Title: 14/15 PSI - Quantum Field Theory II-Lecture 3

Date: Nov 12, 2014 09:00 AM

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Abstract:

Free Field quantization. cont.d.

$\phi(x)$ Real. $X = (t, \vec{x})$ in $M^{1,d-1}$ Minkowski $(- + + +)$

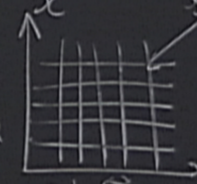
$$S[\phi] = \int d^d x \left[-\frac{1}{2}\left(\partial_\nu \phi \, \partial^\nu \phi\right) - \frac{m^2}{2}\phi^2 \right]$$

Functional Integral ← continuum limit $\epsilon \to 0$ ← discretized space-time lattice $\mathbb{Z}^d$, mesh $= \epsilon$

$$\int \mathcal{D}[\phi] \, \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

old idea

$\vec{x}$



t

site lattice $\vec{i} \in \mathbb{Z}^d$

$X_{\vec{i}} = \epsilon \cdot \vec{i}$

$\phi_{\vec{i}} = \phi(X_{\vec{i}})$

$$D[\phi] = \prod_{\vec{i}\in\mathbb{Z}^d}\left[d\phi_{\vec{i}}\left(\frac{2i\pi\hbar}{\epsilon^{d-2}}\right)^{-1/2}\right]$$

$$D_E[\phi] = \prod_{\vec{i}\in\mathbb{Z}^d} d\phi_i \left(\frac{2\pi\hbar}{\epsilon^{d-2}}\right)^{-1/2}$$

$$S[\phi] \to S_\epsilon[\phi] \qquad \int d^dx \to \sum_{\vec{i}} \epsilon^d$$

$$\partial_\mu \to \text{Finite differences}$$

$$S_E[\phi] = \int d^dx_E \left[\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) + \frac{m^2}{2}\phi^2\right]$$

Euclidean Action

Euclidean Spacetime

$$x_E = (\tau,\vec{x}) = (x^0, x^1 \cdots x^{d-1}) \in \mathbb{R}^d$$

$$(+++\cdots) \qquad g^E_{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\int D_E[\phi]\,\exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$$

Euclidean Funct. Integral

$$\int \mathcal{D}_\epsilon[\phi]\, \exp\left(-\frac{1}{\hbar} S[\phi]\right) \;\propto\; \left[\text{"Det"}\left(-\Delta + m^2\right)\right]^{-1/2} \leftarrow$$

limit when $\epsilon \to 0$, and size L of spacetime $\to \infty$

For a Quadratic Form D_{ab} sym & >0

$$\int \prod_{a=1}^{N} d\phi_a \, \exp\left(-\frac{1}{2}\sum_{a,b} \phi_a \cdot D_{ab}\, \phi_b\right) \qquad \langle \phi_a \phi_b \rangle = \left(D^{-1}\right)_{ab}$$

D^{-1} inverse of the matrix D

$$G_0(x_1, x_2) = \langle \phi(x_1)\, \phi(x_2) \rangle = \text{Matrix element of the inverse of } \left(-\Delta + m^2\right) \text{ for } (x_1, x_2)$$

$$= \hbar \left(\frac{1}{-\Delta + m^2}\right)_{x_1 x_2} = \hbar \langle x_1 | \frac{1}{-\Delta + m^2} | x_2 \rangle$$

Kernel of the operator $\left(-\Delta + m^2\right)^{-1}$, bra-ket notation of ordinary Q.M

$\phi(x)$

space

$\mathbb{R} \to$ d-dim torus

$D = (D_{ab})$ $\quad$ $(a, b = 1, \dots N)$

$\sum_b D_{ab} (D^{-1})_{bc} = \delta_{ac}$

$D \cdot D^{-1} = \mathbb{1}$

$$\left(-\Delta_{x_1} + m^2\right) G_0(x_1, x_2) = \delta^{(d)}(x_1 - x_2)$$

<u>Solution:</u> $\quad G_0(x_1, x_2) = G_0(x_1 - x_2)$

Translation invariance

$[-\Delta + m^2, \partial_\mu] = 0$

$X = (x^0, \dots x^{d-1})$

$K = (k_0, \dots k_{d-1})$

$X \cdot K = x^\mu k_\mu$

$K^2 = k_\mu k^\mu = \sum_\mu k_\mu^2$

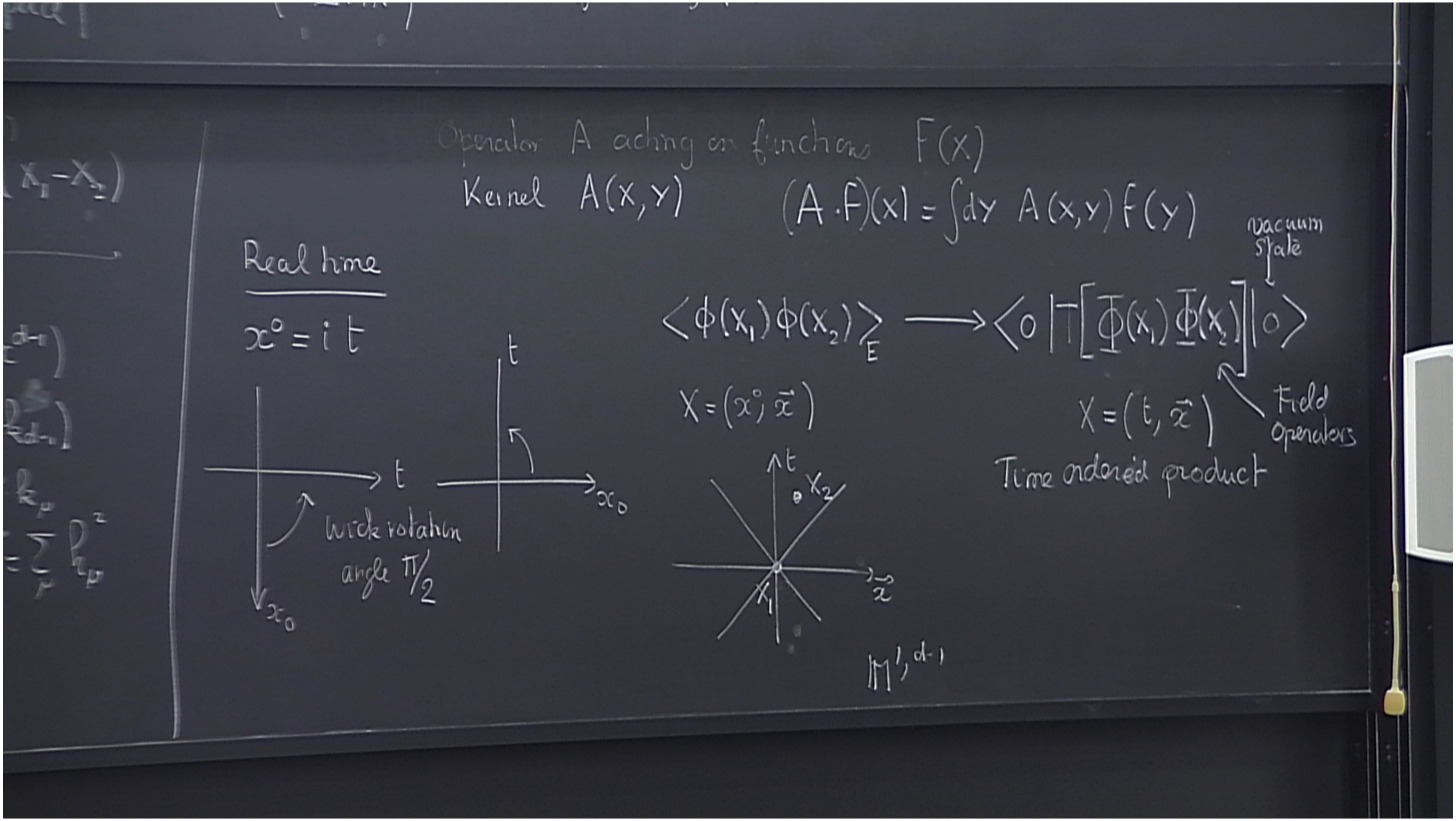
Fourier Transform $\quad \hat{G}_0(K) = \int d^d x \, e^{-iX\cdot K} G_0(X)$

Equation $\quad (K^2 + m^2) \cdot \hat{G}_0(K) = 1$

$$\hat{G}_0(K) = \frac{1}{K^2 + m^2}$$

Lorenzian Function

$$G_0(X) = \int \frac{d^d K}{(2\pi)^d} e^{iX\cdot K} \cdot \frac{1}{K^2 + m^2}$$


Operator A acting on functions F(x)
Kernel A(x, y)
(A . F)(x) = ∫dy A(x, y) f(y)
Real time
x° = i t
Wick rotation
angle π/2
⟨φ(x₁) φ(x₂)⟩_E ⟶ ⟨0| T[Φ(x₁) Φ(x₂)] |0⟩
vacuum state
Field Operators
X = (x°, x⃗)
X = (t, x⃗)
Time ordered product
M^{1,d-1}

$$G_0(K)=\frac{1}{K^2+m^2}$$ Lorenzian Function $$G_0(X)=\int\frac{d^dk}{(2\pi)^d}\,e^{i\cdots}\,\frac{1}{K^2+m^2}$$

Euclidean

$$G_0(x)=\int\frac{dk_0}{2\pi}\int\frac{d\vec{k}}{(2\pi)^{d-1}}\,e^{i(x^0k_0+\vec{x}\cdot\vec{k})}\,\frac{1}{k_0^2+\vec{k}^2+m^2}=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\,e^{i(\omega t+\vec{k}\cdot\vec{x})}\left(\frac{i}{\omega^2-\vec{k}^2-m^2+i\epsilon_+}\right)$$

$X=(x^0,\vec{x})$

$k=(k_0,\vec{k})$

real time frequency

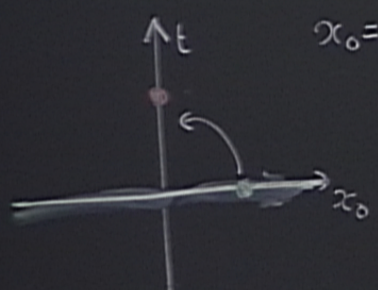
$k_0=\frac{1}{i}\,\omega$

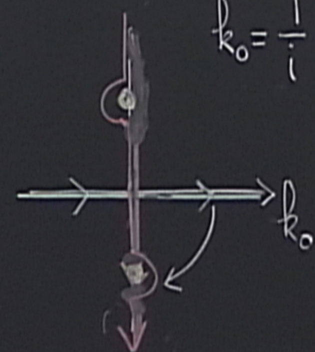
pole at $\omega=\pm\sqrt{\vec{k}^2+m^2}$

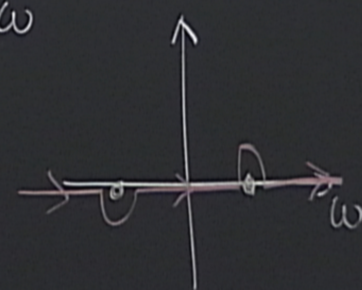
$k_0=\pm i\sqrt{\vec{k}^2+m^2}$

$\epsilon_+\to 0_+$

$x_0=it$







prescription

$\mathbb{M}^{1,d-1}$

$\ldots + \vec{k}\cdot\vec{x})$

$$\left(\frac{i}{\omega^2-\vec{k}^2-m^2+i\,\epsilon_+}\right)$$

$\epsilon_+ \to 0_+ \longrightarrow$ OK with causality

Feynman Propagator
For the Free Field

Euclidean QFT

$\downarrow$

Real time QFT with causality

Short Distance and large distances properties of

2-pt Function = propagator NOT $U(t)$

Euclidean space $G_0(x) = \frac{1}{2\pi}\left(\frac{2\pi|x|}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(|x| m)$

Bessel Function K_ν

$|x| \to \infty$ $G_0(x) \propto \exp(-m|x|)$

exponential decay comes from the poles in Fourier space

Short distances

$|x| \to 0$

$(+ + + \ldots)$ $\quad g^E_{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

Short distances (dimensional analysis)

$|X| \to 0$

$$G_0(x) \simeq \begin{cases} c(d)\,|x|^{2-d} & d > 2 \\ -\frac{1}{2\pi}\cdot \log|x| & d = 2 \\ \frac{1}{2m}\, e^{-m|x|} & d = 1 \end{cases}$$

$d > 2$, $d = 2$: Singular & divergent

$d = 1$: finite

Short distance singularity

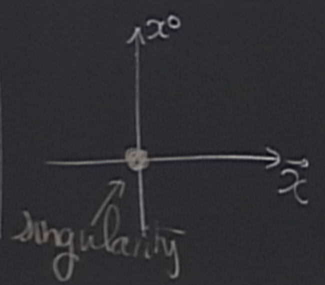
high energy/momentum

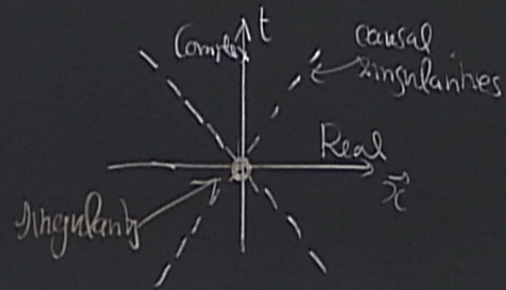
Ultra-Violet singularity (UV)

New feature of QFT

Important feature

Euclidean Space-Time





(dimensional analysis)

$$G_0(x) \simeq \begin{cases} C(d)\,|x|^{2-d} & d>2 \\ -\frac{1}{2\pi}\cdot\log|x| & d=2 \\ \frac{1}{2m}\,e^{-m|x|} & d=1 \end{cases}$$

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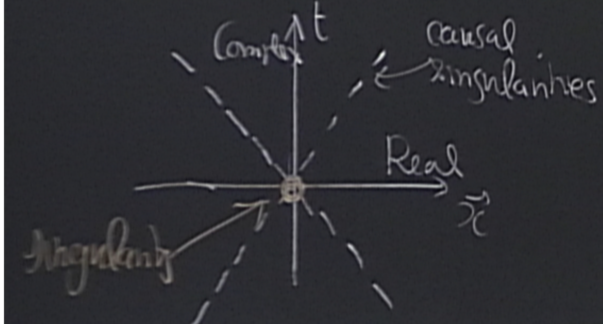
New feature of ΦFT

Important feature

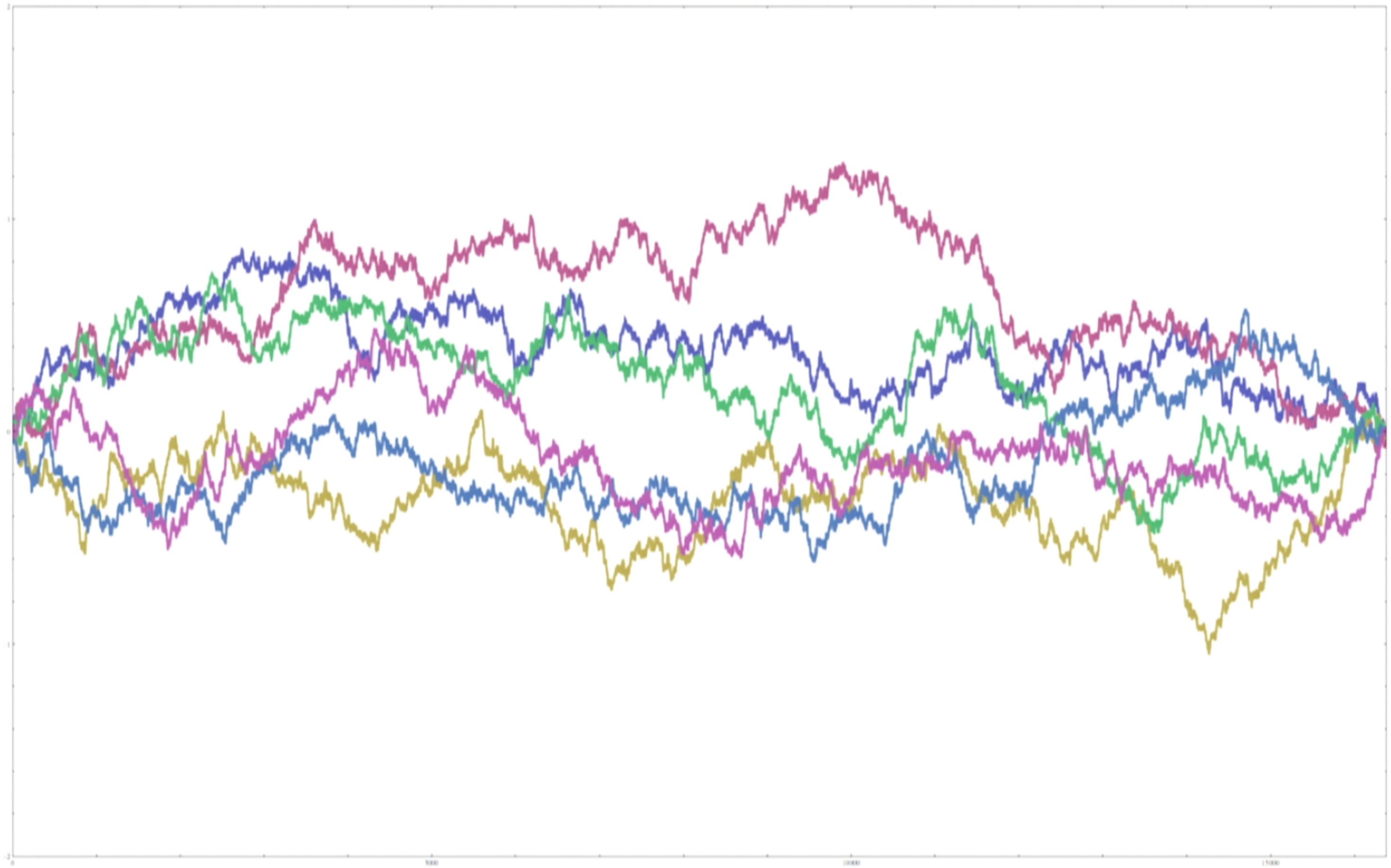
Time



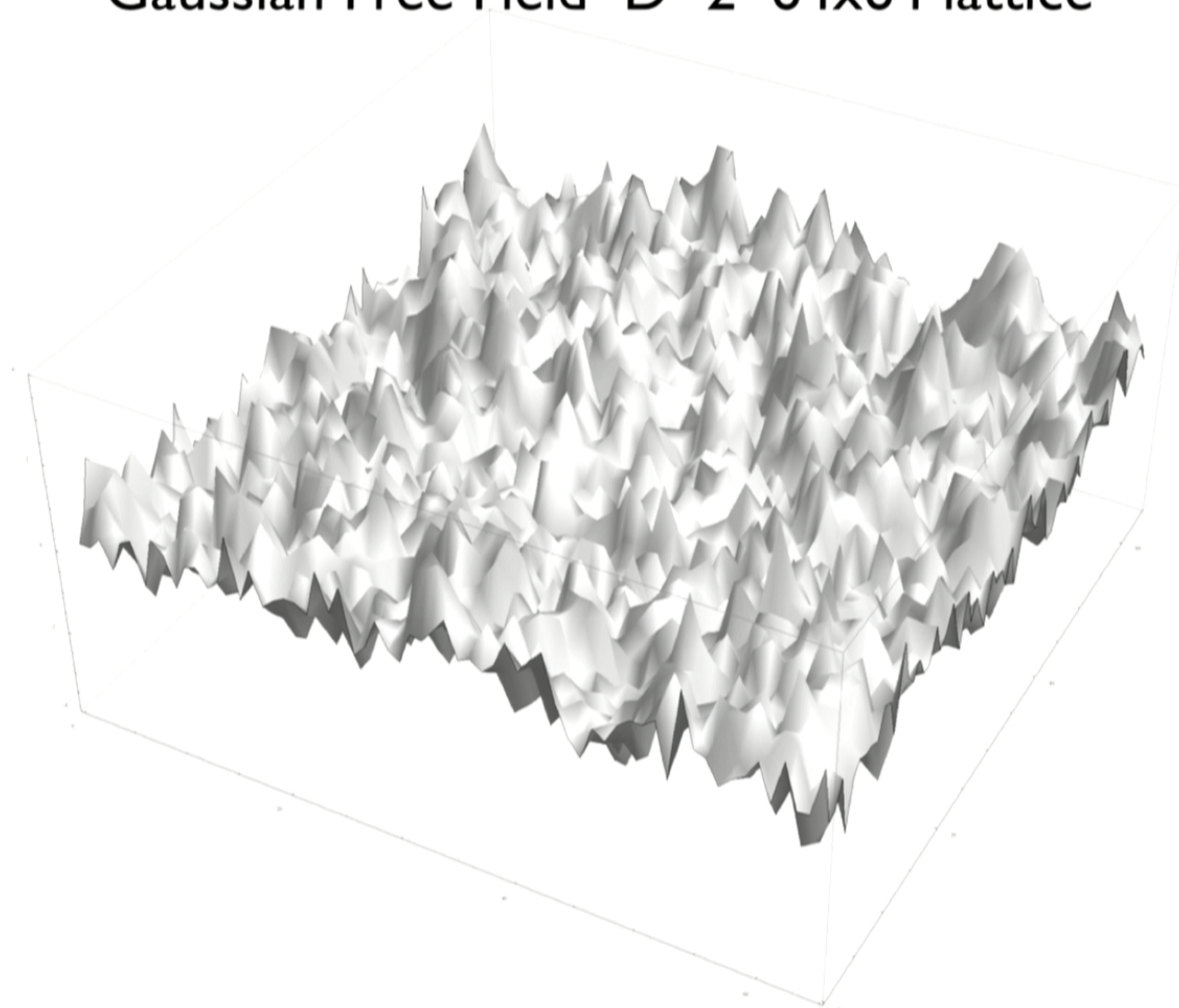
why $G_0(0) = \langle \phi(0)^2 \rangle = \infty$

ϕ is a "wild" object in the functional integral

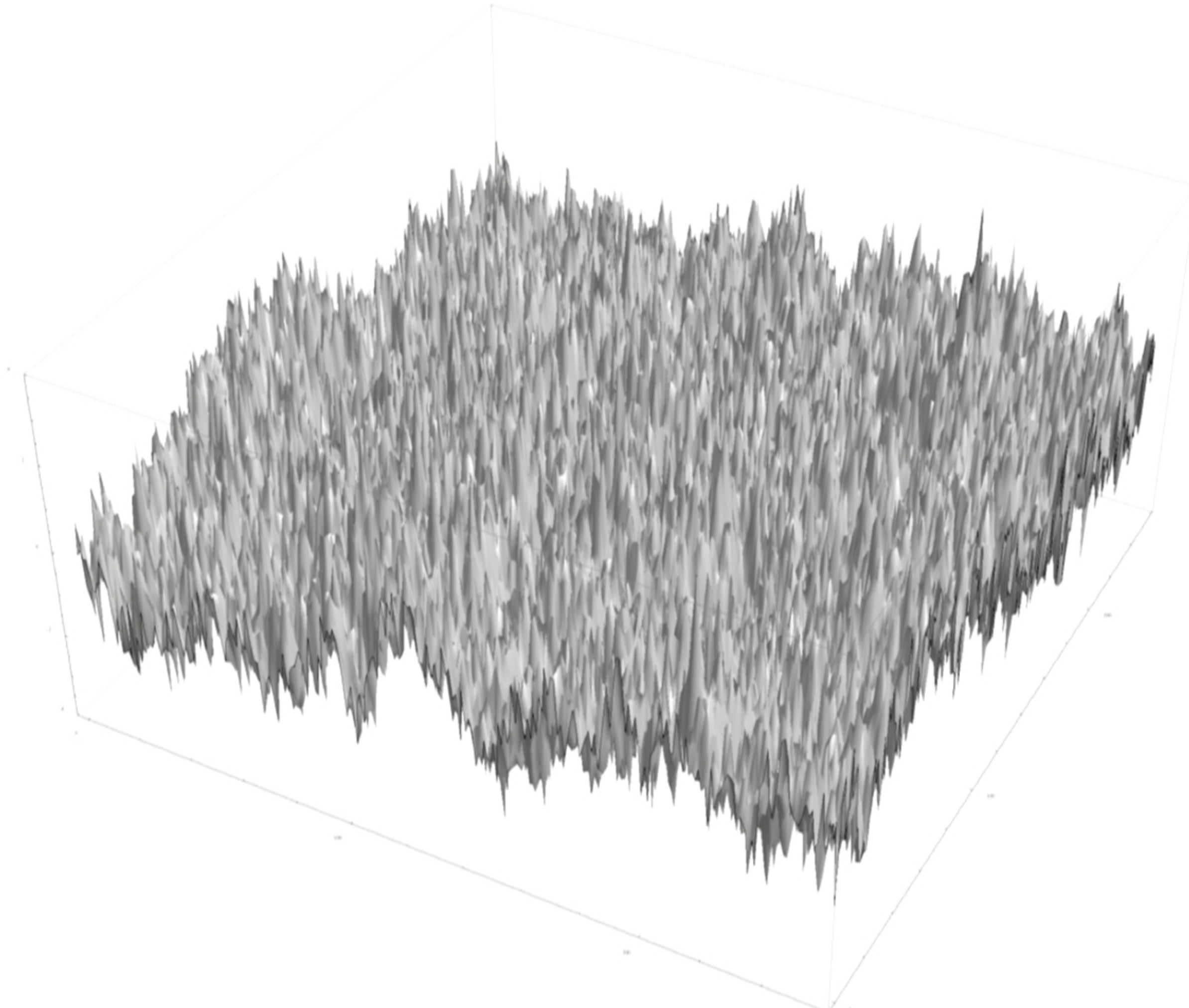
D=1 : Gaussian Free Fields = Random Walk (i.e. Brownian or Wiener Process)



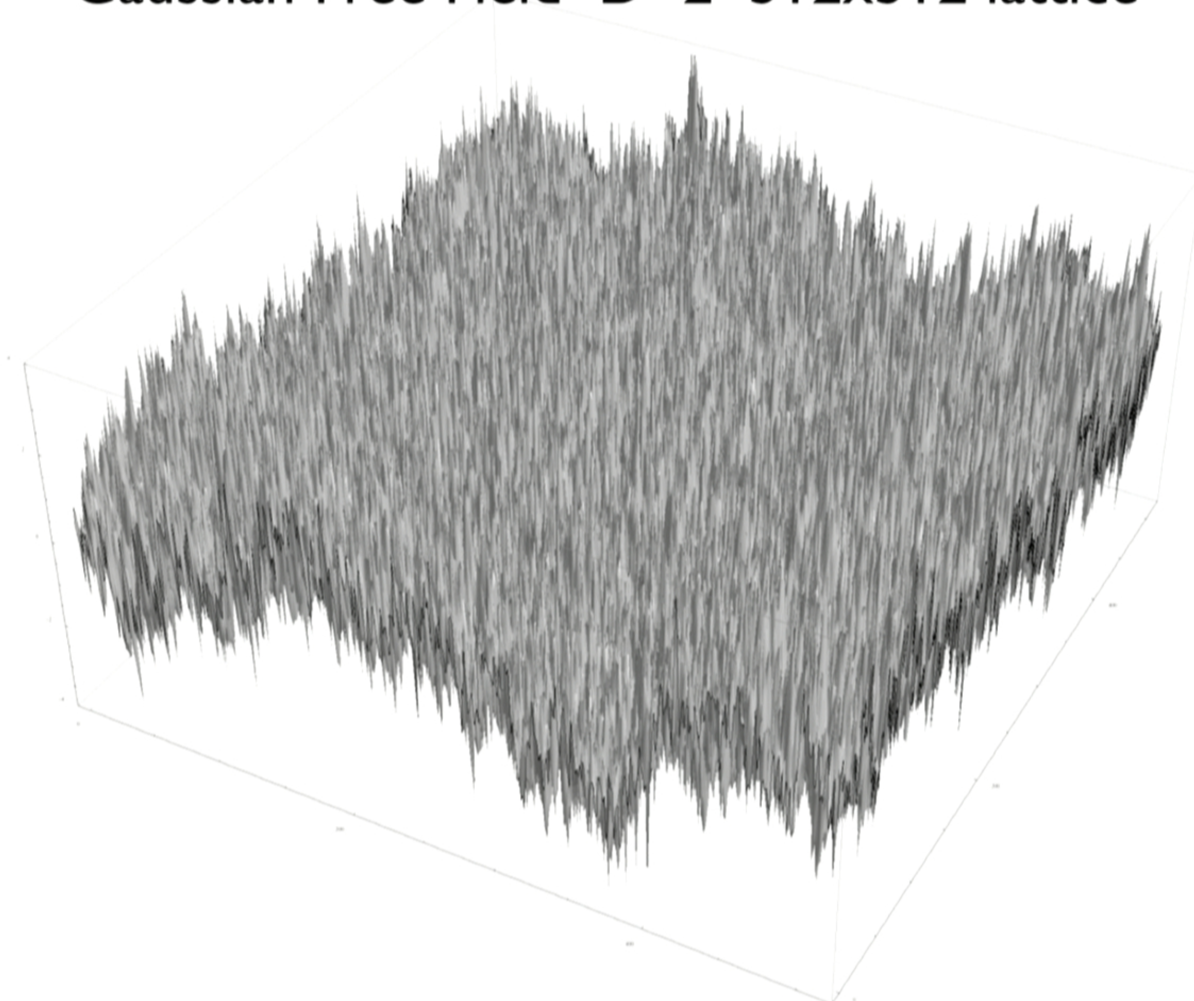
Gaussian Free Field D=2 64x64 lattice



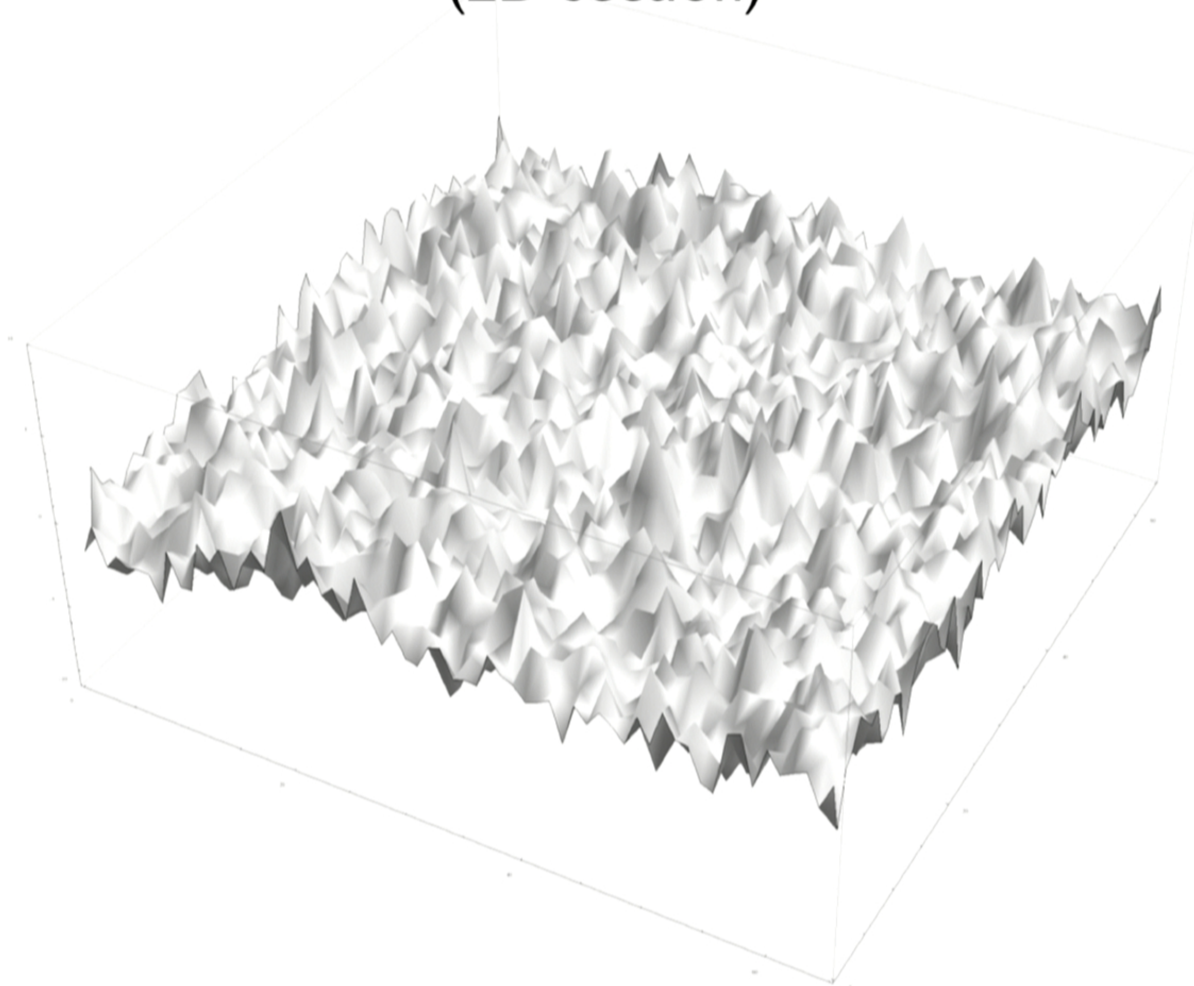
Gaussian Free Field D=2 256x256 lattice



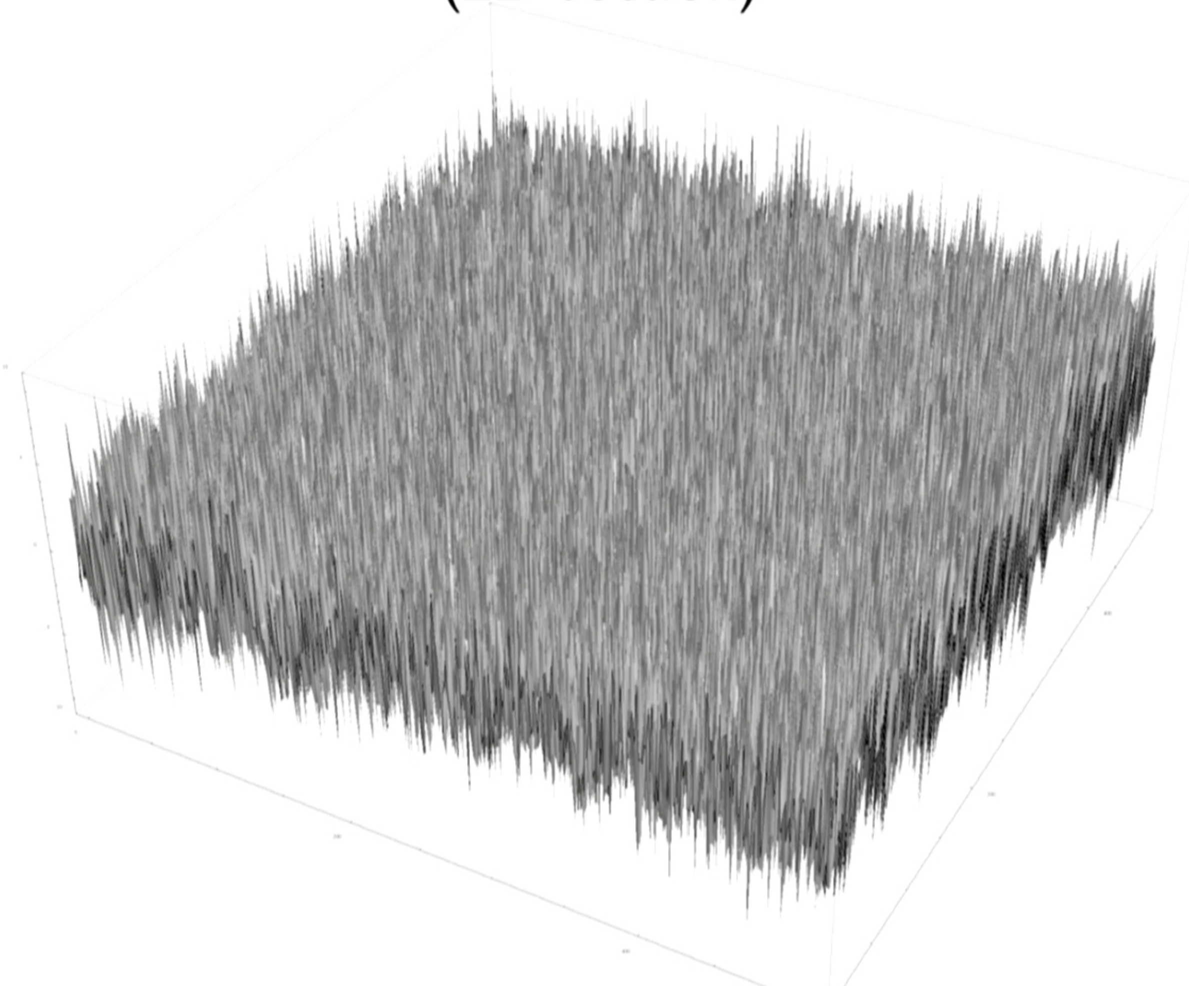
Gaussian Free Field D=2 512x512 lattice



Gaussian Free Field D=3 64x64x64 lattice
(2D section)



Gaussian Free Field D=3 512x512x512 lattice (2D section)



singularity singularity

Higher point functions N-points. N has to be even $N = 2m$

$$\langle \phi(x_1)\cdots\phi(x_N)\rangle = \sum_{\substack{\text{pairing of}\\ x_1\ldots x_{2N}}} \underbrace{\langle\phi(x)\,\phi(x)\rangle\langle\phi(x)\phi(x)\rangle\cdots\langle\phi(x)\phi(x)\rangle}_{M = N/2\ \#\ \text{pairs}}$$

2pt cumulant

Gaussian integral : gaussic distribution

N point correlation

$$\int D[\phi]\exp\left(-\tfrac{1}{2}\,\phi\cdot D\cdot\phi\right)$$

$$\phi\cdot D\cdot\phi = \int dx\,dy\,\phi(x)D(x,y)\phi(y)$$

$$D(x,y) = \frac{1}{\hbar}\left(-\Delta_y + m^2\right)\delta^d(x-y)$$

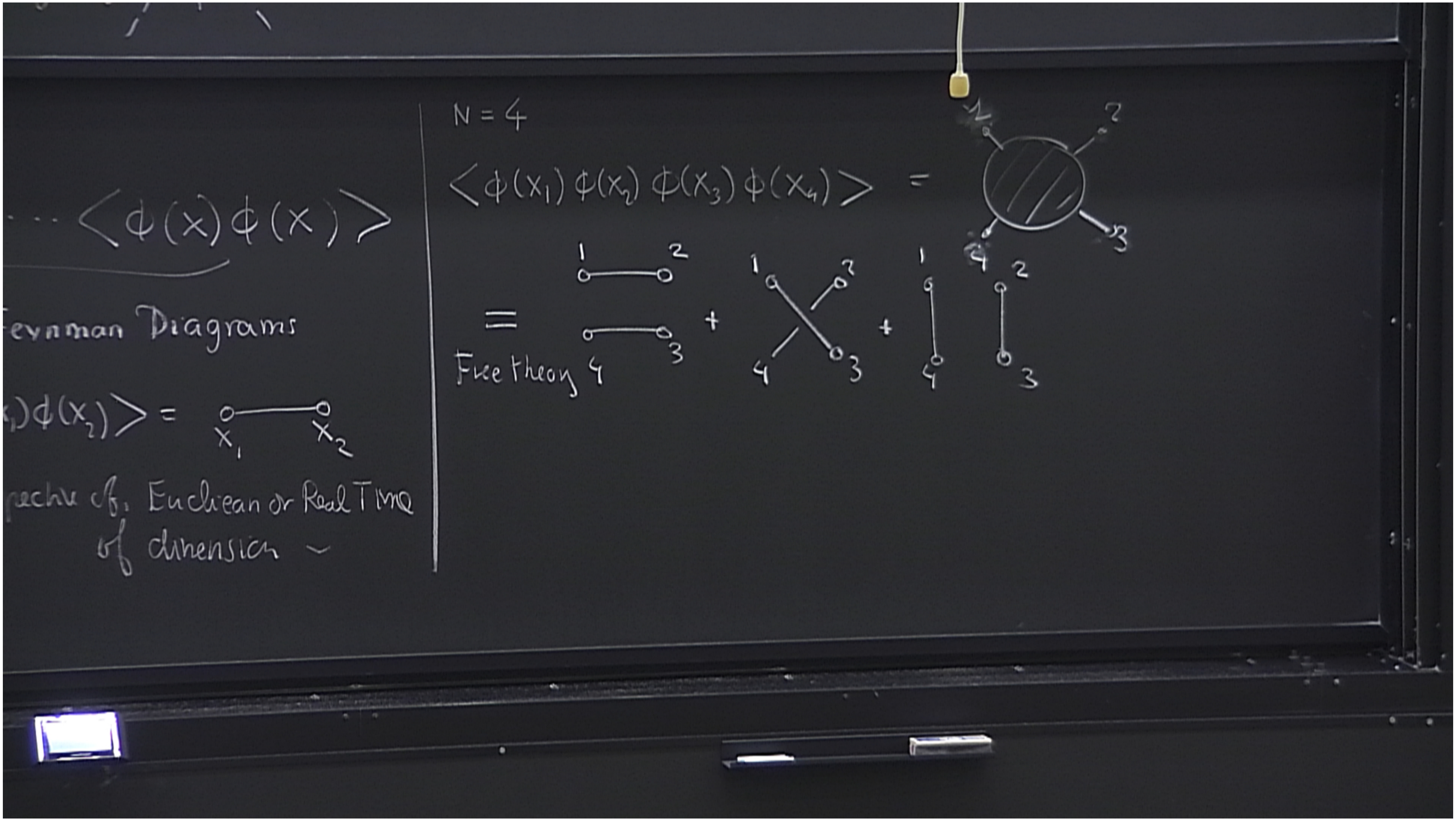
Wick Theorem

Feynman Diagrams

$$\langle\phi(x_1)\phi(x_2)\rangle = x_1\ \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ\ x_2$$

irrespective of Euclidean or Real Time

of dimension ~

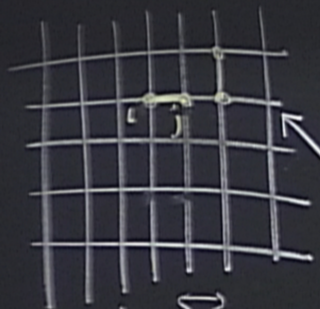


N = 4
<φ(x1) φ(x2) φ(x3) φ(x4)> =
Free theory
<φ(x) φ(x)>
Feynman Diagrams
φ(x2)> =
x1 x2
Euclidean or Real Time
of dimension

1) Functional Integral in discretised Euclidean Space

$\{\phi_i\}$: configuration of a set of oscillators on a lattice coupled linearly

$$Z = \int \prod d\phi_i \, \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$$



$\phi_i = \phi(x_i)$

$\vec{x}_i = \vec{i}\,\epsilon$

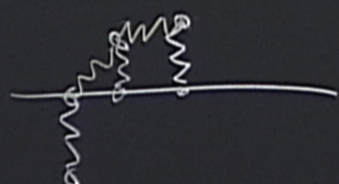
$\mathbb{Z}^d$ lattice, ϵ

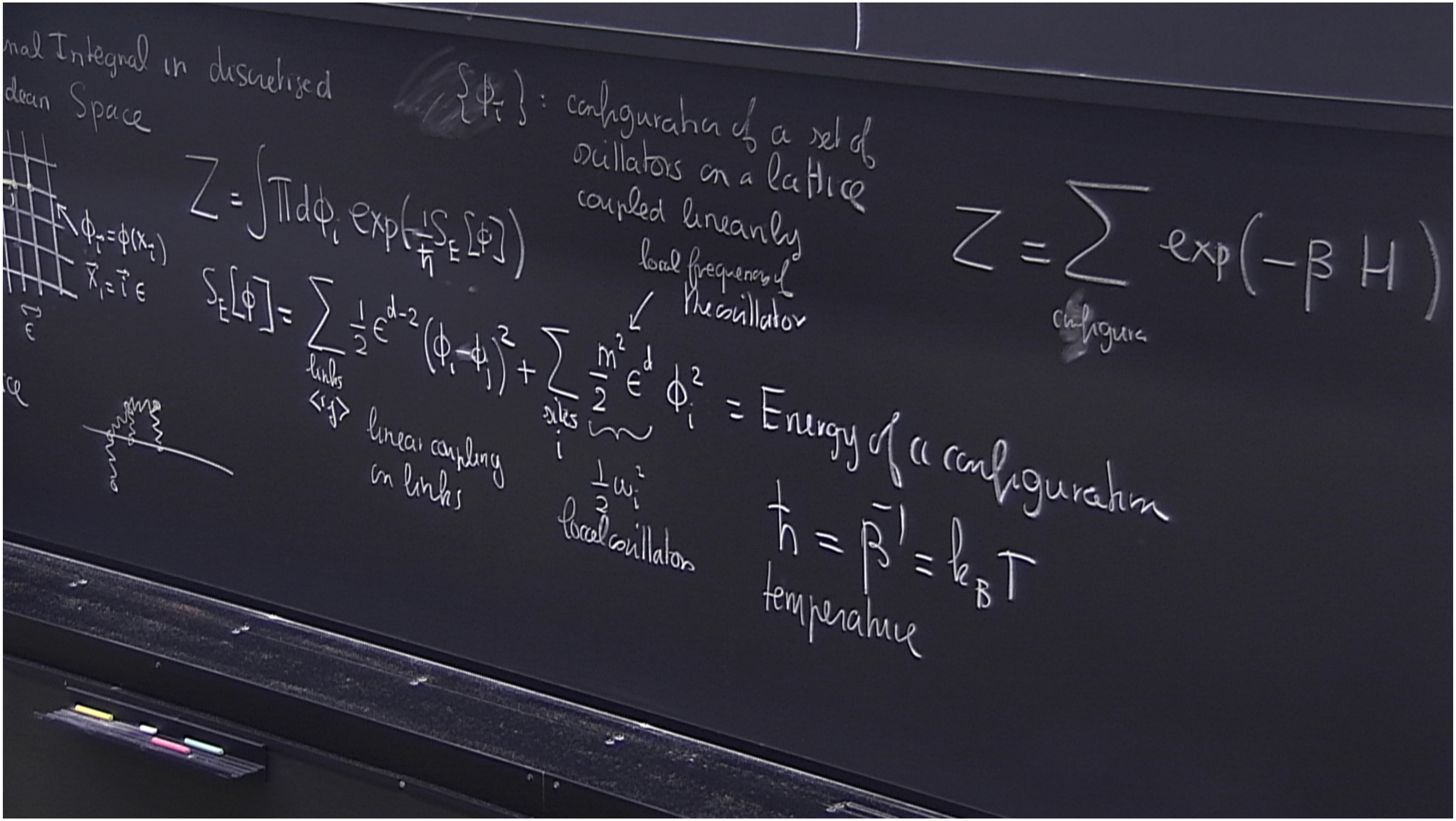
local frequency of the oscillator

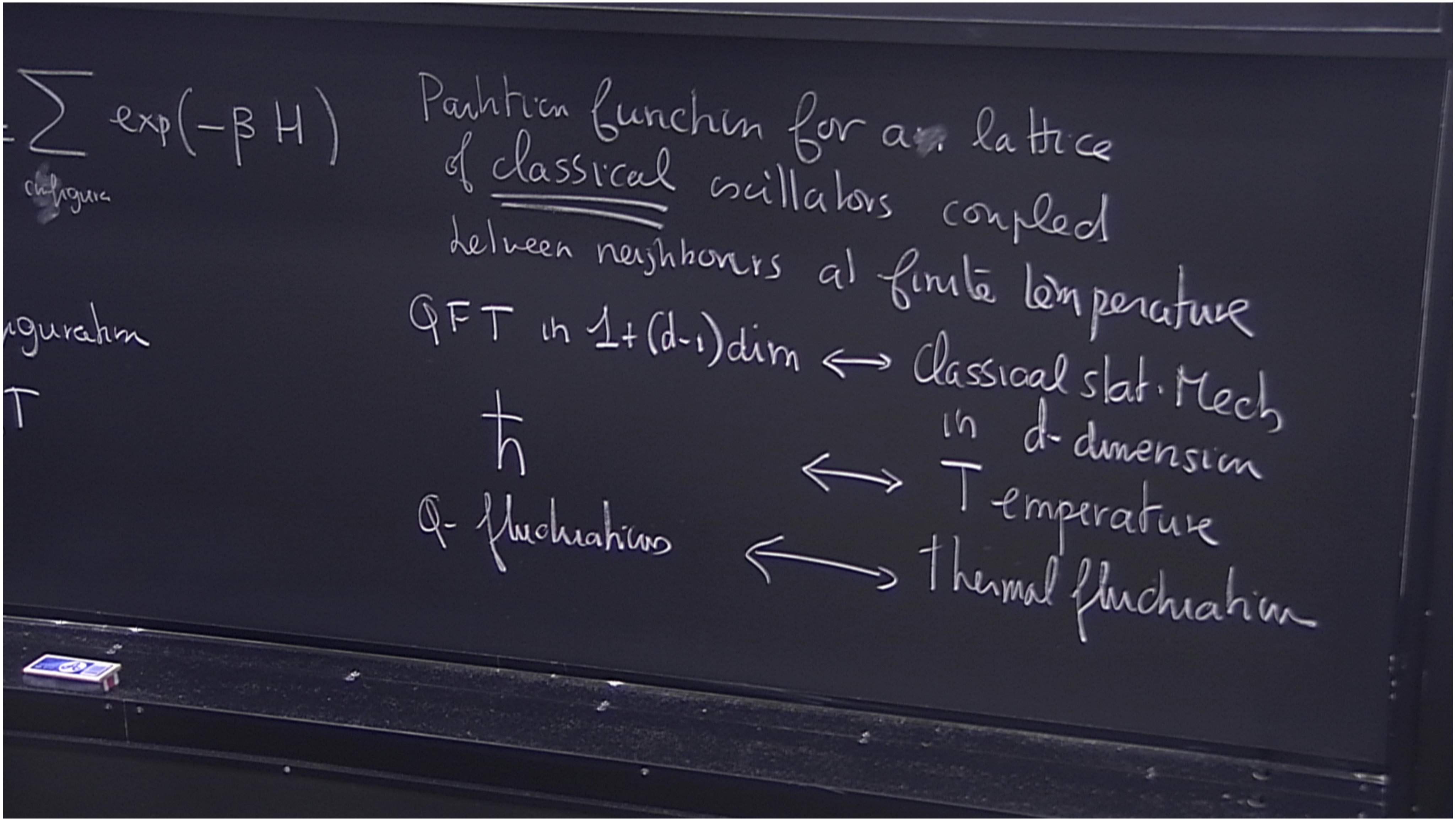
$$S_E[\phi] = \sum_{\substack{\text{links}\\ \langle ij \rangle}} \frac{1}{2}\epsilon^{d-2}(\phi_i - \phi_j)^2 + \sum_{\substack{\text{sites}\\ i}} \underbrace{\frac{m^2}{2}\epsilon^d}_{\frac{1}{2}\omega_i^2} \phi_i^2$$

linear coupling on links

local oscillators




nal Integral in discretised
dean Space
$\{\phi_i\}$: configuration of a set of oscillators on a lattice coupled linearly
$Z = \int \prod d\phi_i \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$
$\phi_i = \phi(x_i)$
$\vec{x}_i = \vec{i}\,\epsilon$
ϵ
$S_E[\phi] = \sum_{\substack{\text{links}\\ \langle ij\rangle}} \frac{1}{2}\epsilon^{d-2}(\phi_i - \phi_j)^2 + \sum_{\substack{\text{sites}\\ i}} \frac{m^2}{2}\epsilon^d \phi_i^2$ = Energy of a configuration
local frequency of the oscillator
linear coupling on links
$\frac{1}{2}\omega_i^2$
local oscillator
ice
$Z = \sum_{\text{configura}} \exp(-\beta H)$
$\hbar = \beta^{-1} = k_B T$
temperature


∑ exp(−β H)
Partition function for a lattice
of classical oscillators coupled
between neighbours at finite temperature
QFT in 1+(d-1)dim ⟷ Classical stat. Mech
in d-dimension
ħ ⟷ Temperature
Q-fluctuations ⟷ thermal fluctuation