Title: 14/15 PSI - Quantum Field Theory II-Lecture 2

Date: Nov 11, 2014 09:00 AM

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Abstract:

Path Integral Quantization.

$$H = \frac{P^2}{2m} + V(q) \qquad 1 \text{ dim} \qquad S = \int_0^t ds \left[\frac{m}{2} \dot{q}(s)^2 - V(q(s)) \right]$$

$$U(t) = \exp\left(\frac{t}{i\hbar} H \right)$$

$$K(q_F, q_I; t) = \langle q_F | U(t) | q_I \rangle_{\text{Schr}} = \int_{\substack{q(0)=q_I \\ q(t)=q_F}} D[q] \exp\left(\frac{i}{\hbar} S[q] \right)$$

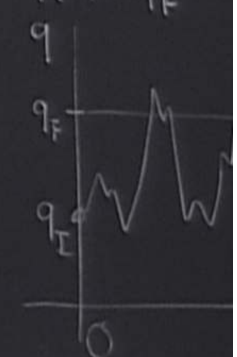
$$= \langle q_F, t | q_I, 0 \rangle_{\text{Heis.}}$$

Operators

$$\int D[q]\, e$$

$q(0) = q_I$

$q(t) = q_F$



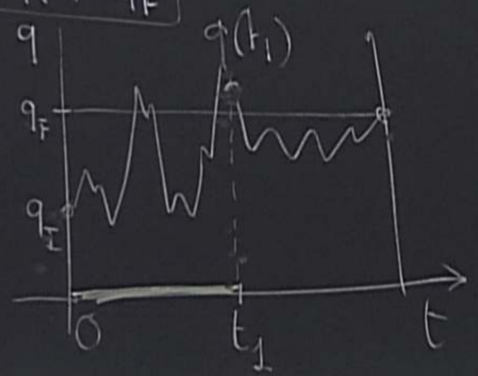
$q(s))]$

Operators & Observables? $0 < t_1 < t$

$$\boxed{\int D[q]\, e^{\frac{i}{\hbar} S[q]}\, q(t_1)} = \int_{-\infty}^{+\infty} dq_1 \langle q_F | U(t_F - t_1) \underbrace{| q_1 \rangle q_1 \langle q_1 |}_{Q \text{ position operator}} U(t_1) | q_I \rangle$$

$q(0) = q_I$

$q(t) = q_F$



$$= \langle q_F | U(t_F - t_1)\, Q\, U(t_1) | q_I \rangle$$

$$= \langle q_F | U(t_F)\, U^{\dagger}(t_1)\, Q\, U(t_1) | q_I \rangle_{\text{Schrödinger}}$$

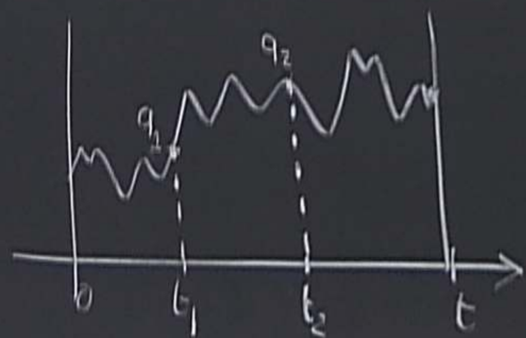
$$\boxed{= \langle q_F, t_F |\, Q(t_1)\, | q_I, 0 \rangle_{\text{Heisenberg}}}$$

Heis. $q(t)=q_F$

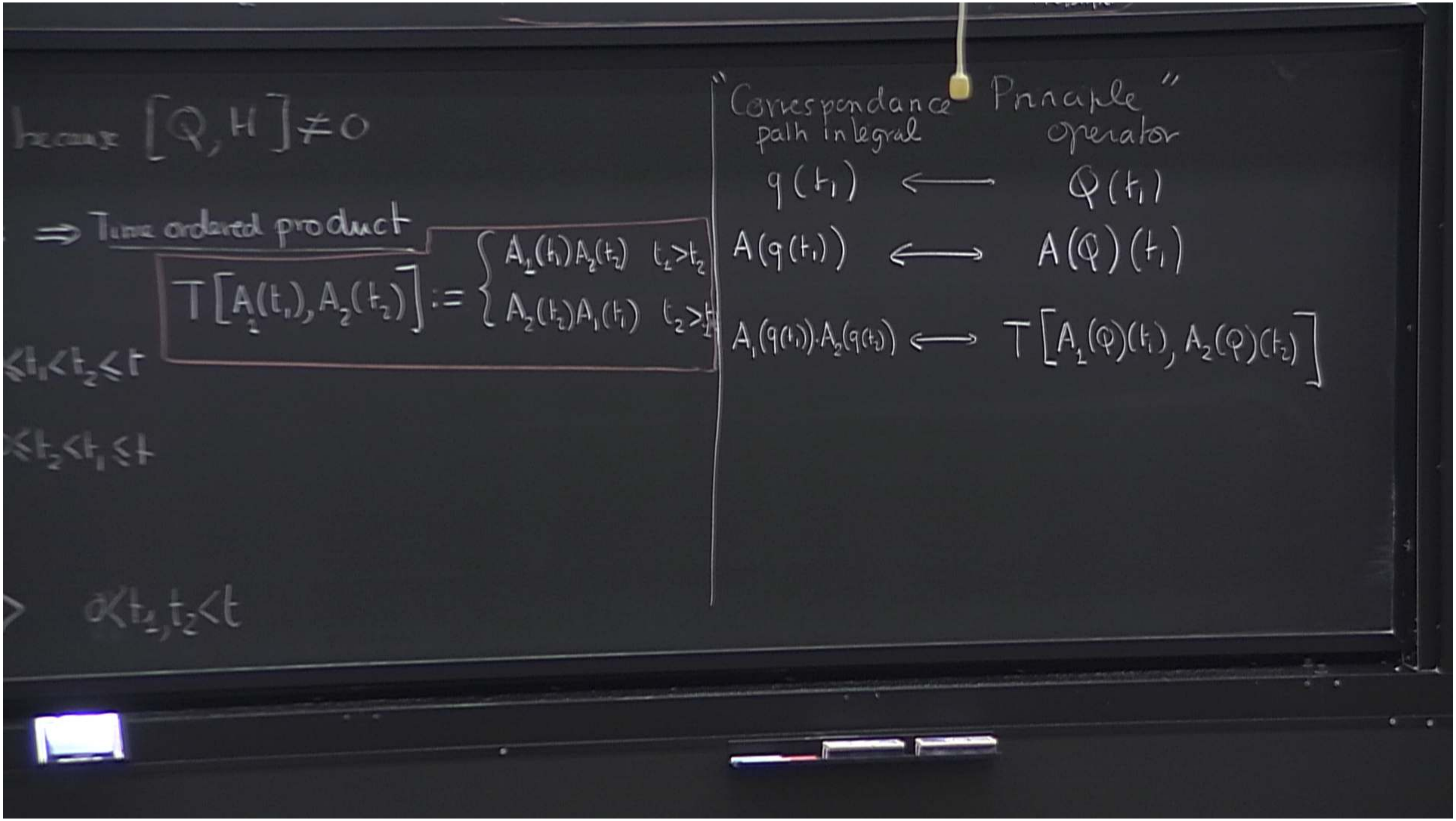
- $A(q_1)$ function $\Longleftrightarrow$ $A[Q_1]$ $\qquad$ $[A_1(t_1), A_2(t_2)] \neq 0$ because $[Q$
- 2 operators ? $\qquad$ path integral orders in time the operators $\Rightarrow$ Time ord

T

$$\int_{\substack{q(0)=q_I \\ q(t)=q_F}} D[q] \exp\left(\frac{i}{\hbar} S[q]\right) \overbrace{A_1(q(t_1)) \cdot A_2(q(t_2))}^{\text{numbers}} = \begin{cases} \langle q_F, t | A_2(Q)(t_2) \cdot A_1(Q)(t_1) | q_I, 0\rangle & \leftarrow \text{if } 0 \leq t_1 < t_2 \leq t \\ \langle q_F, t | A_1(Q)(t_1)\, A_2(Q)(t_2) | q_I, 0\rangle & \text{if } 0 \leq t_2 < t_1 \leq t \end{cases}$$



$$= \langle q_F, t | \, T[A_1(t_1), A_2(t_2)] \, | q_I, 0\rangle \quad 0 < t_1, t_2 <$$


because $[Q,H] \neq 0$
$\Rightarrow$ Time ordered product
$T[A_1(t_1), A_2(t_2)] := \begin{cases} A_1(t_1)A_2(t_2) & t_1>t_2 \\ A_2(t_2)A_1(t_1) & t_2>t_1 \end{cases}$
$\leq t_1<t_2\leq t$
$\leq t_2<t_1\leq t$
$0<t_1,t_2<t$
"Correspondance Principle"
path integral
operator
$q(t_1) \longleftarrow Q(t_1)$
$A(q(t_1)) \longleftrightarrow A(Q)(t_1)$
$A_1(q(t_1))\cdot A_2(q(t_2)) \longleftrightarrow T[A_1(Q)(t_1), A_2(Q)(t_2)]$

Heis. $q(t) = q_F$

- $A(q_1)$ function $\longleftrightarrow$ $A[Q_1]$ $\qquad [A_1(t_1), A_2(t_2)] \neq 0$ because $[Q, H]$
- 2 operators ? $\qquad$ path integral orders in time the operators $\Rightarrow$ Time ordered p

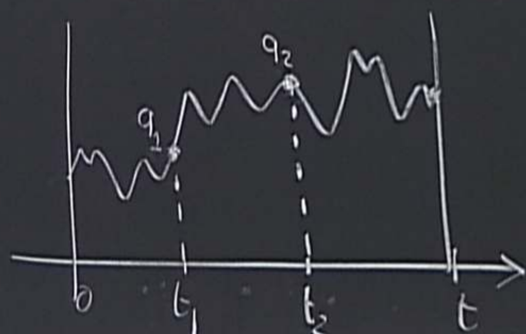
(automatic : build in the formalism)

$\boxed{T[A_1(t_1}$

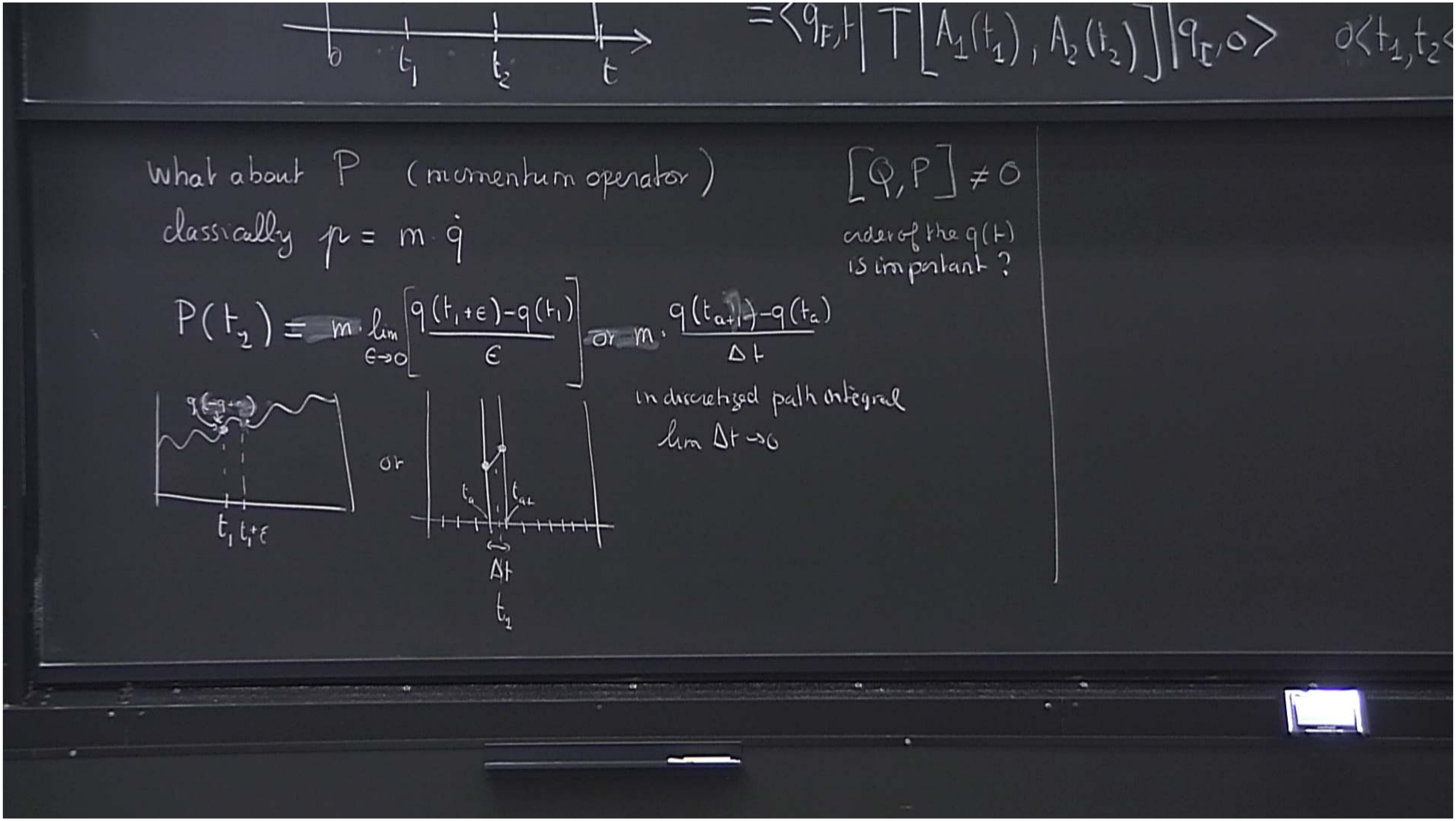
$$\int D[q] \exp\left(\frac{i}{\hbar} S[q]\right) \overbrace{A_1(q(t_1)) \cdot A_2(q(t_2))}^{\text{numbers}} = \begin{cases} \langle q_F, t | A_2(Q)(t_2) \cdot A_1(Q)(t_1) | q_I, 0 \rangle & \leftarrow \text{if } 0 \leq t_1 < t_2 \leq t \\ \langle q_F, t | A_1(Q)(t_1) \cdot A_2(Q)(t_2) | q_I, 0 \rangle & \text{if } 0 \leq t_2 < t_1 \leq t \end{cases}$$

$q(0) = q_I$

$q(t) = q_F$



$$= \langle q_F, t | T[A_1(t_1), A_2(t_2)] | q_I, 0 \rangle \qquad 0 < t_1, t_2 < t$$


= ⟨q_F, t | T [A_1(t_1), A_2(t_2)] | q_I, 0⟩
0 < t_1, t_2
What about P (momentum operator)
classically p = m · q̇
[Q, P] ≠ 0
order of the q(t) is important ?
P(t_2) = m · lim ε→0 [(q(t_1 + ε) − q(t_1)) / ε] or m · (q(t_{a+1}) − q(t_a)) / Δt
in discretized path integral
lim Δt → 0
or
Δt

$0 < t_1, t_2 < t$

$\neq 0$

$q(t)$

ant ?

What about Euclidean path Integrals ?

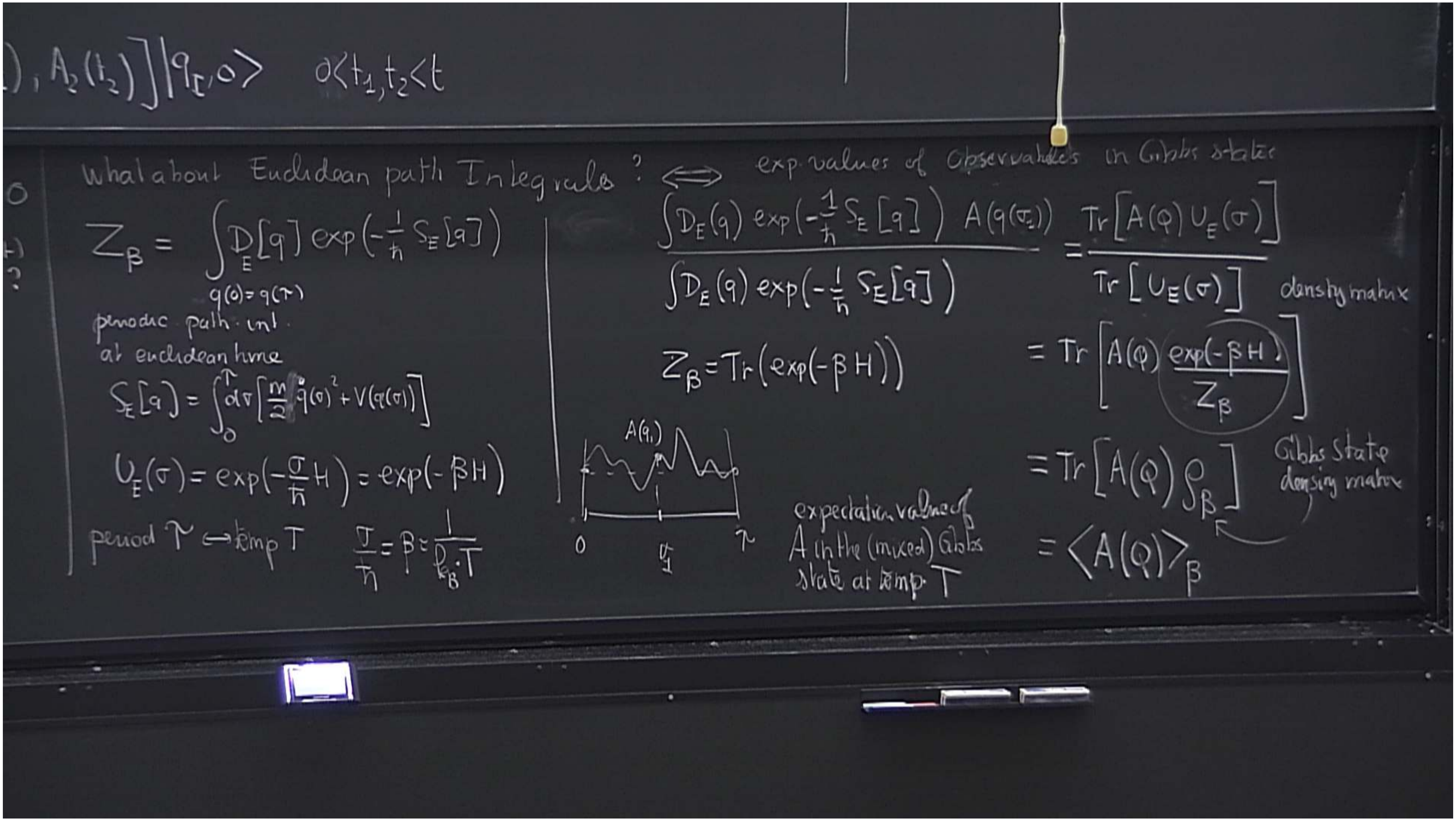
$$Z_\beta = \int_{q(0)=q(\tau)} \mathcal{D}_E[q] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$$

periodic path int.
at euclidean time

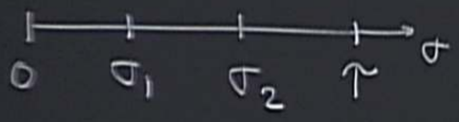
$$S_E[q] = \int_0^{\tau} d\sigma \left[\frac{m}{2} \dot{q}(\sigma)^2 + V(q(\sigma))\right]$$

$$U_E(\sigma) = \exp\left(-\frac{\sigma}{\hbar} H\right)$$

period $\tau \leftrightarrow$ temp T $\quad \frac{\tau}{\hbar} = \beta = \frac{1}{k_B \cdot T}$


$), A_2(t_2)]]|q_I, 0\rangle \quad 0 < t_1, t_2 < t$
What about Euclidean path Integrals ? $\Longleftrightarrow$ exp. values of observables in Gibbs states
$Z_\beta = \int_{q(0)=q(\tau)} D_E[q] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$
periodic path int. at euclidean time
$S_E[q] = \int_0^\tau d\sigma \left[\frac{m}{2}\dot{q}(\sigma)^2 + V(q(\sigma))\right]$
$U_E(\sigma) = \exp\left(-\frac{\sigma}{\hbar}H\right) = \exp(-\beta H)$
period $\tau \leftrightarrow$ temp T $\quad \frac{\tau}{\hbar} = \beta = \frac{1}{k_B \cdot T}$
$\frac{\int D_E(q) \exp\left(-\frac{1}{\hbar} S_E[q]\right) A(q(\sigma_1))}{\int D_E(q) \exp\left(-\frac{1}{\hbar} S_E[q]\right)} = \frac{\mathrm{Tr}[A(Q) U_E(\sigma)]}{\mathrm{Tr}[U_E(\sigma)]}$
density matrix
$Z_\beta = \mathrm{Tr}(\exp(-\beta H))$
$= \mathrm{Tr}\left[A(Q) \frac{\exp(-\beta H)}{Z_\beta}\right]$
$A(q_1)$
$0 \quad \sigma_1 \quad \tau$
$= \mathrm{Tr}[A(Q) \rho_\beta]$
Gibbs State density matrix
expectation value of A in the (mixed) Gibbs state at temp. T
$= \langle A(Q) \rangle_\beta$

We can compute products of operators. definition



$$\boxed{\langle A_2(\sigma_2)\, A_1(\sigma_1)\rangle_\beta = \frac{1}{Z_\beta}\int_{\text{periodic}} D[q]\, e^{-\frac{1}{\hbar} S_E[q]}\, A_1(\sigma_1)\, A_2(\sigma_2)}$$

Euclidean correlation function.

⚠ $U_E(\sigma)$ defined only for $\sigma \geq 0$

so $\sigma_1 < \sigma_2$: take care of the "Euclidean" time ordering

$$\int D_E[q] = \int \prod_{\sigma=0}^{\tau} dq(\sigma)$$

Gibbs state (quantum)

⚠

$$\rho_\beta = \frac{1}{Z_\beta}\exp(-\beta H) = \sum_n |n\rangle\langle n| \frac{\exp(-\beta E_n)}{Z_\beta}$$ ← mixture of pure states $|n\rangle$ with prob $p_n = \frac{1}{Z_\beta}\exp(-\beta E_n)$

$\beta \to \infty$, namely Temp $\to 0$, $\rho_\beta = |0\rangle\langle 0|$ projection operator on the ground state

$E_0 < E_1 < E_2 < \cdots$.

$$Z_\beta = \sum_n \exp(-\beta E_n) \simeq \exp(-\beta E_0)$$

$p_n \searrow 0$ but $p_0 \to 1$

Gibbs state (quantum)

$$\rho_\beta = \frac{1}{Z_\beta} \exp(-\beta H) = \sum_n |n\rangle\langle n| \frac{\exp(-\beta E_n)}{Z_\beta} \leftarrow$$

⚠ mixture of pure states $|n\rangle$ with prob $p_n = \frac{1}{Z_\beta}\exp(-\beta E_n)$

$\beta \to \infty$, namely $\mathrm{Temp} \to 0$, $\rho_\beta = |0\rangle\langle 0|$ projection operator on the ground state

$\|$

period $\tau \to \infty$

$E_0 < E_1 < E_2 < \cdots$

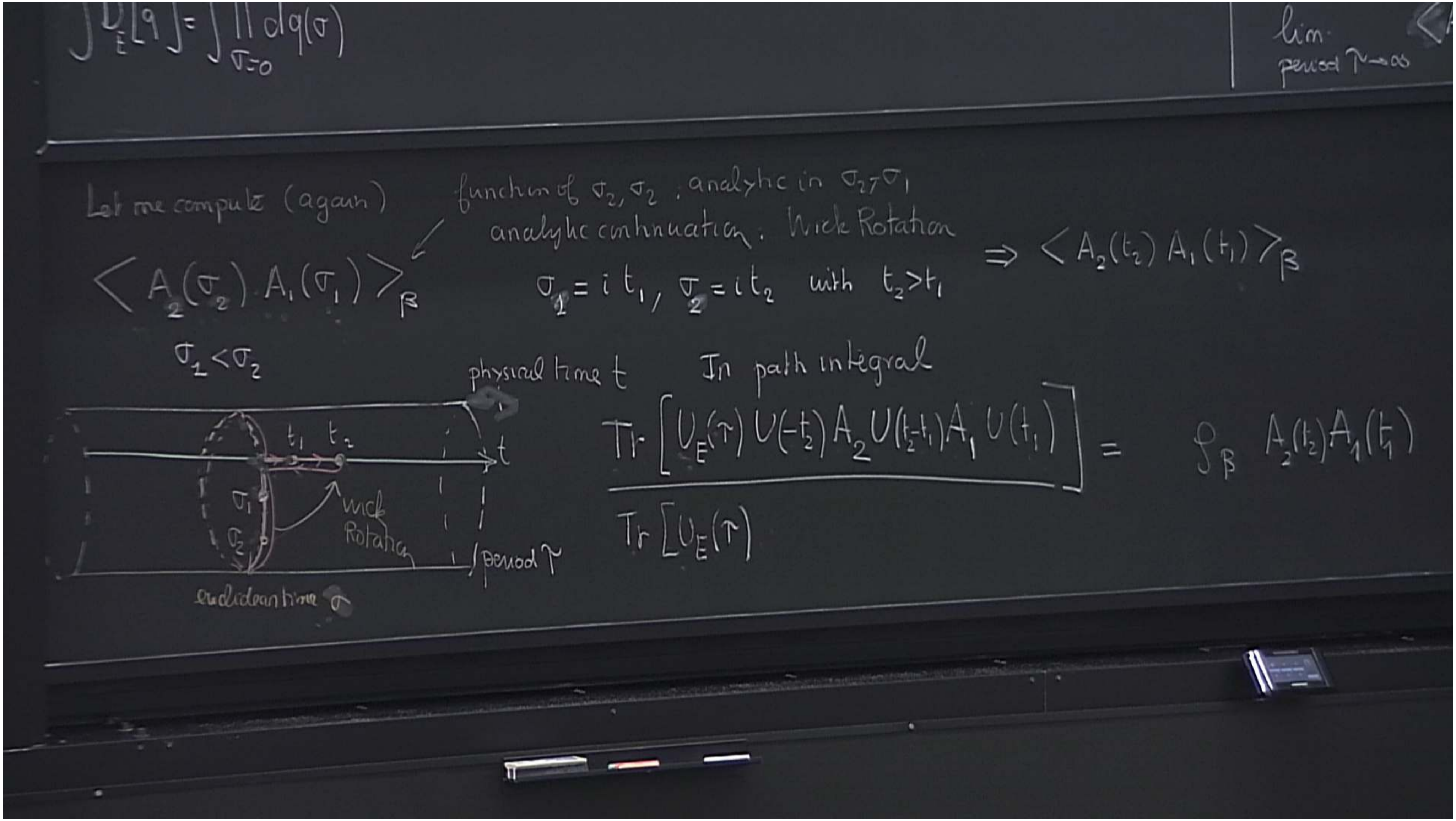
$Z_\beta = \sum_n \exp(-\beta E_n) \simeq \exp(-\beta E_0)$

$p_n \searrow 0$ but $p_0 \to 1$

circle line $\mathbb{R}$

τ Eucl time

$$\lim_{\text{period } \tau \to \infty} \langle A(\varphi)\rangle_\beta \to \langle 0 | A(\varphi) | 0 \rangle$$


$\int D_E[q] = \int \prod_{\sigma=0} dq(\sigma)$
lim period $\tau \to \infty$
Let me compute (again)
function of σ_2, σ_2 ; analytic in $\sigma_2 > \sigma_1$
analytic continuation : Wick Rotation
$\langle A_2(\sigma_2)\, A_1(\sigma_1) \rangle_\beta$
$\sigma_1 = i t_1$, $\sigma_2 = i t_2$ with $t_2 > t_1$
$\Rightarrow \langle A_2(t_2)\, A_1(t_1) \rangle_\beta$
$\sigma_1 < \sigma_2$
physical time t
In path integral
$\frac{\mathrm{Tr}\left[U_E(\tau)\, U(-t_2)\, A_2\, U(t_2-t_1)\, A_1\, U(t_1)\right]}{\mathrm{Tr}\left[U_E(\tau)\right]} = \rho_\beta\, A_2(t_2) A_1(t_1)$
t_1 t_2
t
σ_1
σ_2
wick Rotation
period τ
euclidean time σ

period $\tau \to \infty$

$A_2(t_2)\,A_1(t_1)\rangle_\beta$

expectation value on a Gibbs state of the product of 2 observables A_1 & A_2 at different times t_1 & t_2 = correlation function in time for the quantum system.

$$= \mathrm{Tr}\left[\rho_\beta \; A_2(t_2) A_1(t_1)\right] = \langle T[A_1(t_1), A_2(t_2)]\rangle_\beta .$$

finally · Temp $\searrow 0$ or $\beta, \tau_{period} \nearrow \infty$

$$\langle 0|\, T(A_1(t_1), A_2(t_2))\, |0\rangle$$

ground state or "vacuum" e.v of product of operators

Particles (non-relativistic) $\Longrightarrow$ Field theory (Relativistic Q.M)

Euclide

Free scalar field $\Leftarrow$ 2nd quantization of a Klein Gordon particle

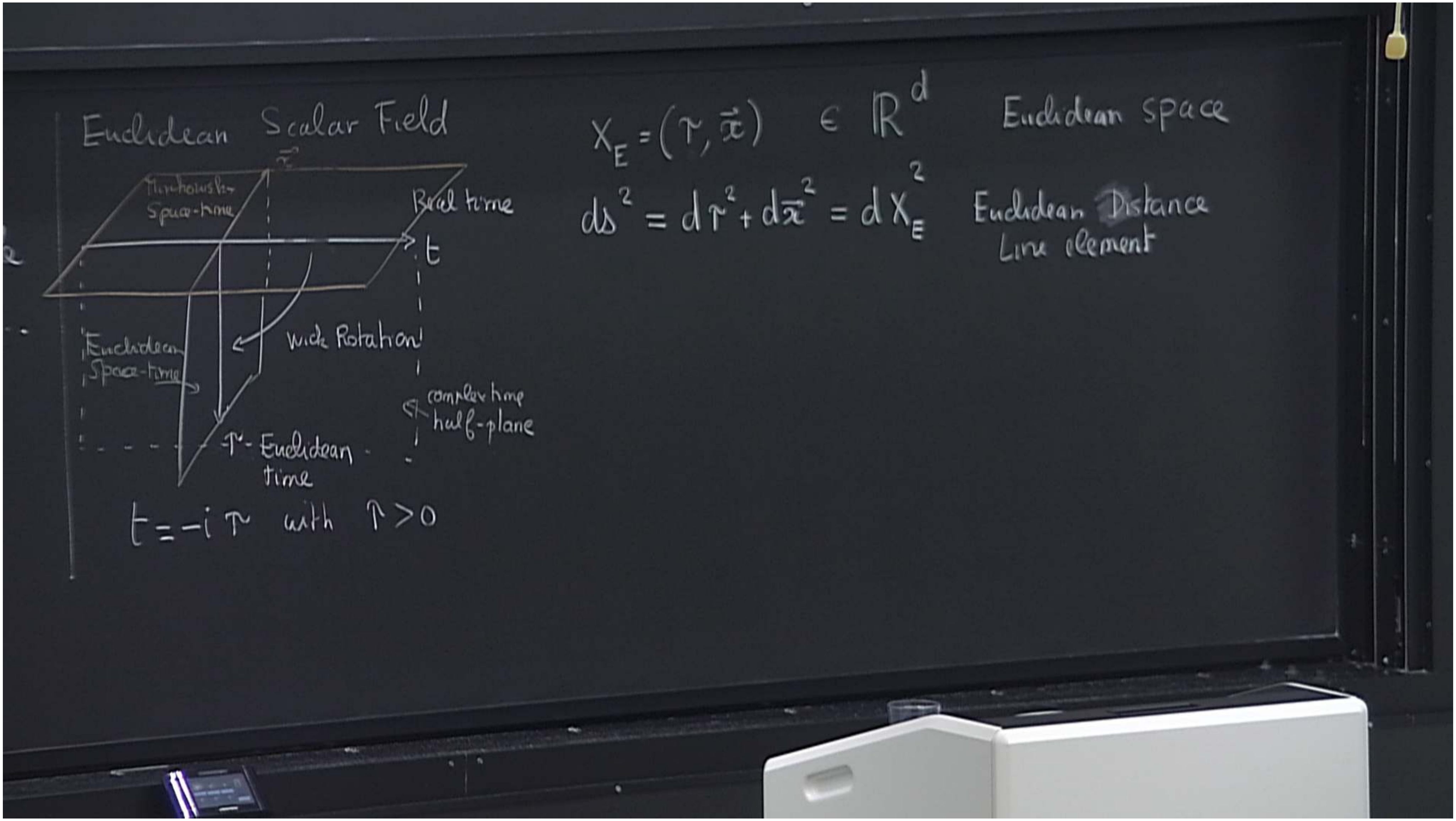
$X = (t, \vec{x})$ 4 or d-vector Weinberg, Misner, Thorne, Wheeler, ...

$ds^2 = -dt^2 + d\vec{x}^{\,2}$ $(-,+,+,+\cdots)$

$\phi(X)$ real field

$$S[\phi] = \int dt \int d^{d-1}\vec{x} \underbrace{\Big[\underbrace{\tfrac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\Big(\tfrac{1}{2}(\vec{\partial}_x \phi)^2 + \tfrac{m^2}{2}\phi^2 \Big)}_{\text{Potential}} \Big]}_{\text{Lagrangian density}}$$

Action at real time


Euclidean Scalar Field
$X_E = (\tau, \vec{x}) \in \mathbb{R}^d$
Euclidean space
$ds^2 = d\tau^2 + d\vec{x}^2 = dX_E^2$
Euclidean Distance
Line element
Minkowski Space-time
Real time
t
$\vec{x}$
Wick Rotation
Euclidean Space-time
complex time half-plane
τ - Euclidean time
$t = -i\tau$ with $\tau > 0$

Particles (non-relativistic) $\Longrightarrow$ Field theory (Relativistic Q.M)

Free scalar field $\Leftarrow$ 2nd quantization of a Klein Gordon particle

$X = (t, \vec{x})$ 4 or d-vector Weinberg, Misner, Thorne, Wheeler, ...

$ds^2 = -dt^2 + d\vec{x}^{\,2}$ $(-,+,+,+\cdots)$ $X \in \mathbb{M}^{1,d-1}$

$\phi(X)$ real field

$$S[\phi] = \int dt\, d^{d-1}\vec{x} \underbrace{\left[\underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\left(\frac{1}{2}(\vec{\partial}_x \phi)^2 + \frac{m^2}{2}\phi^2\right)}_{\text{Potential}} \right]}_{\text{Lagrangian density}}$$

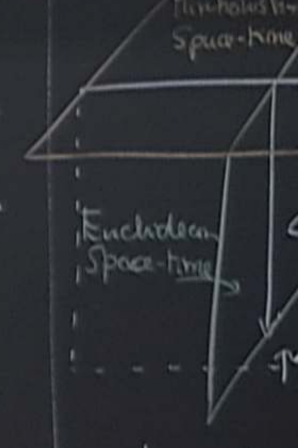
Action at real time

Equation of Motions

$$-\partial_t^2 \phi = -\Delta_{\vec{x}} \phi + m^2 \phi$$

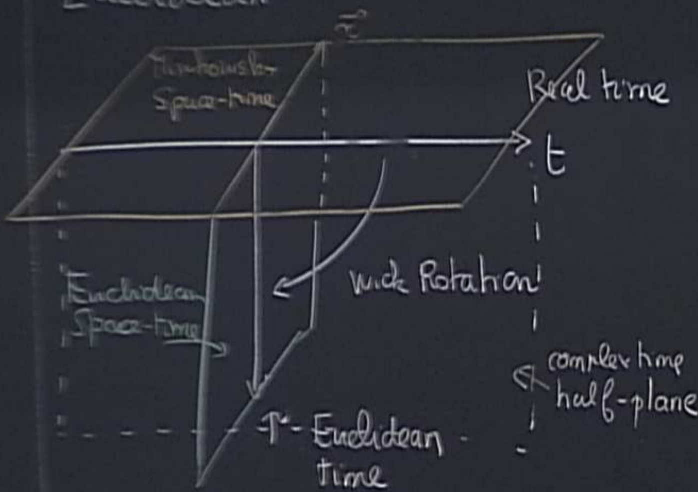
Klein Gordon Equation

Euclidean



$t = -i\tau$

Euclidean Scalar Field



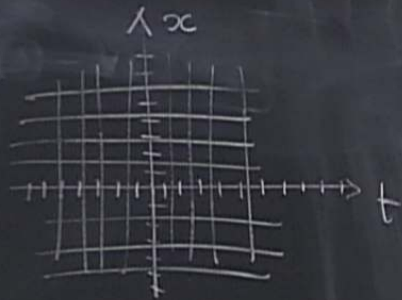
$X_E = (\tau, \vec{x}) \in \mathbb{R}^d$ Euclidean space

$ds^2 = d\tau^2 + d\vec{x}^2 = dX_E^2$ Euclidean Distance Line element

$t = -i\tau$ with $\tau > 0$

$(t_1), A_2(t_2)]]|q_I, 0\rangle \quad 0<t_1,t_2<t$

integral over Field-configurations $\phi(x) = \phi(t,\vec{x})$



discretize time $t \to t_a = a\,\Delta t \quad a\in\mathbb{Z}$

" space $\vec{x}\to \vec{x}_{\vec\imath} = \vec{\imath}\,\Delta x \quad \vec{\imath}\in\mathbb{Z}^{d-1}$

$\vec{X} = \vec{X}_{\vec a} = \vec{a}\cdot\epsilon \quad \vec a \in \mathbb{Z}^d$

$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")

$0 \quad t_1 \quad t_2 \quad t$

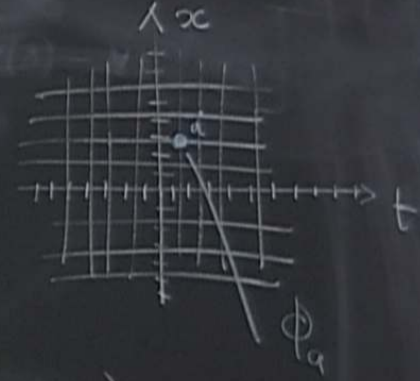
$= \langle q_F, t | T[A_1(t_1), A_2(t_2)] | q_I, 0 \rangle \qquad 0 < t_1, t_2 < t$

Path-Integral Quantization ⇒ integral over Field-configuratio[ns]

$$Z = \int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

How to define this properly?

$$Z = \lim_{\epsilon \to 0} \int \prod_{\vec{a} \in \mathbb{Z}^d} d\phi_{\vec{a}} \left(\quad \right) \exp\left(\frac{i}{\hbar} S_\epsilon[\phi]\right.$$



discretize time

" space

$\Delta t = \Delta x = \epsilon$

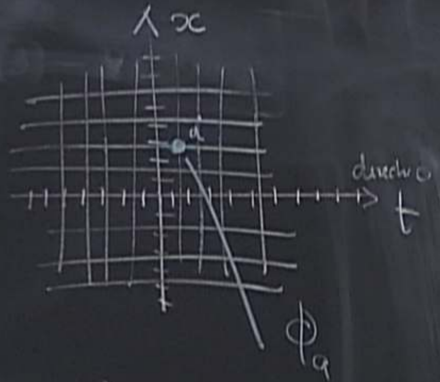
" space-time

$S_\epsilon[\phi] =$

$\ldots t_1), A_2(t_2)\,]\,|q_I, 0\rangle \qquad 0<t_1,t_2<t$

integral over Field-configurations $\phi(x) = \phi(t, \vec{x})$

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_{d-1} \end{pmatrix}$$



discretize time $t \to t_a = a\,\Delta t \quad a \in \mathbb{Z}$

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$\vec{X} = \vec{X}_{\vec{a}} = \vec{a}\cdot\epsilon \quad \vec{a} \in \mathbb{Z}$

$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")

"space-time cut-off"

$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$

$$S_\epsilon[\phi] = \sum_{\vec{a}\in\mathbb{Z}^d} \epsilon^d \;\; \phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_0}$$

$|q_I, 0\rangle \qquad 0<t_1,t_2<t$

...er Field - configurations $\quad \phi(X) = \phi(t,\vec{x})$

discretize time $\quad t \to t_a = a\,\Delta t \quad a \in \mathbb{Z}$

" $\quad$ space $\quad \vec{x} \to \vec{x}_i = i\,\Delta x \quad i \in \mathbb{Z}^{d-1}$

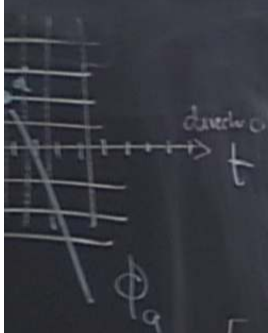
$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")

"space-time cut-off"

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} \quad \vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_\mu = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix} \quad \mu = 1, \ldots, d-1$$

$$\vec{X} = \vec{X}_{\vec{a}} = \vec{a}\cdot\epsilon \qquad \vec{a} \in \mathbb{Z}^d$$

$$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$$ real Field on the lattice



$$= \sum_{\vec{a}\in\mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_0}}{\epsilon}\right)^2 \frac{1}{2} - \left(\sum_{\mu=1}^{d-1} \left(\frac{\phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_\mu}}{\epsilon}\right)^2 \frac{1}{2} + \frac{m^2}{2}\phi_{\vec{a}}^2 \right) \right]$$ discretized action

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Free scalar field $\Leftarrow$ 2nd quantization of a Klein Gordon particle

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$ds^2 = -dt^2 + d\vec{x}^{\,2}$ $(-,+,+,+\cdots)$ $X \in \mathbb{M}^{1,d-1}$

Equation of Motions

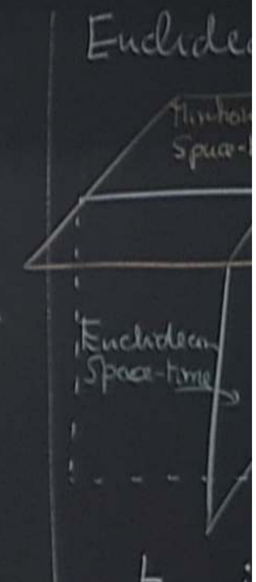
$\phi(X)$ real field

$$S[\phi] = \int \underbrace{dt\, d^{d-1}\vec{x}}_{d^dx} \underbrace{\left[\underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\left(\frac{1}{2}(\vec{\partial}_x \phi)^2 + \frac{m^2}{2}\phi^2 \right)}_{\text{Potential}} \right]}_{\text{Lagrangian density}}$$

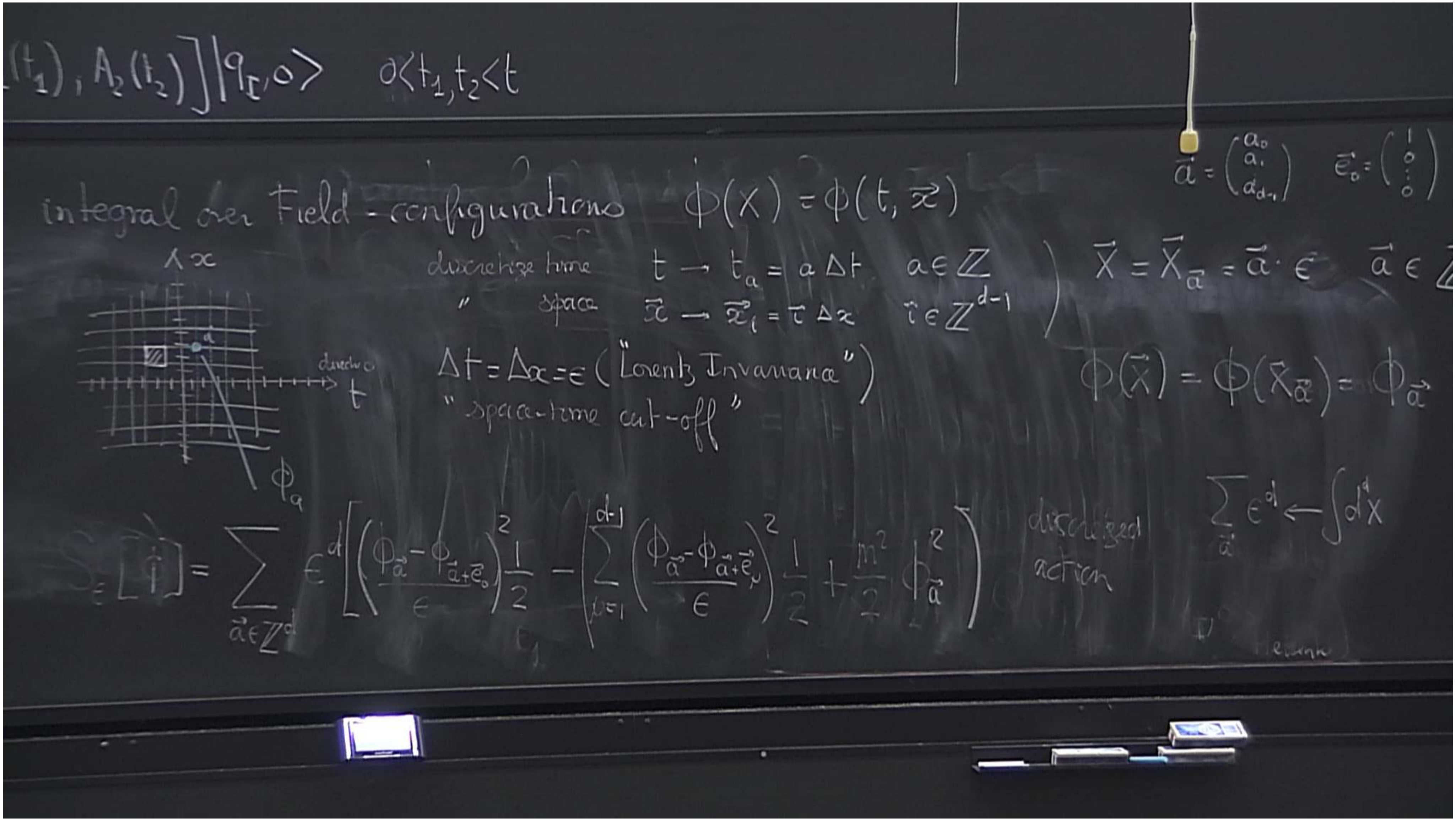
$$-\partial_t^2 \phi = -\Delta_{\vec{x}} \phi + m^2 \phi$$

Klein Gordon Equation

Action at real time



$t = -i$


$(t_1), A_2(t_2)]]|q_I, 0\rangle$ $0<t_1,t_2<t$
integral over Field-configurations $\phi(x) = \phi(t,\vec{x})$
$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix}$ $\vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
x
discretize time $t \to t_a = a\Delta t$ $a \in \mathbb{Z}$
" space $\vec{x} \to \vec{x}_i = \vec{i}\,\Delta x$ $\vec{i} \in \mathbb{Z}^{d-1}$
$\vec{X} = \vec{X}_{\vec{a}} = \vec{a}\cdot\epsilon$ $\vec{a} \in \mathbb{Z}$
direction
t
$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")
" space-time cut-off "
$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$
ϕ_a
$S_\epsilon[\phi] = \sum_{\vec{a}\in\mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_0}}{\epsilon}\right)^2 \frac{1}{2} - \left(\sum_{\mu=1}^{d-1} \left(\frac{\phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_\mu}}{\epsilon}\right)^2 \frac{1}{2} + \frac{m^2}{2}\phi_{\vec{a}}^2 \right) \right]$
discretized action
$\sum_{\vec{a}} \epsilon^d \leftarrow \int d^d x$

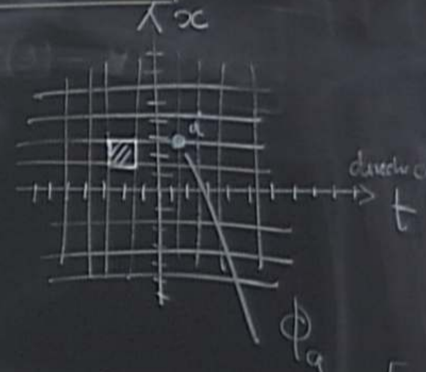
$0 \quad t_1 \quad t_2 \quad t$

$= \langle q_F, t | \, T[A_1(t_1), A_2(t_2)] \, | q_I, 0 \rangle \qquad 0 < t_1, t_2 < t$

Path-Integral Quantization $\Rightarrow$ $\boxed{\text{Functional integral}}$ over Field-configuratio[ns]

$$Z = \int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

How to define this properly?



discretize time

" space

$\Delta t = \Delta x = \epsilon$

" space-time

$$Z = \lim_{\epsilon \to 0} \int \underbrace{\prod_{\vec{a} \in \mathbb{Z}^d} d\phi_{\vec{a}} \left(\frac{2\pi i \hbar}{\epsilon^{d-2}}\right)^{-1/2}}_{D[\phi]} \exp\left(\frac{i}{\hbar} S_\epsilon[\phi]\right)$$

$$S_\epsilon[\phi] = \sum_{\vec{a} \in \mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a}+\vec{e}_0}}{\epsilon}\right)^2 \frac{1}{2} - \ldots \right.$$