Title: 14/15 PSI - Quantum Field Theory II-Lecture 6

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Abstract:

· Effective Action. ϕ^4 $\quad$ ϕ scalar field in D-dim.
(Euclidean case)

$$S[\phi] = \int d^D x \, \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \quad \text{classical action}$$

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr}\left[\mathrm{Log}\left[S''[\varphi]\right]\right] + o(\hbar^2)$$

φ background Field Variable

$S''[\varphi]$ Hessian operator action on a space $\mathcal{L}(\mathbb{R}^D)$ of function on $\mathbb{R}^D$

$$\psi \longrightarrow S'' \cdot \psi = \chi \qquad \chi(x) = \int \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)} \cdot \psi(y) \, d^D y$$

Field

Space

$$S''[\varphi] = -\Delta + m^2 + \frac{g}{2}\varphi^2$$

$$X(x) = -\Delta_x \psi(x) + m^2 \psi(x) + \frac{g}{2}\varphi^2(x)\cdot\psi(x)$$

$$\mathrm{Tr}\left[\mathrm{Log}\left[-\Delta+m^2+\frac{g}{2}\varphi^2\right]\right] = \mathrm{Tr}\left[\mathrm{Log}\left[(-\Delta+m^2)\times\left(\mathbb{1}+\frac{g}{2}\,\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]\right]$$

$$= \mathrm{Tr}\left[\mathrm{Log}\left(-\Delta+m^2\right)\right] + \mathrm{Tr}\left[\mathrm{Log}\left(\mathbb{1}+\frac{g}{2}\,\frac{1}{-\Delta+m^2}\cdot\varphi^2\right)\right]$$

↑ Free Field contribution

Term 0

Term 1

$$\text{Term } 1 = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{g}{2}\right)^k \mathrm{Tr}\Big[\underbrace{\frac{1}{-\Delta+m^2}\cdot\varphi^2\cdot\frac{1}{-\Delta+m^2}\cdot\varphi^2\cdot\ldots\cdot\frac{1}{-\Delta+m^2}\cdot\varphi^2}_{k\text{-times}}\Big]$$

$x \text{———} y \xleftarrow{FT} \dfrac{1}{k^2+m^2}$

Feymann diagrams.

$\varphi^2(x)$

$x=y$

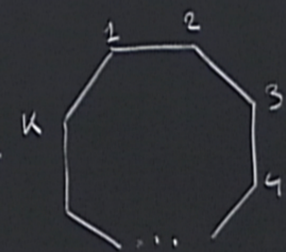
(x)

$$= \int \prod_{i=1}^{K} dx_i\, dy_i \cdot G_0(x_1,y_1)\,\delta(y_1,x_2)\,\varphi^2(y_1)\cdot G_0(x_2,y_2)\cdot\delta(y_2,x_3)\,\varphi^2(y_2)\cdots\; G_0(x_K,y_K)\,\delta(y_K,x_1)\,\varphi^2(y_K)$$

x_1

with $\Gamma_{(K)}(x_1,\dots x_K) = G_0(x_1-x_2)\,G_0(x_2-x_3)\cdots G_0(x_K-x_1)$

$_3)\,\varphi^2(x_3)\cdots\; G_0(x_K,x_1)\,\varphi^2(x_1)$

$=$ 

closed loop
of propagators

$x_K\; \varphi^2(x_1)\cdots\varphi^2(x_K)\;\Gamma_{(K)}(x_1\cdots x_K)$

$$\text{Kernel}\left(\frac{1}{-\Delta+m^2}\right)_{xy} = G_0(x,y) = \underset{x}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\underset{y}{\circ} \xleftarrow{FT} \frac{1}{k^2+m^2}$$

Feymann diagrams.

$$\text{Kernel}\,(\varphi^2)_{xy} = \delta(x-y)\,\varphi^2(x) \qquad \overset{\varphi^2(x)}{\underset{x=y}{\circ}}$$

$$\mathrm{Tr}\left[\left(\frac{1}{-\Delta+m^2}\varphi^2\right)^k\right] = \int dx_1 \left(\cdots\right)^k_{x_1,x_1} = \int \prod_{i=1}^{k} dx_i\, dy_i \cdot G_0(x_1,y_1)\,\delta(y_1,x_2)\,\varphi^2(y_1)\cdot G_0(x_2,y_2)\cdot \delta(y_2,x_3)\,\varphi^2(y_2)\cdots$$

$$= \int \prod_{i=1}^{k} dx_k\; G_0(x_1,x_2)\varphi^2(x_2)\,G_0(x_2,x_3)\varphi^2(x_3)\cdots\; G_0(x_k,x_1)\,\varphi^2(x_1)$$

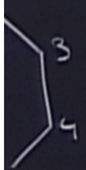
with $\Gamma_{(k)}(x_1,\dots x_k) = G$

$= {}^k$

$$\text{Term } 1 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{g}{2}\right)^k \int d^Dx_1\dots d^Dx_k\; \varphi^2(x_1)\cdots\varphi^2(x_k)\;\Gamma_{(k)}(x_1\dots x_k)$$

$)\,\delta(y_k, x_1)\,\varphi^2(y_k)$

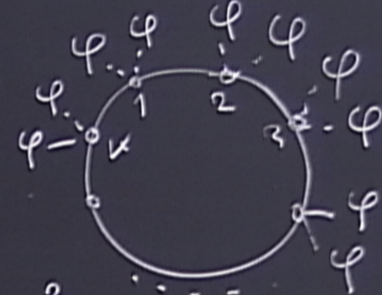
$G_0(x_2-x_3)\cdots G_0(x_k-x_1)$

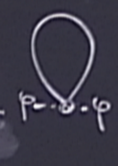
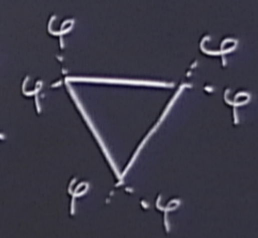


closed loop of propagators

Feynman diagram representation of $\Gamma[\varphi]$

$$\frac{1}{2}\,\mathrm{Tr}\,\mathrm{Log}(-\Delta+m^2)+\sum_{K=1}^{\infty}-\frac{1}{2K}\left(\frac{-g}{2}\right)^{K}$$



$= -\frac{1}{2}$  $+\ \frac{g}{4}$  $-\ \frac{g^2}{16}$  $+\ \frac{g^3}{48}$ 

$+\ \cdots$

↑ diagrammatic representation

sum of contributions of irreducible 1 loop diagrams term $\hbar$

$$\text{Kernel}\left(\frac{1}{-\Delta+m^2}\right)_{xy} = G_0(x,y) = \underset{x}{\circ}\!\!-\!\!-\!\!-\!\!\underset{y}{\circ} \xleftarrow{FT} \frac{1}{k^2+m^2}$$

Feymann diagrams.

$$\text{Kernel}\left(\varphi^2\right)_{xy} = \delta(x-y)\,\varphi^2(x)$$

$\varphi^2(x)$

$x=y$

$$\mathrm{Tr}\left[\left(\frac{1}{-\Delta+m^2}\varphi^2\right)^K\right] = \int dx_1 \left(\cdots\right)^K_{x_1,x_1} = \int \prod_{i=1}^{K} dx_i\, dy_i \cdot G_0(x_1,y_1)\,\delta(y_1,x_2)\,\varphi^2(y_1)\cdot G_0(x_2,y_2)\cdot \delta(y_2,x_3)\,\varphi^2(y_2)\cdots$$

$$= \int \prod_{i=1}^{K} dx_K\; G_0(x_1,x_2)\,\varphi^2(x_2)\,G_0(x_2,x_3)\,\varphi^2(x_3)\cdots G_0(x_K,x_1)\,\varphi^2(x_1)$$

with $\Gamma_{(K)}(x_1,\ldots x_K) = G_0(x_1$

$$\text{Term } 1 = \sum_{K=1}^{\infty} \frac{(-1)^{K-1}}{K}\left(\frac{g}{2}\right)^K \int d^Dx_1\ldots d^Dx_K\, \varphi^2(x_1)\cdots\varphi^2(x_K)\,\Gamma_{(K)}(x_1\ldots x_K)\,\frac{1}{K!}$$

$= {}^K$

$= $ K-point irreducible function

notation $\underset{x_1 \quad x_2}{\circ\!\!-\!\!\!-\!\!\!-\!\!\circ} = G_0(x_1-x_2)$ $\quad$ $\underset{x_1 \quad x_2}{\circ\!\!-\!\!|\!\!-} = \delta(x_1-x_2)$

$$(x_1, x_2) = (-\Delta + m^2)_{x_1, x_2} + \hbar\, \frac{g}{2}\, \delta(x_1-x_2) \quad x_1 = x_2$$

$$(x_1 \ldots x_4) = g\, \delta(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_1-x_4) \quad x_1\, x_2\, x_3\, x_4 \quad + \hbar\,(-g)\,\frac{3}{2}\Big[\ \ldots\ \Big]$$

$$\underline{\delta(x_1-x_4)\,\delta(x_2-x_3)\,G_0(x_1-x_2)^2}$$

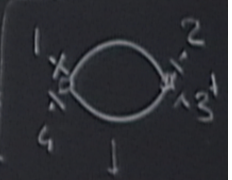
$C \ldots$

$$\frac{\delta\,\Gamma[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_k)} = \Gamma_{(k)}(x_1 \ldots x_k) \quad = \text{k-point irreducible function}$$

notation $\underset{x_1}{\circ}\!\!-\!\!-\!\!\underset{x_2}{\circ} = G(x_1, \ldots$

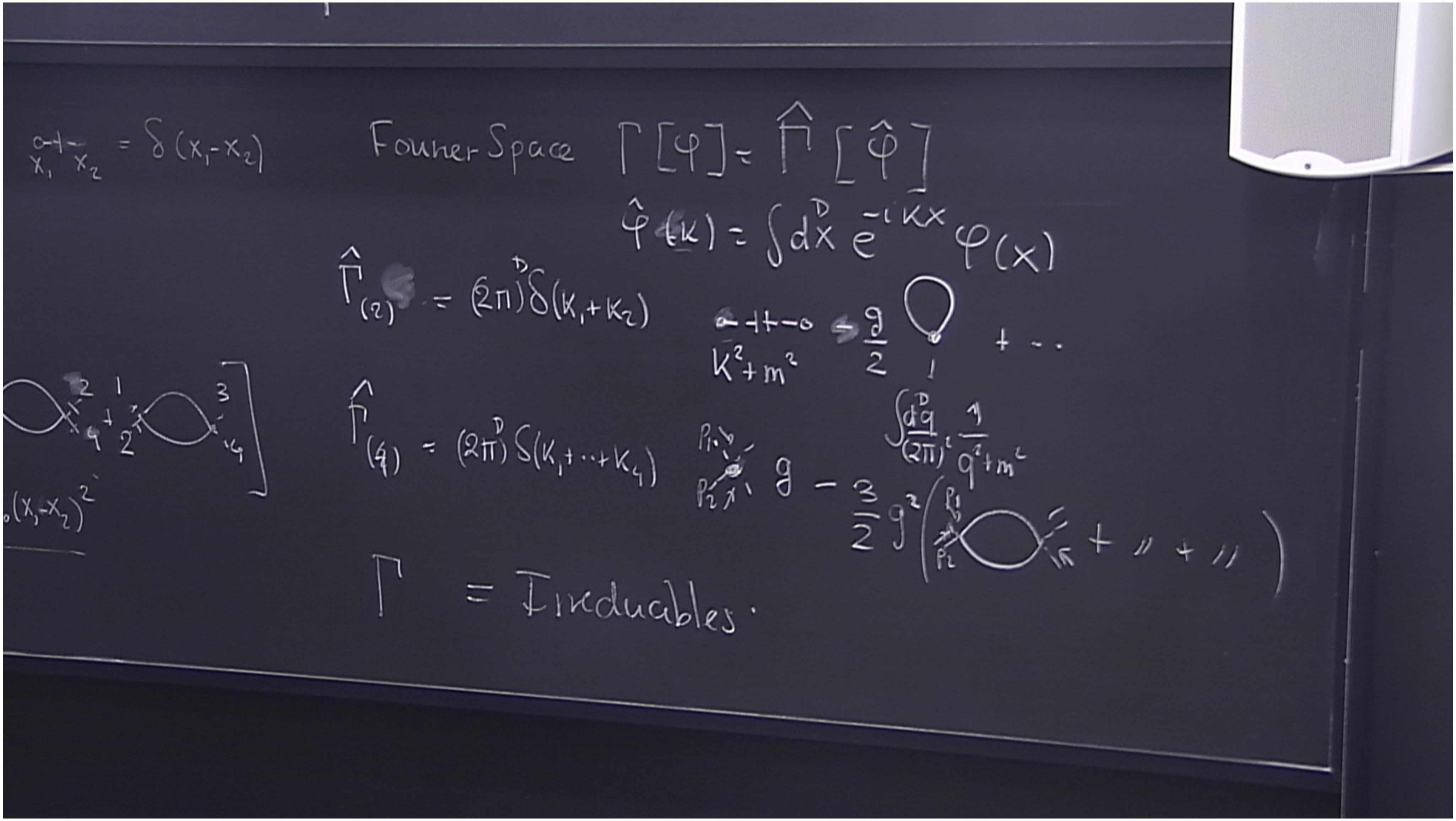
2pt. Irreducible function $\quad \Gamma_{(2)}(x_1, x_2) = \left(-\Delta + m^2\right)_{x_1, x_2} + \hbar\, \frac{g}{2}\, \delta(x_1 - x_2)$

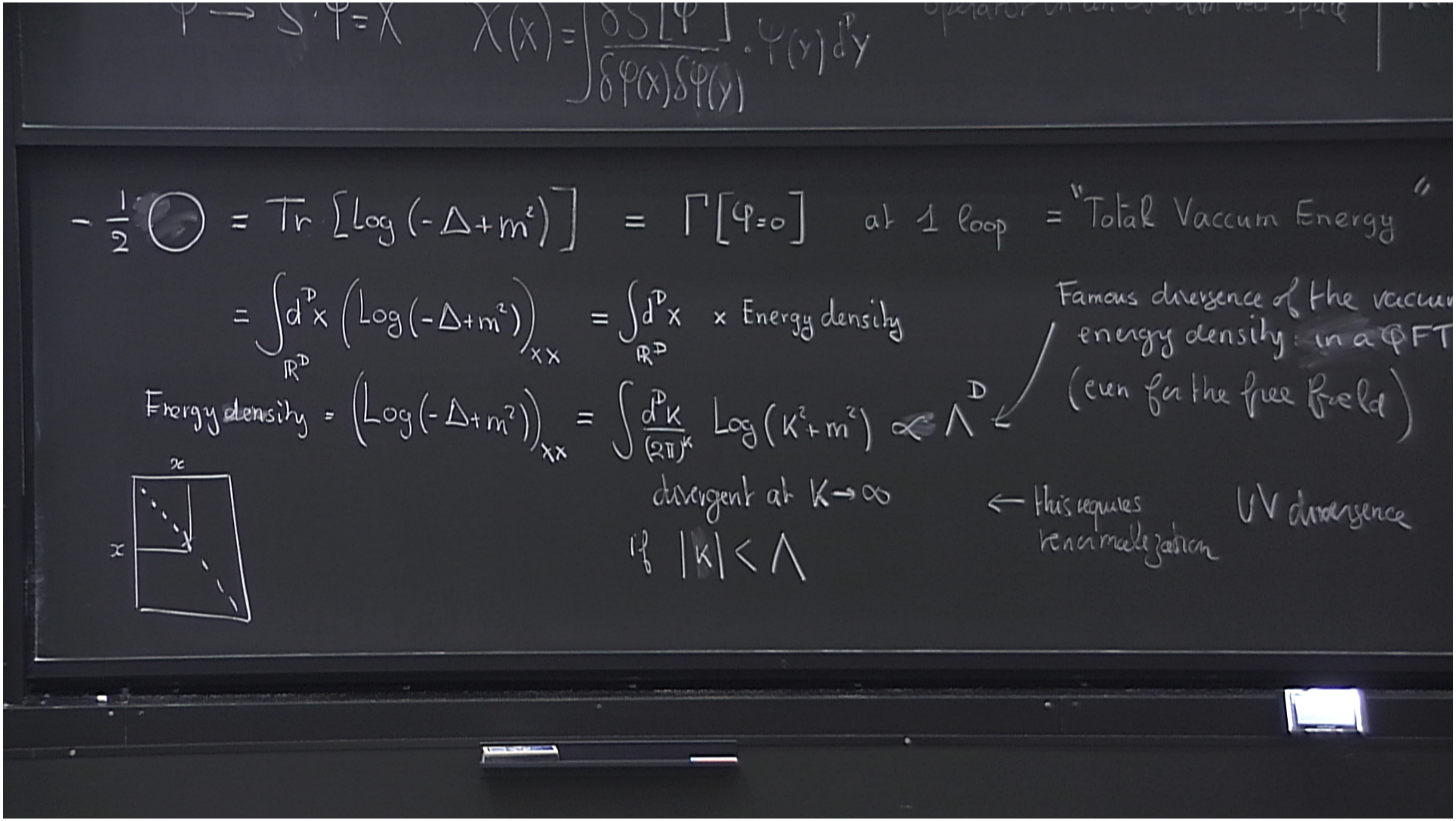
4pt. $\quad \Gamma_{(4)}(x_1 \ldots x_4) = g\, \delta(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_1-x_4) + \hbar\,(-g)\,\frac{3}{2}\Big[$

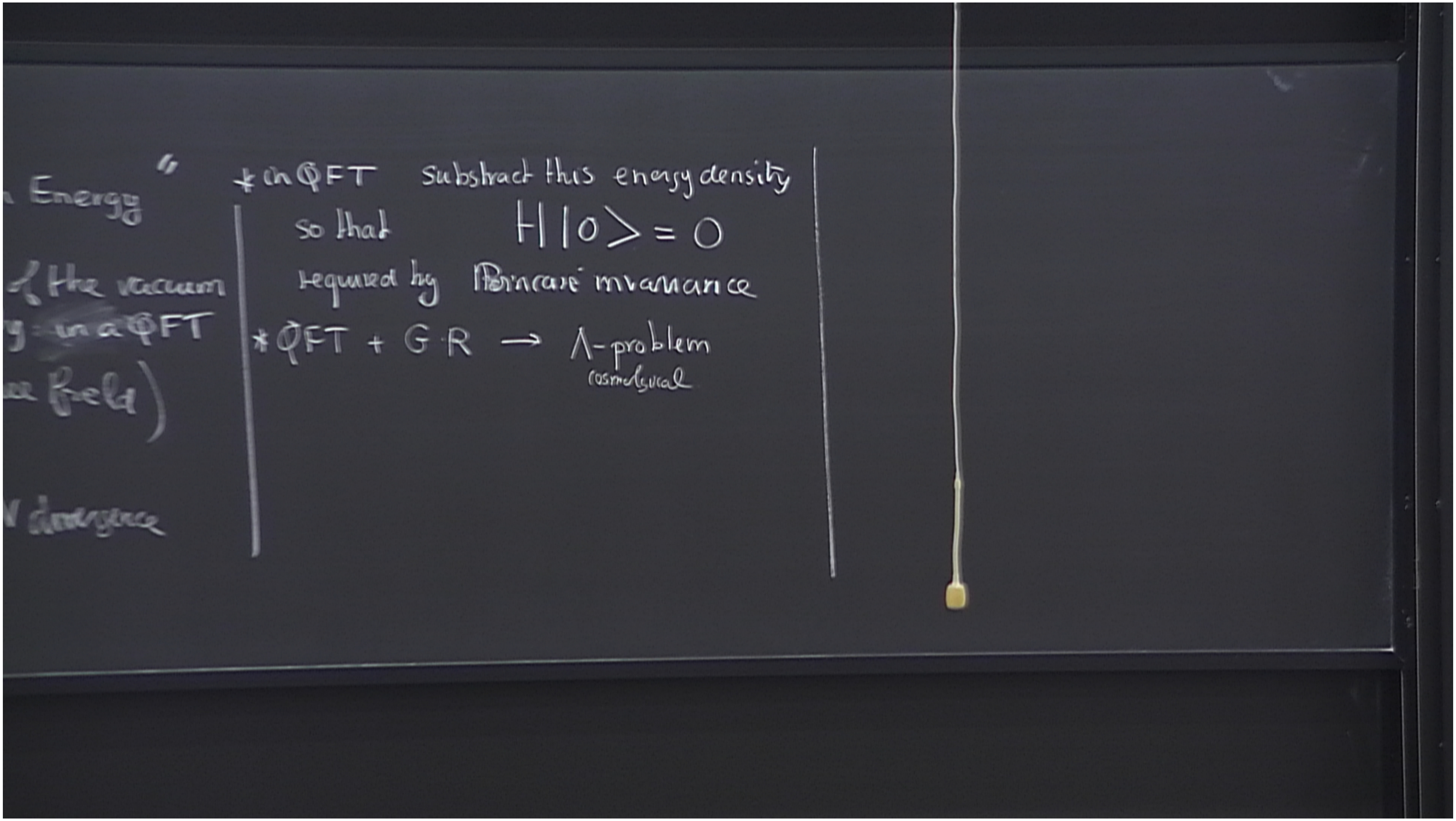


$\delta(x_1 - x_4)\,\delta(x_2 - x_3)$

etc...


= δ(x1-x2)
Fourier Space Γ[φ] = Γ̂[φ̂]
φ̂(k) = ∫d^D x e^{-ikx} φ(x)
Γ̂(2) = (2π)^D δ(k1+k2)
1/(k²+m²)
g/2
+ ...
Γ̂(4) = (2π)^D δ(k1+...+k4)
∫ d^D q/(2π)^D 1/(q²+m²)
g - 3/2 g² (+ " + ")
Γ = Irreducables.


$-\frac{1}{2}$ ◯ $= \mathrm{Tr}\left[\mathrm{Log}(-\Delta+m^2)\right] = \Gamma[\varphi=0]$ at 1 loop = "Total Vaccum Energy"
$= \int_{\mathbb{R}^D} d^Dx \left(\mathrm{Log}(-\Delta+m^2)\right)_{xx} = \int_{\mathbb{R}^D} d^Dx \times$ Energy density
Famous divergence of the vacuum energy density in a φFT (even for the free field)
Energy density $= \left(\mathrm{Log}(-\Delta+m^2)\right)_{xx} = \int \frac{d^Dk}{(2\pi)^k} \mathrm{Log}(k^2+m^2) \propto \Lambda^D$
divergent at $k \to \infty$
if $|k| < \Lambda$
← this requires renormalization
UV divergence
x
x


Energy"
of the vacuum
in a QFT
field)
divergence
* in QFT substract this energy density
so that H|0> = 0
required by Poincaré invariance
* QFT + G.R → Λ-problem
cosmological

divergent at $k \to \infty$ ← this requires renormalization UV divergence

if $|k| < \Lambda$

Renormalization of ϕ^4 (at 1 loop order) : can we build a UV-finite theory with finite observables ?

1] UV regulator · regularization procedure

1a in X space → Lattice — Loose Poincaré Invariance

Calculationally convenient · Spin 0 & Spin 1

Fermions (Spin ½) & Supersymmetry ⚠

↘ do the calculation $|x| > a$ Unitarity & Causality

1.b in P-space (momentum) change the propagator

Scale Λ large $\Lambda \gg$ masses

ⓐ $\frac{1}{p^2+m^2} \longrightarrow \frac{1}{p^2+m^2}\,\theta(\Lambda-|p|)$ Sharp cut-off

ⓑ $\frac{1}{p^2+m^2}\left(\frac{\Lambda^2}{p^2+\Lambda^2}\right)^M$

Pauli-Villars Regul.

Lorentz invariance is OK

$\frac{1}{p^2+m^2}\,\frac{\Lambda^2}{p^2+\Lambda^2} \simeq \left[\frac{1}{p^2+m^2} - \frac{1}{p^2+\Lambda^2}\right]$ −1 sign

$\Lambda^2 \gg m^2$ physical mass m pa

Is start from Regularized theory $|k| < \Lambda$

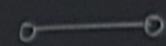
1st step: massless interacting theory in $D=4$

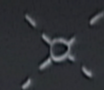
what is the mass ?

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

path integral

A, B, C are parameters to be determined..

 $= \frac{1}{Ap^2+B}$

 $= C$

2 pa

func

$$S[\phi] = \int d^4x \, \frac{A}{2} \partial_\nu\phi \partial^\nu\phi + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$$

renormalized action, function of a renormalized field ϕ

$$= \frac{1}{Ap^2+B}$$

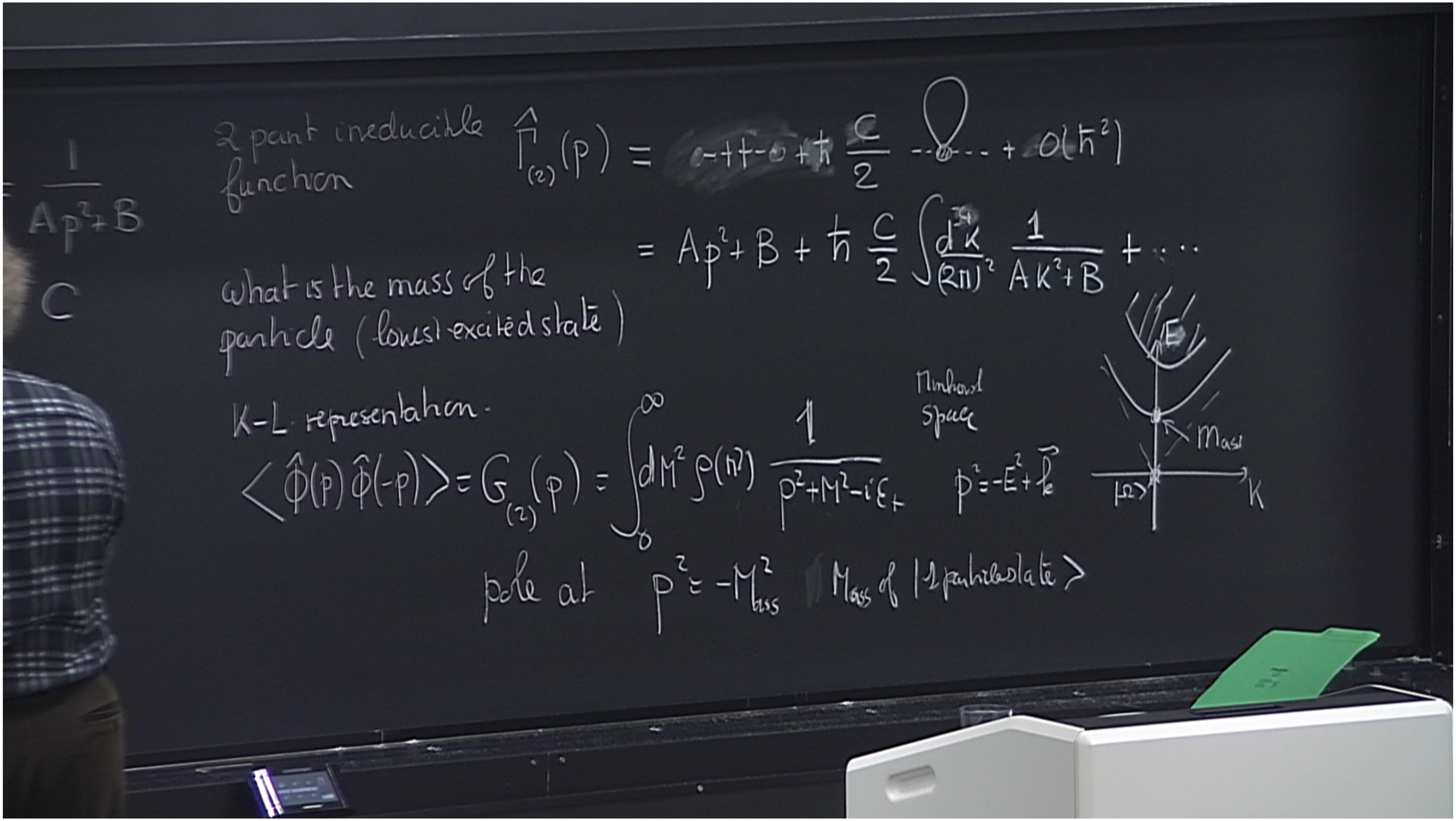
$$= C$$

2 point irreducible function

$$\hat{\Gamma}_{(2)}(p) = \quad + \hbar \frac{C}{2} \quad + O(\hbar^2)$$

$$= Ap^2 + B + \hbar \frac{C}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{Ak^2+B} + \cdots$$

what is the mass of the particle (lowest excited state)



$\frac{1}{Ap^2+B}$
C
2 point irreducible function
$\hat{\Gamma}_{(2)}(p) = \hbar \frac{C}{2} + O(\hbar^2)$
$= Ap^2+B+\hbar\frac{C}{2}\int\frac{d^3k}{(2\pi)^2}\frac{1}{Ak^2+B} + \ldots$
what is the mass of the particle (lowest excited state)
K-L representation.
Minkowski space
$\langle \hat{\phi}(p)\hat{\phi}(-p)\rangle = G_{(2)}(p) = \int_0^\infty dM^2 \rho(M^2)\frac{1}{p^2+M^2-i\epsilon_+}$
$p^2=-E^2+\vec{k}$
E
mass
$|\Omega\rangle$
K
pole at $p^2=-M^2_{mass}$
Mass of |1 particle state>

Is start from Regularized theory $|k| < \Lambda$

1st step: massless interacting theory in $D=4$

what is the mass ?

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$ path integral

A, B, C are parameters to be determined..

$$S[\phi] = \int d^4x \, \frac{A}{2} \partial_\mu \phi \partial^\mu \phi + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4$$

renormalized action, function of a renormalized field ϕ

$$= \frac{1}{Ap^2 + B}$$

$$= C$$

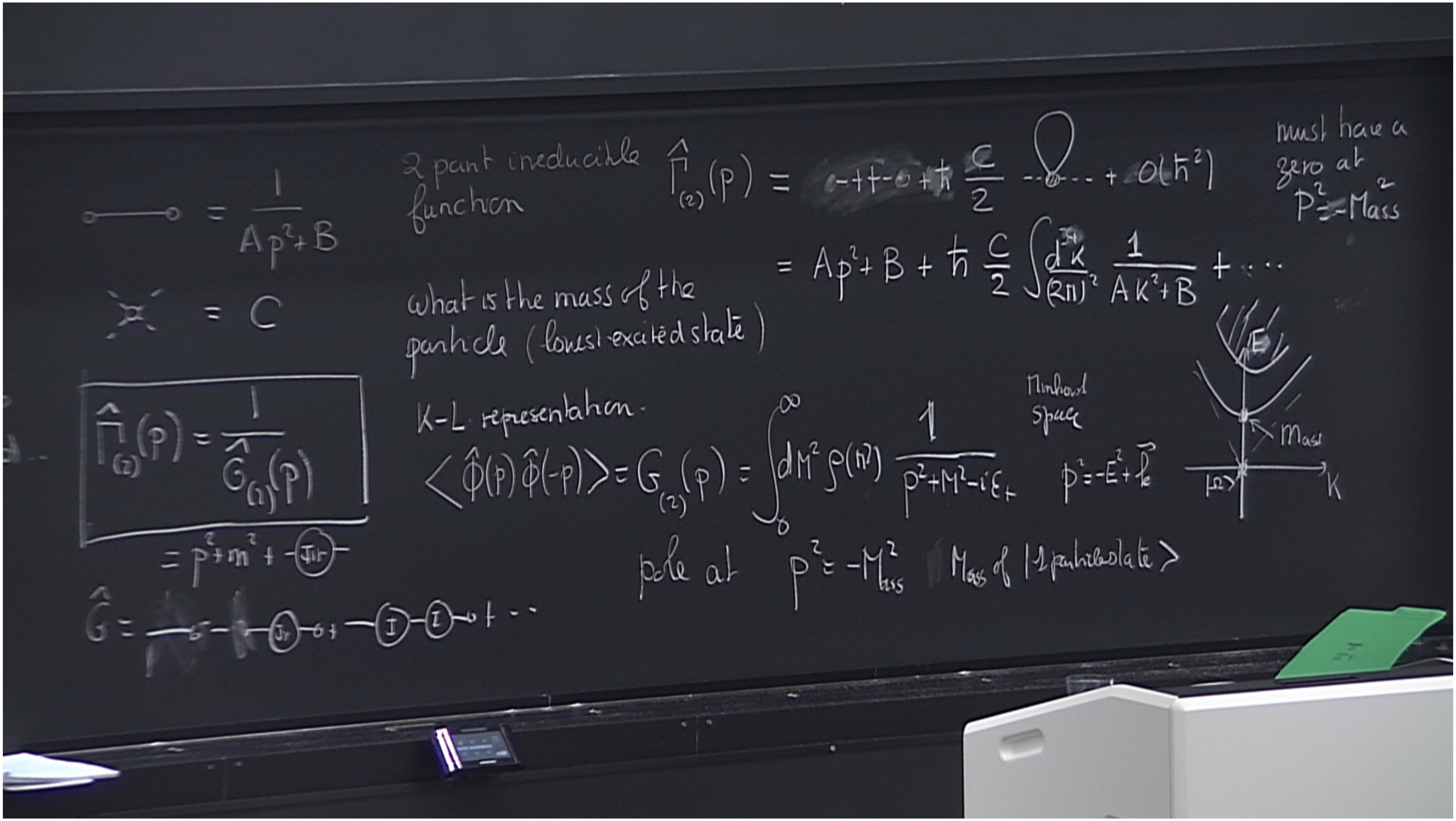
$$\hat{\Gamma}_{(2)}(p) = \frac{1}{\hat{G}_{(2)}(p)}$$

2 point function

what is particl

K-L. re

$\langle \hat{\phi}($


2 point irreducible function
Γ̂(2)(p) = + ħ C/2 + O(ħ²)
= Ap²+B + ħ C/2 ∫ d³k/(2π)³ 1/(Ak²+B) + ...
must have a zero at p² = -Mass²
= 1/(Ap²+B)
= C
what is the mass of the particle (lowest excited state)
Γ̂(2)(p) = 1/Ĝ(2)(p)
= p²+m² + Jir
Ĝ = Jr + I I + ...
K-L representation
Minkowski space
⟨φ̂(p) φ̂(-p)⟩ = G(2)(p) = ∫0^∞ dM² ρ(M²) 1/(p²+M²-iε+)
p² = -E² + k²
mass
|Ω⟩
E
K
pole at p² = -M²mass
Mass of |1 particle state⟩

if $|k| < \Lambda$

renormalization

$$\Gamma_{(2)}(p) = A p^2 + B + \hbar \frac{C}{2} \bigcirc$$

$$\bigcirc = T = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{A k^2 + B} \sim \Lambda^2$$

"tadpole diagram"

$$M^2_{phys} = M^2_{ass} = \left(B + \hbar \frac{C}{2} T\right) A^{-1}$$ instead of $\frac{A}{B}$ "bare mass" or classical mass

mass is renormalized (not surprising)

renormalization in ∞ $\Lambda \nearrow \infty$

cloud of virtual particles

mass

ball in a fluid

$M^2_{phys} = 0$ $\quad B + \hbar \frac{C}{2} T = 0$ $\quad \Gamma_2(p) = A p^2 + O(\hbar^2)$

massless theory $\quad$ adjust B (finetuning)

Same as for the free massless theory because T independent of p

$$\frac{1}{AK^2+B} \quad \sim \quad \Lambda^2$$

ast

ma fluid

as for

ree massless theory

T independent of P

Take $\boxed{A = 1}$

$$B = -\hbar \frac{c}{2} T(B,\Lambda)$$

$$T(B) = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{K^2+B}$$

$$\Gamma_{(2)} = p^2 + O(\hbar^2)$$

$$\boxed{B = -\hbar \frac{c}{2} T(0,\Lambda)} + O(\hbar^2)$$

counterterm