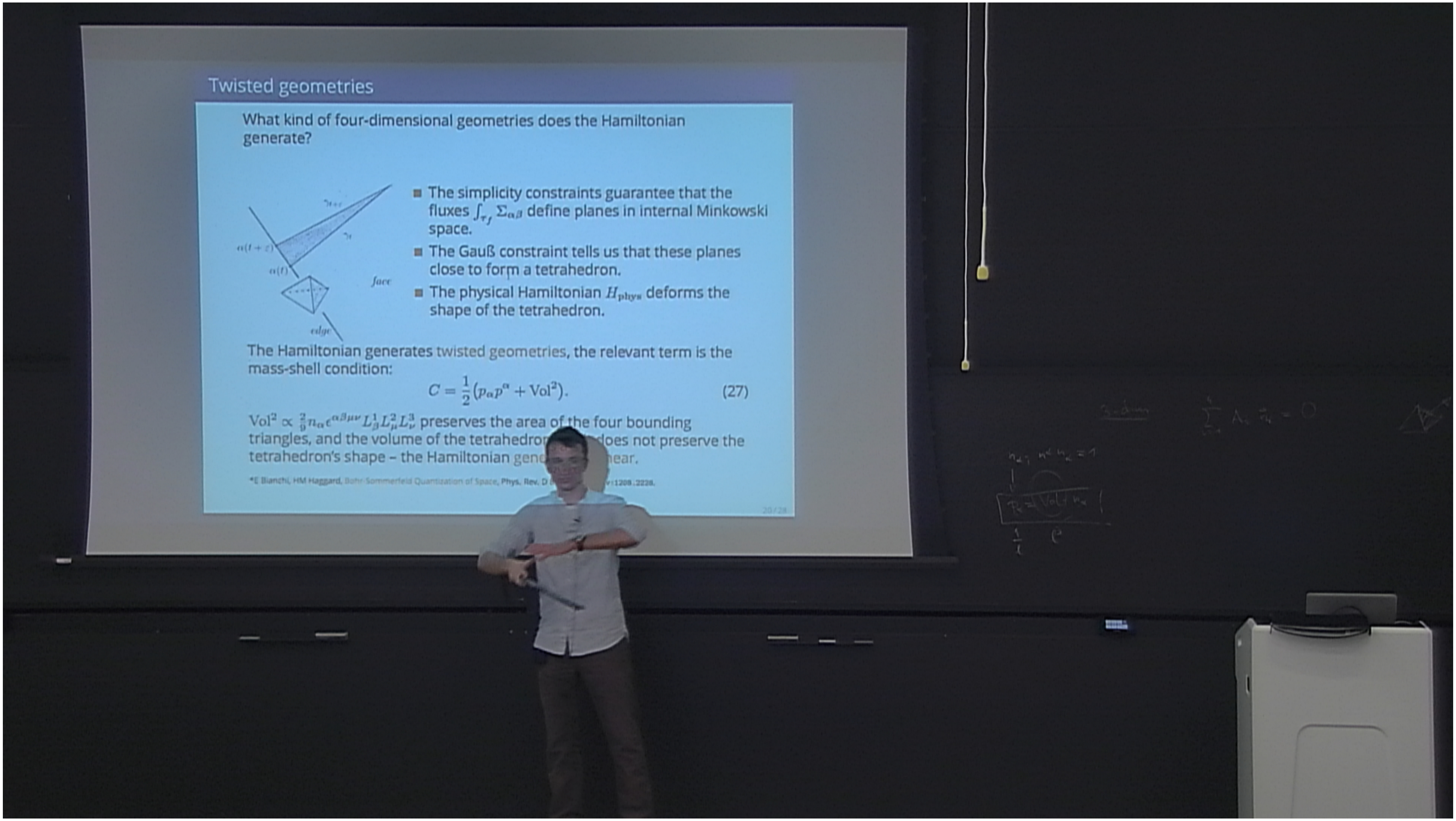

Hamiltonian formulation
The Hamiltonian:
$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + z^f W_{ef} + \bar{z}^f \bar{W}_{ef} + \lambda^f V_f \right) + N C_e,$ (22)
generates the t-evolution along the edges of the discretization:
$\frac{\mathrm{d}}{\mathrm{d}t} \omega_f^A = \{H, \omega_f^A\}.$ (23)
The fundamental Poisson brackets are:
face
edge


Dirac analysis
- The Hamiltonian preserves all constraints provided $z_f = 0$.
- There are no secondary constraints.
Physical Hamiltonian
$$H_{\text{phys}} = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \left(\zeta^f \Delta_f + \bar{\zeta}^f \bar{\Delta}_f + \lambda^f V_f \right) + NC. \quad (25)$$
second-class simplicity constraint: $W_{ef} = p_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \overset{!}{=} 0,$
first-class simplicity constraint: $V_f = \frac{i}{\beta + i} \pi^f_A \omega^A_f + \text{cc.} \overset{!}{=} 0,$
area-matching condition (first-class): $\Delta_f = \underline{\pi}^f_A \underline{\omega}^A_f - \pi^f_A \omega^A_f \overset{!}{=} 0,$
mass-shell condition (first-class): $C_e = \frac{1}{2}\left(p^e_\alpha p^\alpha_e + \mathrm{Vol}^2(e)\right) \overset{!}{=} 0,$
$SU(2)_n$ Gauß constraint (first-class): $G^e_\alpha = \sum_{f:\partial f \supset e} \tau^{AB}{}_\alpha \omega^f_A \pi^f_B + \text{cc.}$
Notation:
$\tau^A{}_{B\alpha}$ are the $SU(2)_n$ generators: $[\tau_\alpha, \tau_\beta] = n^\mu \epsilon_{\mu\alpha\beta}{}^\nu \tau_\nu$.
$\mathrm{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \text{cc.}$



Twisted geometries
What kind of four-dimensional geometries does the Hamiltonian generate?
face
edge
The simplicity constraints guarantee that the fluxes $\int_{\tau_f} \Sigma_{\alpha\beta}$ define planes in internal Minkowski space.
The Gauß constraint tells us that these planes close to form a tetrahedron.
The physical Hamiltonian H_{phys} deforms the shape of the tetrahedron.
The Hamiltonian generates twisted geometries, the relevant term is the mass-shell condition:
$C = \frac{1}{2}(p_\alpha p^\alpha + \mathrm{Vol}^2)$. (27)
$\mathrm{Vol}^2 \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$ preserves the area of the four bounding triangles, and the volume of the tetrahedron does not preserve the tetrahedron's shape – the Hamiltonian