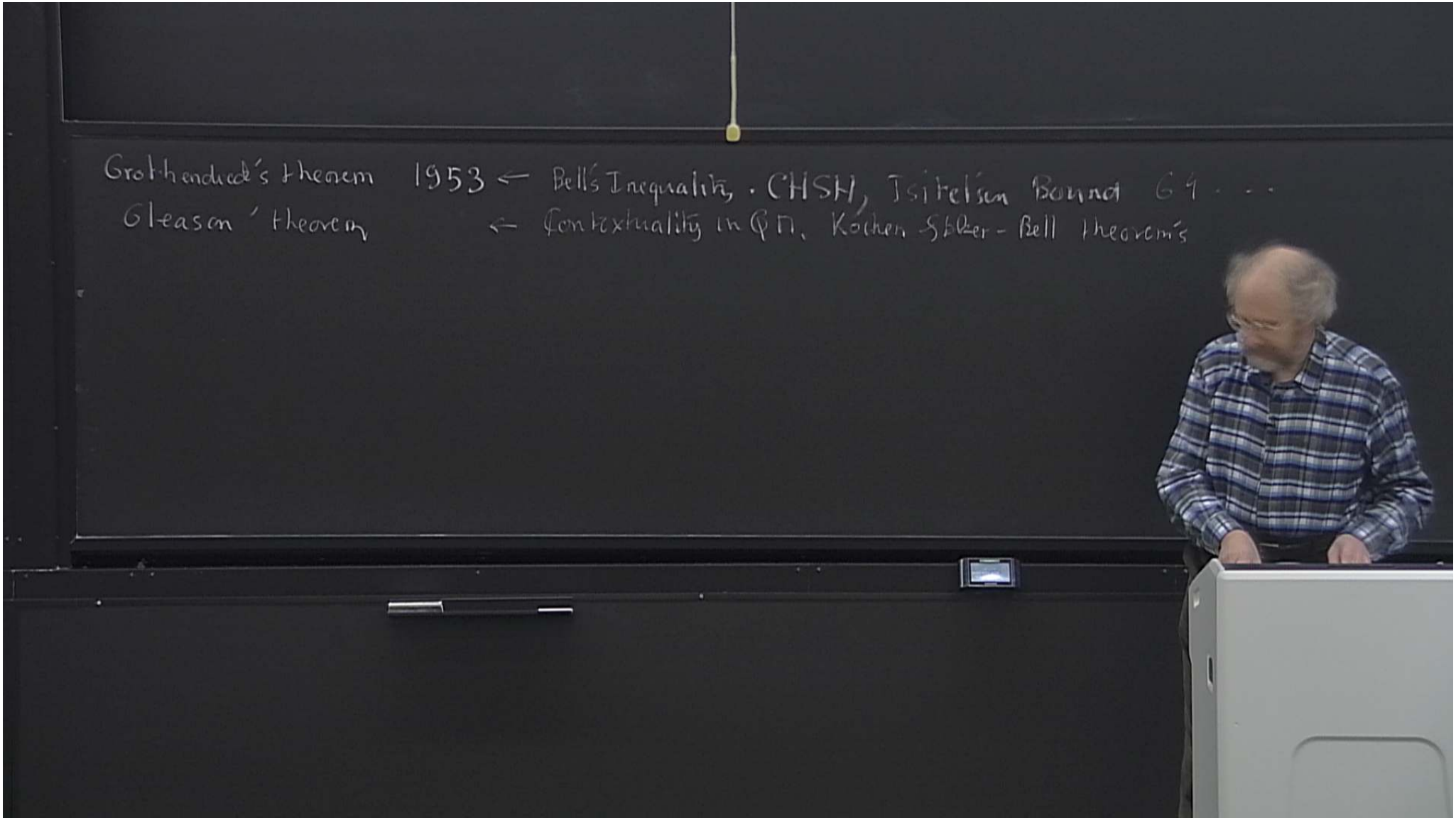
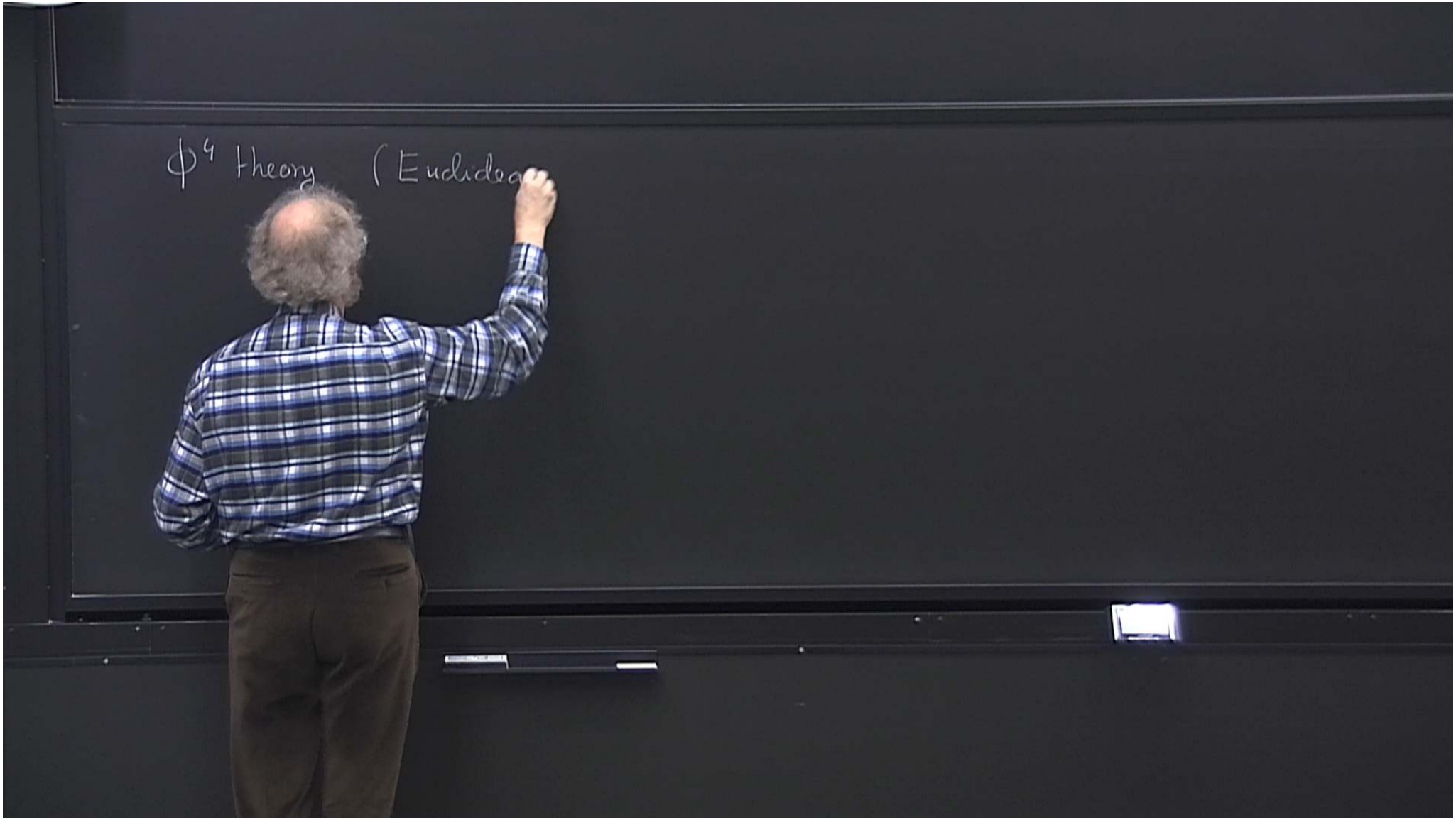
Title: 14/15 PSI - Quantum Field Theory II-Lecture 5

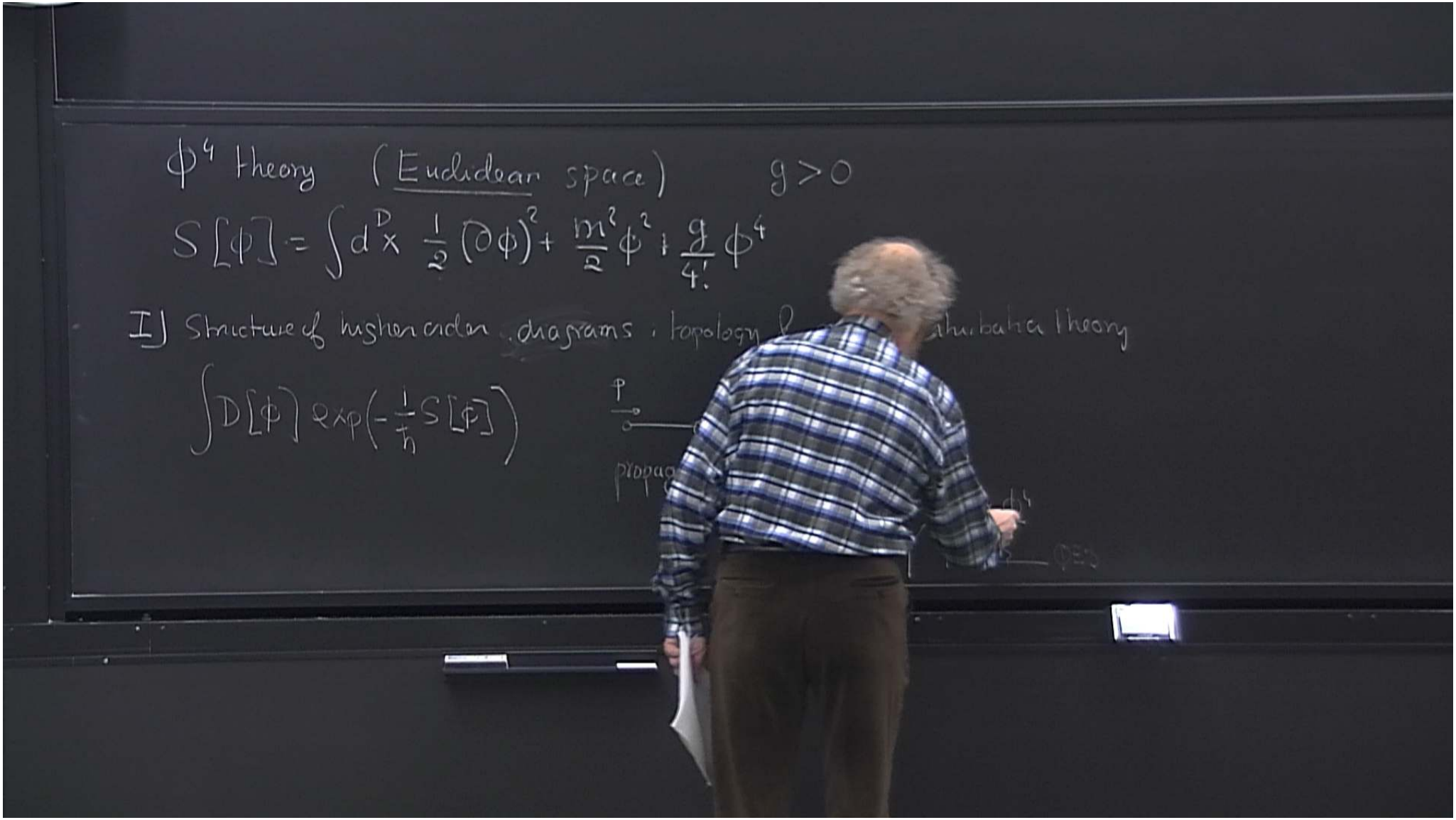
Date: Nov 14, 2014 09:00 AM

URL: http://pirsa.org/14110010

Abstract:


Grothendieck's theorem 1953 ← Bell's Inequality, CHSH, Tsirelson Bound 64 ...
Gleason's theorem ← Contextuality in QT. Kochen-Specker - Bell theorem's

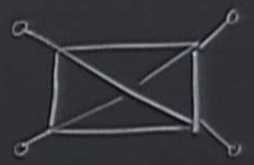

ϕ^4 Theory (Euclidean


ϕ^4 theory (Euclidean space) $g>0$
$S[\phi] = \int d^D x \, \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$
II Structure of higher order diagrams, topology
perturbation theory
$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$
p
propag
ϕ^4
QED

ϕ^4 theory (Euclidean space) $g > 0$

$$S[\phi] = \int d^D x \, \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

general diagrams:



I) Structure of higher order diagrams : topology & order of p[illegible] a theory

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

propagator $\frac{\hbar}{p^2 + m^2}$

ϕ^4 theory (Euclidean space) $g > 0$

$$S[\phi] = \int d^D x \, \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

$\langle \phi \dots \phi \rangle$ (N) order g^K

general diagrams:

N external vertices
K internal vertices
order $\hbar^M$

Structure of higher order diagrams: topology & order of perturbation theory

$$\int D[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)$$

propagator $\frac{\hbar}{p^2+m^2}$

$\frac{g}{\hbar}$

interaction vertex ϕ^3

ϕ^3 ξ $\phi^{E=3}$

ϕ^4 theory (Euclidean space) $g > 0$ $\langle \phi \dots \phi \rangle$ (N) order g^K

$$S[\phi] = \int d^D x \, \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

general diagrams:

L lines

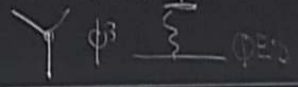
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$

... higher order diagrams : topology & order of perturbation theory

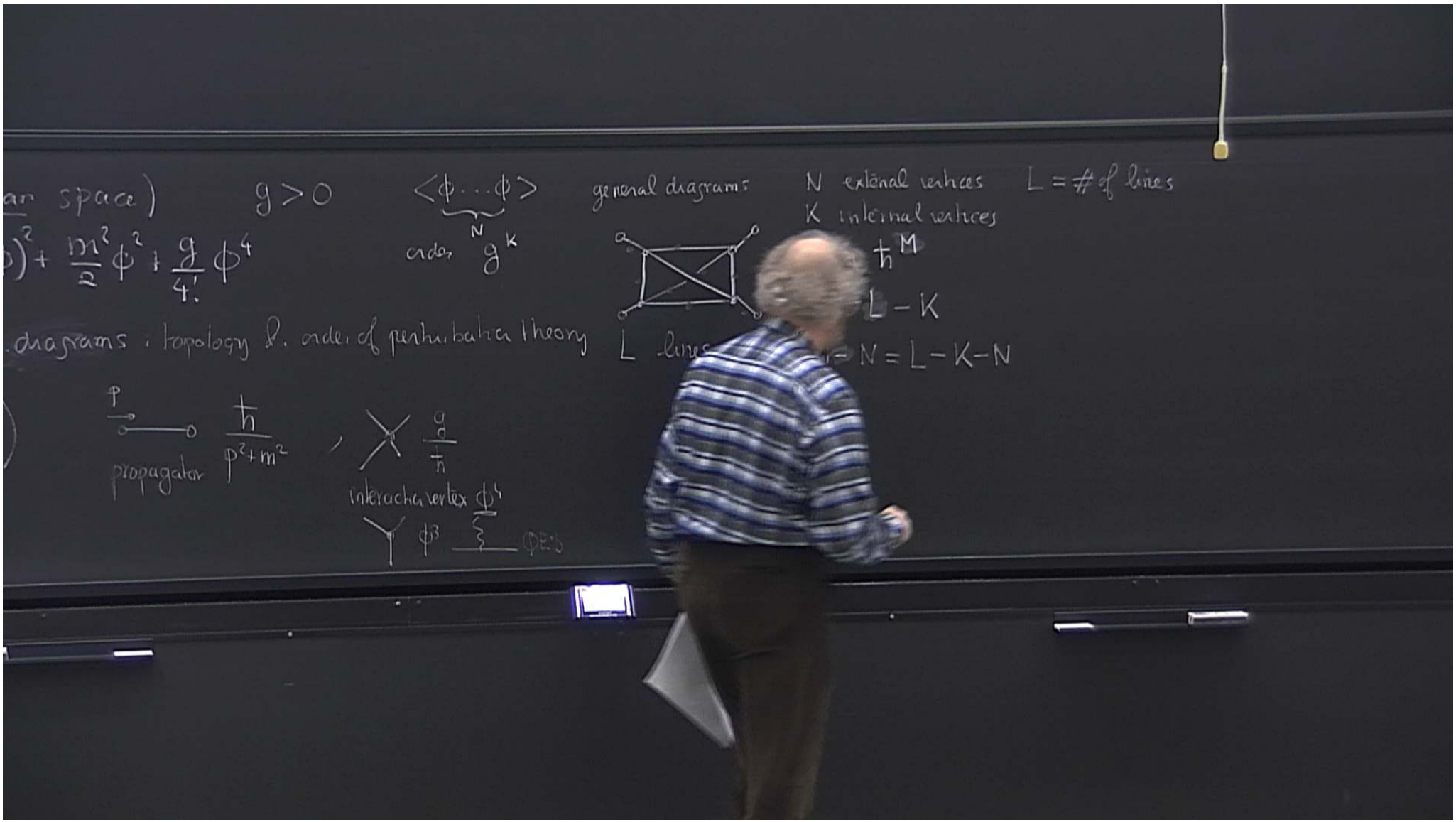
$\exp\left(-\frac{1}{\hbar} S[\phi]\right)$

propagator $\frac{\hbar}{p^2 + m^2}$

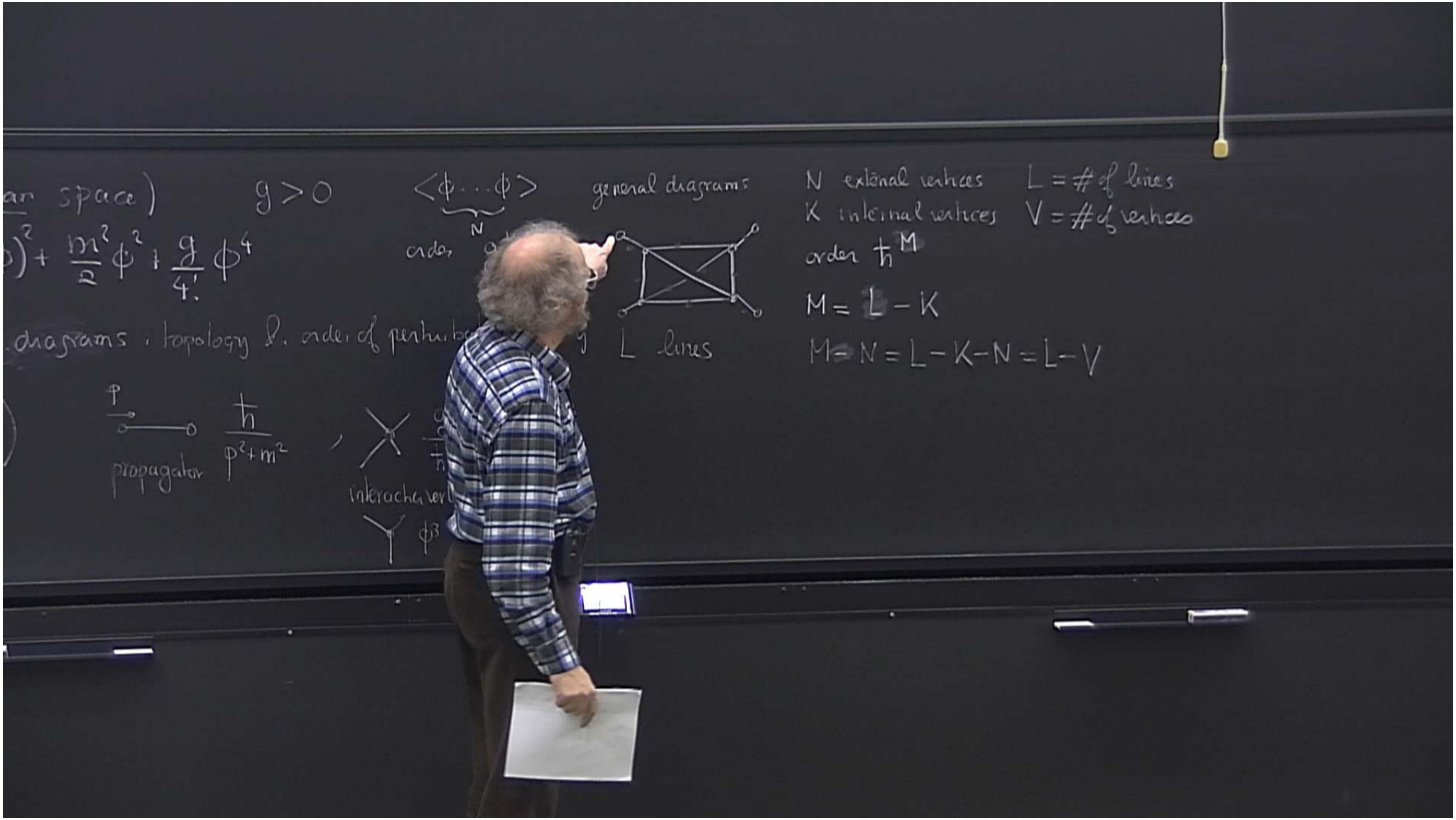
interaction vertex ϕ^3



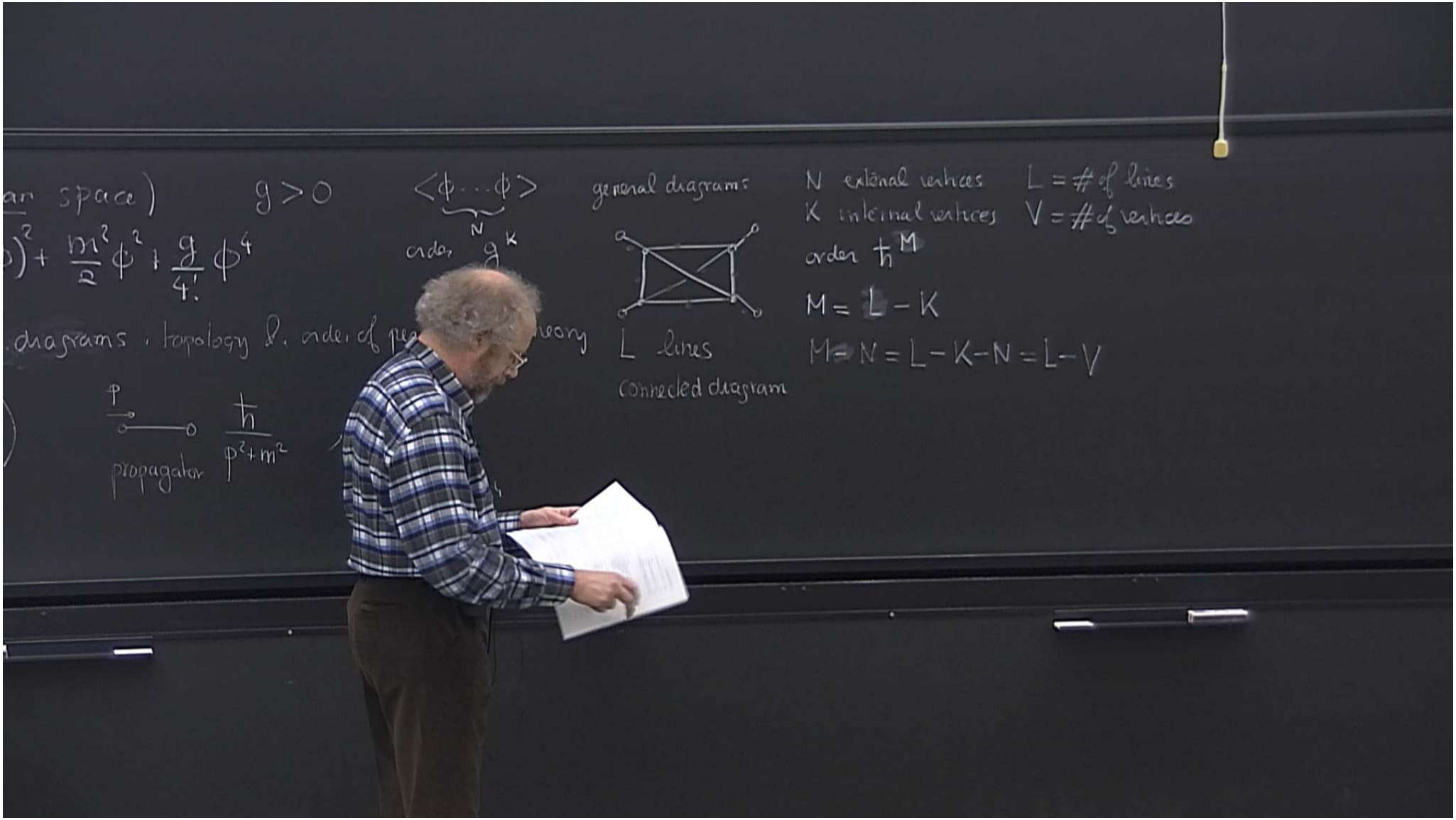
ϕ^3 $\phi^{E>3}$

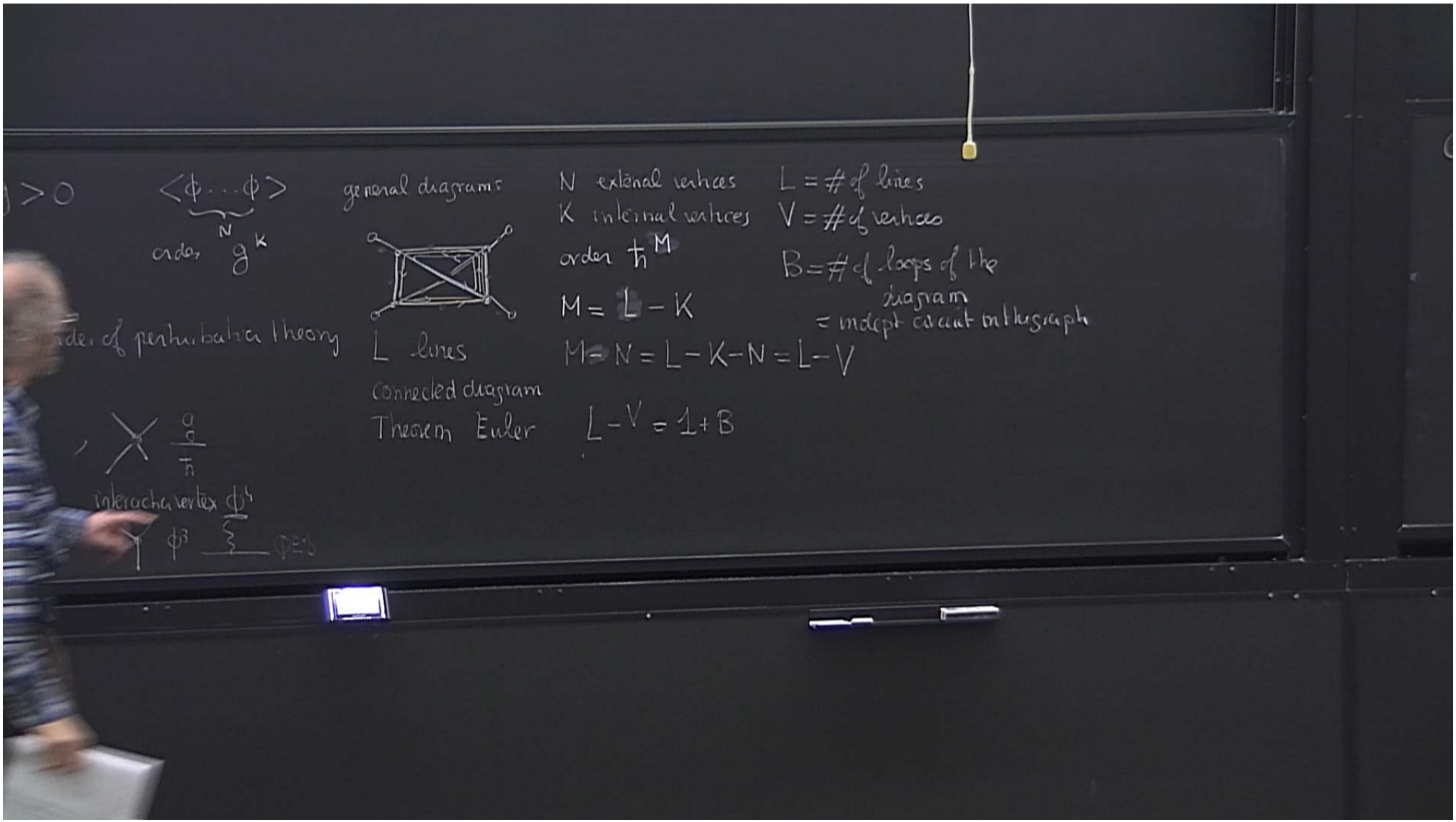


space)
g > 0
<φ...φ>
N
order g^K
general diagrams
N external vertices
K internal vertices
L = # of lines
ħ^M
L − K
diagrams : topology & order of perturbation theory
L lines
N = L − K − N
p
ħ / (p² + m²)
propagator
g / ħ
interaction vertex φ⁴
φ³
QED

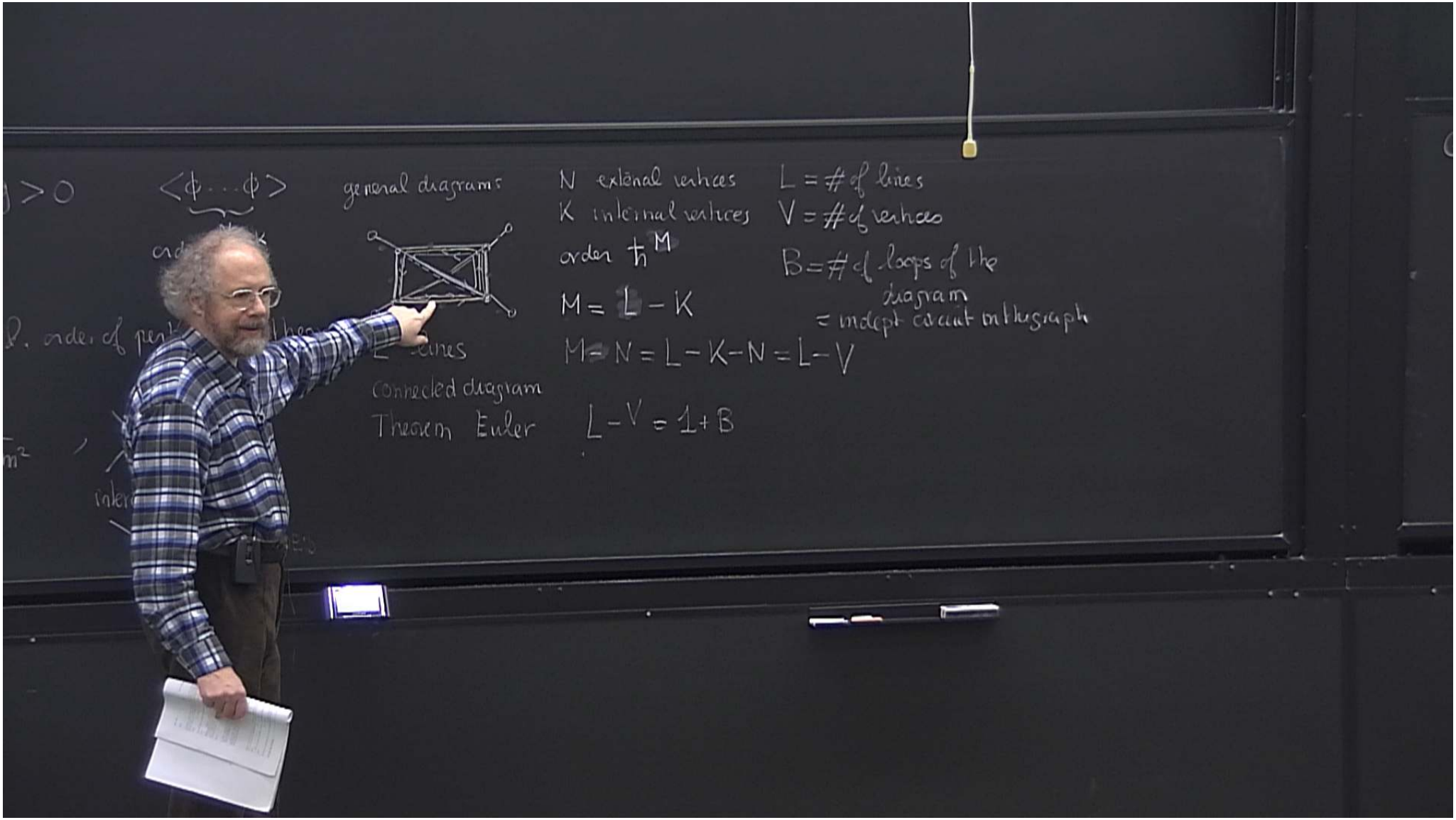


g > 0
general diagrams
N external vertices
L = # of lines
K internal vertices
V = # of vertices
order ħ^M
M = L − K
M − N = L − K − N = L − V
L lines
propagator


ar space)
$g > 0$
$\langle \phi \dots \phi \rangle$
N
order g^K
$\frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$
general diagrams:
N external vertices
K internal vertices
order $\hbar^M$
L = # of lines
V = # of vertices
$M = L - K$
$M = N = L - K - N = L - V$
L lines
connected diagram
diagrams, topology &, order of
p
propagator
$\frac{\hbar}{p^2+m^2}$



$g > 0$
$\langle \underbrace{\phi \cdots \phi}_{N} \rangle$
order g^K
general diagrams
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$
$M - N = L - K - N = L - V$
L = # of lines
V = # of vertices
B = # of loops of the diagram
= indept circuit in the graph
order of perturbation theory
L lines
connected diagram
Theorem Euler
$L - V = 1 + B$
interaction vertex ϕ^4
ϕ^3
QED


general diagrams:
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$
$M = N = L - K - N = L - V$
L = # of lines
V = # of vertices
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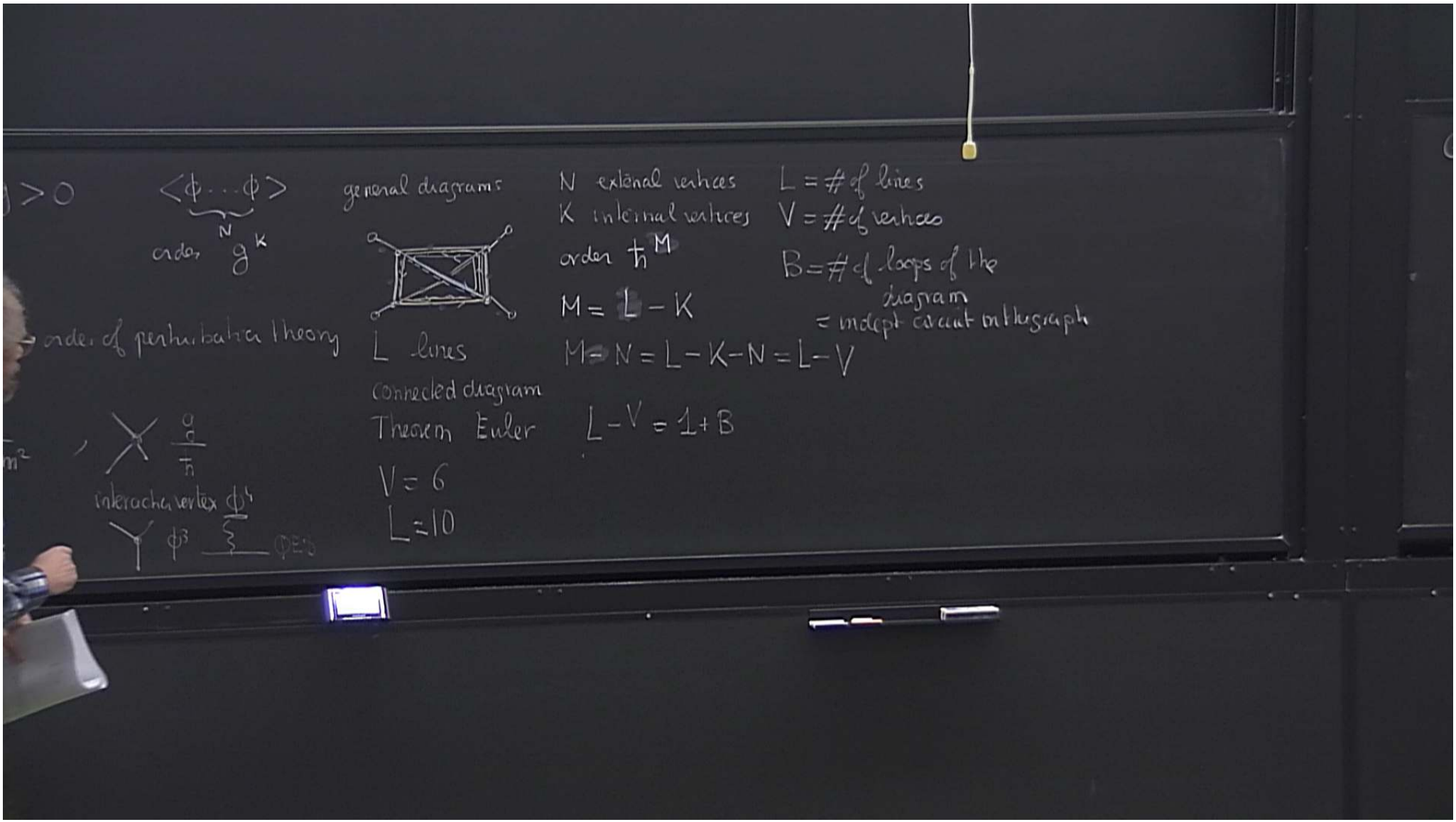
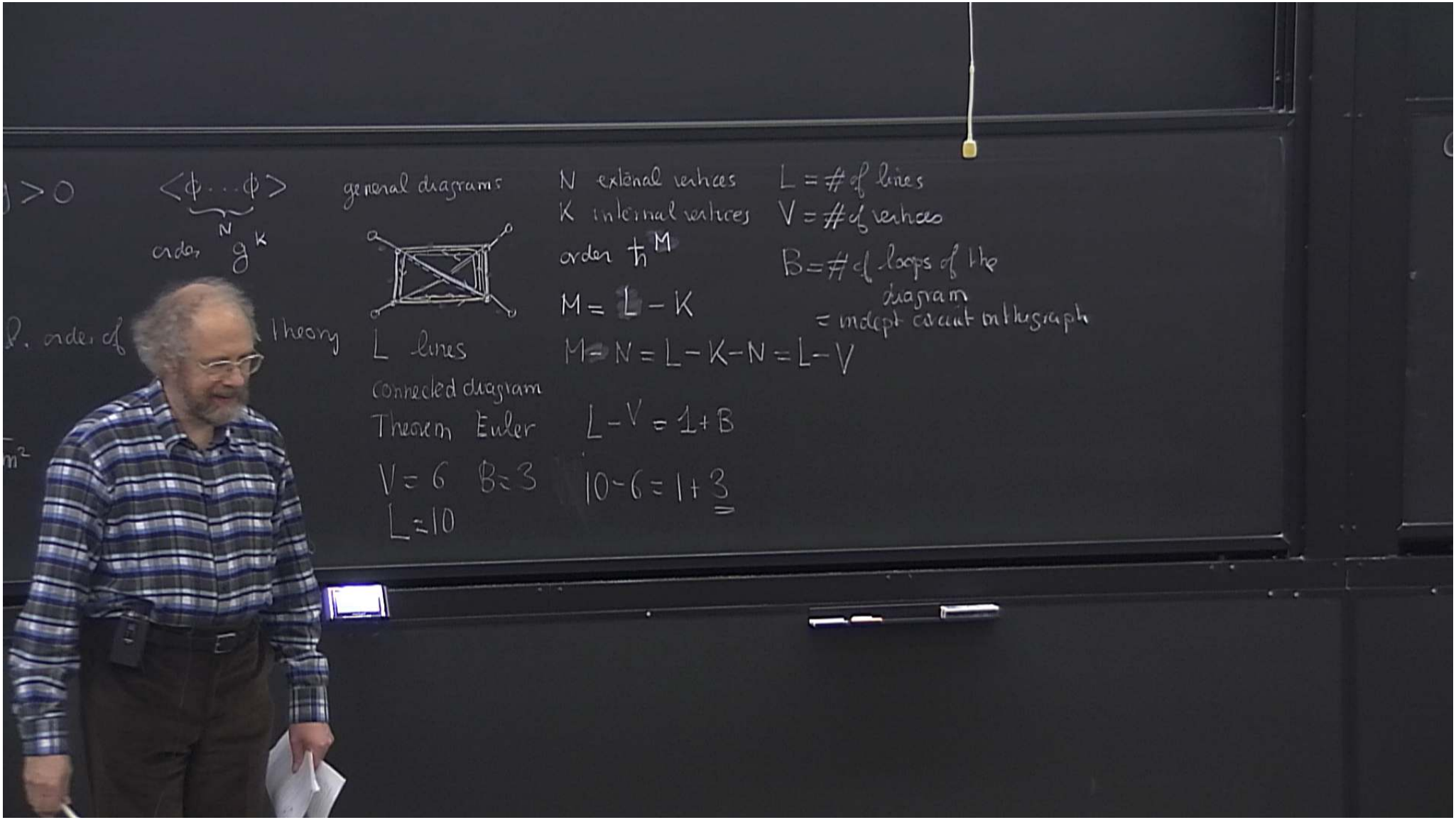
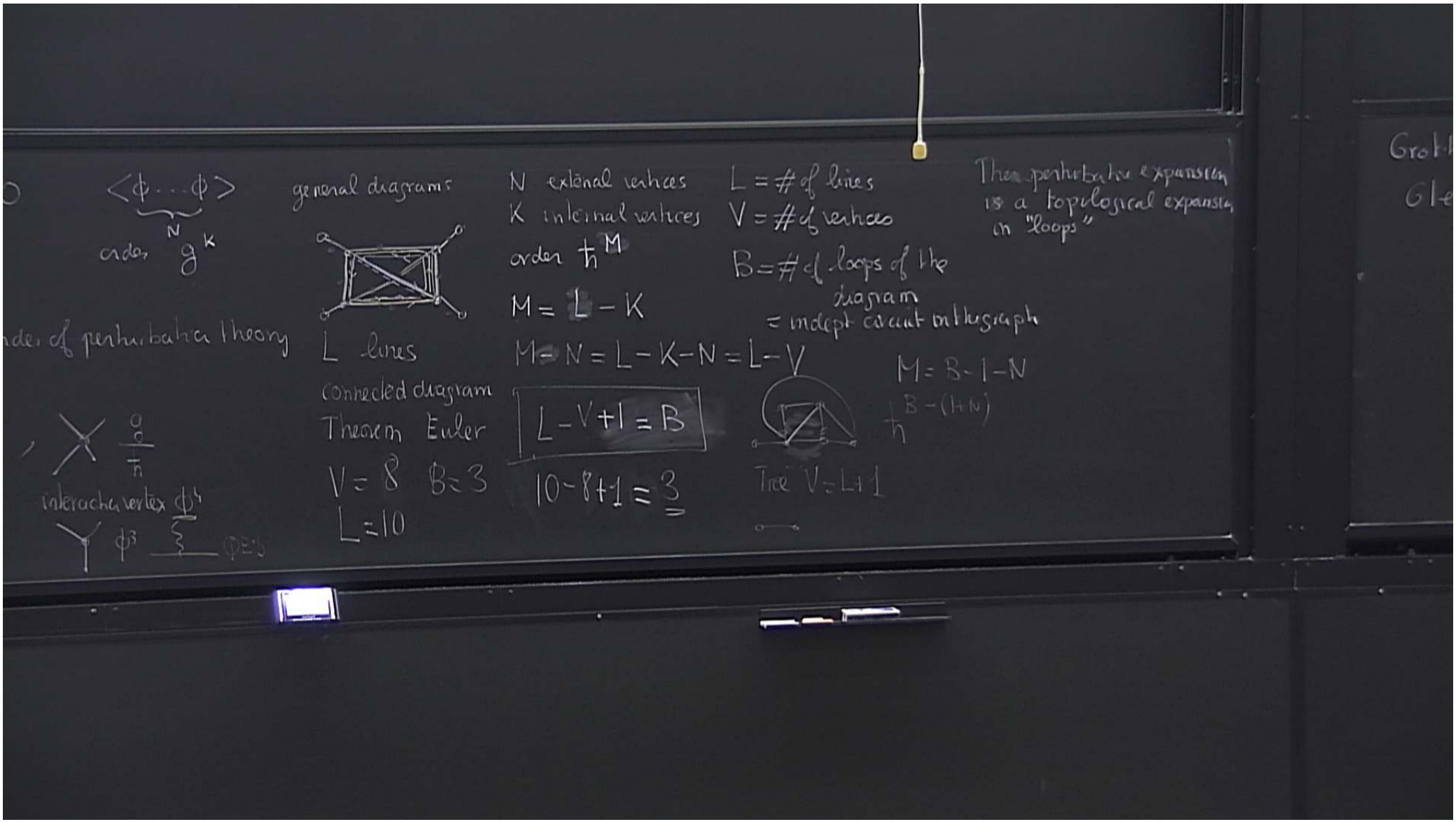
= indept circuit in the graph
connected diagram
Theorem Euler
$L - V = 1 + B$

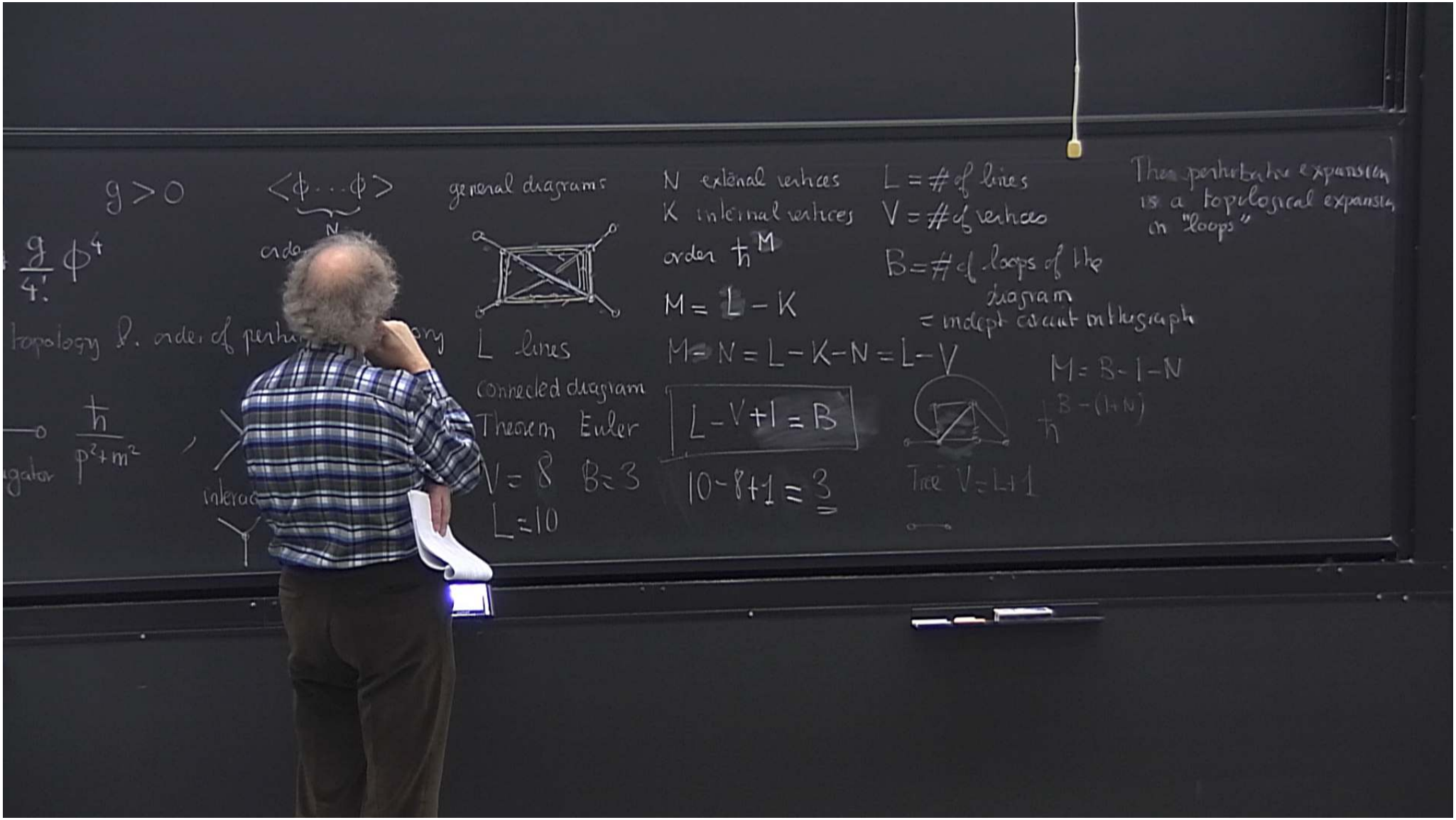


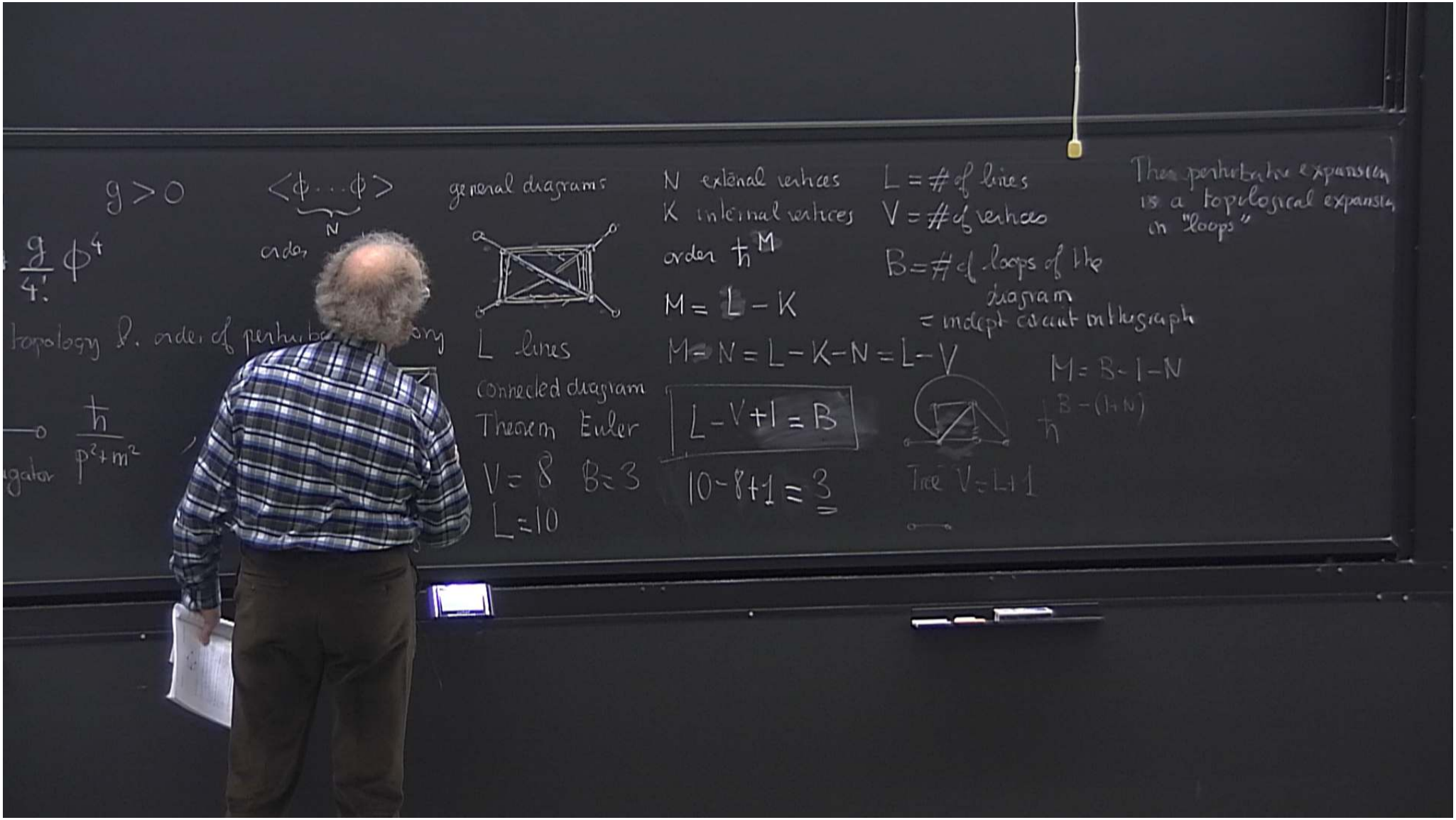
$g > 0$
$\langle \underbrace{\phi \cdots \phi}_{N} \rangle$
order g^K
order of perturbation theory
$\frac{g}{\hbar}$
interaction vertex ϕ^4
ϕ^3
general diagrams
L lines
connected diagram
Theorem Euler
$V = 6$
$L = 10$
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$
$M - N = L - K - N = L - V$
$L - V = 1 + B$
$L = \#$ of lines
$V = \#$ of vertices
$B = \#$ of loops of the diagram
= indept circuit in the graph

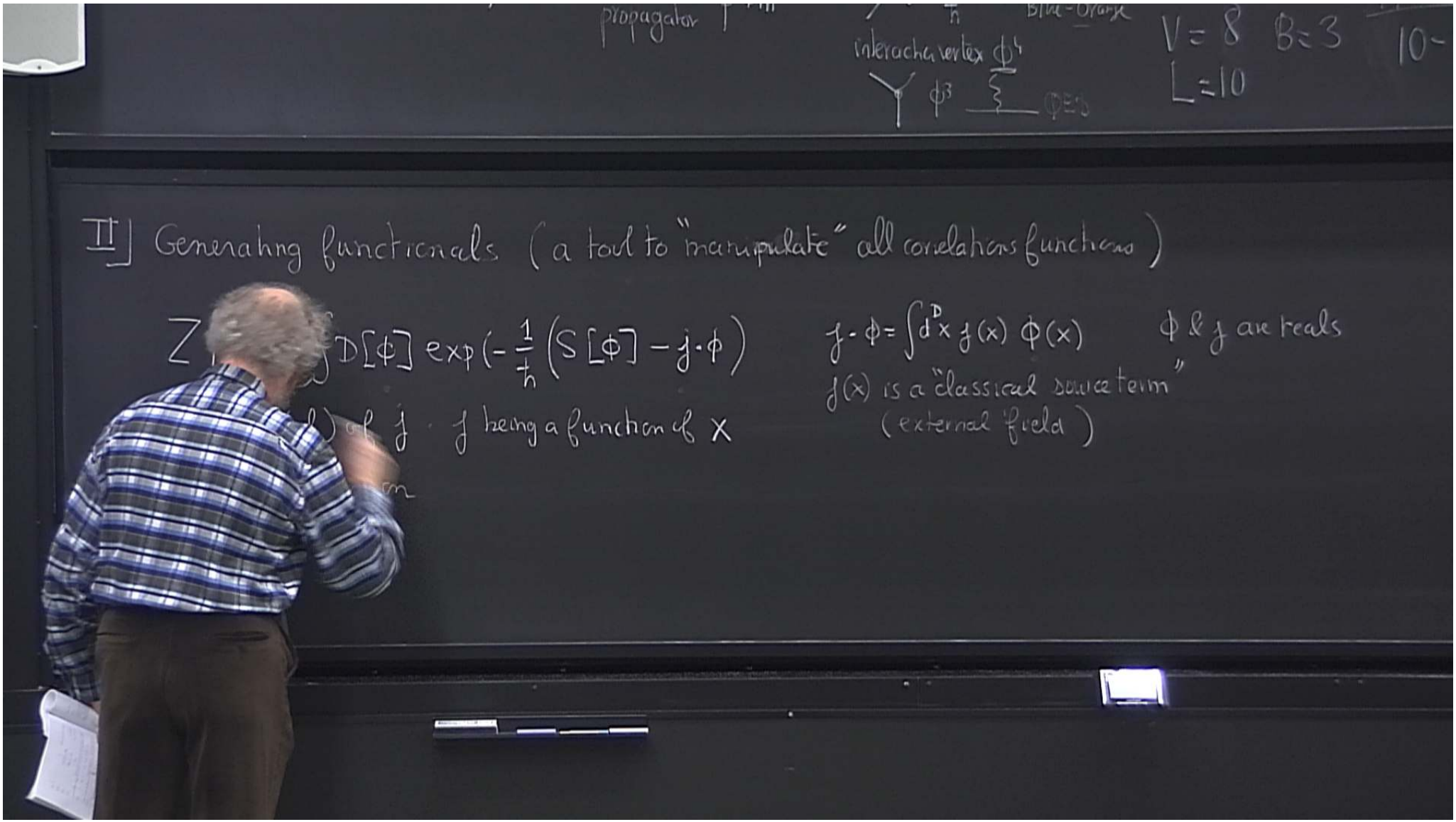

$g > 0$
$\langle \phi \ldots \phi \rangle$
N
order g^K
R. order of
theory
m^2
general diagrams:
N external vertices
K internal vertices
order $\hbar^M$
L lines
connected diagram
Theorem Euler
V = 6 B = 3
L = 10
L = # of lines
V = # of vertices
B = # of loops of the diagram
= indept circuit on the graph
$M = L - K$
$M = N = L - K - N = L - V$
$L - V = 1 + B$
$10 - 6 = 1 + 3$

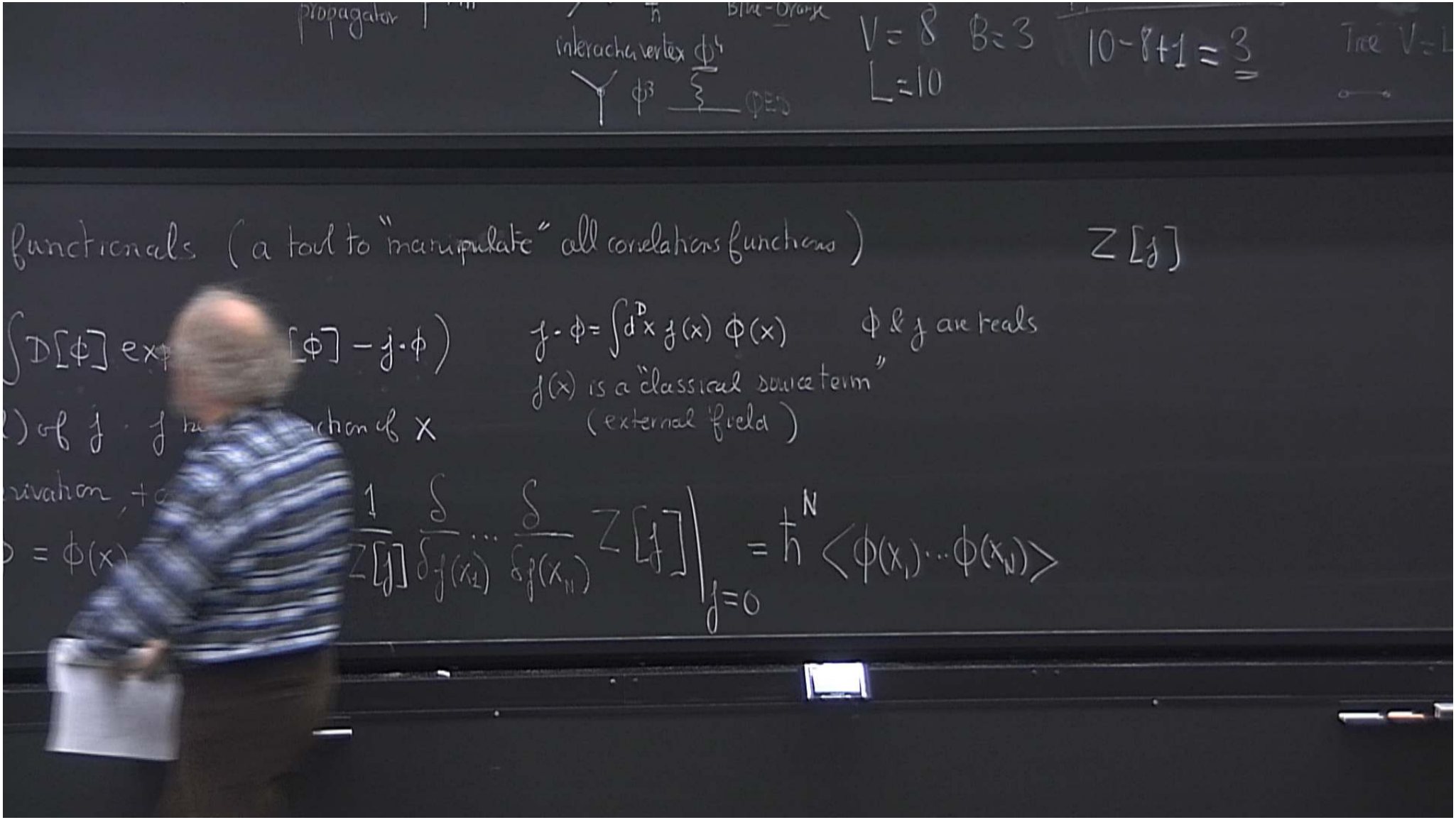


$\langle \underbrace{\phi \cdots \phi}_{N} \rangle$
order g^K
nder of perturbation theory
$\frac{g}{\hbar}$
interaction vertex ϕ^4
ϕ^3
general diagrams
L lines
connected diagram
Theorem Euler
$V = 8$ $B = 3$
$L = 10$
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$
$M - N = L - K - N = L - V$
$\boxed{L - V + 1 = B}$
$10 - 8 + 1 = 3$
$L = \#$ of lines
$V = \#$ of vertices
$B = \#$ of loops of the diagram
= indept circuit in the graph
$M = B - 1 - N$
$\hbar^{B-(1+N)}$
Tree $V = L + 1$
The perturbative expansion is a topological expansion in "loops"


$g > 0$
$\langle \phi \dots \phi \rangle$
N
general diagram:
$\frac{g}{4!}\phi^4$
N external vertices
K internal vertices
order $\hbar^M$
$M = L - K$
$M = N = L - K - N = L - V$
L = # of lines
V = # of vertices
B = # of loops of the diagram = indept circuit in the graph
The perturbative expansion is a topological expansion in "loops"
L lines
connected diagram
Theorem Euler
$L - V + 1 = B$
$10 - 8 + 1 = 3$
V = 8 B = 3
L = 10
$M = B - 1 - N$
$\hbar^{B-(1+N)}$
Tree $V = L + 1$
$\frac{\hbar}{p^2 + m^2}$


g > 0
<φ...φ>
N
order
general diagrams
N external vertices
K internal vertices
order ħ^M
M = L − K
L = # of lines
V = # of vertices
B = # of loops of the diagram
= indept circuit in the graph
The perturbative expansion is a topological expansion in "loops"
L lines
connected diagram
Theorem Euler
V = 8 B = 3
L = 10
M − N = L − K − N = L − V
L − V + 1 = B
10 − 8 + 1 = 3
M = B − 1 − N
ħ^{B−(1+N)}
Tree V = L + 1
topology & order of perturb
ħ / (p² + m²)


propagator
interaction vertex ϕ^4
ϕ^3
QED
V = 8 B = 3
L = 10
II] Generating functionals (a tool to "manipulate" all correlations functions)
$D[\phi] \exp(-\frac{1}{\hbar}(S[\phi] - j\cdot\phi)$
$j\cdot\phi = \int d^D x\, j(x)\, \phi(x)$
ϕ & j are reals
$j(x)$ is a "classical source term"
(external field)
j being a function of x

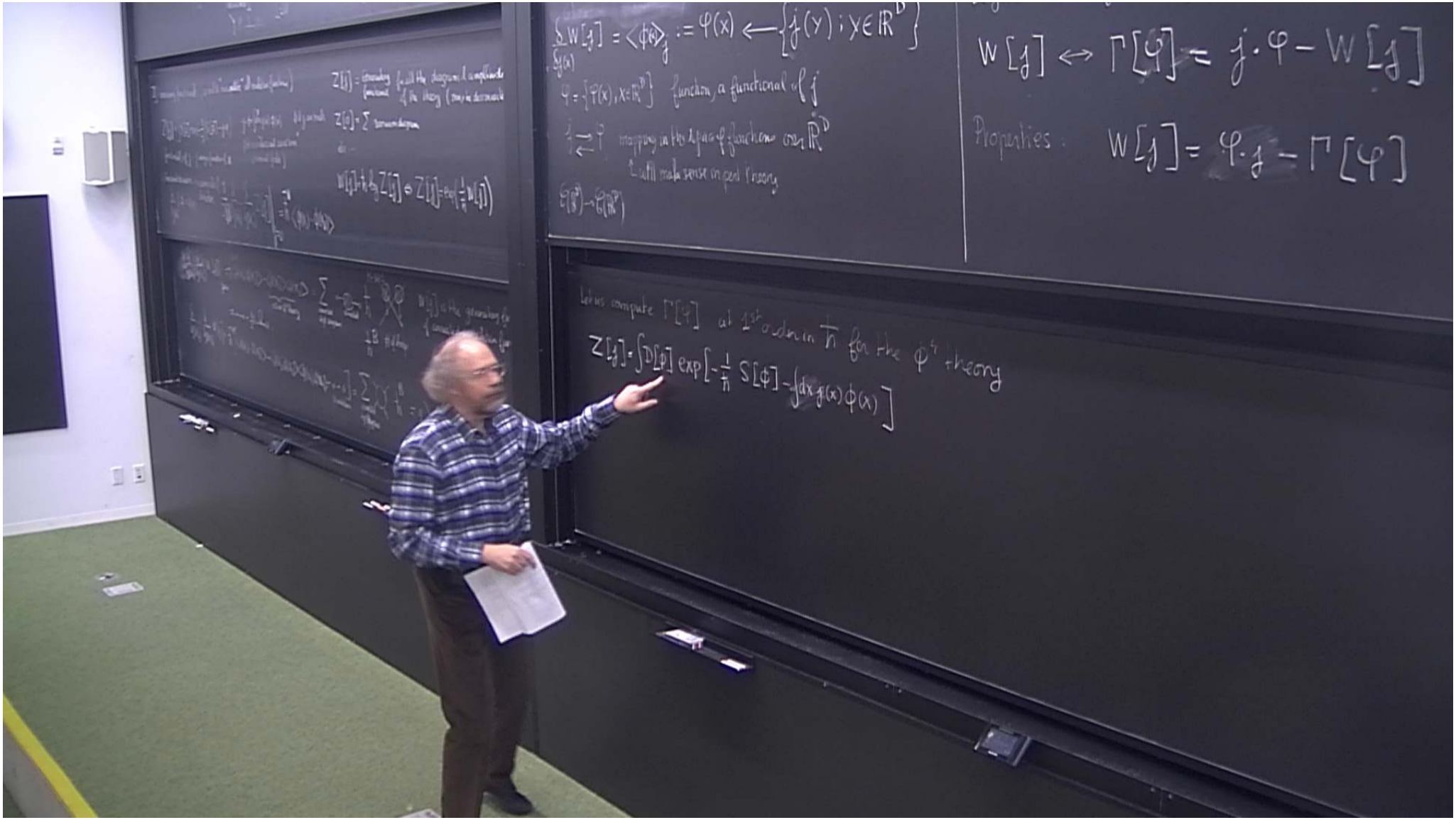

propagator
interaction vertex ϕ^3
ϕ^3
QED
$V=8$ $B=3$
$L=10$
$10-8+1=3$
Tree
functionals (a tool to "manipulate" all correlations functions)
$Z[j]$
$\int \mathcal{D}[\phi]$
$j\cdot\phi = \int d^D x\, j(x)\,\phi(x)$
ϕ & j are reals
$j(x)$ is a "classical source term"
(external field)
$\frac{1}{Z[j]}\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_N)} Z[j]\Big|_{j=0} = \hbar^N \langle \phi(x_1)\cdots\phi(x_N)\rangle$

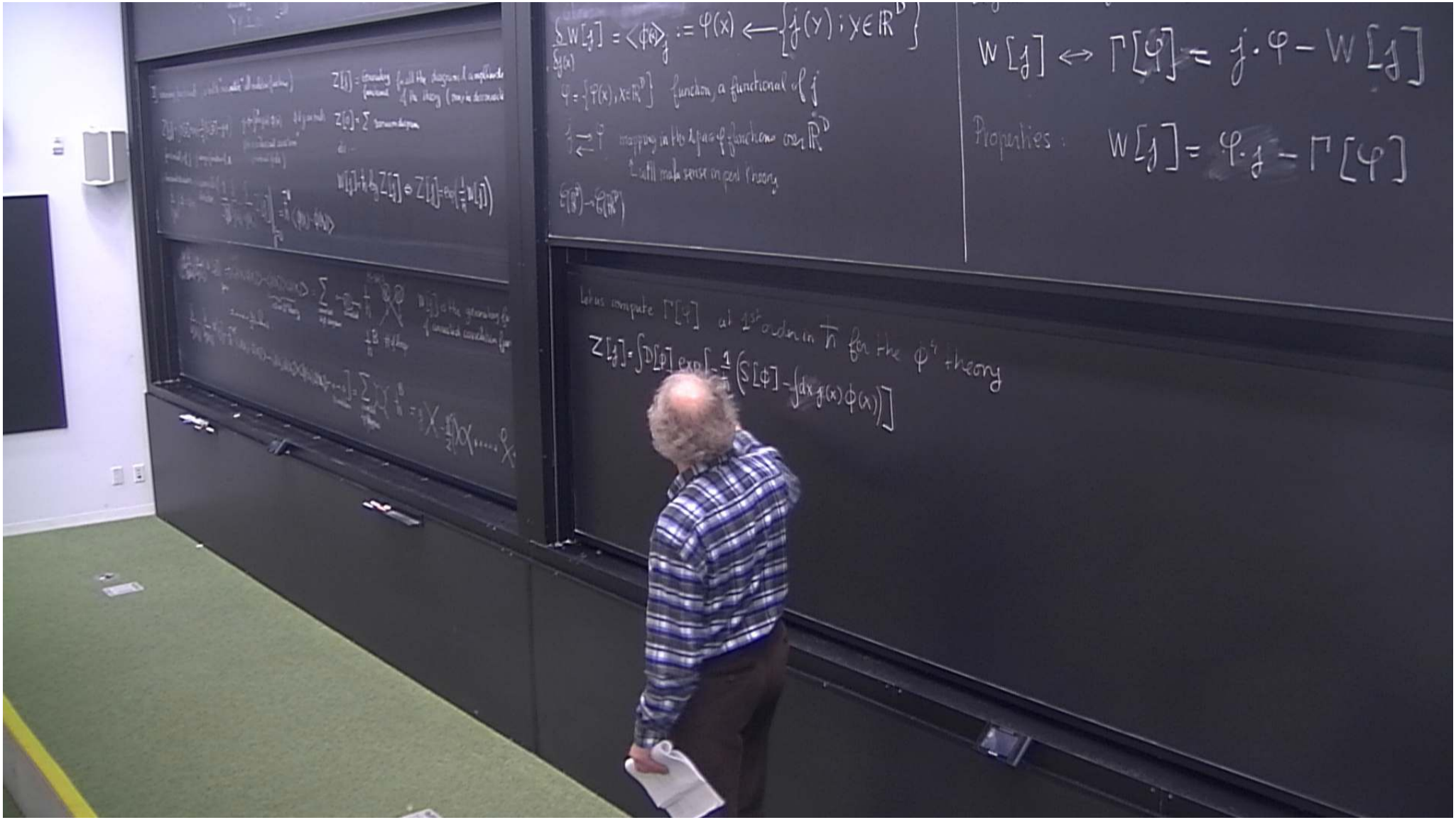
Effective action $\Gamma[\varphi]$ — φ will be a classical field $\varphi(x)$ function $\mathbb{R}^D \to \mathbb{R}$ or $M^{(1,D-1)}$
(background field)

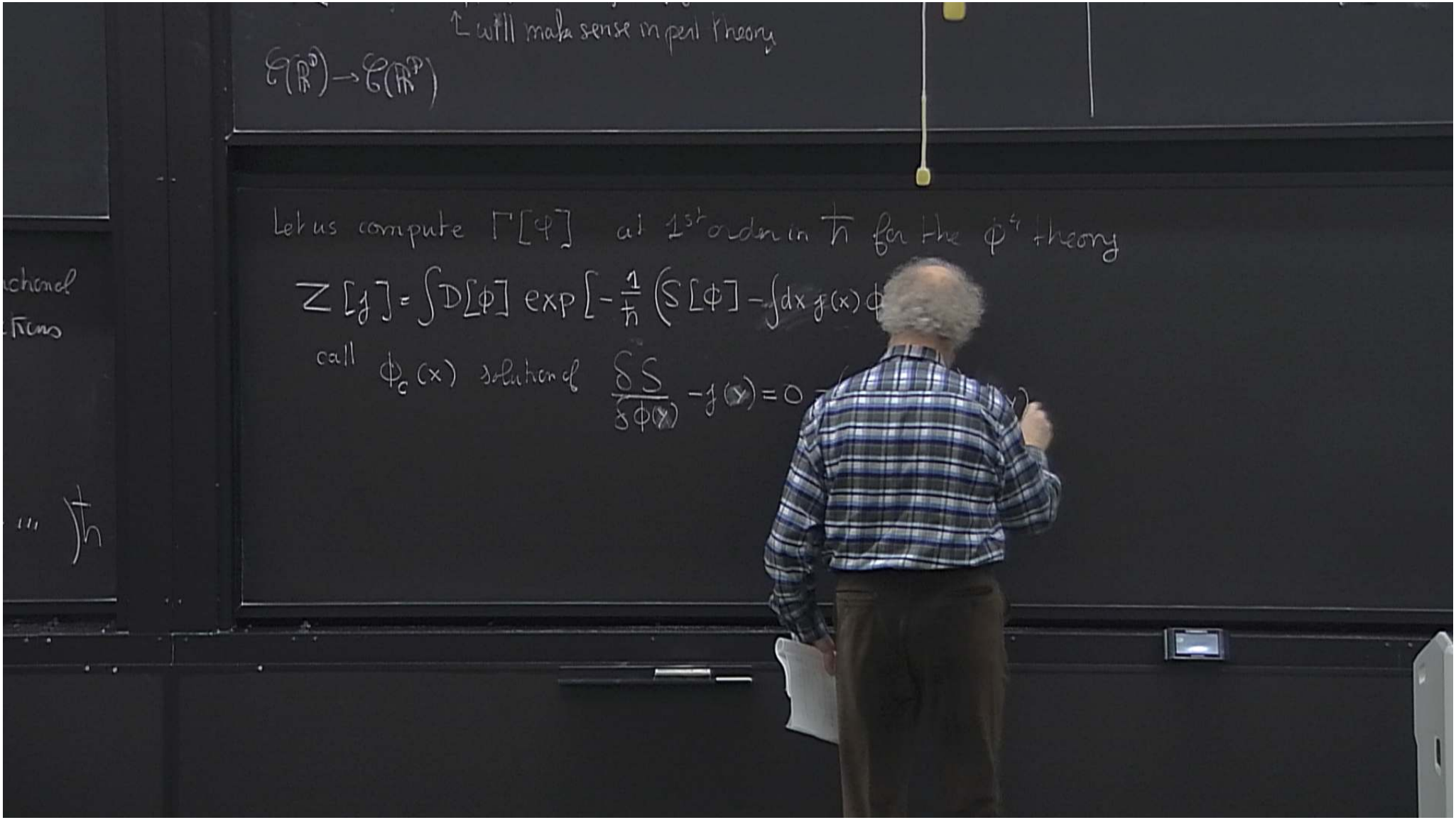
$$\frac{\delta W[j]}{\delta j(x)} = \langle \phi(x) \rangle_j := \varphi(x) \longleftarrow \{ j(y) ; y \in \mathbb{R}^D \}$$

$\varphi = \{ \varphi(x), x \in \mathbb{R}^D \}$ function, a functional of j

$j \longrightarrow \varphi$ mapping in the space of functions on $\mathbb{R}^D$


$\frac{\delta W[j]}{\delta j(x)} = \langle \phi(x) \rangle_j := \varphi(x) \longleftarrow \{ j(y) ; y \in \mathbb{R}^D \}$
$\varphi = \{ \varphi(x), x \in \mathbb{R}^D \}$ function, a functional of j
$\rightleftarrows$ mapping in the space of functions over $\mathbb{R}^D$
$\Gamma[\varphi]$
$W[j] \leftrightarrow \Gamma[\varphi] = j \cdot \varphi - W[j]$
Properties: $W[j] = \varphi \cdot j - \Gamma[\varphi]$
Let us compute $\Gamma[\varphi]$ at 1st order in $\hbar$ for the ϕ^4 theory
$Z[j] = \int \mathcal{D}[\phi] \exp\left[-\frac{1}{\hbar} S[\phi] - \int dx\, j(x) \phi(x) \right]$


W[j] ↔ Γ[φ] = j·φ − W[j]
Properties : W[j] = φ·j − Γ[φ]
Let us compute Γ[φ] at 1st order in ħ for the φ⁴ theory


↑ will make sense in perl theory
$\mathcal{E}(\mathbb{R}^D) \to \mathcal{E}(\mathbb{R}^D)$
Let us compute $\Gamma[\varphi]$ at 1st order in $\hbar$ for the φ^4 theory
$Z[J] = \int \mathcal{D}[\phi] \exp\left[-\frac{1}{\hbar}\left(S[\phi] - \int dx\, J(x)\phi\right.\right.$
call $\phi_c(x)$ solution of $\frac{\delta S}{\delta \phi(x)} - J(x) = 0$

↑ will make sense in perturbation theory

$\mathcal{C}(\mathbb{R}^D) \to \mathcal{C}(\mathbb{R}^D)$

Let us compute $\Gamma[\varphi]$ at 1st order in $\hbar$ for the ϕ^4 theory

$$Z[j] = \int \mathcal{D}[\phi] \exp\left[-\frac{1}{\hbar}\left(S[\phi] - \int dx\, j(x)\phi(x)\right)\right]$$

call ϕ_c solution of $\frac{\delta S}{\delta \phi(x)} - j(x) = 0 = \left(-\Delta_y + m^2\right)\phi_c(y) + \frac{g}{3!}\phi_c^3(y) - j(x) = 0$

ϕ_c is a functional of j, classical object : saddle point of the functional integral

$\phi = \phi_c +$

call ϕ_c solution of $\frac{\delta S}{\delta \phi(x)} - j(x) = 0 = \left(-\Delta_y + m^2\right)\phi_c(y) + \frac{g}{3!}\phi_c^3(y) - j(x) = 0$

$Z[j] = \exp\left(-\frac{1}{\hbar}\left(S[\phi_c] - j \cdot \phi_c\right)\right)$

ϕ_c is a functional of j, classical object, saddle point of the functional integral

$S''[\phi_c]$ = operator whose Kernel

$= \left[-\Delta_x + m^2 + \frac{g}{2}\phi_c^2\right]$

$\phi = \phi_c + \hbar^{1/2}\tilde{\phi}$, integrate over the $\tilde{\phi}$

$\left(S''[\phi_c]\cdot\psi\right)(x) = -\Delta_x\psi(x) + m^2\psi(x) + \frac{g}{2}\phi_c^2(x)\cdot\psi(x)$

linear operator acting on functions ψ on $\mathbb{R}^D$

call ϕ_c solution of $\frac{\delta S}{\delta \phi(y)} - j(y) = 0 = (-\Delta_y + m^2)\phi_c(y) + \frac{g}{3!}\phi_c^3(y) - j(y) = 0$

ϕ_c is a functional of j. classical object : saddle point of the functional integral

$\phi = \phi_c + \hbar^{1/2}\tilde{\phi}$, integrate over the $\tilde{\phi}$

$Z[j] = \exp\left[-\frac{1}{\hbar}\left(\right.\right.$

$S''[\phi_c]$ = operator

$= [-\Delta_x + m$

$\frac{1}{\hbar} W[j])$

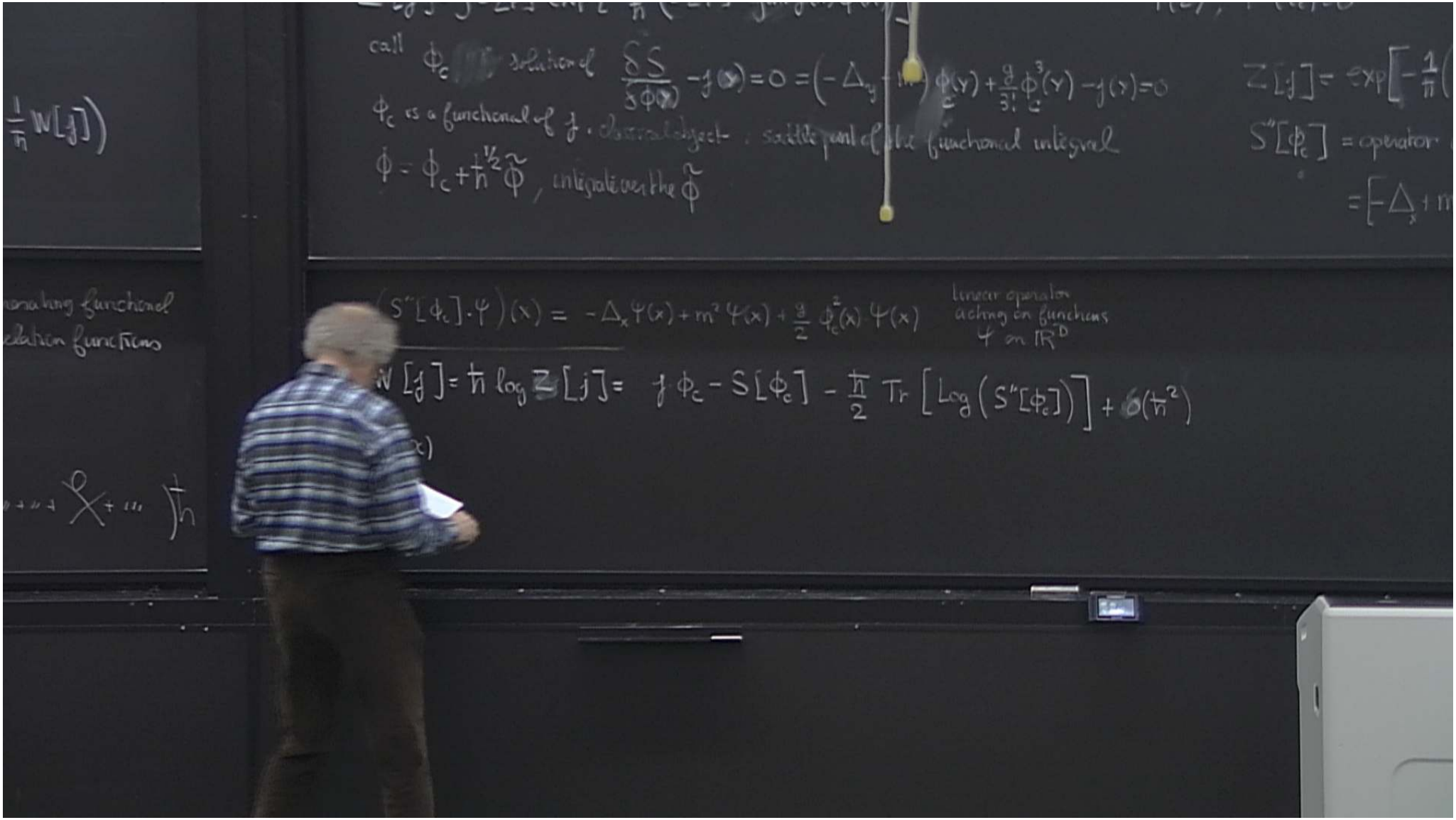
...erating functional

...elation functions

$(S''[\phi_c]\cdot\psi)(x) = -\Delta_x \psi(x) + m^2\psi(x) + \frac{g}{2}\phi_c^2(x)\,\psi(x)$

linear operator acting on functions ψ on $\mathbb{R}^D$

$W[j] = \hbar \log Z[j] = j\phi_c - S[\phi_c] - \frac{\hbar}{2}\mathrm{Tr}\left[\mathrm{Log}\left(S''[\phi_c]\right)\right] + O(\hbar^2)$


call ϕ_c solution of $\frac{\delta S}{\delta \phi(x)} - j(x) = 0 = (-\Delta_y + m^2)\phi(y) + \frac{g}{3!}\phi^3(y) - j(y) = 0$
ϕ_c is a functional of j. classical object : saddle point of the functional integral
$\phi = \phi_c + \hbar^{1/2}\tilde{\phi}$, integrate over the $\tilde{\phi}$
$Z[j] = \exp\left[-\frac{1}{\hbar}($
$S''[\phi_c]$ = operator
$= [-\Delta_x + m$
$(S''[\phi_c]\cdot\psi)(x) = -\Delta_x\psi(x) + m^2\psi(x) + \frac{g}{2}\phi_c^2(x)\cdot\psi(x)$
linear operator acting on functions ψ on $\mathbb{R}^D$
$W[j] = \hbar \log Z[j] = j\phi_c - S[\phi_c] - \frac{\hbar}{2}\mathrm{Tr}\left[\mathrm{Log}\left(S''[\phi_c]\right)\right] + \mathcal{O}(\hbar^2)$
$\frac{1}{\hbar}W[j])$
generating functional
relation functions

call ϕ_c solution of $\frac{\delta S}{\delta \phi(x)} - j(x) = 0 = \left(-\Delta_y + m^2\right)\phi(y) + \frac{g}{3!}\phi^3(y) - j(x) = 0$

ϕ_c is a functional of j. classical object : saddle point of the functional integral

$\phi = \phi_c + \hbar^{1/2}\tilde{\phi}$, integrate over the $\tilde{\phi}$

$Z[j] = \exp\left[-\frac{1}{\hbar}\right.$

$S''[\phi_c]$ = operator

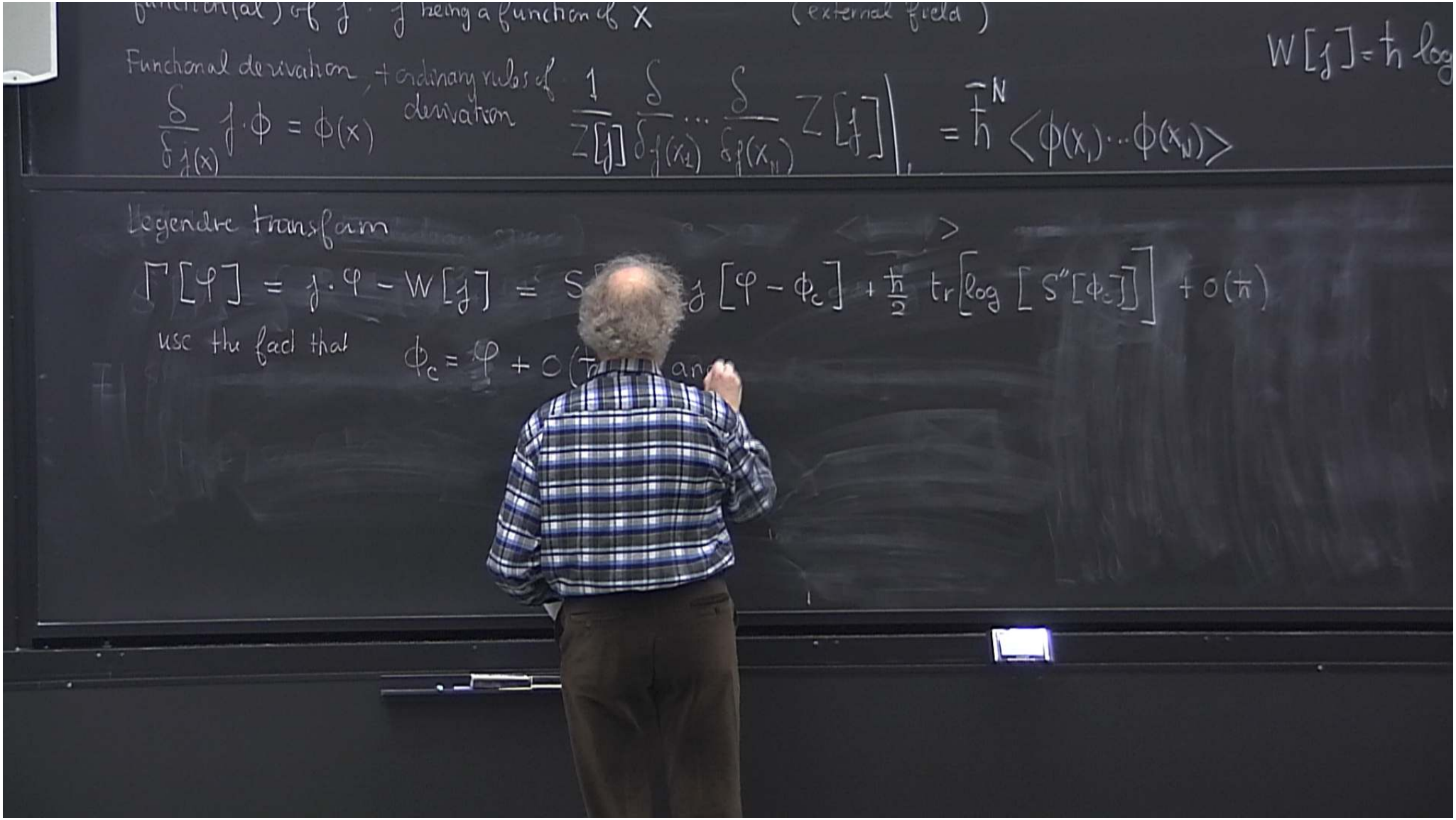
$= \left[-\Delta_x + m\right.$

$\left(S''[\phi_c]\cdot\varphi\right)(x) = -\Delta_x\varphi(x) + m^2\varphi(x) + \frac{g}{2}\phi_c^2(x)\cdot\varphi(x)$ linear operator acting on functions φ on $\mathbb{R}^D$

$W[j] = \hbar \log Z[j] = j\phi_c - S[\phi_c] - \frac{\hbar}{2}\mathrm{Tr}\left[\mathrm{Log}\left(S''[\phi_c]\right)\right] + \mathcal{O}(\hbar^2)$

$\varphi(x) = \frac{\delta W[j]}{\delta j(x)} = \phi_c(x) + \int d^D y \, \frac{\delta \phi_c(y)}{\delta j(x)} \cdot \left[\frac{\delta}{\delta \phi(y)}\left[j\cdot\phi - S[\phi] - \frac{\hbar}{2}\mathrm{Tr}\left[\mathrm{Log}\left(S''[\phi]\right)\right]\right]\right]_{\phi=\phi_c}$

\varphi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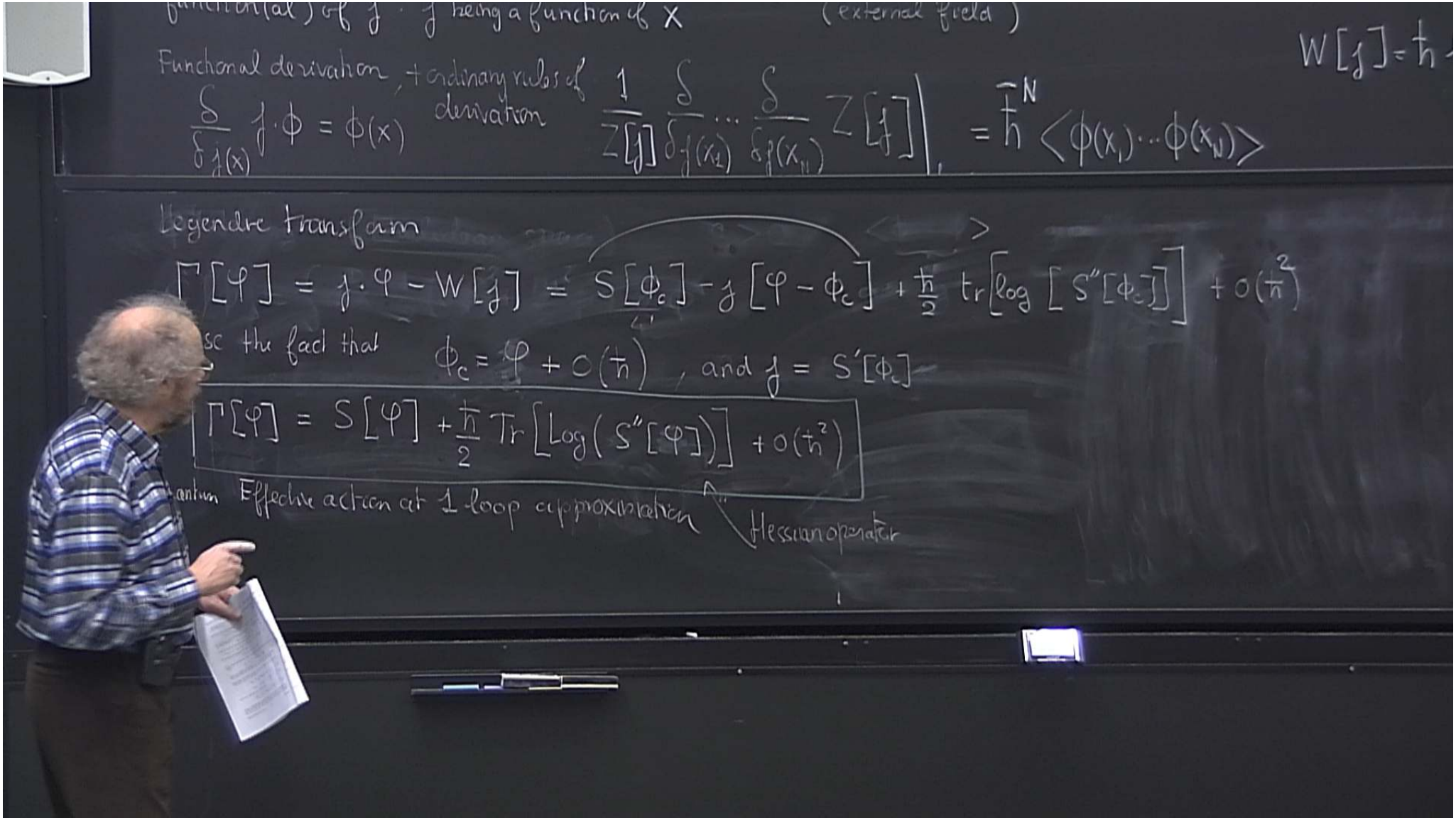
Functional derivation, + ordinary rules of derivation
Legendre transform
use the fact that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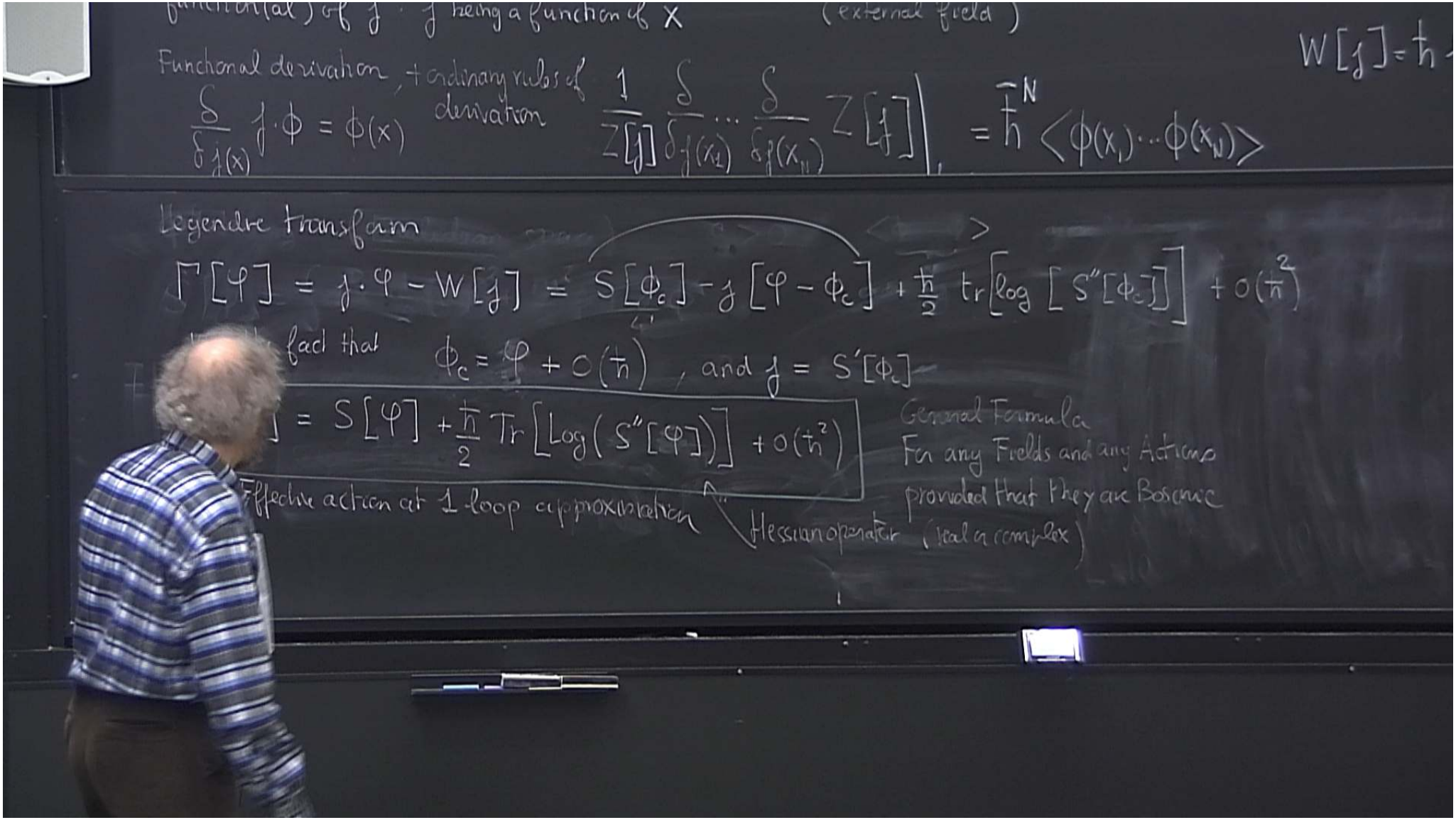
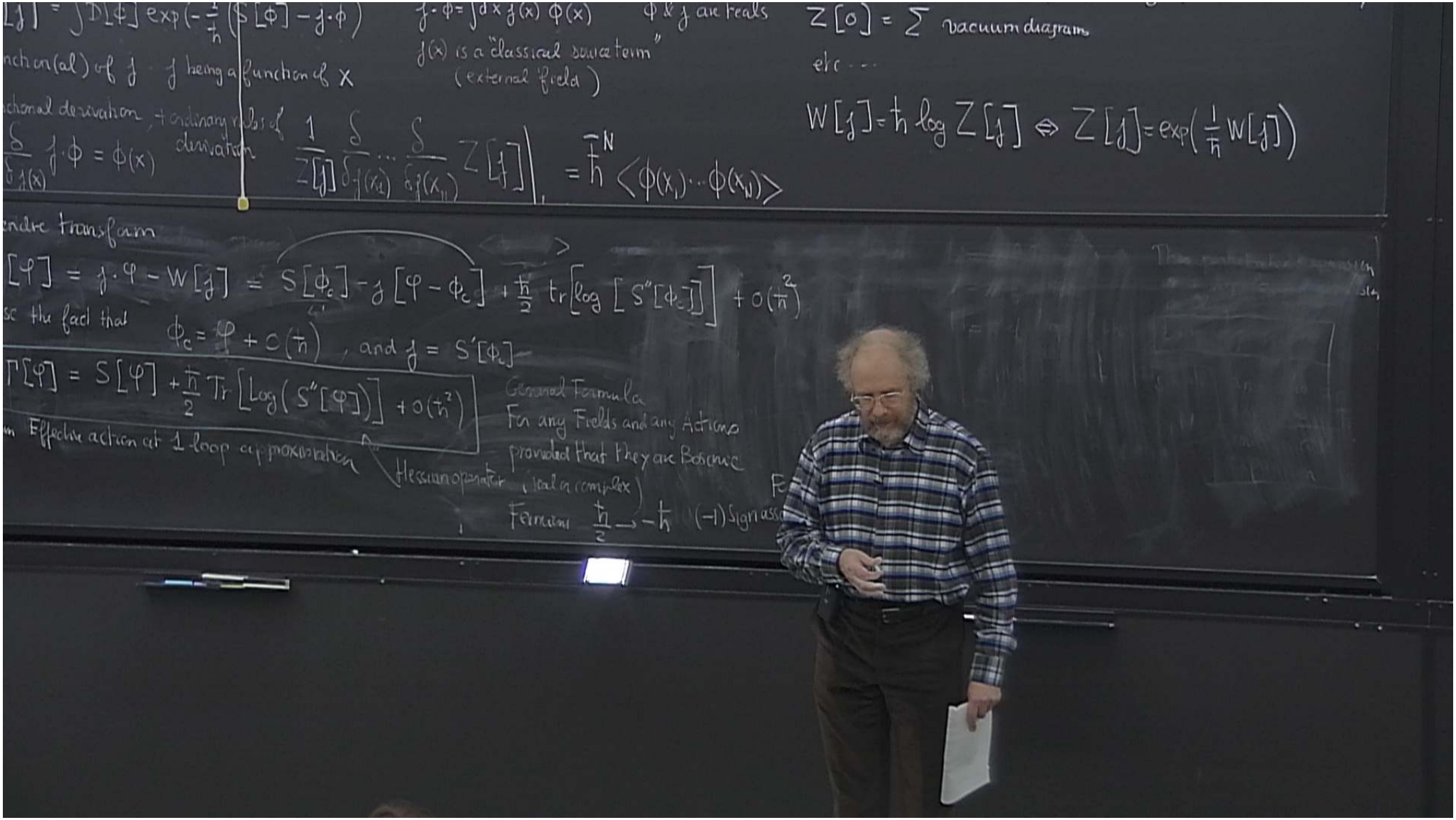
Functional derivation, + ordinary rules of derivation
$\frac{\delta}{\delta j(x)} j \cdot \phi = \phi(x)$
$\frac{1}{Z[j]} \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j] \Big|_{j=0} = \hbar^N \langle \phi(x_1) \cdots \phi(x_N) \rangle$
Legendre transform
$\Gamma[\varphi] = j \cdot \varphi - W[j] = S[\phi_c] - j[\varphi - \phi_c] + \frac{\hbar}{2} \mathrm{tr}\left[\log\left[S''[\phi_c]\right]\right] + O(\hbar^2)$
use the fact that $\phi_c = \varphi + O(\hbar)$, and $j = S'[\phi_c]$
$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] + O(\hbar^2)$
Quantum Effective action at 1-loop approximation
Hessian operator


(external field)
Functional derivation, + ordinary rules of derivation
$\frac{\delta}{\delta j(x)} j \cdot \phi = \phi(x)$
$\frac{1}{Z[j]} \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z[j] \Big|_{j=0} = \hbar^N \langle \phi(x_1) \cdots \phi(x_N) \rangle$
Legendre transform
$\Gamma[\varphi] = j \cdot \varphi - W[j] = S[\phi_c] - j[\varphi - \phi_c] + \frac{\hbar}{2} \mathrm{tr}\left[\log\left[S''[\phi_c]\right]\right] + O(\hbar^2)$
fact that $\phi_c = \varphi + O(\hbar)$, and $j = S'[\phi_c]$
$\Gamma = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] + O(\hbar^2)$
Effective action at 1-loop approximation
Hessian operator
General Formula
For any Fields and any Actions
provided that they are Bosonic
(real or complex)


$Z[0] = \sum$ vacuum diagram
etc ...
$W[J] = \hbar \log Z[J] \Leftrightarrow Z[J] = \exp\left(\frac{1}{\hbar} W[J]\right)$
$J \cdot \phi = \int dx\, J(x)\, \phi(x)$
$J(x)$ is a "classical source term" (external field)
$\frac{\delta}{\delta J(x)} J \cdot \phi = \phi(x)$
$\frac{1}{Z[J]} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} Z[J] \Big|_{J=0} = \hbar^N \langle \phi(x_1) \cdots \phi(x_N) \rangle$
$\Gamma[\varphi] = J \cdot \varphi - W[J] = S[\phi_c] - J[\varphi - \phi_c] + \frac{\hbar}{2} \mathrm{tr}\left[\log\left[S''[\phi_c]\right]\right] + O(\hbar^2)$
$\phi_c = \varphi + O(\hbar)$, and $J = S'[\phi_c]$
$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr}\left[\mathrm{Log}\left(S''[\varphi]\right)\right] + O(\hbar^2)$
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For any Fields and any Actions
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Fermions $\frac{\hbar}{2} \to -\hbar$ $(-1)^{\text{Signature}}$