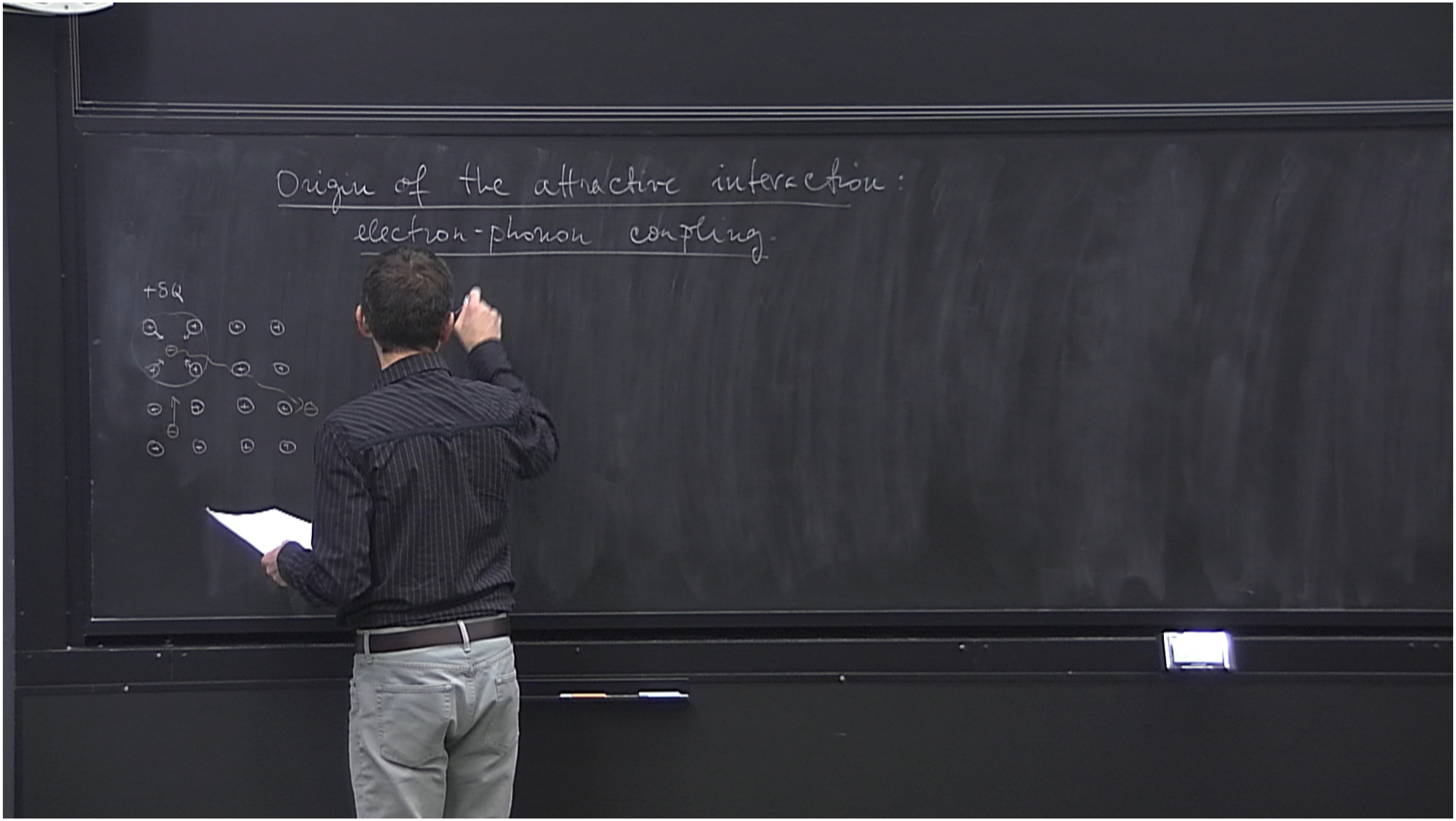
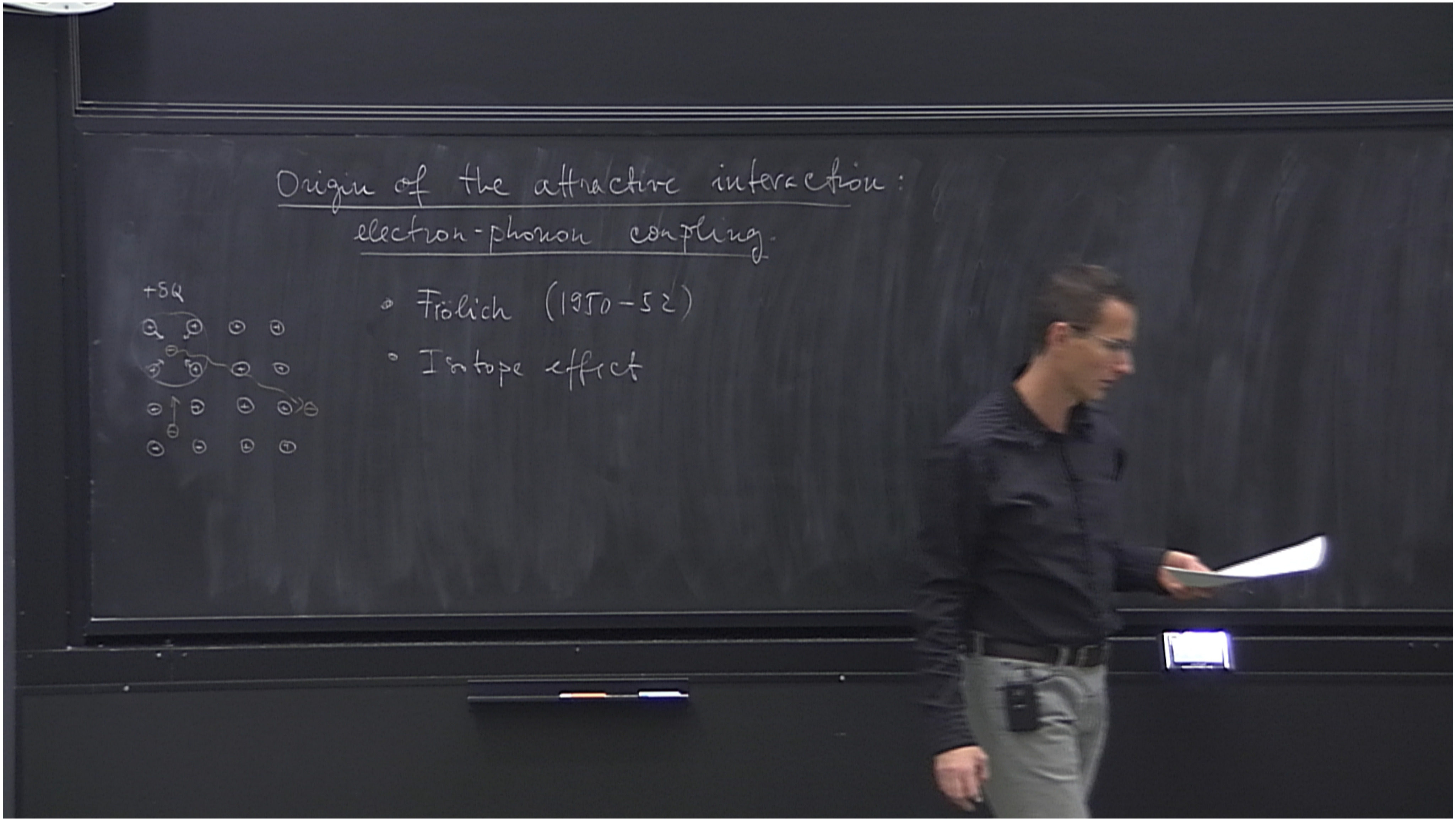
Title: Condensed Matter-14

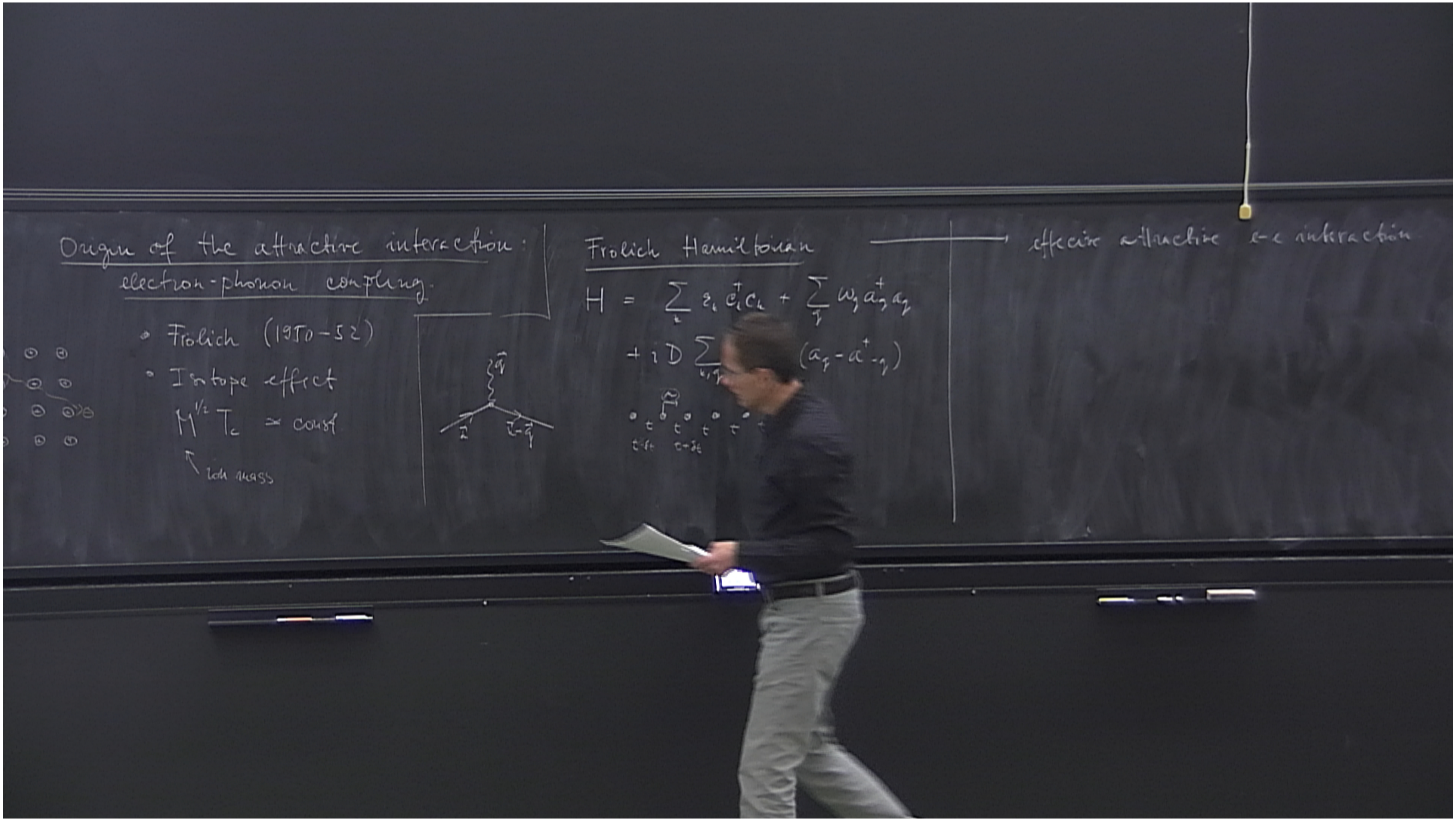
Date: Nov 27, 2014 10:30 AM

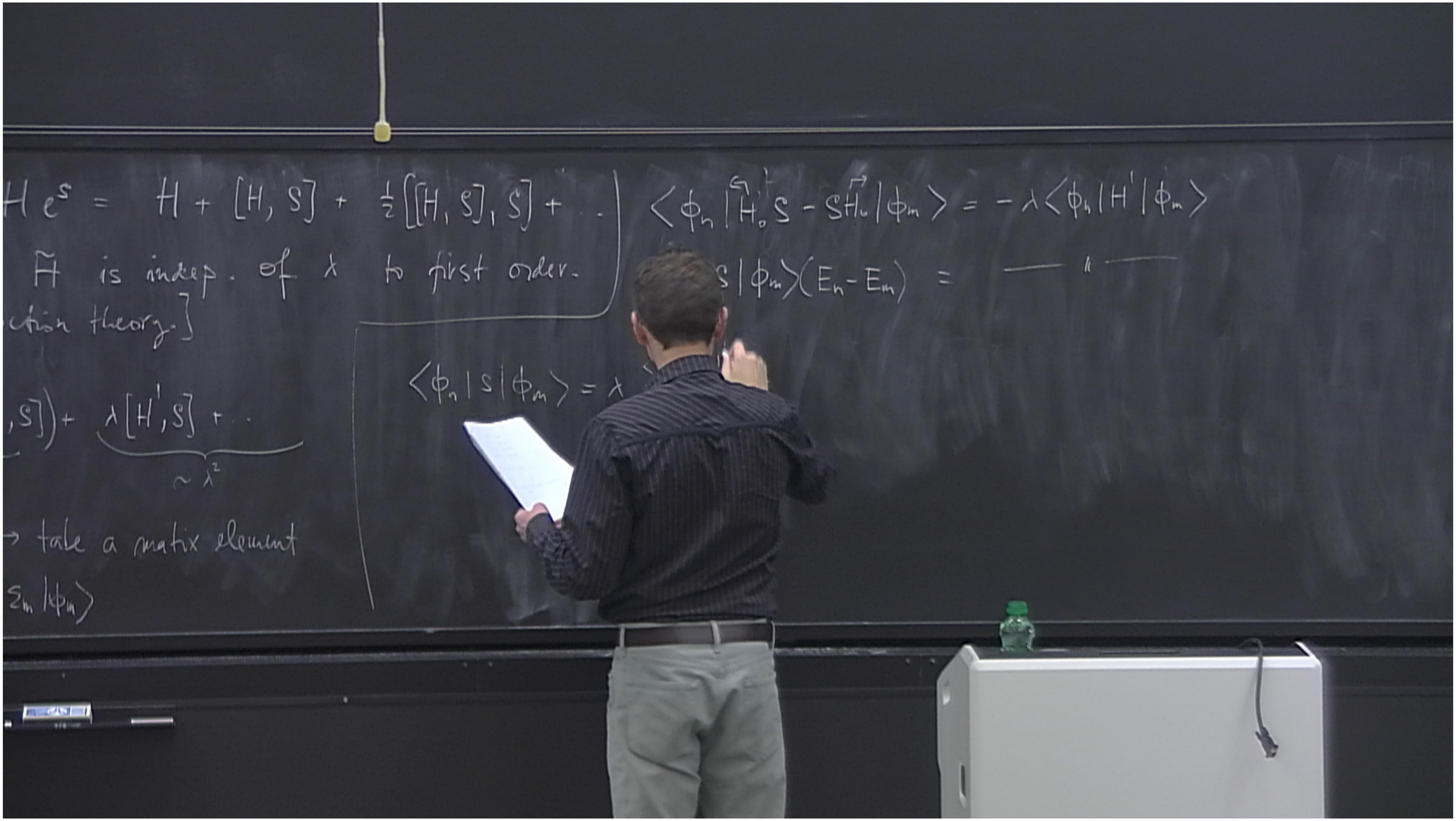
URL: http://pirsa.org/14110042

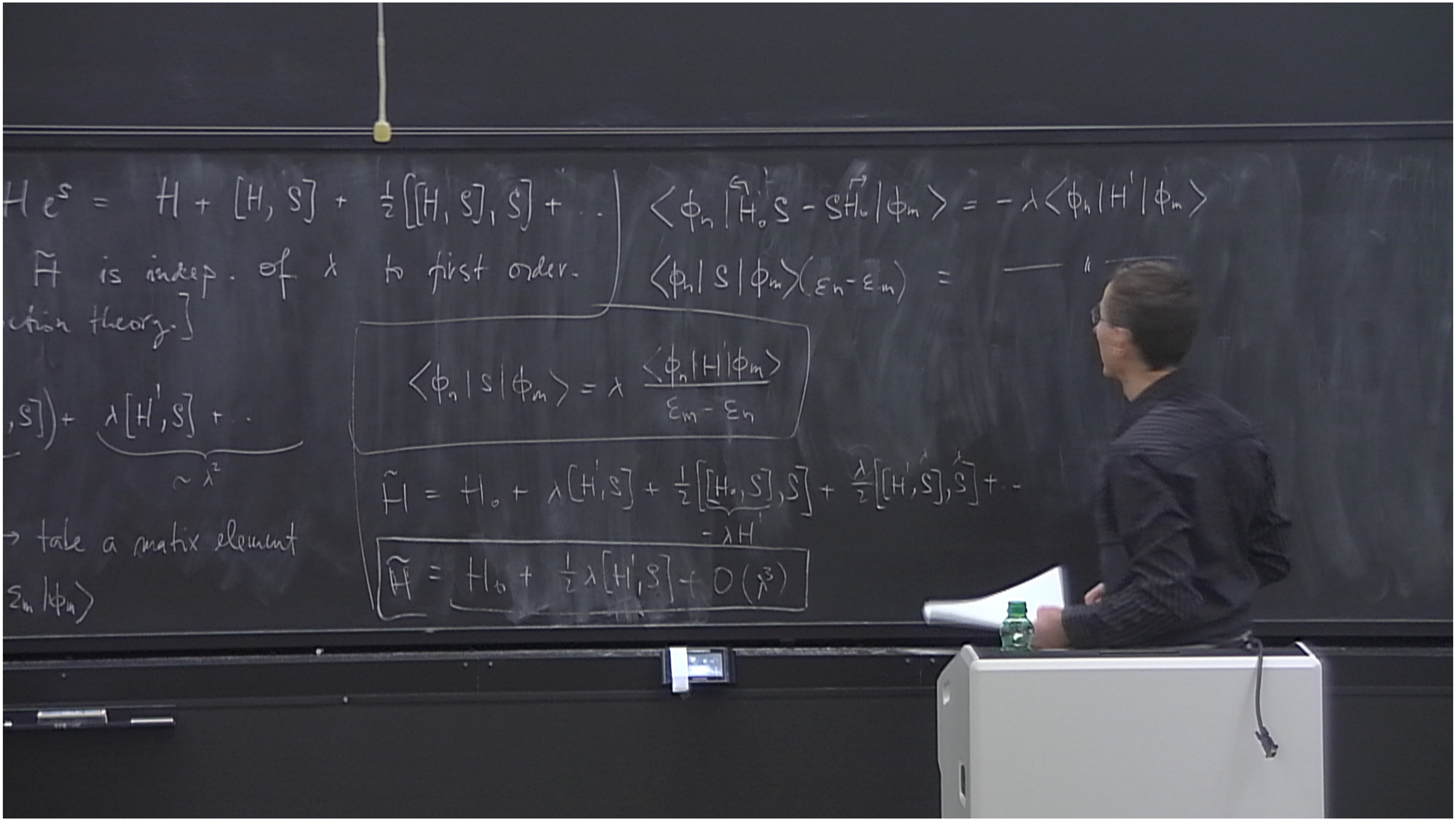
Abstract:

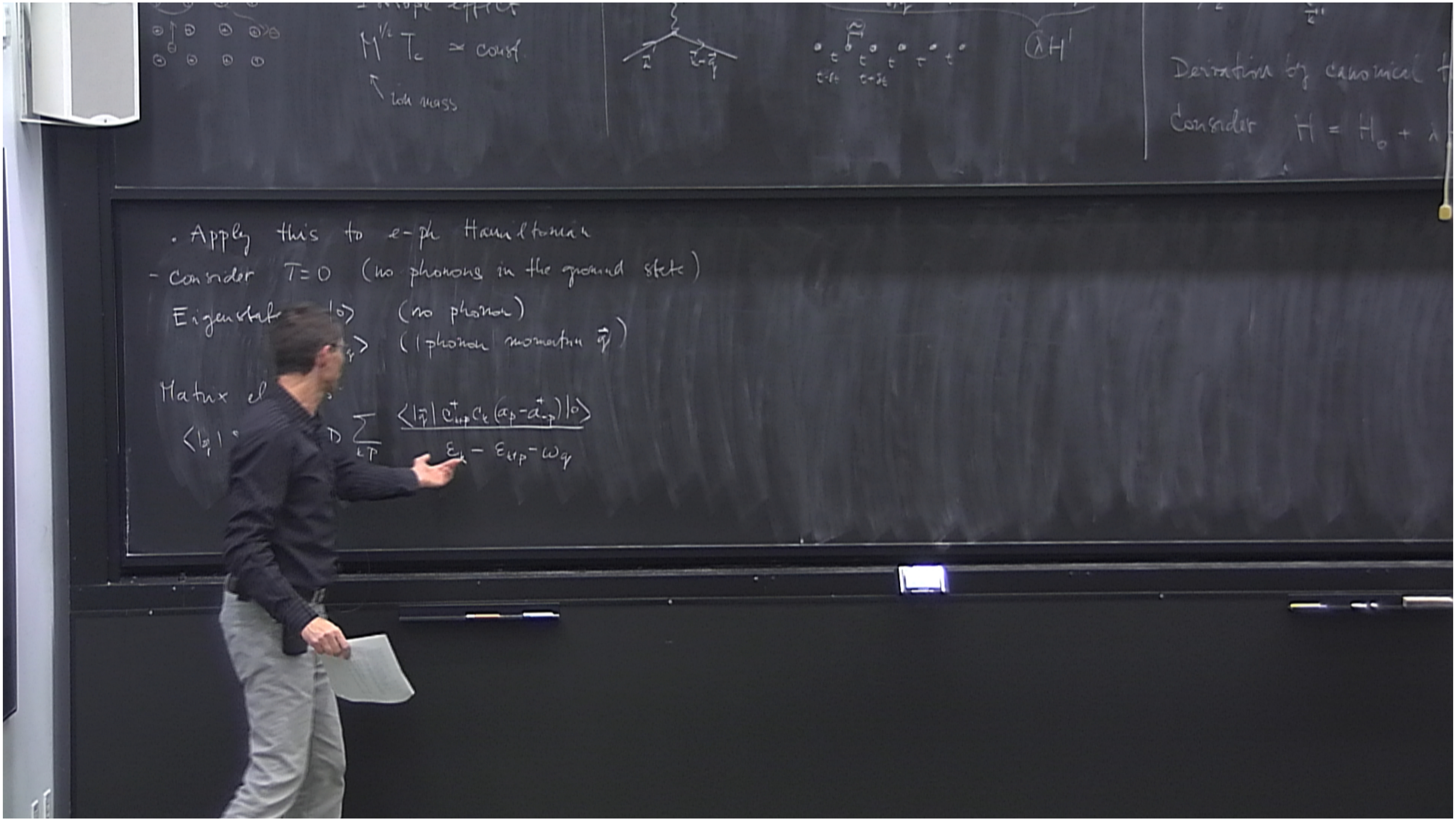

Origin of the attractive interaction:
electron-phonon coupling.
+δQ


Origin of the attractive interaction:
electron-phonon coupling.
+δQ
• Frölich (1950-52)
• Isotope effect

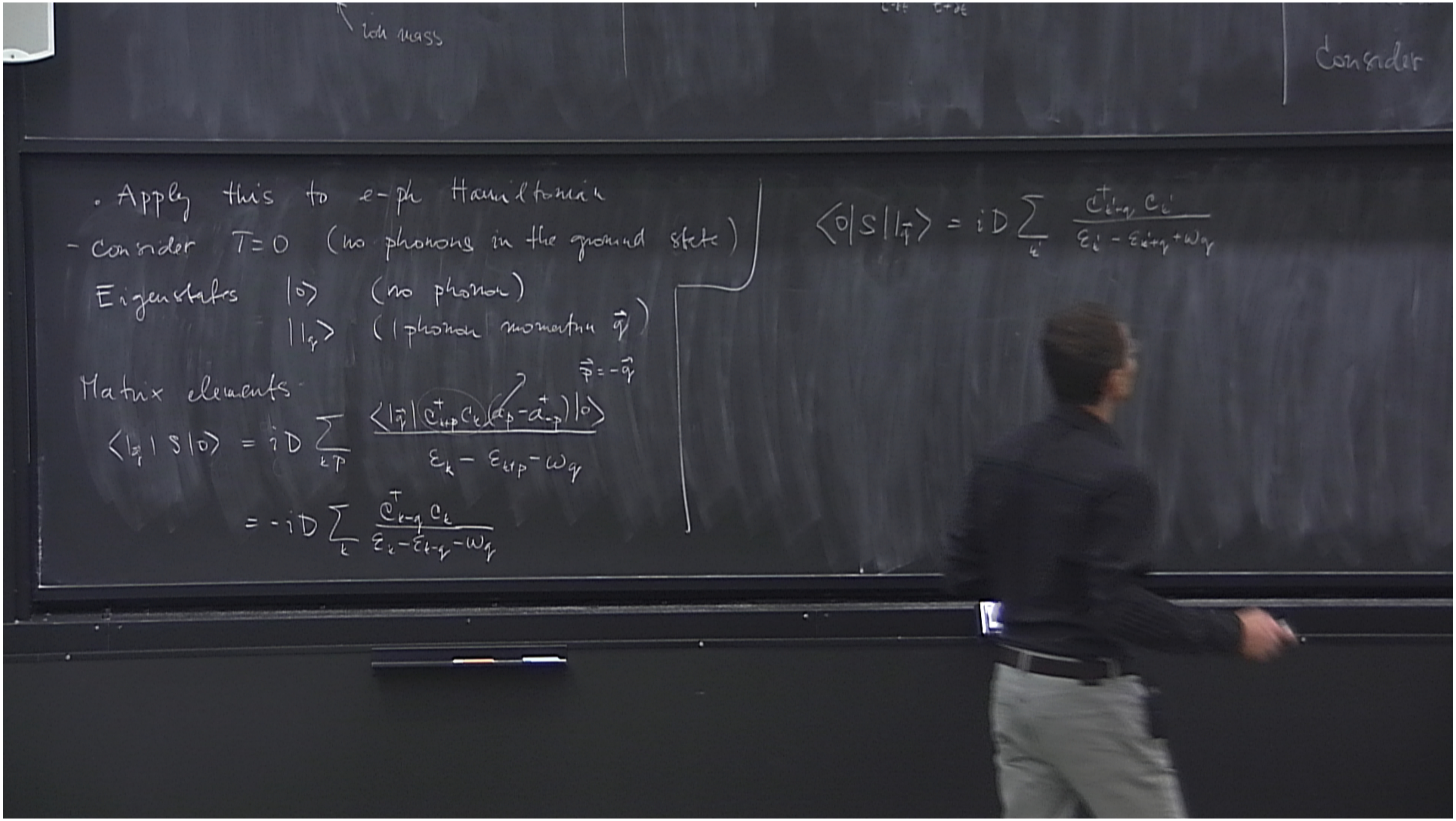

Origin of the attractive interaction:
electron-phonon coupling.
Frolich (1950-52)
Isotope effect
ion mass
Frolich Hamiltonian
effective attractive e-e interaction

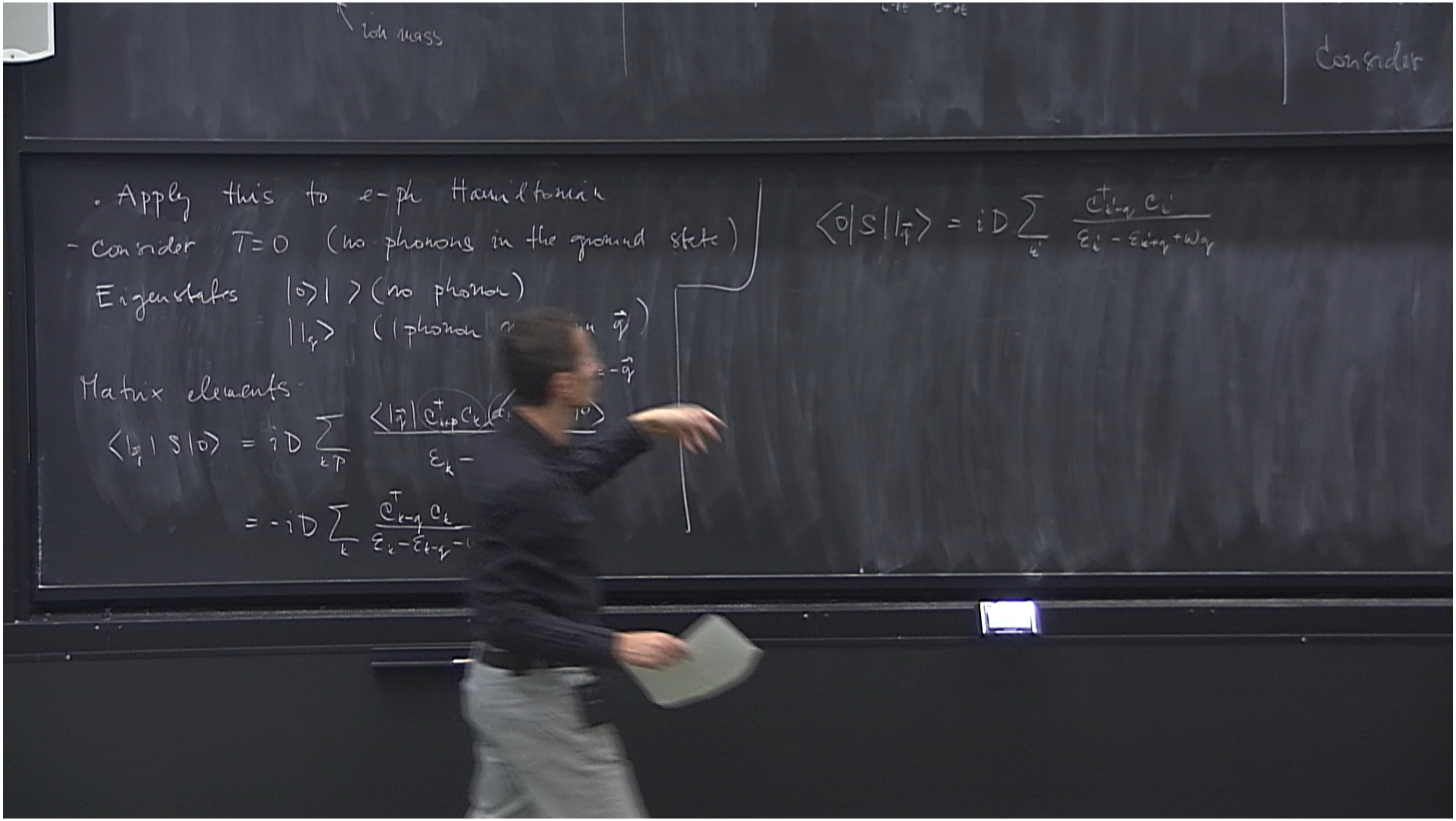

H e^S = H + [H, S] + ½[[H, S], S] + . .
H̃ is indep. of λ to first order.
ction theory.]
, S]) + λ[H', S] + . .
~ λ²
→ take a matrix element
E_m |φ_m⟩
⟨φ_n | H_0 S - S H_0 | φ_m⟩ = -λ⟨φ_n | H' | φ_m⟩
S | φ_m⟩(E_n - E_m) =
⟨φ_n | S | φ_m⟩ = λ


H̃ is indep. of λ to first order.
take a matrix element

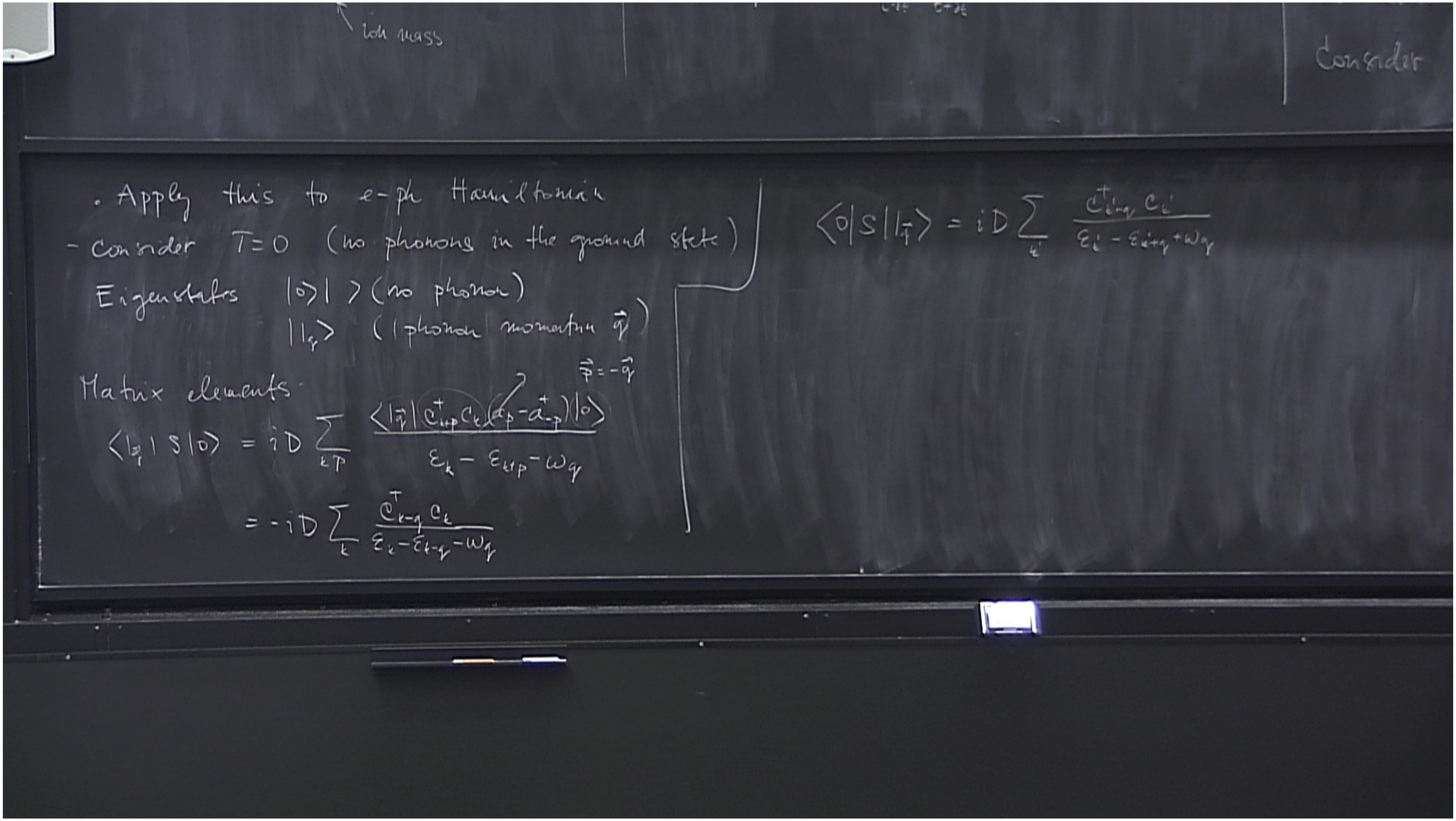


• Apply this to e-ph Hamiltonian
- Consider T=0 (no phonons in the ground state)
(no phonon)
(1 phonon momentum q)
ion mass

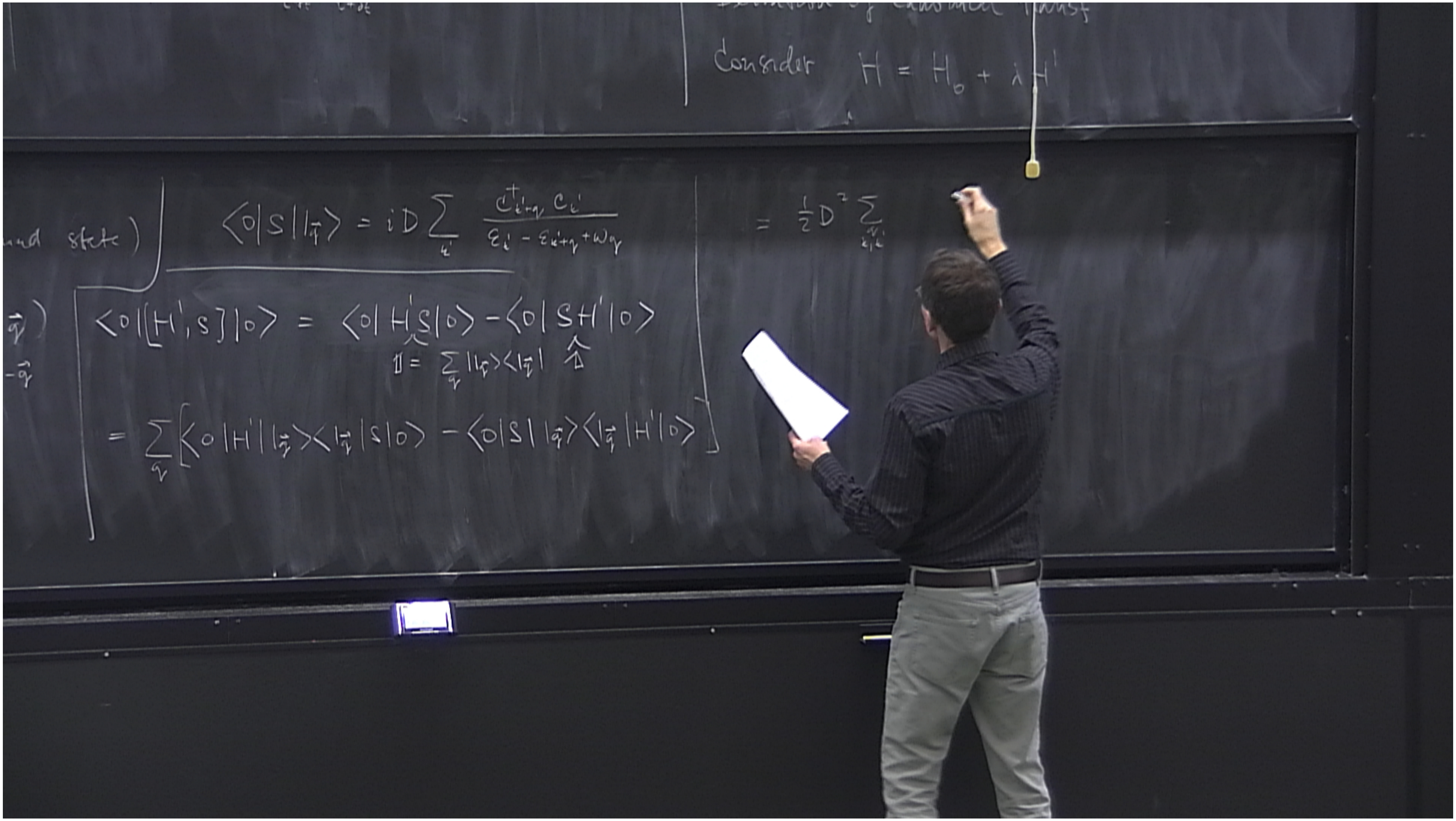

ion mass
Consider
. Apply this to e-ph Hamiltonian
- Consider T=0 (no phonons in the ground state)
Eigenstates |0> (no phonon)
|1_q> (1 phonon momentum q)
p = -q
Matrix elements
<1_q | S | 0> = iD Σ_{kp} <1_q| c†_{k+p} c_k (a_p - a†_{-p}) |0> / (ε_k - ε_{k+p} - ω_q)
= -iD Σ_k c†_{k-q} c_k / (ε_k - ε_{k-q} - ω_q)
<0|S|1_q> = iD Σ_{k'} c†_{k'+q} c_{k'} / (ε_{k'} - ε_{k'+q} + ω_q)

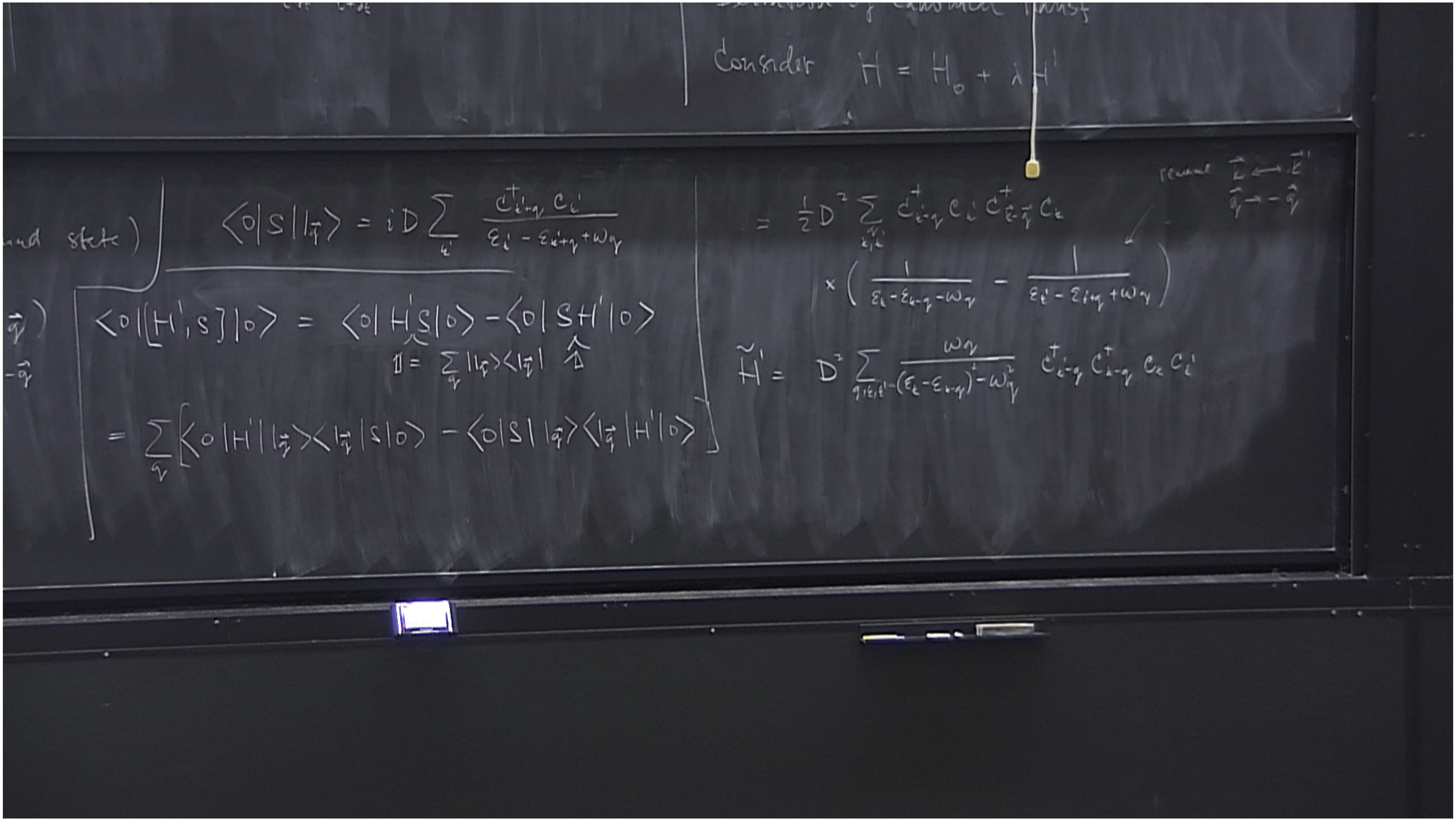


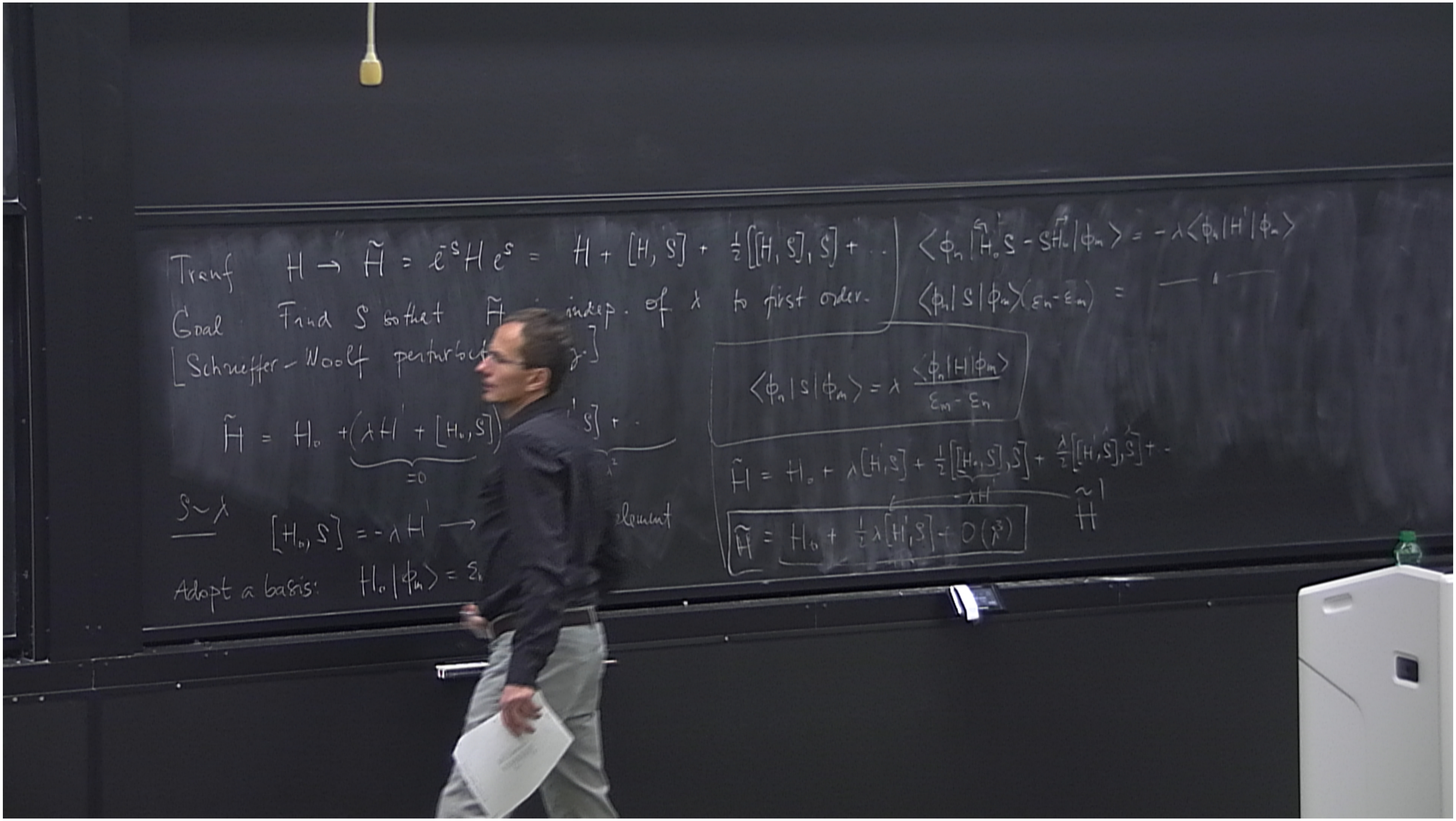
Consider
. Apply this to e-ph Hamiltonian
- Consider T=0 (no phonons in the ground state)
Eigenstates |0>| > (no phonon)
|1_q> (1 phonon
Matrix elements
<0|S|1_q> = iD ∑_k' c†_{k'-q} c_k' / (ε_k' - ε_{k'+q} + ω_q)

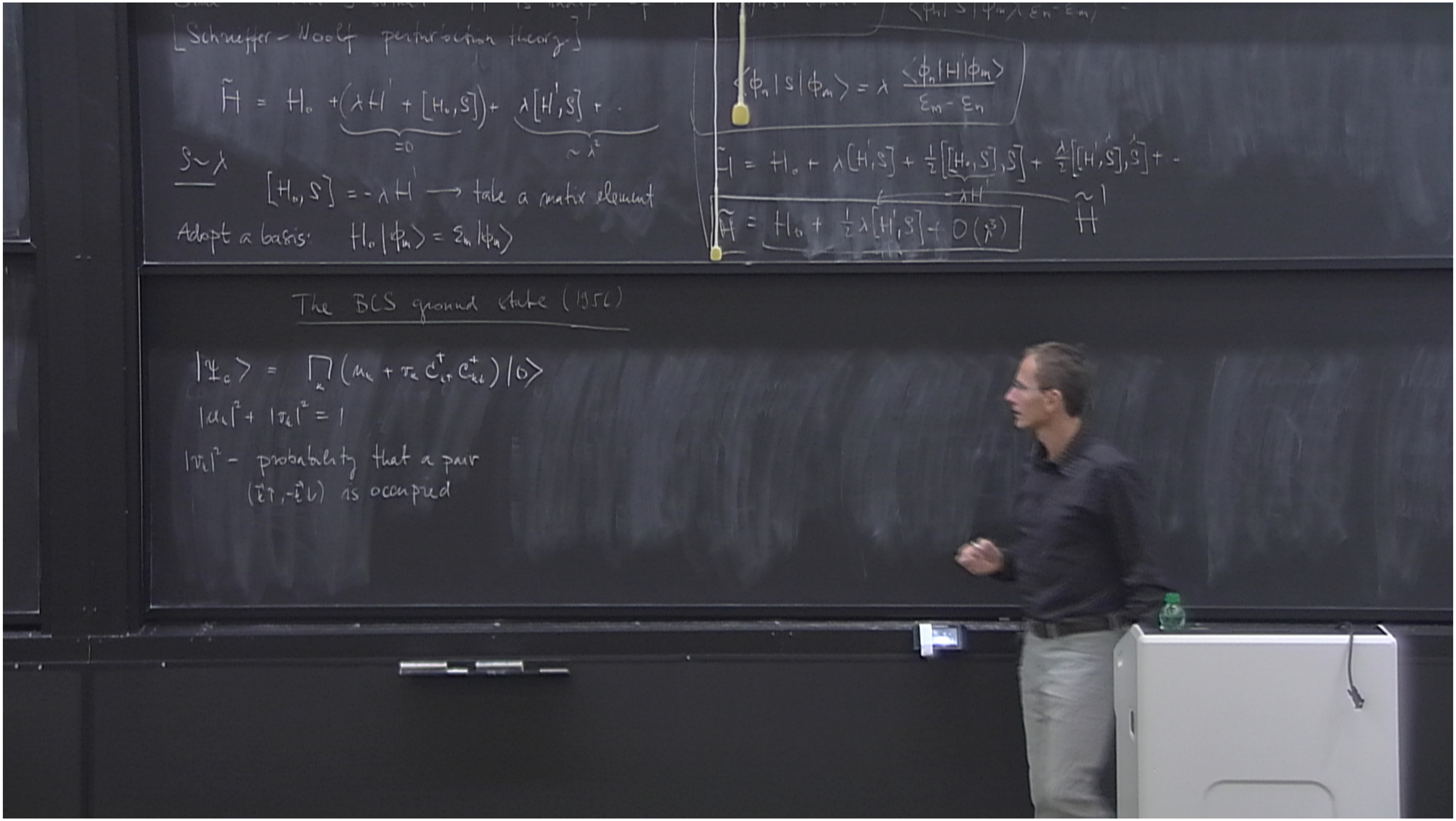


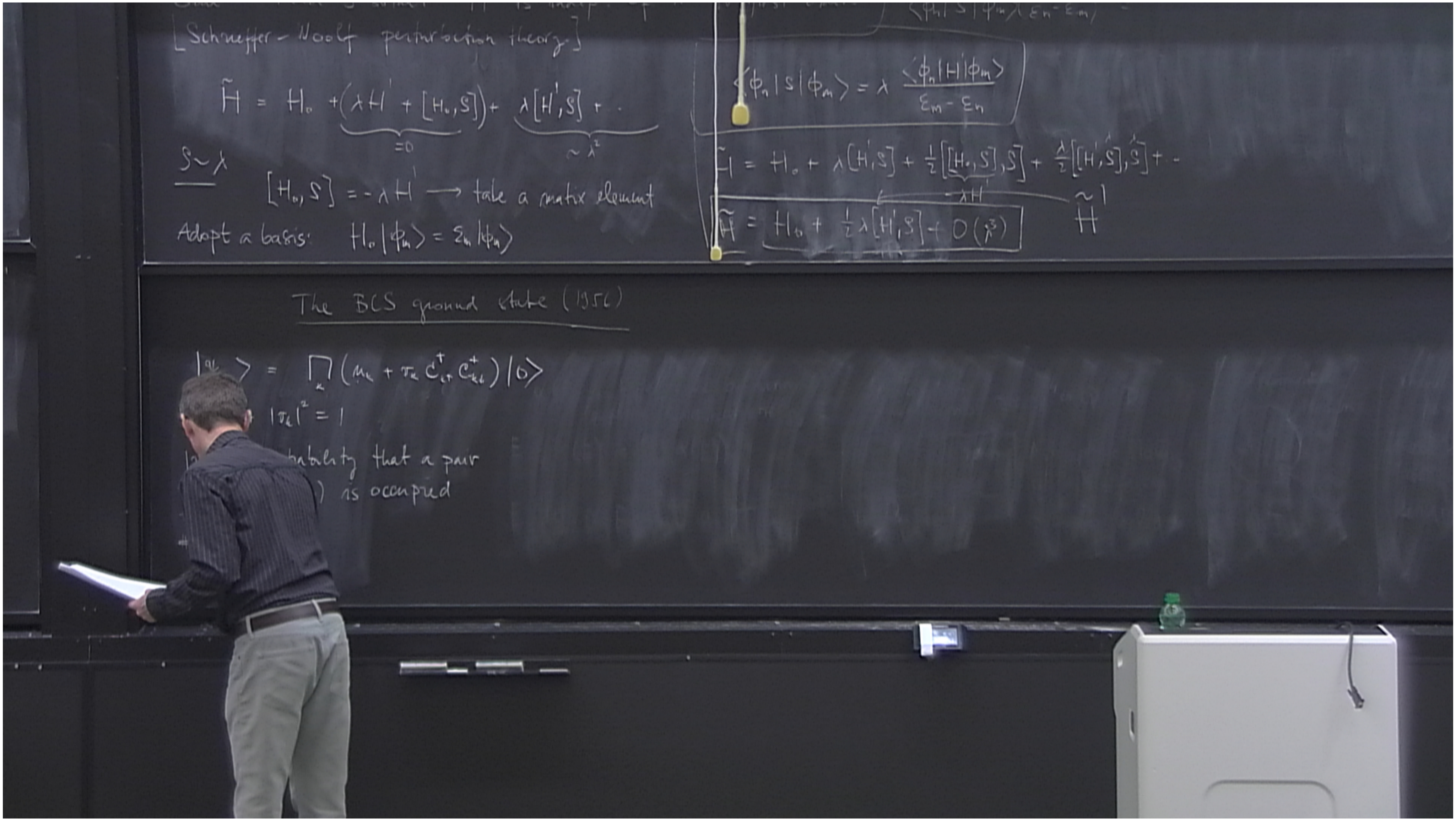
10h mass
Consider
• Apply this to e-ph Hamiltonian
- Consider T=0 (no phonons in the ground state)
Eigenstates $|0\rangle|\ \rangle$ (no phonon)
$|1_q\rangle$ (1 phonon momentum $\vec{q}$)
$\vec{p} = -\vec{q}$
Matrix elements
$\langle 1_q | S | 0 \rangle = iD \sum_{kp} \frac{\langle 1_{\vec{q}} | c^\dagger_{k+p} c_k (a_p - a^\dagger_{-p}) | 0 \rangle}{\varepsilon_k - \varepsilon_{k+p} - \omega_q}$
$= -iD \sum_k \frac{c^\dagger_{k-q} c_k}{\varepsilon_k - \varepsilon_{k-q} - \omega_q}$
$\langle 0 | S | 1_{\vec{q}} \rangle = iD \sum_{k'} \frac{c^\dagger_{k'+q} c_{k'}}{\varepsilon_{k'} - \varepsilon_{k'+q} + \omega_q}$

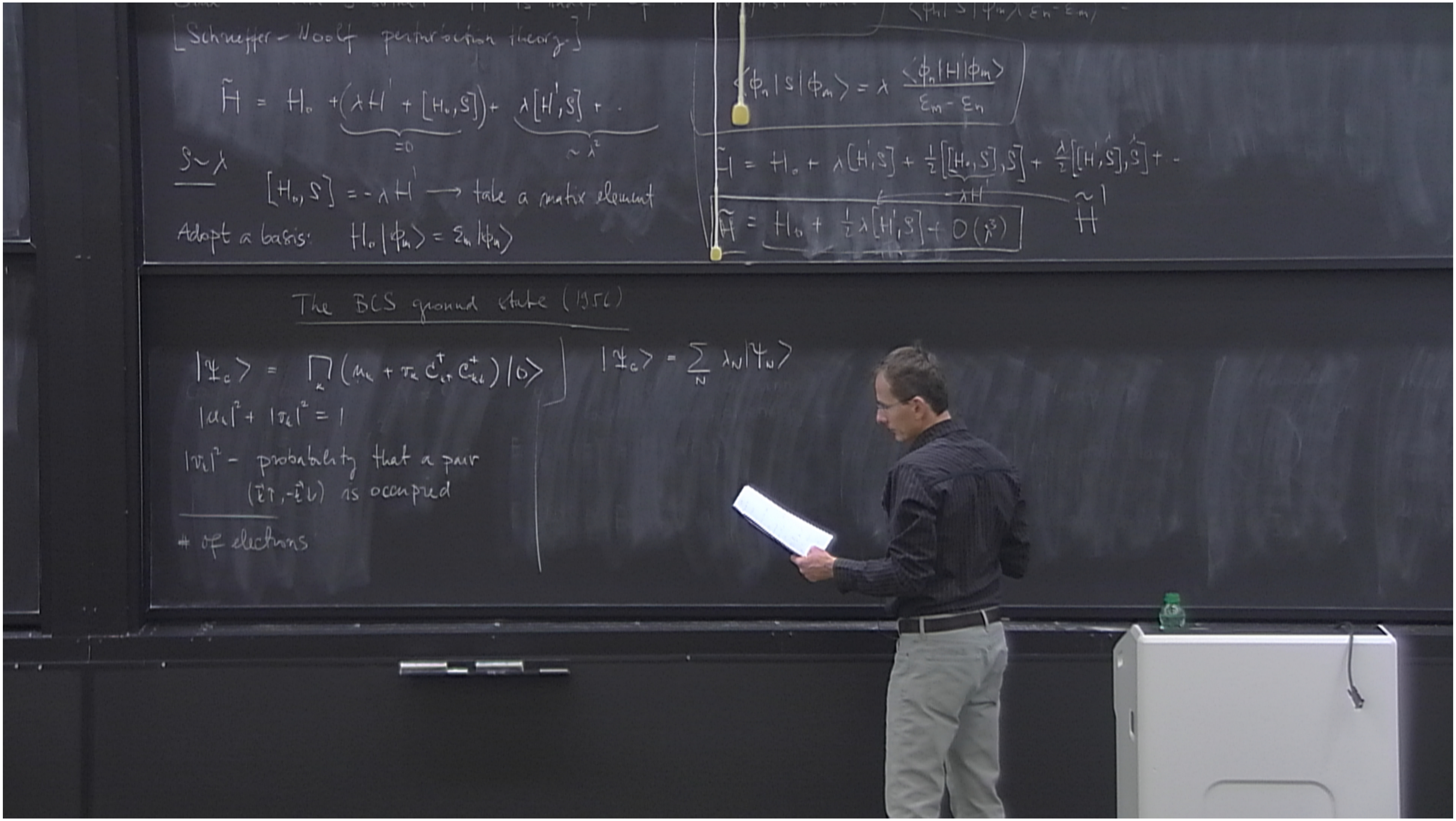

Consider H = H0 + λH'

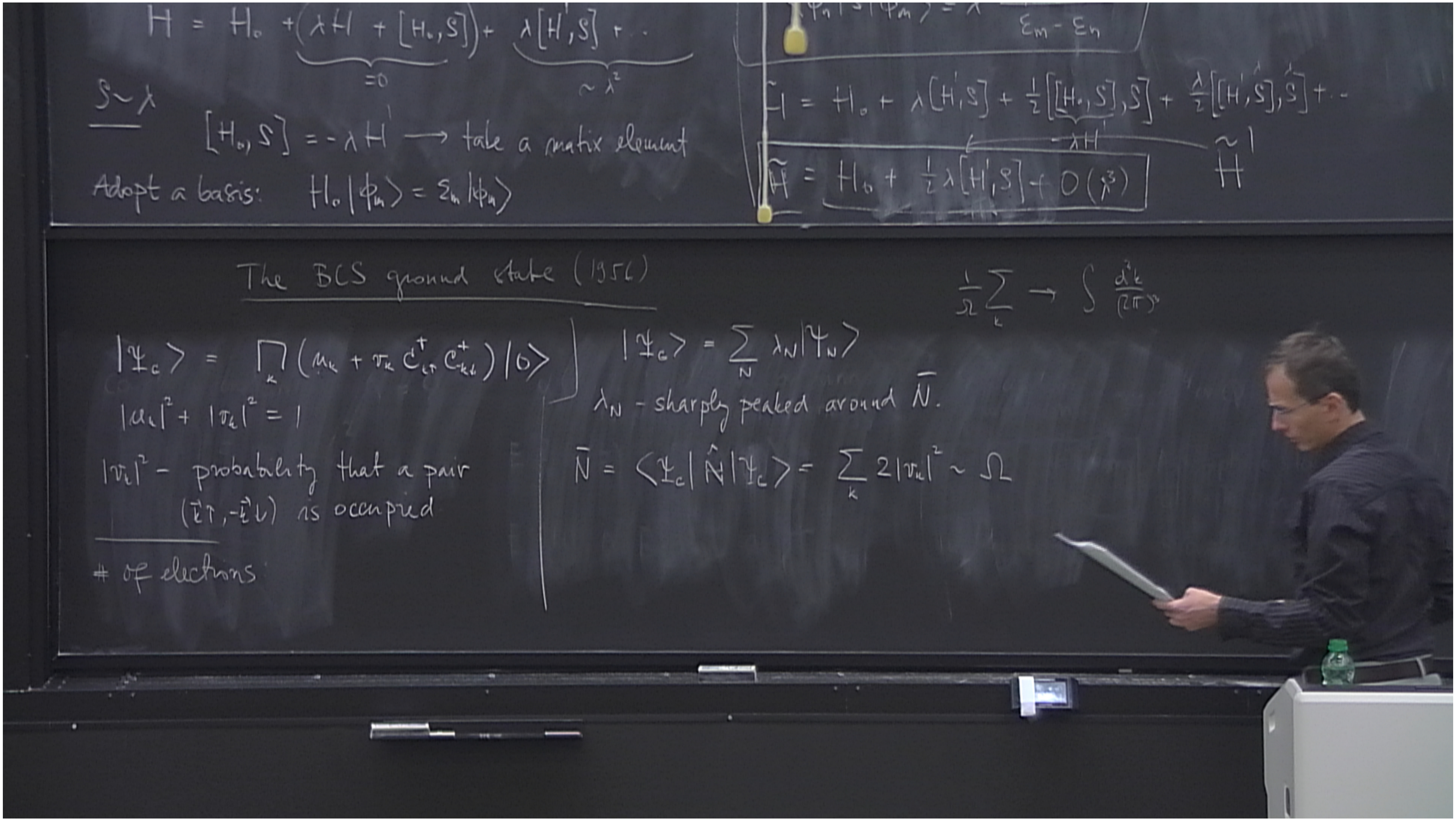

Consider $H = H_0 + \lambda H'$
$\langle 0|S|\vec{q}\rangle = iD\sum_{k'} \frac{c^\dagger_{k'+q}\, c_{k'}}{\varepsilon_{k'} - \varepsilon_{k'+q} + \omega_q}$
$\langle 0|[H', S]|0\rangle = \langle 0|H'S|0\rangle - \langle 0|SH'|0\rangle$
$\mathbb{1} = \sum_q |\vec{q}\rangle\langle\vec{q}|$
$= \sum_q \left[\langle 0|H'|\vec{q}\rangle\langle\vec{q}|S|0\rangle - \langle 0|S|\vec{q}\rangle\langle\vec{q}|H'|0\rangle\right]$
$= \frac{1}{2}D^2 \sum_{q,k,k'} c^\dagger_{k'+q}\, c_{k'}\, c^\dagger_{k-q}\, c_k$
$\times \left(\frac{1}{\varepsilon_k - \varepsilon_{k-q} - \omega_q} - \frac{1}{\varepsilon_{k'} - \varepsilon_{k'+q} + \omega_q} \right)$
rename $\vec{k} \leftrightarrow \vec{k}'$
$\vec{q} \to -\vec{q}$
$\tilde{H}' = D^2 \sum_{q,k,k'} \frac{\omega_q}{(\varepsilon_k - \varepsilon_{k-q})^2 - \omega_q^2}\, c^\dagger_{k'-q}\, c^\dagger_{k-q}\, c_k\, c_{k'}$

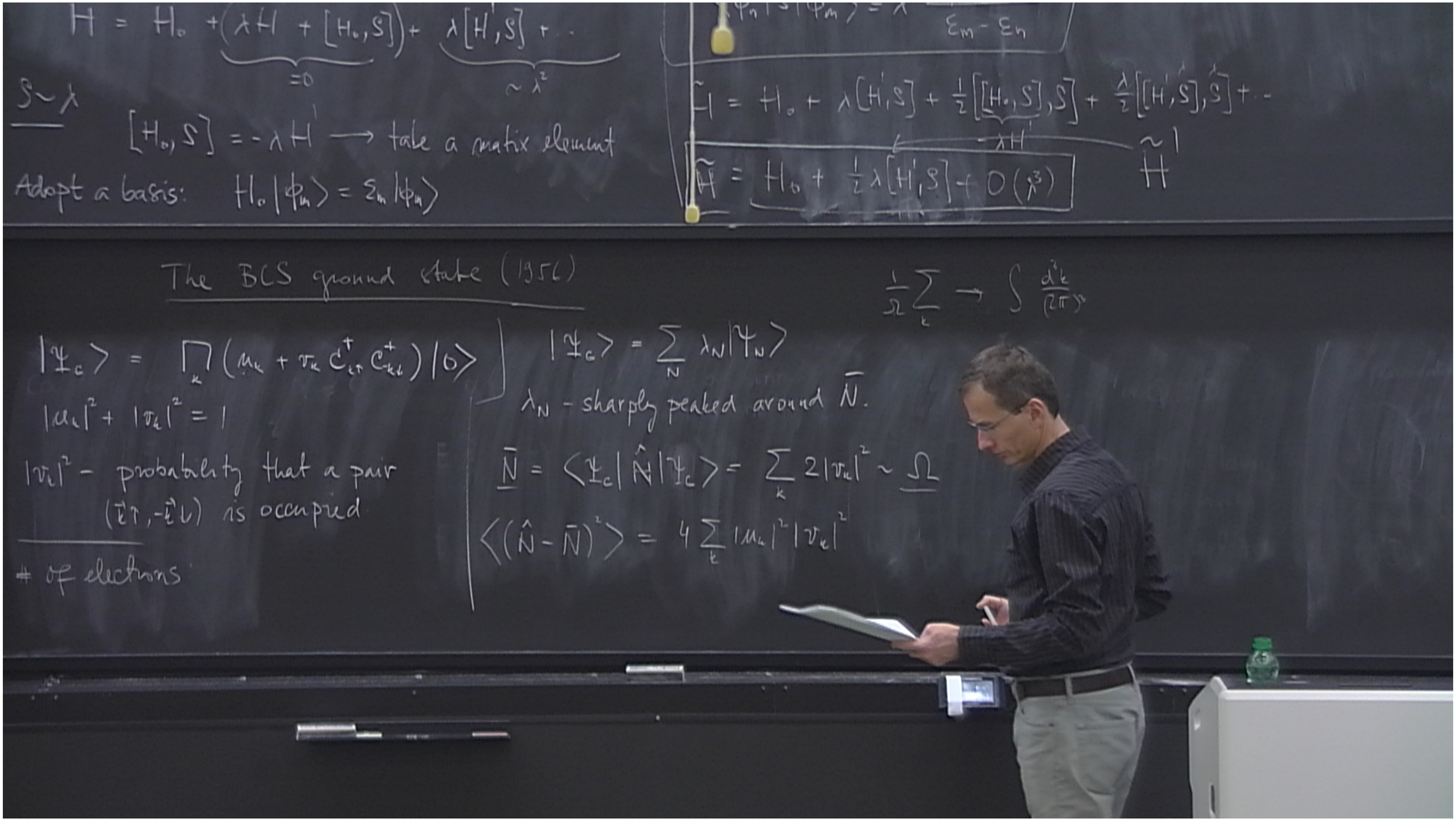

Tranf H → H̃ = e^{-S} H e^{S} = H + [H, S] + ½[[H, S], S] + ...
Goal Find S so that H̃ indep. of λ to first order.
[Schrieffer-Woolf
H̃ = H_0 + (λH' + [H_0, S])
=0
S~λ
[H_0, S] = -λH' →
element
Adopt a basis: H_0|φ_m⟩ =
⟨φ_n|H_0 S - S H_0|φ_m⟩ = -λ⟨φ_n|H'|φ_m⟩
⟨φ_n|S|φ_m⟩(ε_n - ε_m) =
⟨φ_n|S|φ_m⟩ = λ ⟨φ_n|H'|φ_m⟩ / (ε_m - ε_n)
H̃ = H_0 + λ[H', S] + ½[[H_0, S], S] + λ/2[[H', S], S] + ...
H̃ = H_0 + ½λ[H', S] + O(λ^3)

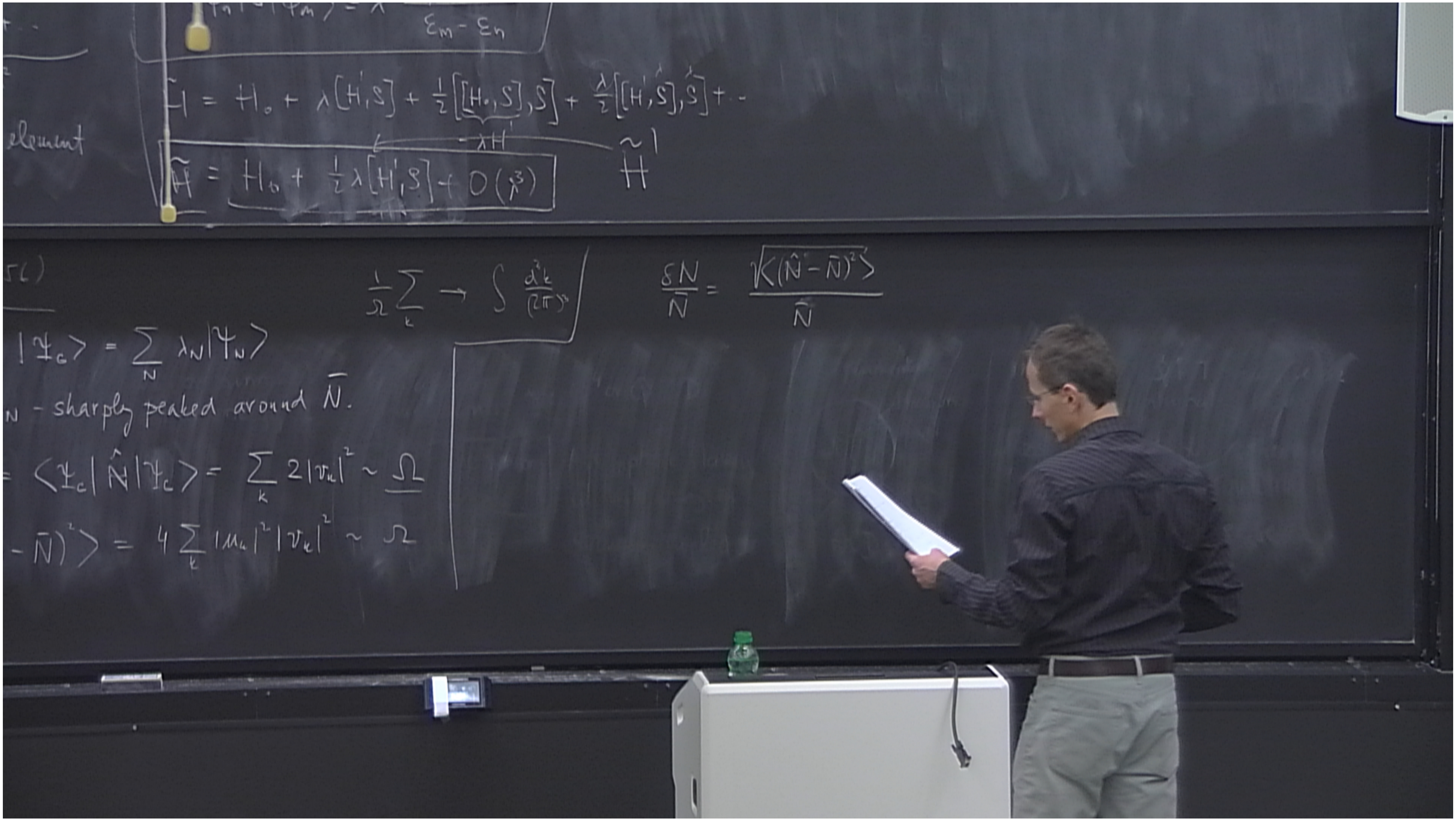

[Schrieffer - Woolf perturbation theory]
$\tilde{H} = H_0 + (\lambda H' + [H_0, S]) + \lambda[H', S] + \ldots$
$= 0$
$\sim \lambda^2$
$S \sim \lambda$
$[H_0, S] = -\lambda H' \longrightarrow$ take a matrix element
Adopt a basis: $H_0 |\phi_m\rangle = \varepsilon_m |\phi_m\rangle$
$\langle \phi_n | S | \phi_m \rangle = \lambda \frac{\langle \phi_n | H' | \phi_m \rangle}{\varepsilon_m - \varepsilon_n}$
$\tilde{H} = H_0 + \lambda[H', S] + \frac{1}{2}[[H_0, S], S] + \frac{\lambda}{2}[[H', S], S] + \ldots$
$-\lambda H'$
$\tilde{H} = H_0 + \frac{1}{2}\lambda[H', S] + O(\lambda^3)$
The BCS ground state (1956)
$|\Psi_G\rangle = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |0\rangle$
$|u_k|^2 + |v_k|^2 = 1$
$|v_k|^2$ - probability that a pair
$(\vec{k}\uparrow, -\vec{k}\downarrow)$ is occupied

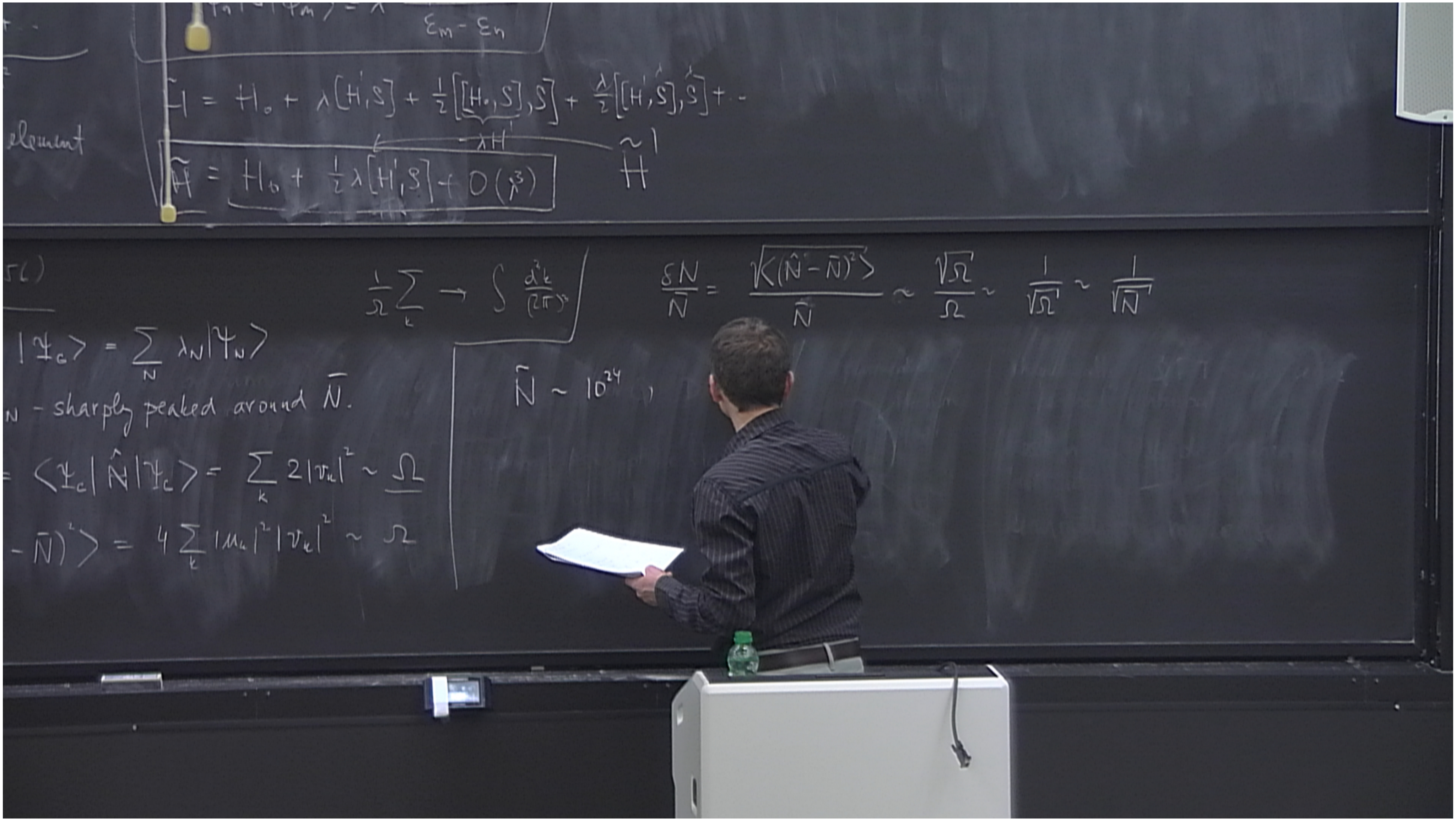

[Schrieffer - Wolff perturbation theory]
take a matrix element
Adopt a basis:
The BCS ground state (1956)
probability that a pair
is occupied

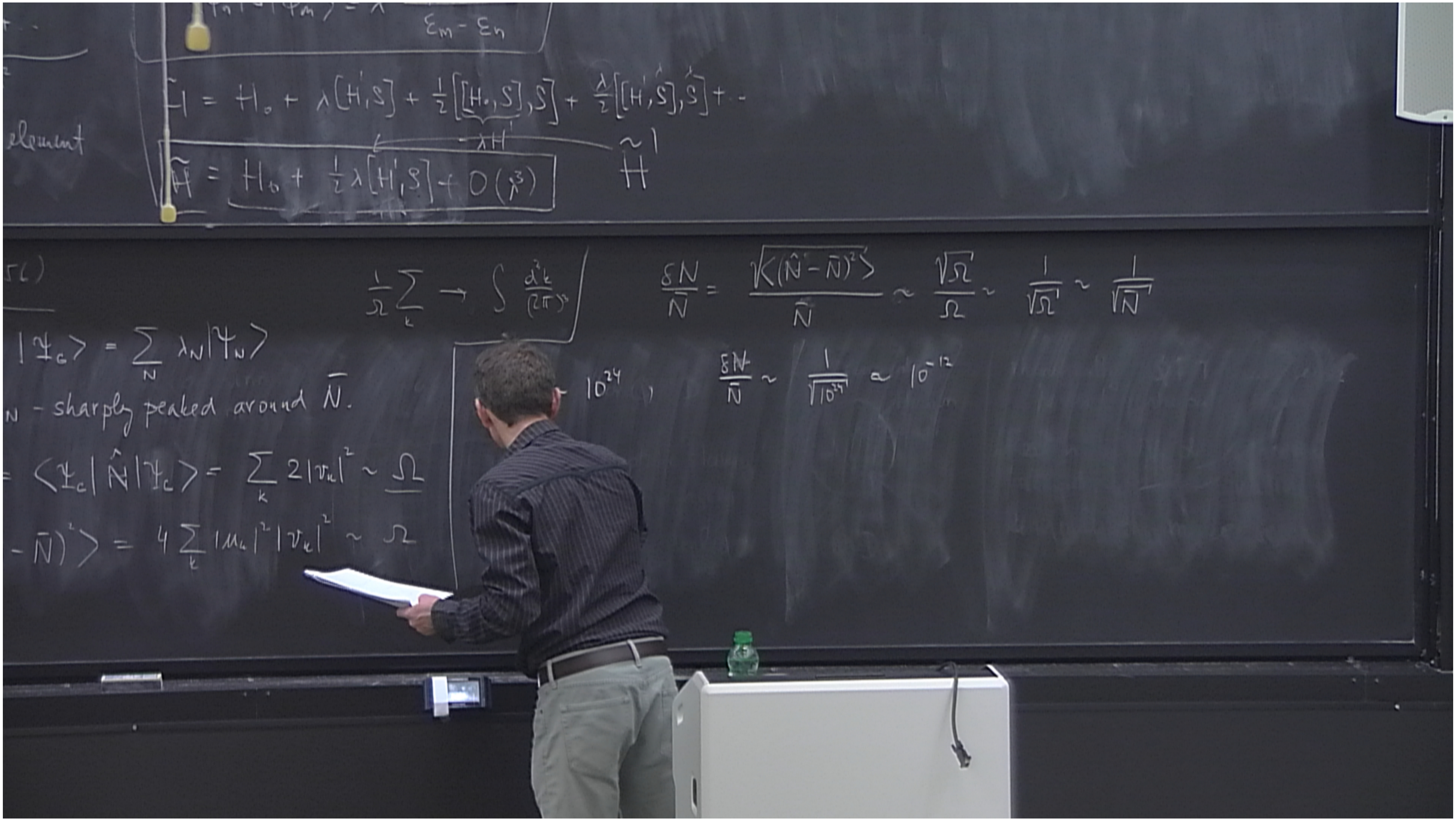

[Schrieffer – Wolff perturbation theory]
H̃ = H₀ + (λH' + [H₀, S]) + λ[H', S] + ...
=0
~λ²
S ~ λ
[H₀, S] = −λH' ⟶ take a matrix element
Adopt a basis: H₀|φₙ⟩ = εₙ|φₙ⟩
⟨φₙ|S|φₘ⟩ = λ ⟨φₙ|H'|φₘ⟩ / (εₘ − εₙ)
H̃ = H₀ + ½ λ[H', S] + O(λ³)
The BCS ground state (1956)
|Ψ_G⟩ = ∏ₖ (uₖ + vₖ c†ₖ↑ c†₋ₖ↓)|0⟩
|Ψ_G⟩ = Σ_N λ_N |Ψ_N⟩
|uₖ|² + |vₖ|² = 1
|vₖ|² – probability that a pair (k↑, −k↓) is occupied
# of electrons

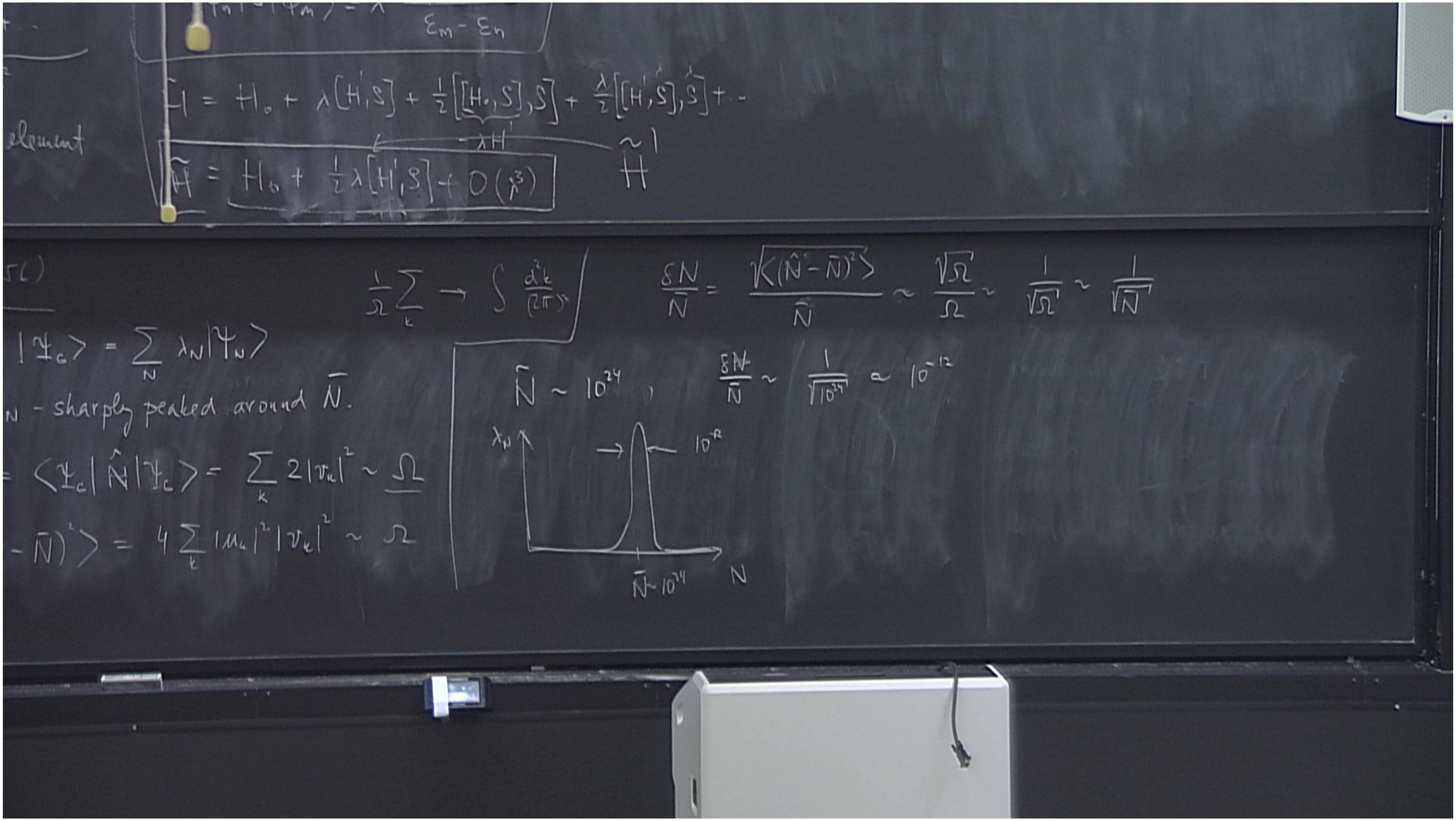

H = H₀ + (λH' + [H₀,S]) + λ[H',S] + ...
=0
~ λ²
S ~ λ
[H₀, S] = -λH' → take a matrix element
Adopt a basis: H₀|φₘ⟩ = εₘ|φₘ⟩
εₘ - εₙ
H̃ = H₀ + λ[H',S] + ½[[H₀,S],S] + λ/2[[H',S],S] + ..
-λH'
H̃ = H₀ + ½λ[H',S] + O(λ³)
H̃'
The BCS ground state (1956)
1/Ω Σₖ → ∫ d³k/(2π)³
|Ψc⟩ = ∏ₖ (uₖ + vₖ c†ₖ↑ c†₋ₖ↓)|0⟩
|Ψc⟩ = Σ_N λ_N|Ψ_N⟩
λ_N - sharply peaked around N̄.
|uₖ|² + |vₖ|² = 1
|vₖ|² - probability that a pair (k↑, -k↓) is occupied
N̄ = ⟨Ψc| N̂ |Ψc⟩ = Σₖ 2|vₖ|² ~ Ω
# of electrons

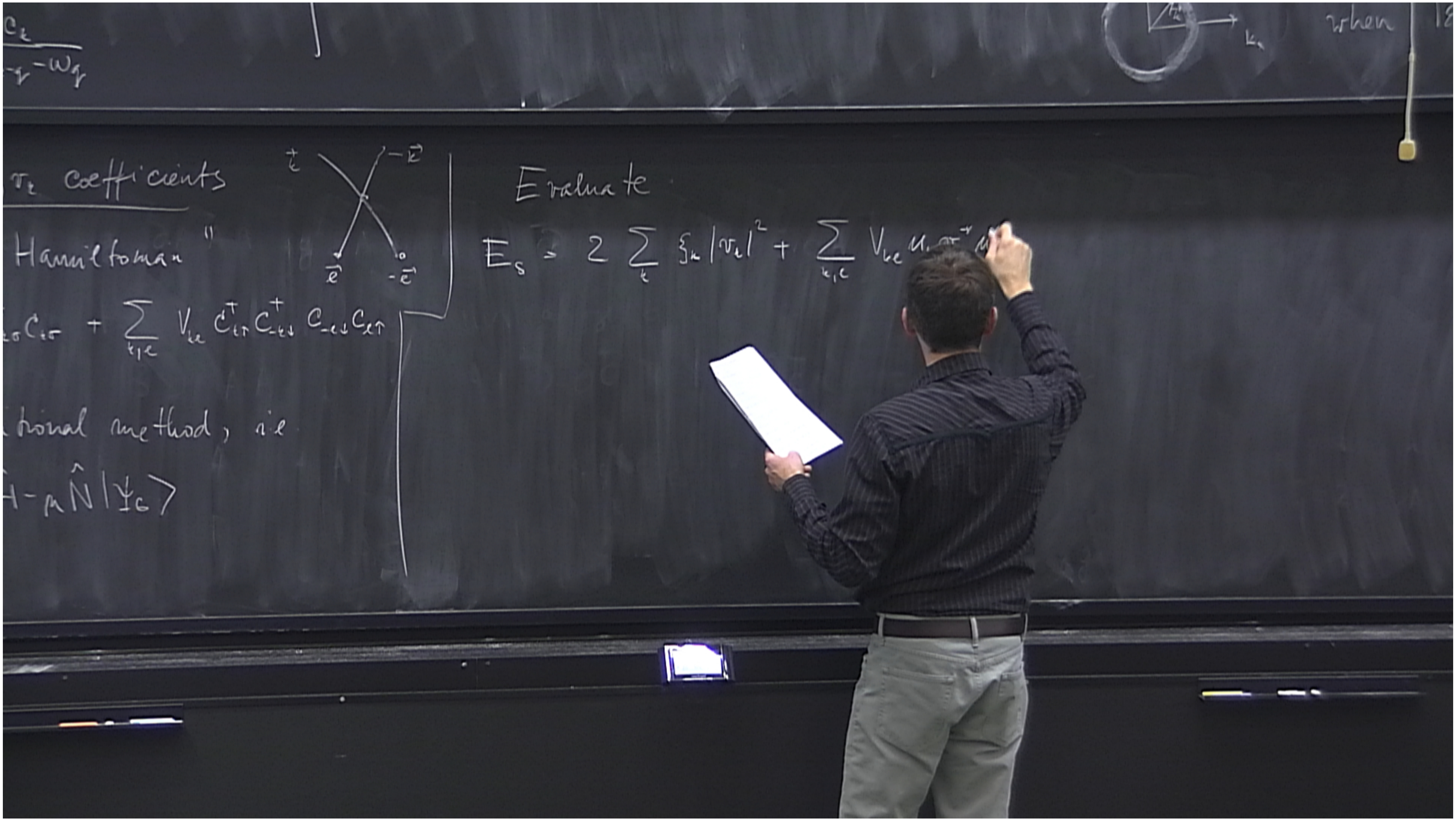

H = H₀ + (λH' + [H₀,S]) + λ[H',S] + ..
=0
~ λ²
S ~ λ
[H₀, S] = -λH' ⟶ take a matrix element
Adopt a basis: H₀|φₘ⟩ = εₘ|φₘ⟩
εₘ - εₙ
H̃ = H₀ + λ[H',S] + ½[[H₀,S],S] + λ/2[[H',S],S] + ..
-λH'
H̃ = H₀ + ½λ[H',S] + O(λ³)
H̃'
The BCS ground state (1956)
1/Ω Σₖ → ∫ d³k/(2π)³
|Ψ_c⟩ = Πₖ (uₖ + vₖ c†ₖ↑ c†₋ₖ↓)|0⟩
|uₖ|² + |vₖ|² = 1
|vₖ|² - probability that a pair (k↑, -k↓) is occupied
# of electrons
|Ψ_c⟩ = Σ_N λ_N|Ψ_N⟩
λ_N - sharply peaked around N̄.
N̄ = ⟨Ψ_c|N̂|Ψ_c⟩ = Σₖ 2|vₖ|² ~ Ω
⟨(N̂ - N̄)²⟩ = 4 Σₖ |uₖ|²|vₖ|²

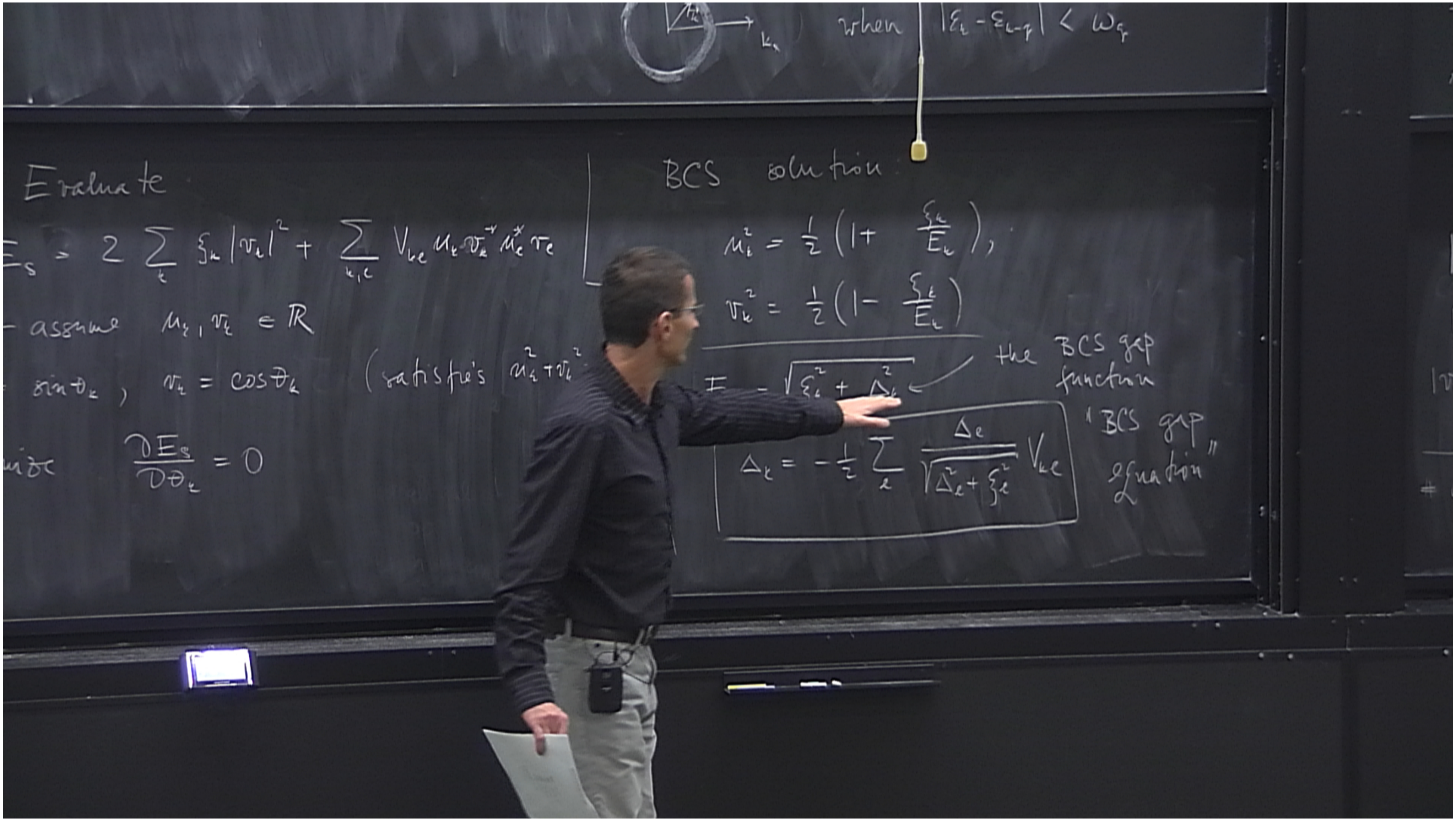

sharply peaked around N̄.

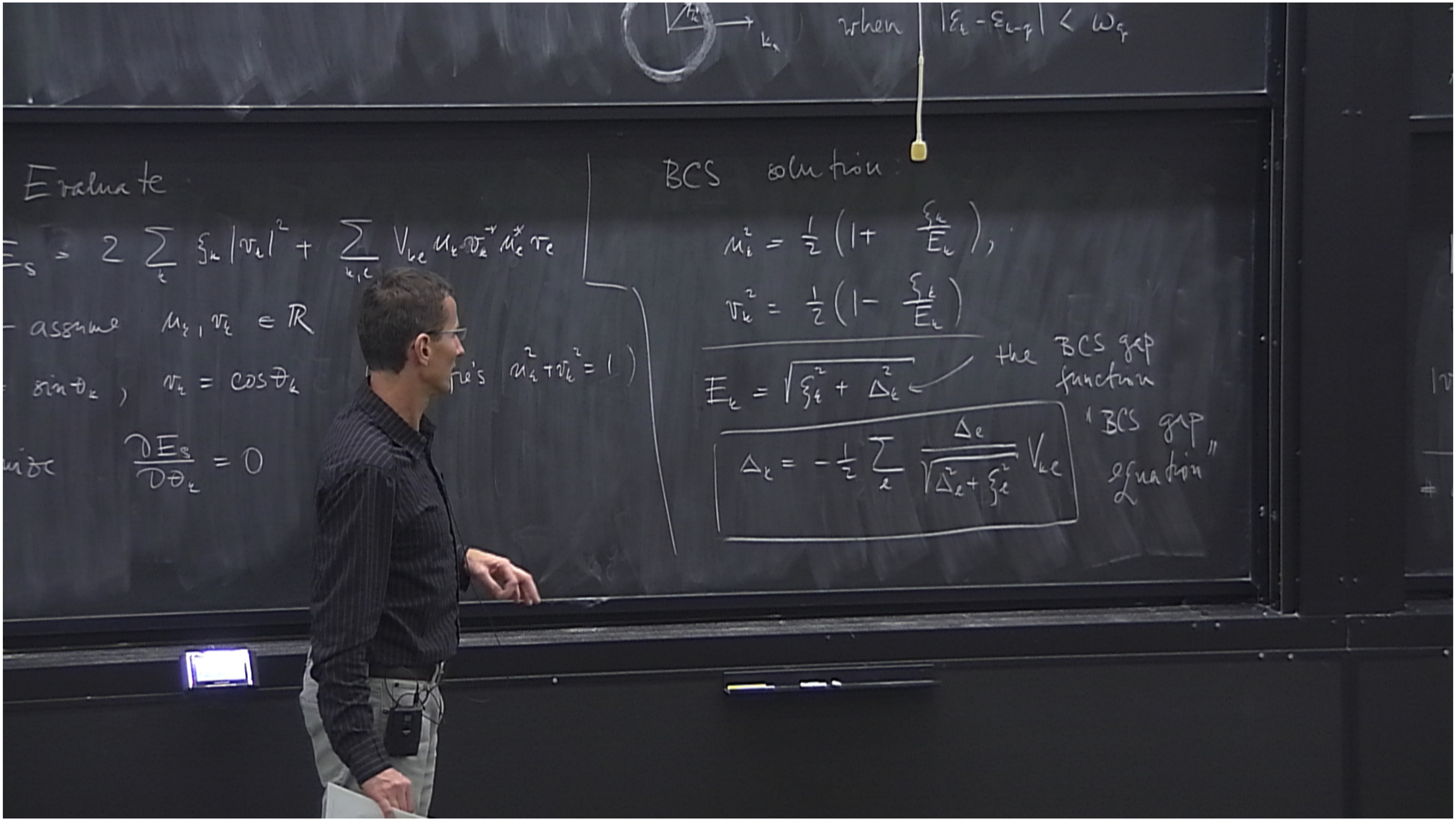

element
sharply peaked around N̄.
N̄ ~ 10^24

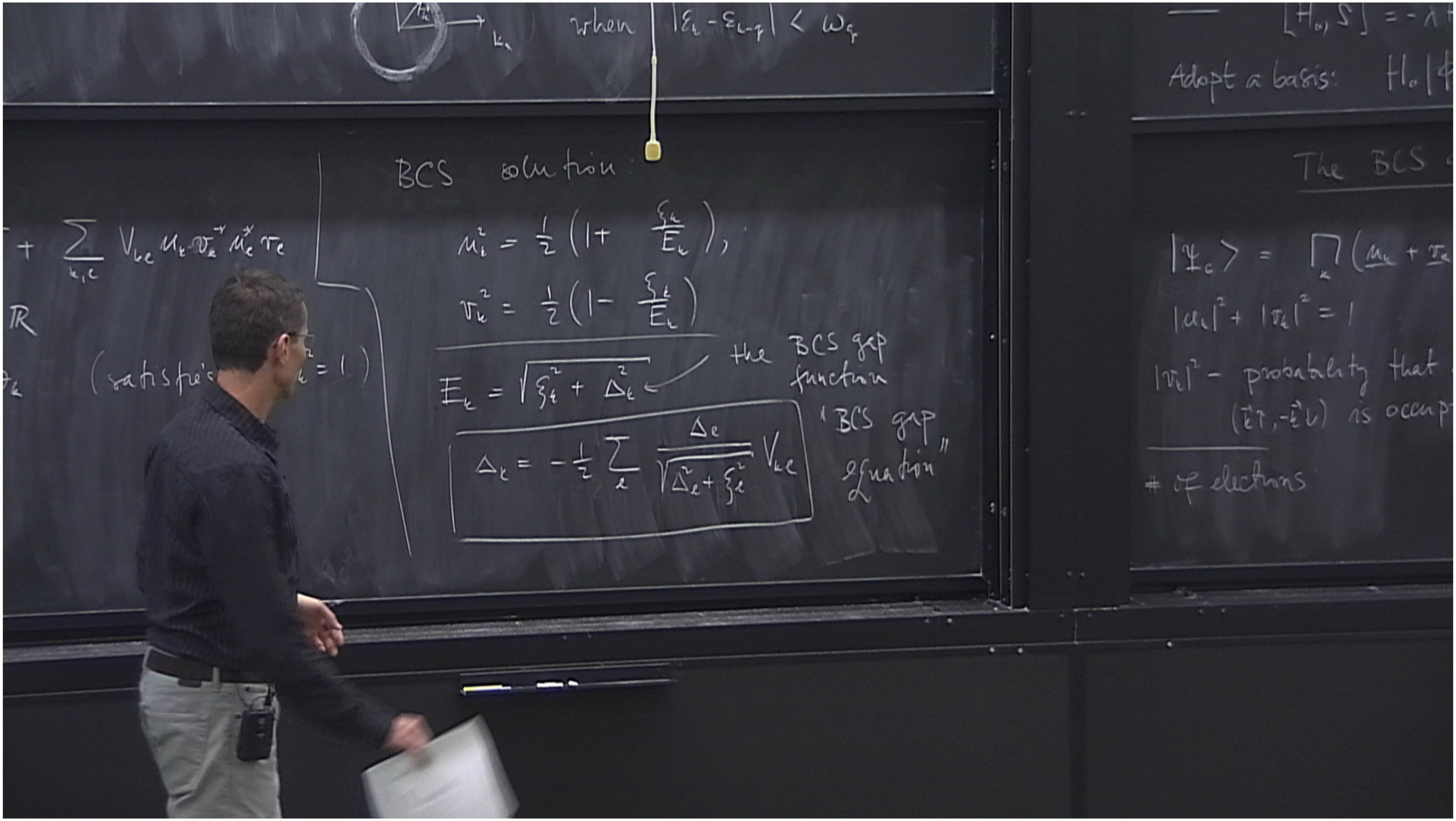


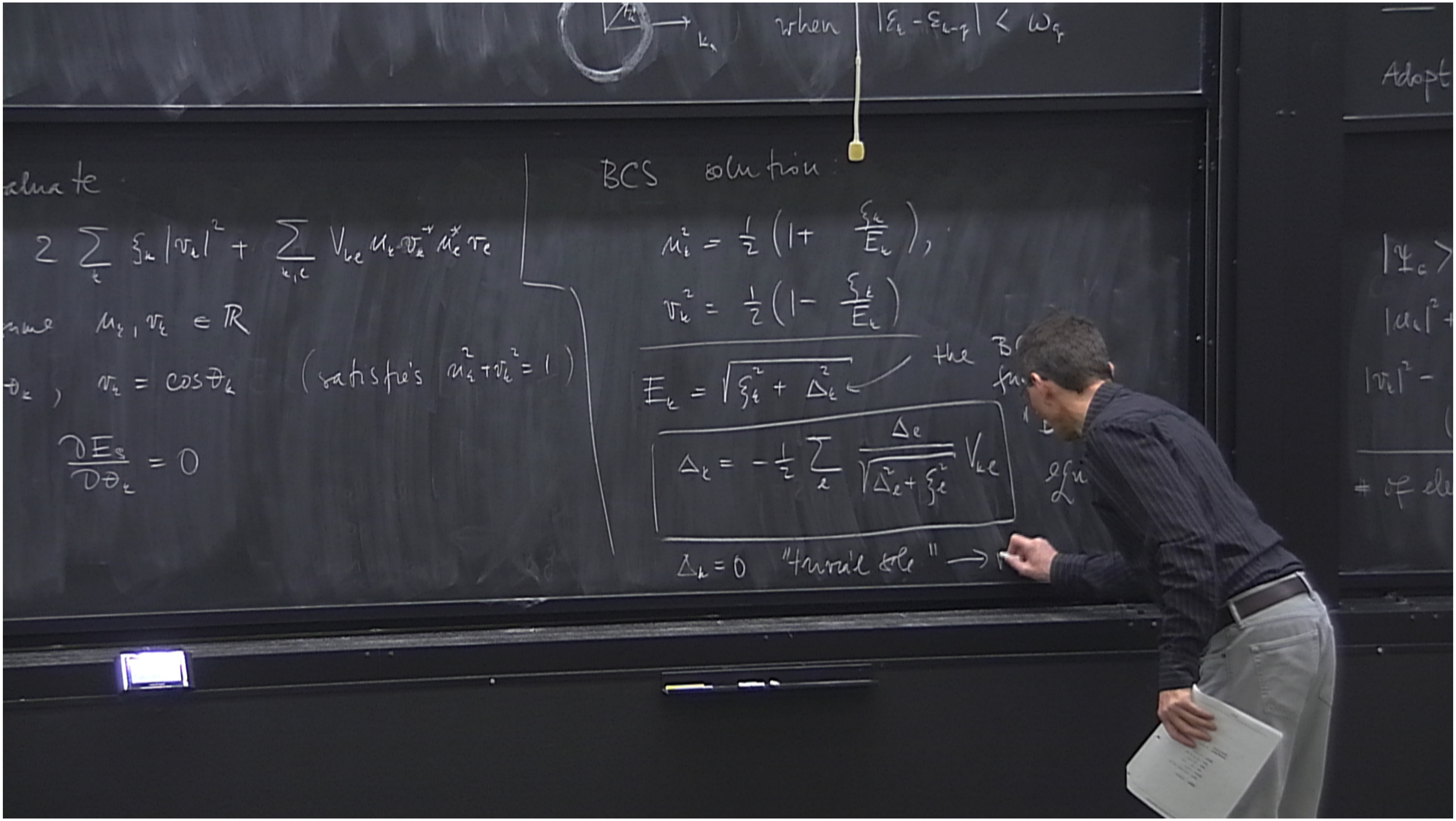

$\varepsilon_m - \varepsilon_n$
$H = H_0 + \lambda[H', S] + \frac{1}{2}[[H_0, S], S] + \frac{\lambda}{2}[[H', S], S] + \dots$
$-\lambda H'$
element
$\tilde{H} = H_0 + \frac{1}{2}\lambda[H', S] + O(\lambda^3)$
$\frac{1}{\Omega}\sum_k \to \int \frac{d^3k}{(2\pi)^3}$
$\frac{\delta N}{\bar{N}} = \frac{\sqrt{\langle(\hat{N} - \bar{N})^2\rangle}}{\bar{N}} \sim \frac{\sqrt{\Omega}}{\Omega} \sim \frac{1}{\sqrt{\Omega}} \sim \frac{1}{\sqrt{\bar{N}}}$
$|\Psi_c\rangle = \sum_N \lambda_N |\Psi_N\rangle$
$\bar{N} \sim 10^{24}$, $\frac{\delta N}{\bar{N}} \sim \frac{1}{\sqrt{10^{24}}} \sim 10^{-12}$
N - sharply peaked around $\bar{N}$.
λ_N
10^{-12}
$= \langle\Psi_c|\hat{N}|\Psi_c\rangle = \sum_k 2|v_k|^2 \sim \Omega$
$-\bar{N})^2\rangle = 4\sum_k |u_k|^2|v_k|^2 \sim \Omega$
$\bar{N} \sim 10^{24}$
N

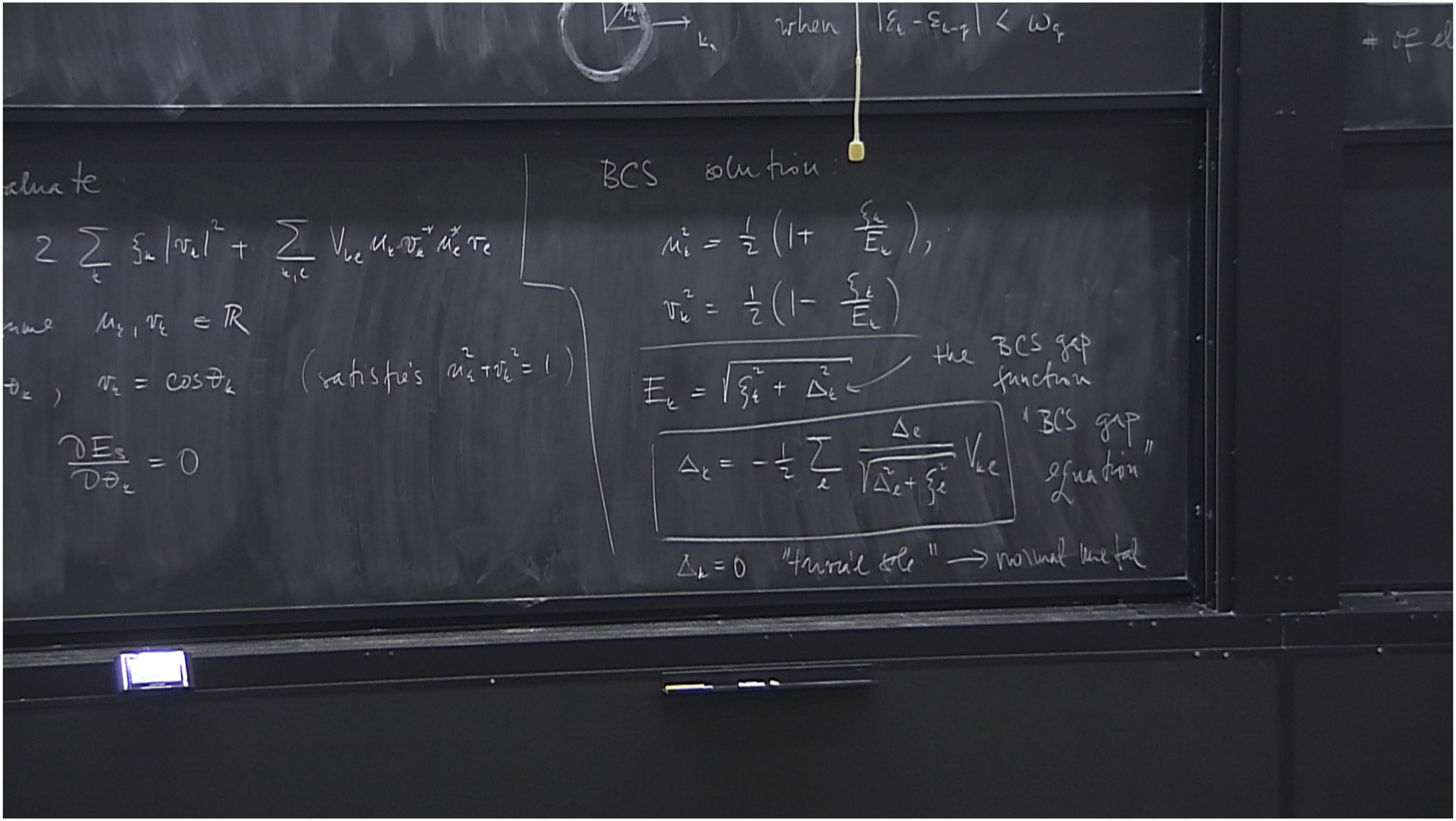


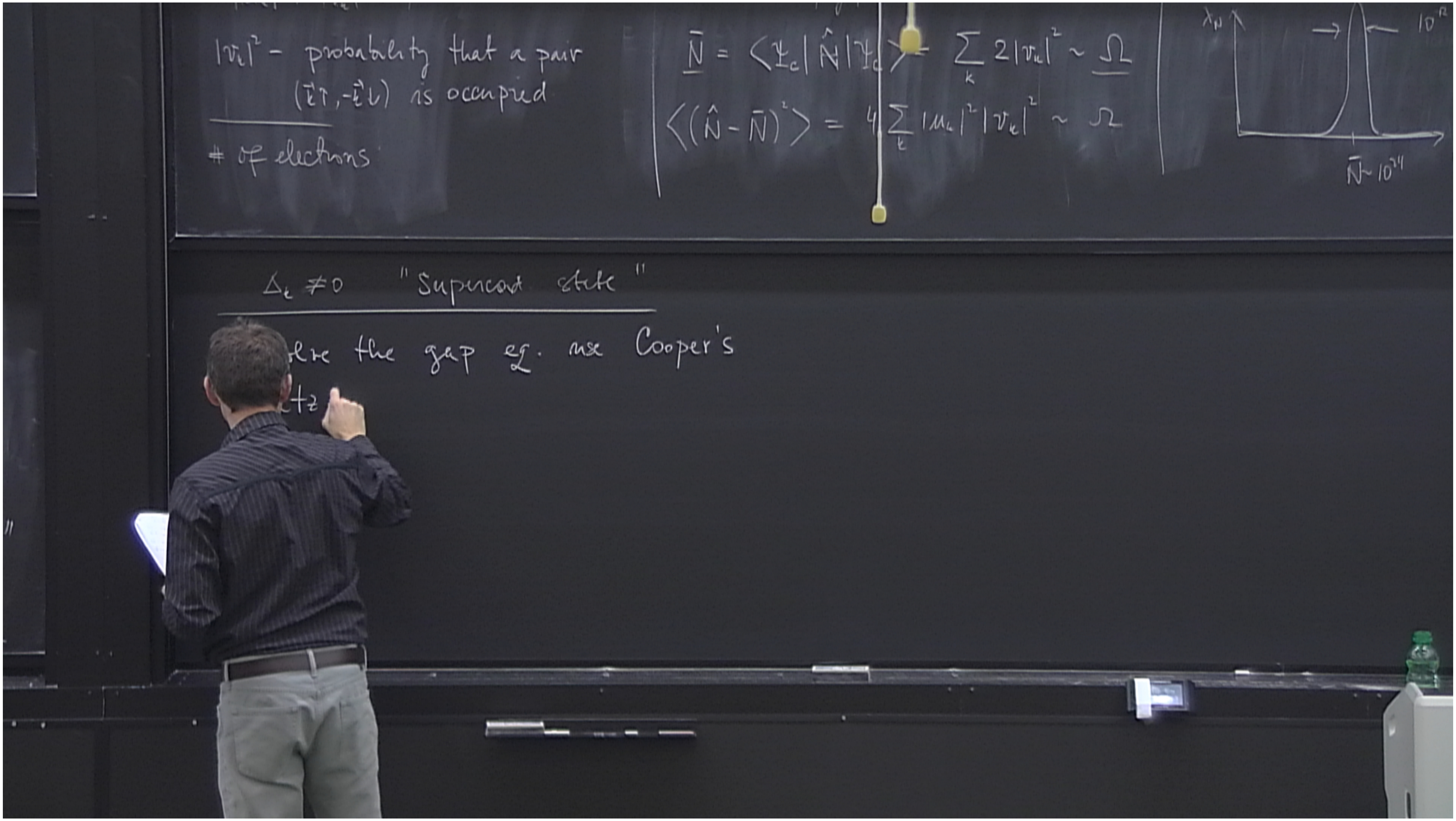

Evaluate
BCS solution
the BCS gap function
"BCS gap equation"


when
Evaluate
assume
BCS solution:
the BCS gap function
"BCS gap equation"


when $|\varepsilon_k - \varepsilon_{k-q}| < \omega_q$
Adopt a basis:
BCS solution:
$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right),$
$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$
$E_k = \sqrt{\xi_k^2 + \Delta_k^2}$
the BCS gap function
$\Delta_k = -\frac{1}{2}\sum_l \frac{\Delta_l}{\sqrt{\Delta_l^2 + \xi_l^2}} V_{kl}$
"BCS gap equation"
The BCS
$|\Psi_0\rangle = \prod_k (u_k + v_k$
$|u_k|^2 + |v_k|^2 = 1$
$|v_k|^2$ - probability that
$(\vec{k}\uparrow, -\vec{k}\downarrow)$ is occup
# of electrons


when
BCS solution:
the
Adopt


when |ε_k − ε_{k−q}| < ω_q
BCS solution:
u_k^2 = 1/2 (1 + ξ_k/E_k),
v_k^2 = 1/2 (1 − ξ_k/E_k)
E_k = √(ξ_k^2 + Δ_k^2)
the BCS gap function
Δ_k = −1/2 Σ_l Δ_l/√(Δ_l^2 + ξ_l^2) V_kl
"BCS gap equation"
Δ_k = 0 "trivial sol" → normal metal
2 Σ_k ξ_k |v_k|^2 + Σ_{k,l} V_kl u_k v_k^* u_l^* v_l
u_k, v_k ∈ ℝ
v_k = cos θ_k
(satisfies u_k^2 + v_k^2 = 1)
∂E_s/∂θ_k = 0

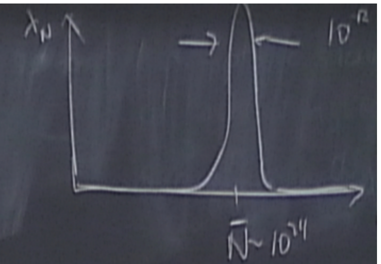

|v_k|^2 - probability that a pair (k↑, -k↓) is occupied
# of electrons
N̄ = ⟨Ψ_c| N̂ |Ψ_c⟩ = Σ_k 2|v_k|^2 ~ Ω
⟨(N̂ - N̄)^2⟩ = 4 Σ_k |u_k|^2 |v_k|^2 ~ Ω
10^12
N̄ ~ 10^24
Δ_k ≠ 0 "Supercond state"
the gap eq. use Cooper's

$|v_k|^2$ - probability that a pair $(\vec{k}\uparrow, -\vec{k}\downarrow)$ is occupied

\# of electrons

$$\bar{N} = \langle \Psi_c|\hat{N}|\Psi_c\rangle = \sum_k 2|v_k|^2 \sim \Omega$$

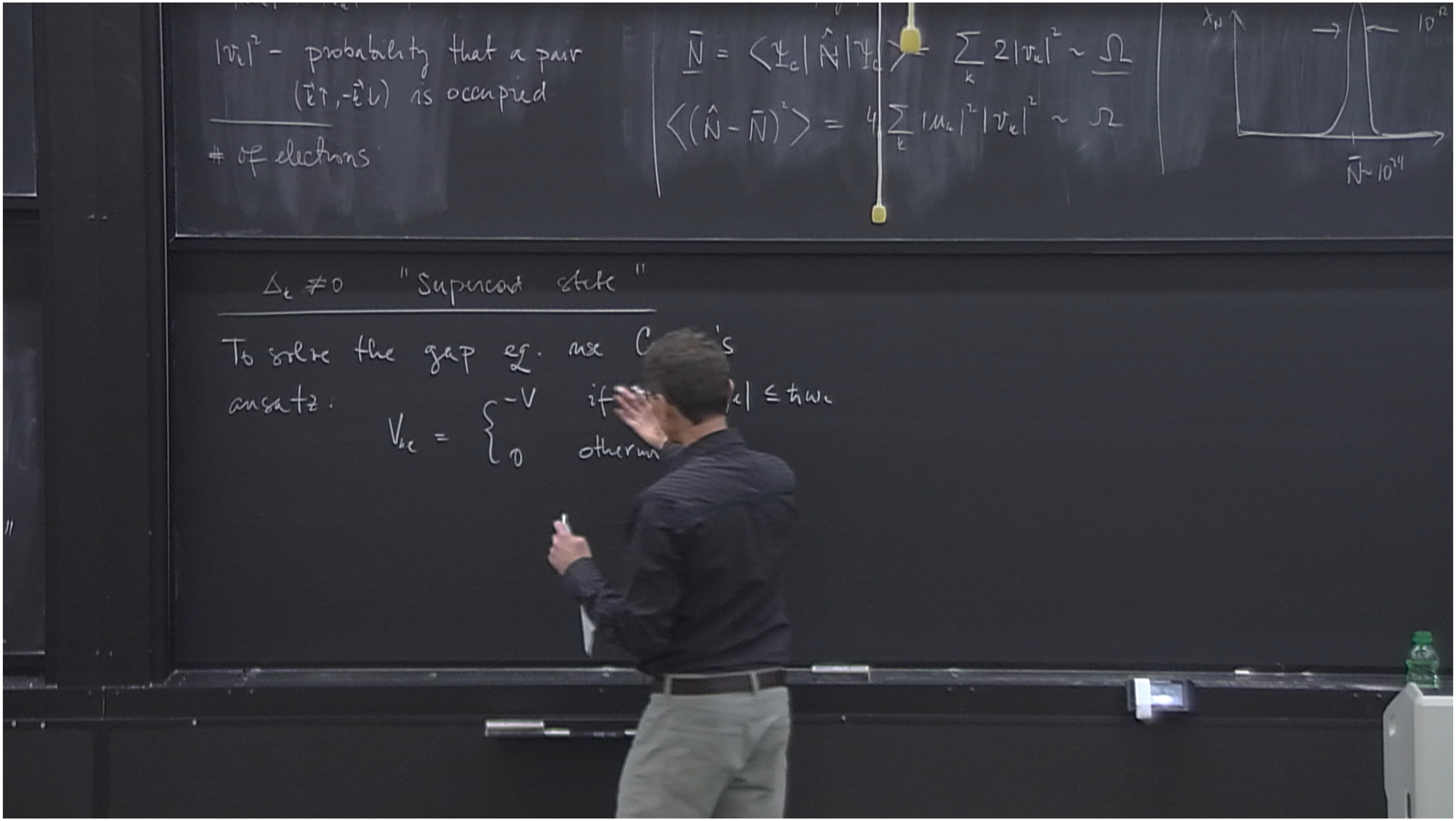
$$\langle(\hat{N} - \bar{N})^2\rangle = 4\sum_k |u_k|^2|v_k|^2 \sim \Omega$$



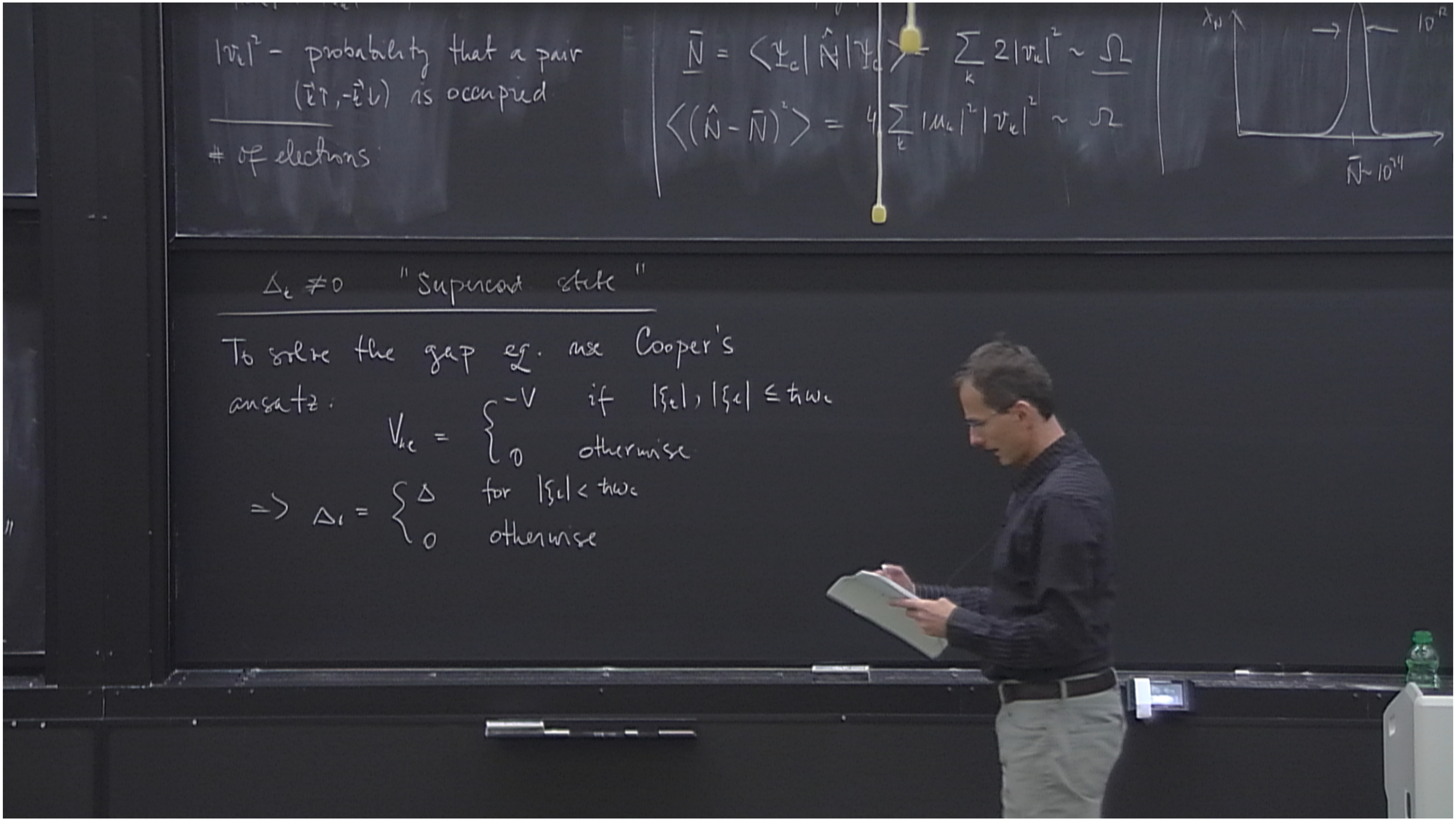
$\Delta_k \neq 0$ "Supercond. state"

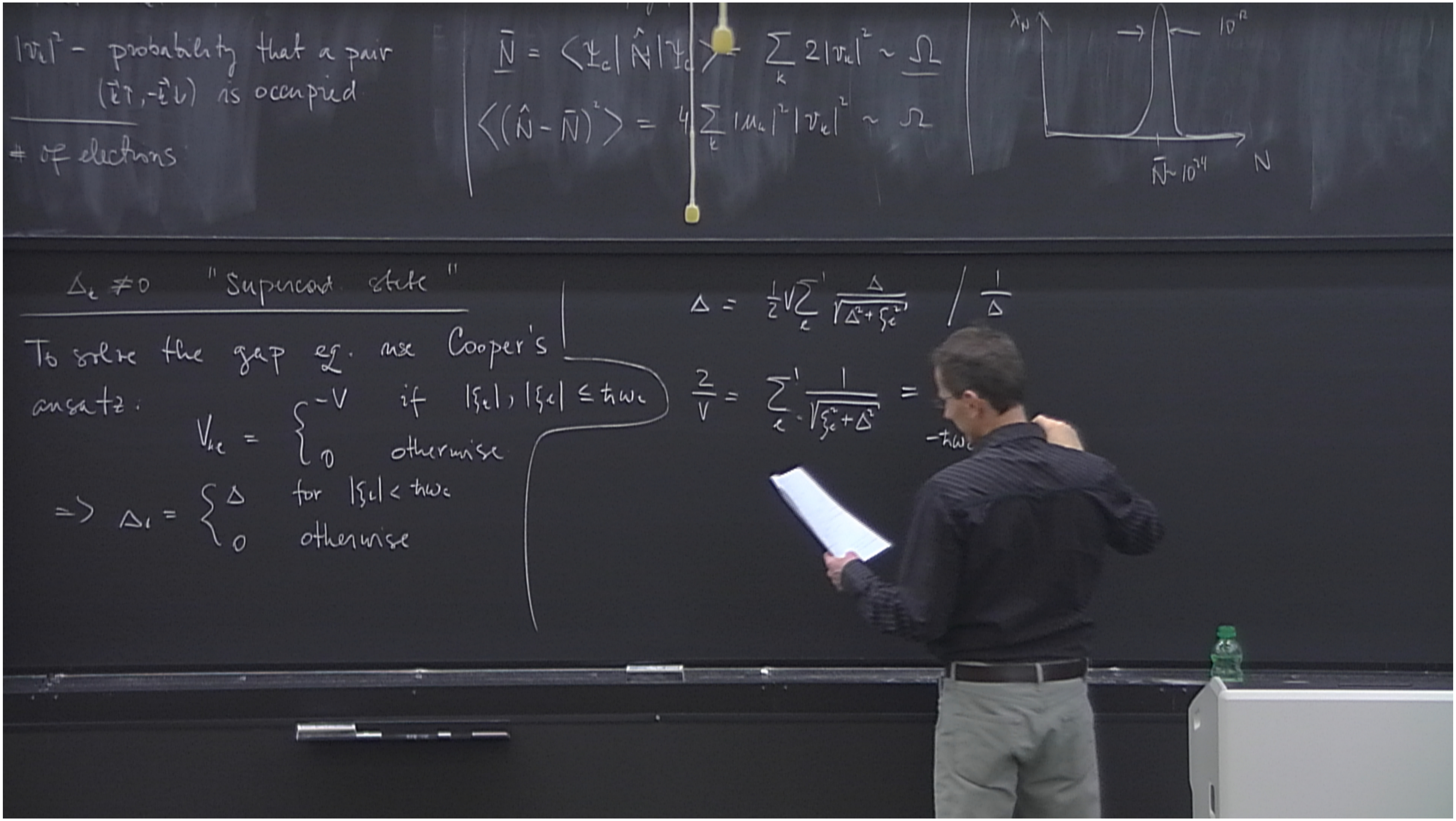
To solve the gap eq. use Cooper's ansatz:

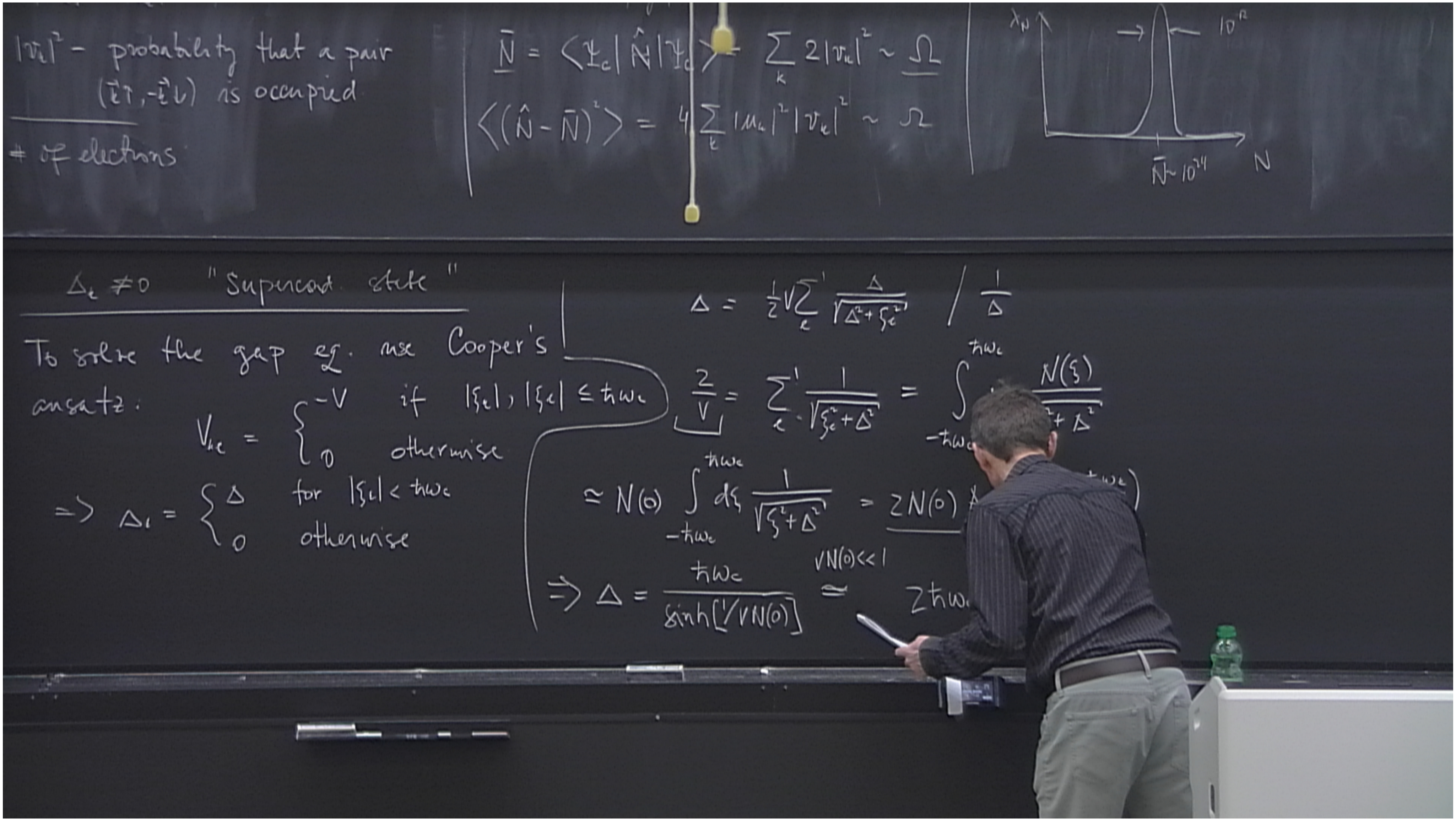
$$V_{kk'} = \begin{cases} -V & \text{if } |\xi_k|, |\xi_{k'}| \le \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$



|v_k|² – probability that a pair
(k↑, -k↓) is occupied
# of electrons
N̄ = ⟨Ψ_c| N̂ |Ψ_c⟩ = Σ_k 2|v_k|² ~ Ω
⟨(N̂ - N̄)²⟩ = 4 Σ_k |u_k|² |v_k|² ~ Ω
N̄ ~ 10²⁴
Δ_k ≠ 0 "Supercond state"
To solve the gap eq. use
ansatz:
V_kk' = { -V if
0 otherwise


$|v_k|^2$ - probability that a pair $(\vec{k}\uparrow, -\vec{k}\downarrow)$ is occupied
# of electrons
$\bar{N} = \langle \Psi_c | \hat{N} | \Psi_c \rangle = \sum_k 2|v_k|^2 \sim \Omega$
$\langle (\hat{N} - \bar{N})^2 \rangle = 4 \sum_k |u_k|^2 |v_k|^2 \sim \Omega$
10^{12}
$\bar{N} \sim 10^{24}$
$\Delta_k \neq 0$ "Supercond. state"
To solve the gap eq. use Cooper's ansatz:
$V_{kk'} = \begin{cases} -V & \text{if } |\xi_k|, |\xi_{k'}| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$
$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{for } |\xi_k| < \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$


|v_k|² - probability that a pair (k↑, -k↓) is occupied
# of electrons
N̄ = ⟨Ψ_c| N̂ |Ψ_c⟩ = Σ_k 2|v_k|² ~ Ω
⟨(N̂ - N̄)²⟩ = 4 Σ_k |u_k|² |v_k|² ~ Ω
λ_N
10¹²
N̄ ~ 10²⁴
N
Δ_k ≠ 0 "Supercond. state"
To solve the gap eq. use Cooper's ansatz:
V_kk' = { -V if |ξ_k|, |ξ_k'| ≤ ħω_c ; 0 otherwise
=> Δ_k = { Δ for |ξ_k| < ħω_c ; 0 otherwise
Δ = ½ V Σ'_k Δ / √(Δ² + ξ_k²) / 1/Δ
2/V = Σ'_k 1 / √(ξ_k² + Δ²) =
-ħω_c



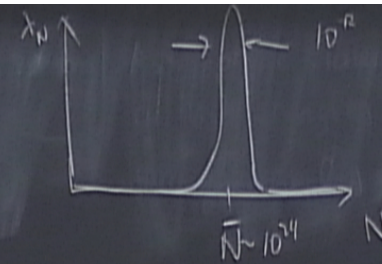
|v_k|^2 - probability that a pair (k↑, -k↓) is occupied
# of electrons
N̄ = ⟨Ψ_c| N̂ |Ψ_c⟩ = Σ_k 2|v_k|^2 ~ Ω
⟨(N̂ - N̄)^2⟩ = 4 Σ_k |u_k|^2 |v_k|^2 ~ Ω
λ_N
10^12
N̄ ~ 10^24
N
Δ_k ≠ 0 "Supercond. state"
To solve the gap eq. use Cooper's ansatz:
V_kk' = { -V if |ξ_k|, |ξ_k'| ≤ ħω_c ; 0 otherwise
=> Δ_k = { Δ for |ξ_k| < ħω_c ; 0 otherwise
Δ = ½ V Σ'_k Δ / √(Δ^2 + ξ_k^2) / (1/Δ)
2/V = Σ'_k 1/√(ξ_k^2 + Δ^2) = ∫_{-ħω_c}^{ħω_c} N(ξ)
≃ N(0) ∫_{-ħω_c}^{ħω_c} dξ 1/√(ξ^2 + Δ^2) = 2N(0)
VN(0) << 1
⇒ Δ = ħω_c / sinh[1/VN(0)] ≃ 2ħω_c

$|v_k|^2$ – probability that a pair $(\vec{k}\uparrow, -\vec{k}\downarrow)$ is occupied

\# of electrons

$$\bar{N} = \langle \Psi_c | \hat{N} | \Psi_c \rangle = \sum_k 2|v_k|^2 \sim \Omega$$

$$\langle (\hat{N} - \bar{N})^2 \rangle = 4 \sum_k |u_k|^2 |v_k|^2 \sim \Omega$$



$\Delta_k \neq 0$ "Supercond. state"

To solve the gap eq. use Cooper's ansatz:

$$V_{kk'} = \begin{cases} -V & \text{if } |\xi_k|, |\xi_{k'}| \le \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

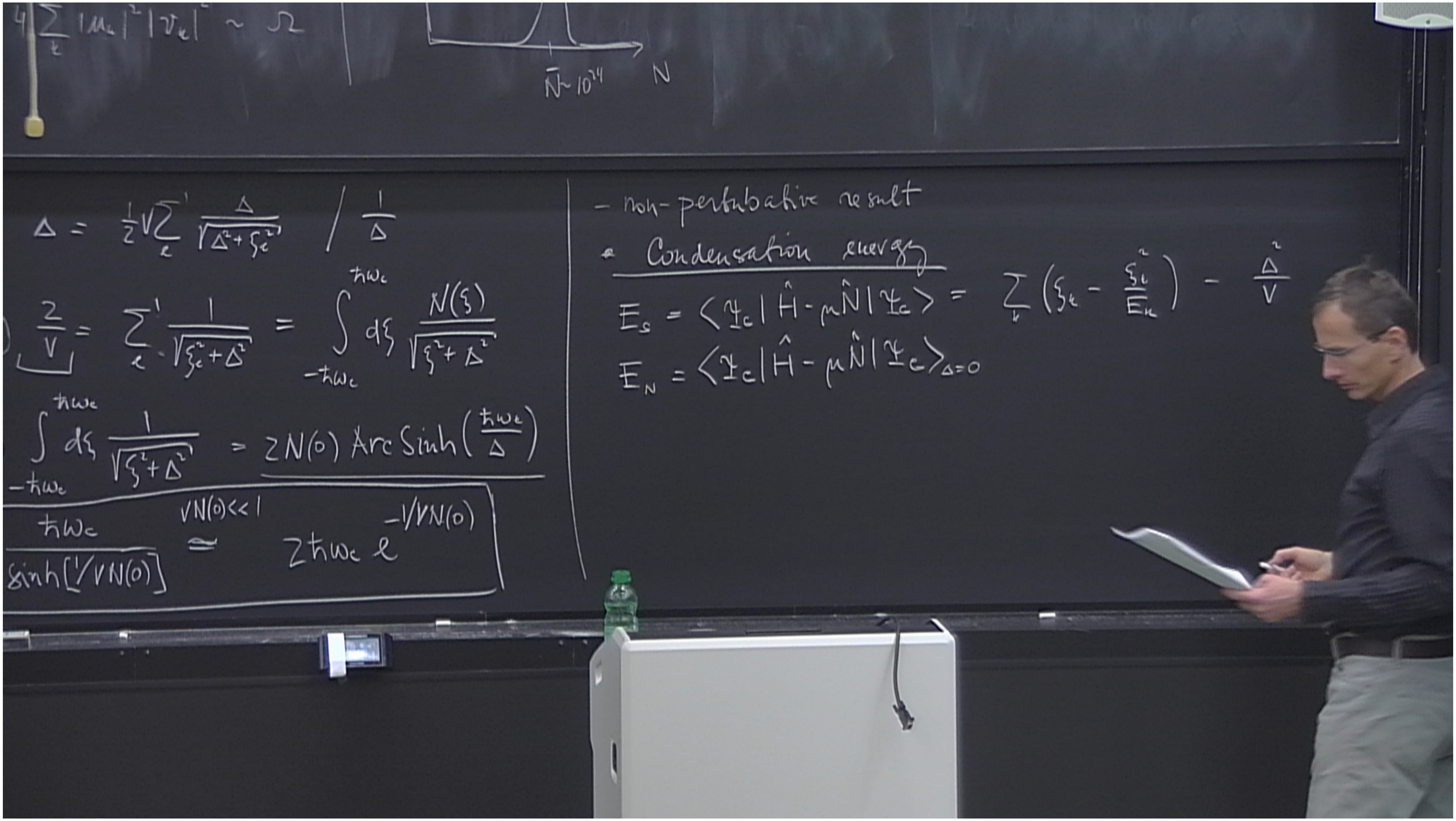
$$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{for } |\xi_k| < \hbar\omega_c \\ 0 & \text{otherwise} \end{cases}$$

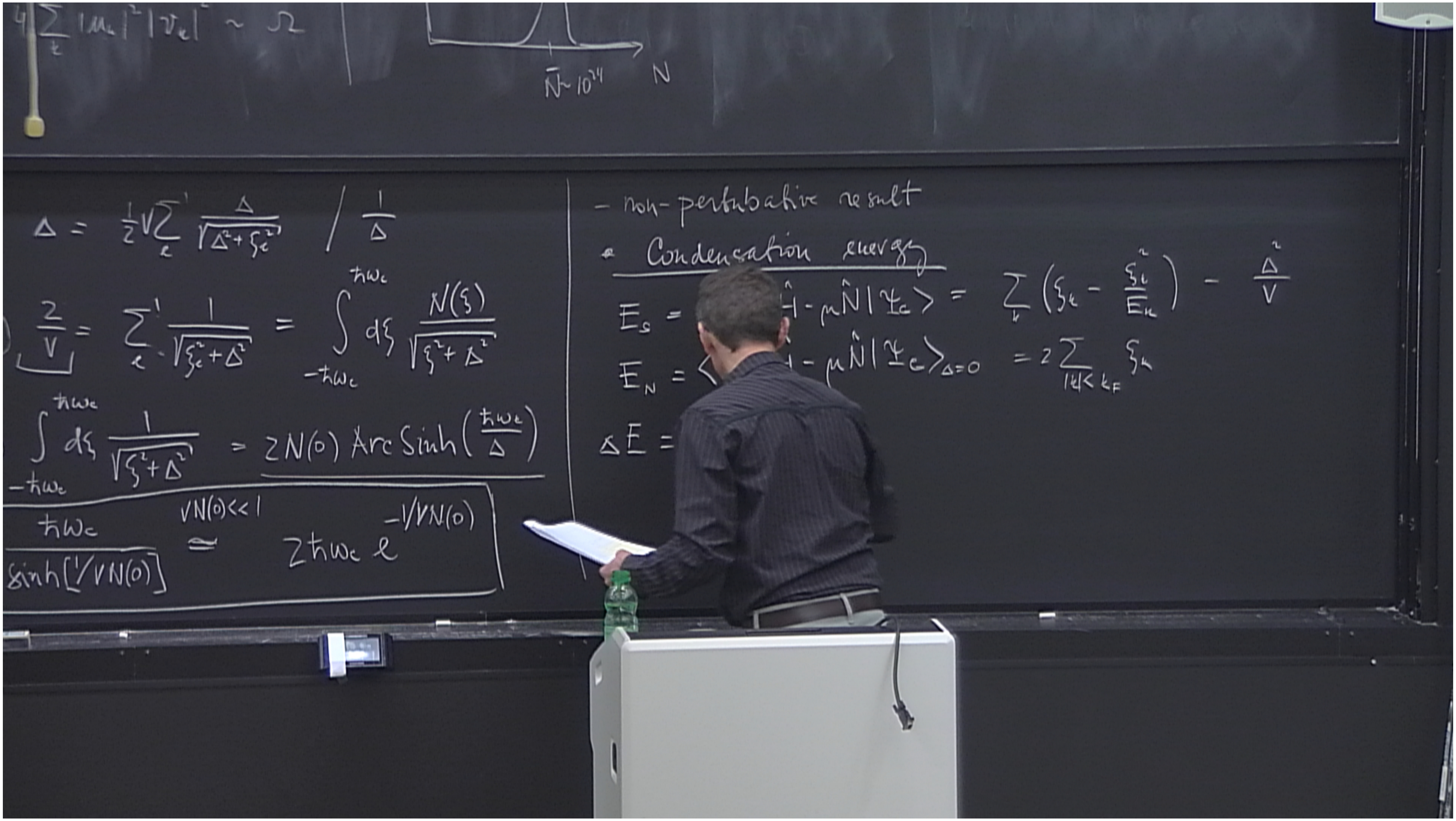
$$\Delta = \frac{1}{2} V {\sum_k}' \frac{\Delta}{\sqrt{\Delta^2 + \xi_k^2}} \quad \Big/ \ \frac{1}{\Delta}$$

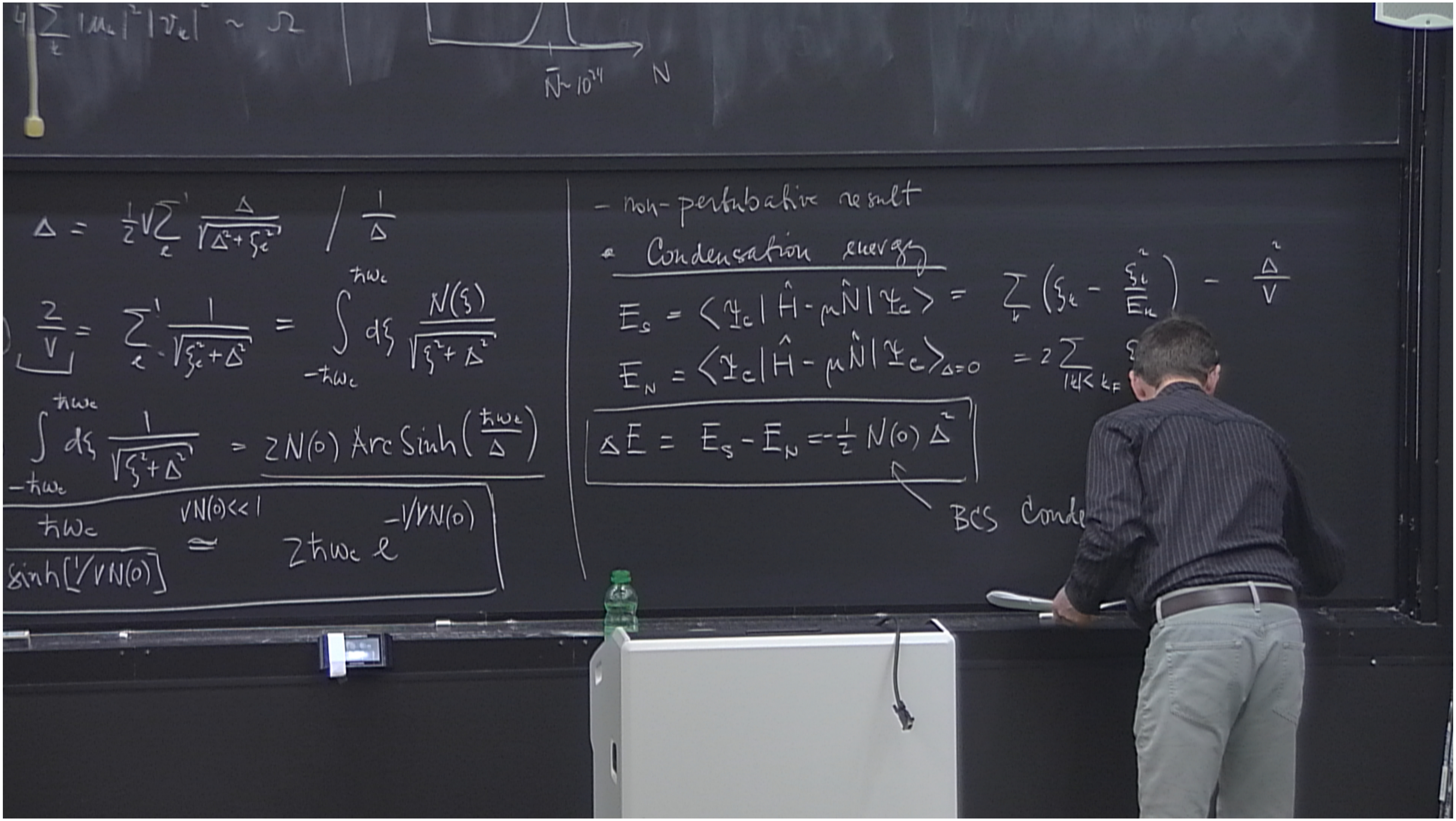
$$\frac{2}{V} = {\sum_k}' \frac{1}{\sqrt{\xi_k^2 + \Delta^2}} = \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \, \frac{N(\xi)}{\sqrt{\xi^2 + \Delta^2}}$$

$$\simeq N(0) \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \, \frac{1}{\sqrt{\xi^2 + \Delta^2}} = 2N(0) \, \mathrm{ArcSinh}\left(\frac{\hbar\omega_c}{\Delta}\right)$$

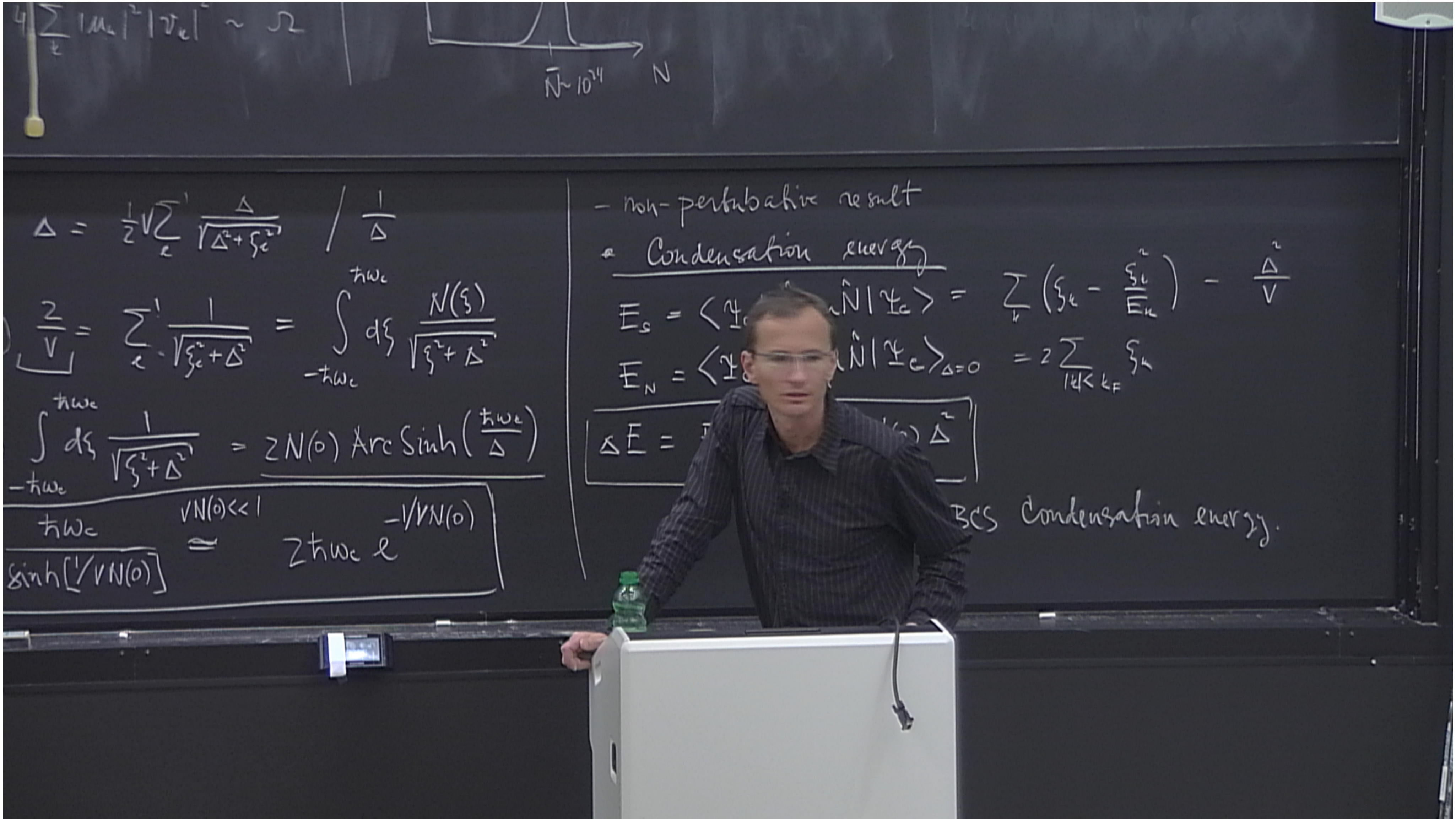
$$\Rightarrow \boxed{\Delta = \frac{\hbar\omega_c}{\sinh[1/VN(0)]} \overset{VN(0) \ll 1}{\simeq} 2\hbar\omega_c \, e^{-1/VN(0)}}$$


- non-perturbative result
Condensation energy
E_s = ⟨Ψ_c | Ĥ - μN̂ | Ψ_c⟩ = Σ_k (ξ_k - ξ_k²/E_k) - Δ²/V
E_N = ⟨Ψ_c | Ĥ - μN̂ | Ψ_c⟩_{Δ=0}
Δ = ½ V Σ'_k Δ/√(Δ² + ξ_k²)
2/V = Σ'_k 1/√(ξ_k² + Δ²) = ∫_{-ħω_c}^{ħω_c} dξ N(ξ)/√(ξ² + Δ²)
∫_{-ħω_c}^{ħω_c} dξ 1/√(ξ² + Δ²) = 2N(0) ArcSinh(ħω_c/Δ)
ħω_c / sinh[1/VN(0)] ≃ 2ħω_c e^{-1/VN(0)}
VN(0) << 1





- non-perturbative result
Condensation energy
BCS Conde


- non-perturbative result
Condensation energy
BCS Condensation energy.