

Title: 14/15 PSI - Quantum Field Theory II-Lecture 2

Date: Nov 11, 2014 09:00 AM

URL: <http://pirsa.org/14110007>

Abstract:

Path-Integral Quantization

$$H = \frac{p^2}{2m} + V(q) \quad 1 \text{ dim}$$

$$S = \int_0^t ds \left[\frac{m}{2} \dot{q}(s)^2 - V(q(s)) \right]$$

$$U(t) = \exp\left(\frac{t}{i\hbar} H\right)$$

$$K(q_F, q_I; t) = \langle q_F | U(t) | q_I \rangle_{\text{Schr}} = \int_{q(0)=q_I}^{q(t)=q_F} \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right)$$

$$= \langle q_F, t | q_I, 0 \rangle_{\text{Heis}}$$

Operators

$$\int \mathcal{D}[q] e$$

$$q(0) = q_I$$

$$q(t) = q_F$$

q

q_F

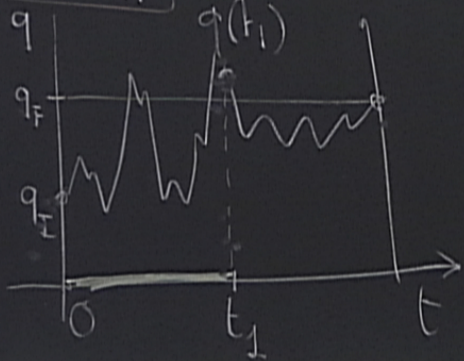
q_I



Operators & Observables? $0 < t_1 < t$

$$\int D[q] e^{\frac{i}{\hbar} S[q]} q(t_1) = \int_{-\infty}^{+\infty} dq_1 \langle q_F | U(t_F - t_1) | q_1 \rangle \underbrace{\langle q_1 | U(t_1) | q_I \rangle}_{Q \text{ position operator}}$$

$$\begin{aligned} q(0) &= q_I \\ q(t) &= q_F \end{aligned}$$



$$= \langle q_F | U(t_F - t_1) Q U(t_1) | q_I \rangle$$

$$= \langle q_F | U(t_F) U^\dagger(t_1) Q U(t_1) | q_I \rangle$$

Schrödinger

$$= \langle q_F, t_F | Q(t_1) | q_I, 0 \rangle$$

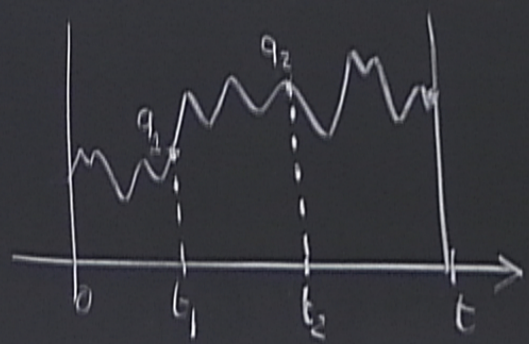
Heisenberg

Hors. $q(t) = q_F$

- $A(q_1)$ function $\leftrightarrow A[Q_1]$ $[A_1(t_1), A_2(t_2)] \neq 0$ because $[Q$
- 2 operators ? path integral orders in time the operators \Rightarrow Time ord

$$\int \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right) A_1(q(t_1)) \cdot A_2(q(t_2)) = \begin{cases} \langle q_F, t | A_2(q)(t_2) A_1(q)(t_1) | q_I, 0 \rangle & \text{if } 0 \leq t_1 < t_2 \leq t \\ \langle q_F, t | A_1(q)(t_1) A_2(q)(t_2) | q_I, 0 \rangle & \text{if } 0 \leq t_2 < t_1 \leq t \end{cases}$$

$q(0) = q_I$
 $q(t) = q_F$



$$= \langle q_F, t | T[A_1(t_1), A_2(t_2)] | q_I, 0 \rangle \quad 0 < t_1, t_2 < t$$

because $[Q, H] \neq 0$

⇒ Time ordered product

$$T[A_1(t_1), A_2(t_2)] := \begin{cases} A_1(t_1)A_2(t_2) & t_1 > t_2 \\ A_2(t_2)A_1(t_1) & t_2 > t_1 \end{cases}$$

$t_1 < t_2 < t$

$t_2 < t_1 < t$

$t_2 < t_1 < t$

"Correspondance Principle"
path integral operator

$$q(t_1) \longleftrightarrow Q(t_1)$$

$$A(q(t_1)) \longleftrightarrow A(Q)(t_1)$$

$$A_1(q(t_1)) \cdot A_2(q(t_2)) \longleftrightarrow T[A_1(Q)(t_1), A_2(Q)(t_2)]$$

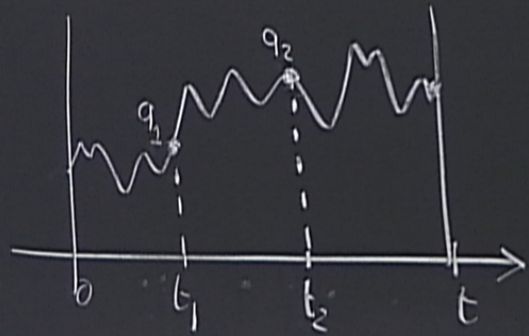
Heis. $q(t) = q_F$

- $A(q_1)$ function $\leftrightarrow A[Q_1]$ $[A_1(t_1), A_2(t_2)] \neq 0$ because $[Q, H]$
- 2 operators ?
 numbers path integral orders in time the operators \Rightarrow Time ordered p
(automatic; build in the formalism)

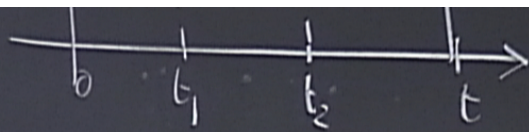
$$\int \mathcal{D}[q] \exp\left(\frac{i}{\hbar} S[q]\right) A_1(q(t_1)) \cdot A_2(q(t_2)) = \begin{cases} \langle q_F, t | A_2(q)(t_2) \cdot A_1(q)(t_1) | q_I, 0 \rangle & \text{if } 0 \leq t_1 < t_2 \leq t \\ \langle q_F, t | A_1(q)(t_1) \cdot A_2(q)(t_2) | q_I, 0 \rangle & \text{if } 0 \leq t_2 < t_1 \leq t \end{cases}$$

$$= \langle q_F, t | T[A_1(t_1), A_2(t_2)] | q_I, 0 \rangle \quad 0 < t_1, t_2 < t$$

$q(0) = q_I$
 $q(t) = q_F$



$$T[A_1(t_1), A_2(t_2)]$$

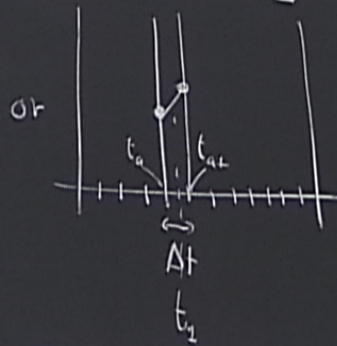
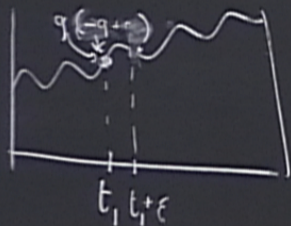


$$= \langle q_F, t | T [A_1(t_1), A_2(t_2)] | q_I, 0 \rangle \quad 0 < t_1, t_2 <$$

What about P (momentum operator)

classically $p = m \dot{q}$

$$P(t_2) = m \cdot \lim_{\epsilon \rightarrow 0} \left[\frac{q(t_1 + \epsilon) - q(t_1)}{\epsilon} \right] \text{ or } m \cdot \frac{q(t_{a+1}) - q(t_a)}{\Delta t}$$



in discretized path integral
 $\lim \Delta t \rightarrow 0$

$$[Q, P] \neq 0$$

order of the $q(t)$
 is important?

$\psi(t_1), \psi(t_2) \rangle \langle \psi, \psi \rangle$ $0 < t_1, t_2 < t$

What about Euclidean path Integrals?

$$Z_{\beta} = \int_{\substack{E \\ q(0)=q(\tau)}} \mathcal{D}[q] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$$

periodic path int.
at euclidean time

$$S_E[q] = \int_0^{\tau} d\sigma \left[\frac{m}{2} \dot{q}(\sigma)^2 + V(q(\sigma)) \right]$$

$$U_E(\sigma) = \exp\left(-\frac{\sigma}{\hbar} H\right)$$

period $\tau \leftrightarrow \text{temp } T$ $\frac{\sigma}{\hbar} = \beta = \frac{1}{k_B \cdot T}$

$$), A_2(t_2)] |q_{\tau, 0}\rangle \quad 0 < t_1, t_2 < \tau$$

What about Euclidean path Integrals? \Leftrightarrow exp. values of observables in Gibbs state

$$Z_{\beta} = \int_{q(0)=q(\tau)} \mathcal{D}_E[q] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$$

periodic path int.
at euclidean time

$$S_E[q] = \int_0^{\tau} d\sigma \left[\frac{m}{2} \dot{q}(\sigma)^2 + V(q(\sigma)) \right]$$

$$U_E(\sigma) = \exp\left(-\frac{\sigma}{\hbar} H\right) = \exp(-\beta H)$$

period $\tau \Leftrightarrow k_B T$ $\frac{\tau}{\hbar} = \beta = \frac{1}{k_B T}$

$$\frac{\int \mathcal{D}_E(q) \exp\left(-\frac{1}{\hbar} S_E[q]\right) A(q(\sigma_2))}{\int \mathcal{D}_E(q) \exp\left(-\frac{1}{\hbar} S_E[q]\right)} = \frac{\text{Tr}[A(Q) U_E(\sigma)]}{\text{Tr}[U_E(\sigma)]}$$

density matrix

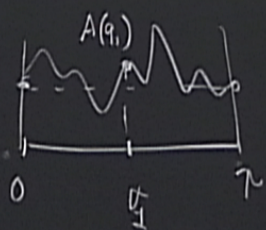
$$Z_{\beta} = \text{Tr}(\exp(-\beta H))$$

$$= \text{Tr} \left[A(Q) \frac{\exp(-\beta H)}{Z_{\beta}} \right]$$

$$= \text{Tr} [A(Q) \rho_{\beta}]$$

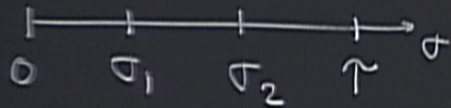
Gibbs state density matrix

$$= \langle A(Q) \rangle_{\beta}$$



expectation value of
A in the (mixed) Gibbs
state at temp. T

We can compute products of operators. definition



$$\langle A_2(\sigma_2) A_1(\sigma_1) \rangle_{\beta} = \frac{1}{Z_{\beta}} \int_{\text{periodic}} D[q] e^{-\frac{1}{\hbar} S_E[q]} A_1(\sigma_1) A_2(\sigma_2)$$

Euclidean correlation function.

Δ $U_E(\sigma)$ defined only for $\sigma \geq 0$

so $\sigma_1 < \sigma_2$: take care of the "Euclidean" time ordering

$$\int D_E[q] = \int_{\sigma=0}^{\tau} \prod dq(\sigma)$$

Gibbs state (quantum)

$$\rho_B = \frac{1}{Z_\beta} \exp(-\beta H) = \sum_n |n\rangle\langle n| \frac{\exp(-\beta E_n)}{Z_\beta}$$

$$\beta \rightarrow \infty, \text{ namely } \text{Temp} \rightarrow 0, \quad \rho_B = |0\rangle\langle 0|$$

Δ
↓
mixture of pure states $|n\rangle$ with prob $p_n = \frac{1}{Z_\beta} \exp(-\beta E_n)$

projection operator on the ground state

$$E_0 < E_1 < E_2 < \dots$$

$$Z_\beta = \sum_n \exp(-\beta E_n) \simeq \exp(-\beta E_0)$$

$$p_n \gg 0 \text{ but } p_0 \simeq 1$$

Gibbs state (quantum)

$$\rho_B = \frac{1}{Z_\beta} \exp(-\beta H) = \sum_n |n\rangle\langle n| \frac{\exp(-\beta E_n)}{Z_\beta}$$

Δ
 \downarrow
 mixture of pure states $|n\rangle$ with prob $p_n = \frac{1}{Z_\beta} \exp(-\beta E_n)$

$\beta \rightarrow \infty$, namely Temp $\rightarrow 0$, $\rho_B = |0\rangle\langle 0|$

projection operator on the ground state

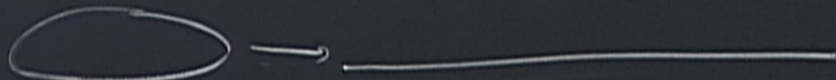
$$E_0 < E_1 < E_2 < \dots$$

period $\tau \rightarrow \infty$

$$Z_\beta = \sum_n \exp(-\beta E_n) \simeq \exp(-\beta E_0)$$

circle

line \mathbb{R}



τ Eucl time

$p_n \gg 0$ but $p_0 \rightarrow 1$

$$\lim_{\text{period } \tau \rightarrow \infty} \langle A(\varphi) \rangle_\beta \rightarrow \langle 0 | A(\varphi) | 0 \rangle$$

$$\int_{\mathcal{D}} U_E[q] = \int_{\mathcal{D}} \mathcal{L} dq(\sigma)$$

lim. period $\tau \rightarrow \infty$

Let me compute (again)

function of σ_2, σ_1 ; analytic in σ_2, σ_1
analytic continuation. Wick Rotation

$$\langle A_2(\sigma_2) A_1(\sigma_1) \rangle_{\mathcal{B}}$$

$$\sigma_1 = it_1, \sigma_2 = it_2 \text{ with } t_2 > t_1$$

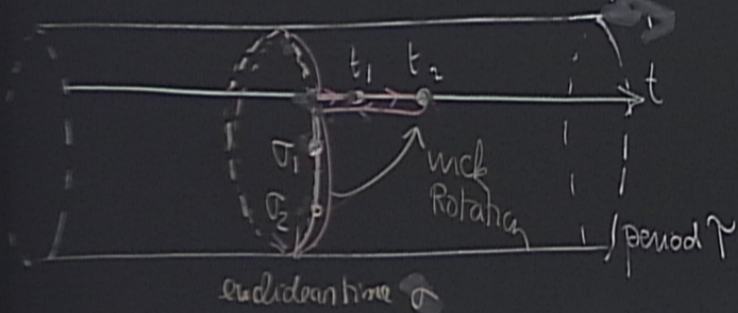
$$\Rightarrow \langle A_2(t_2) A_1(t_1) \rangle_{\mathcal{B}}$$

$$\sigma_2 < \sigma_1$$

physical time t

In path integral

$$\frac{\text{Tr} \left[U_E(\tau) U(-t_2) A_2 U(t_2 - t_1) A_1 U(t_1) \right]}{\text{Tr} [U_E(\tau)]} = \int_{\mathcal{B}} A_2(t_2) A_1(t_1)$$



period $T \rightarrow \infty$

$$\langle A_2(t_2) A_1(t_1) \rangle_{\beta}$$

expectation value on a Gibbs state of the product of 2 observables A_1 & A_2 at different times t_1 & t_2

$A_1, A_2 =$ correlation function in time for the quantum system.

$$= \text{Tr} \left[\rho_{\beta} A_2(t_2) A_1(t_1) \right] = \langle T[A_1(t_1), A_2(t_2)] \rangle_{\beta}$$

finally \cdot Temp $\rightarrow 0$ or $\beta, T_{\text{period}} \rightarrow \infty$

$$\langle 0 | T(A_1(t_1), A_2(t_2)) | 0 \rangle$$

ground state or "vacuum" e.v of product of operators

Particles (non-relativistic) \Rightarrow Field theory (Relativistic Q.M)

Free scalar field \Leftarrow 2nd quantization of a Klein Gordon particle

$X = (t, \vec{x})$ 4 or d-vector Weinberg, Misner, Thorne, Wheeler, ...

$$ds^2 = -dt^2 + d\vec{x}^2 \quad (-, +, +, + \dots)$$

$\phi(x)$ real field

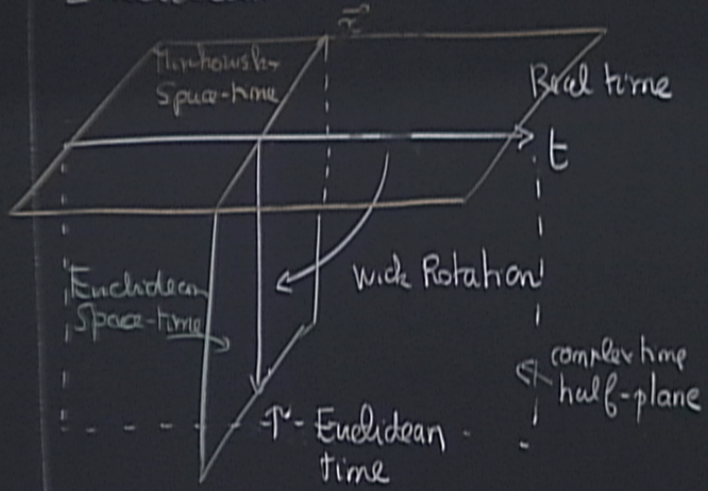
$$S[\phi] = \int dt d\vec{x} \left[\underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\left(\frac{1}{2}(\vec{\partial}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right)}_{\text{Potential}} \right]$$

Lograngian density

Action at
real time

Euclidean

Euclidean Scalar Field



$$t = -i\tau \text{ with } \tau > 0$$

$$X_E = (\tau, \vec{x}) \in \mathbb{R}^d$$

$$ds^2 = d\tau^2 + d\vec{x}^2 = dX_E^2$$

Euclidean space

Euclidean Distance
Line element

Particles (non-relativistic) \Rightarrow Field theory (Relativistic Q.M)

Free scalar field \Leftarrow 2nd quantization of a Klein Gordon particle

$X = (t, \vec{x})$ 4 or d-vector Weinberg, Misner, Thorne, Wheeler, ...

$ds^2 = -dt^2 + d\vec{x}^2$ $(-, +, +, + \dots)$ $X \in \mathbb{M}^{1, d-1}$

Equation of Motion

$\phi(x)$ real field

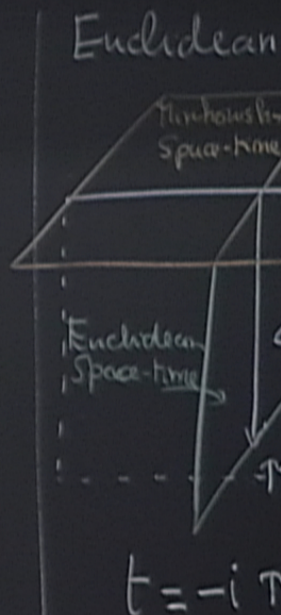
$$S[\phi] = \int dt \int d\vec{x} \left[\underbrace{\frac{1}{2}(\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\left(\frac{1}{2}(\vec{\partial}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right)}_{\text{Potential}} \right]$$

$$-\partial_t^2 \phi = -\Delta_x \phi + m^2 \phi$$

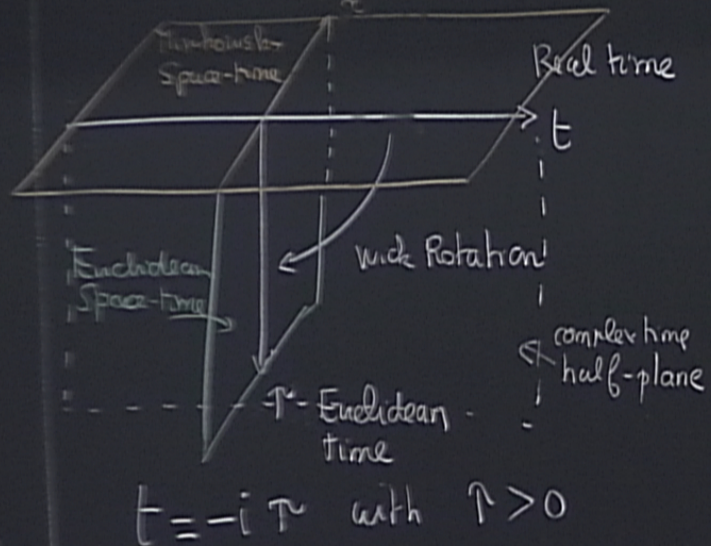
Klein Gordon Equation

Action at
real time

Lograngian density



Euclidean Scalar Field



$$X_E = (\tau, \vec{x}) \in \mathbb{R}^d$$

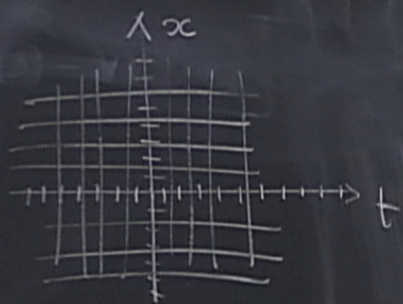
$$ds^2 = d\tau^2 + d\vec{x}^2 = dX_E^2$$

Euclidean space

Euclidean Distance
Line element

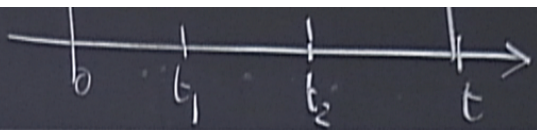
$$[A_1(t_1), A_2(t_2)] |q, 0\rangle \quad 0 \langle t_1, t_2 \rangle t$$

integral over Field-configurations $\phi(x) = \phi(t, \vec{x})$



discretize time $t \rightarrow t_a = a \Delta t \quad a \in \mathbb{Z}$
 " space $\vec{x} \rightarrow \vec{x}_i = \vec{i} \Delta x \quad \vec{i} \in \mathbb{Z}^{d-1}$
 $\Delta t = \Delta x = \epsilon$ ("Lorentz Invariant")

$$\vec{X} = \vec{X}_{\vec{a}} = \vec{a} \cdot \epsilon \quad \vec{a} \in \mathbb{Z}^d$$



$$= \langle q_{F,t} | T [A_1(t_1), A_2(t_2)] | q_{I,0} \rangle \quad 0 < t_1, t_2 < t$$

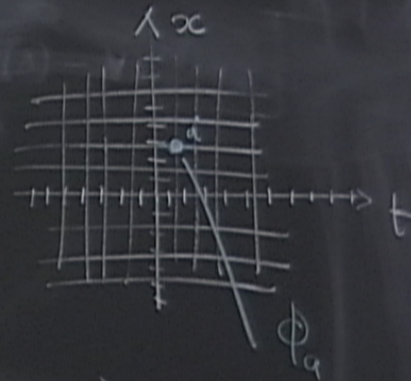
Path-Integral Quantization \Rightarrow

$$Z = \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

How to define this properly?

$$Z = \lim_{\epsilon \rightarrow 0} \int \prod_{\vec{a} \in \mathbb{Z}^d} d\phi_{\vec{a}} \exp\left(\frac{i}{\hbar} S_{\epsilon}[\phi]\right)$$

integral over Field-configuration



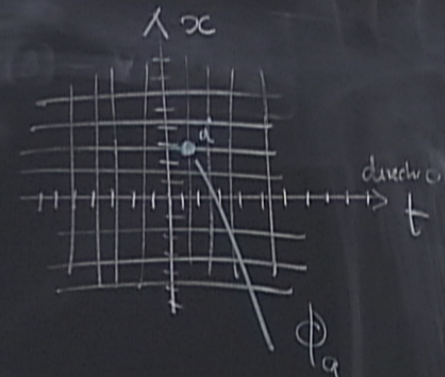
discretize time
" space
 $\Delta t = \Delta x = \epsilon$
" space-time

$$S_{\epsilon}[\phi] =$$

$$t_1), A_2(t_2) \rangle |p_{\Gamma}, 0\rangle \quad 0 < t_1, t_2 < t$$

integral over Field-configurations $\phi(X) = \phi(t, \vec{x})$

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix}$$



discretize time $t \rightarrow t_a = a \Delta t \quad a \in \mathbb{Z}$
 " space $\vec{x} \rightarrow \vec{x}_i = \vec{i} \Delta x \quad \vec{i} \in \mathbb{Z}^{d-1}$

$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")
 "space-time cut-off"

$$\vec{X} = \vec{X}_{\vec{a}} = \vec{a} \cdot \epsilon \quad \vec{a} \in \mathbb{Z}^d$$

$$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$$

$$S_{\epsilon}[\phi] = \sum_{\vec{a} \in \mathbb{Z}^d} \epsilon^d \phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_0}$$

$$|p_T, 0\rangle \quad 0 < t_1, t_2 < t$$

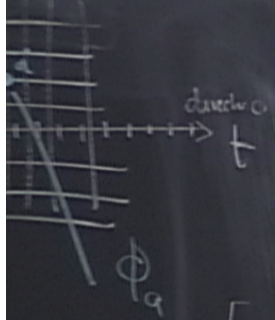
Free Field configurations

$$\phi(x) = \phi(t, \vec{x})$$

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} \quad \vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_\mu = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \mu = 1, \dots, d-1$$

discretize time $t \rightarrow t_a = a \Delta t \quad a \in \mathbb{Z}$
 " space $\vec{x} \rightarrow \vec{x}_i = \vec{i} \Delta x \quad \vec{i} \in \mathbb{Z}^{d-1}$

$$\vec{X} = \vec{X}_{\vec{a}} = \vec{a} \cdot \epsilon \quad \vec{a} \in \mathbb{Z}^d$$



$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")
 "space-time cut-off"

$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$ real field on the lattice

$$\sum_{\vec{a} \in \mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_0}}{\epsilon} \right)^2 \frac{1}{2} - \left(\sum_{\mu=1}^{d-1} \left(\frac{\phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_\mu}}{\epsilon} \right)^2 \frac{1}{2} + \frac{m^2}{2} \phi_{\vec{a}}^2 \right) \right]$$

discretized action

Particles (non-relativistic) \Rightarrow Field theory (Relativistic Q.M)

Free scalar field \Leftarrow 2nd quantization of a Klein Gordon particle

$X = (t, \vec{x})$ 4 or d-vector Weinberg, Misner, Thorne, Wheeler, ...

$ds^2 = -dt^2 + d\vec{x}^2$ $(-, +, +, + \dots)$ $X \in M^{1, d-1}$

Equation of Motion

$\phi(x)$ real field

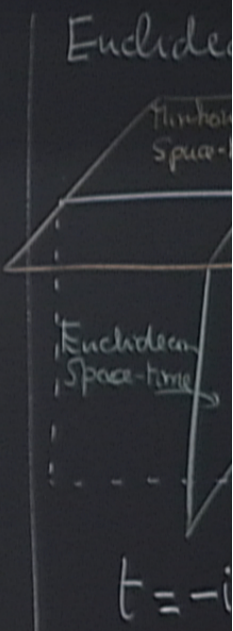
$$S[\phi] = \int \frac{dt d\vec{x}}{d^d x} \left[\underbrace{\frac{1}{2} (\partial_t \phi)^2}_{\text{Kinetic}} - \underbrace{\left(\frac{1}{2} (\vec{\partial}_x \phi)^2 + \frac{m^2}{2} \phi^2 \right)}_{\text{Potential}} \right]$$

Lagrangian density

$$-\partial_t^2 \phi = -\Delta_x \phi + m^2 \phi$$

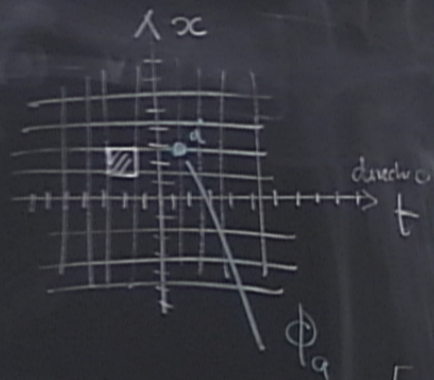
Klein Gordon Equation

Action at real time



$$[A_1(t_1), A_2(t_2)] |p_{\Gamma}, 0\rangle \quad 0 < t_1, t_2 < t$$

integral over Field-configurations $\phi(X) = \phi(t, \vec{x})$



discretize time $t \rightarrow t_a = a \Delta t \quad a \in \mathbb{Z}$
 " space $\vec{x} \rightarrow \vec{x}_i = \vec{r} \Delta x \quad \vec{r} \in \mathbb{Z}^{d-1}$

$\Delta t = \Delta x = \epsilon$ ("Lorentz Invariance")
 "space-time cut-off"

$$\vec{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} \quad \vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vec{X} = \vec{X}_{\vec{a}} = \vec{a} \cdot \vec{e} \quad \vec{a} \in \mathbb{Z}^d$$

$$\phi(\vec{X}) = \phi(\vec{X}_{\vec{a}}) = \phi_{\vec{a}}$$

$$S_{\epsilon}[\vec{\phi}] = \sum_{\vec{a} \in \mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_0}}{\epsilon} \right)^2 \frac{1}{2} - \left(\sum_{\mu=1}^{d-1} \left(\frac{\phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_\mu}}{\epsilon} \right)^2 \frac{1}{2} + \frac{m^2}{2} \phi_{\vec{a}}^2 \right) \right]$$

discretized action

$$\sum_{\vec{a}} \epsilon^d \leftarrow \int d^d X$$

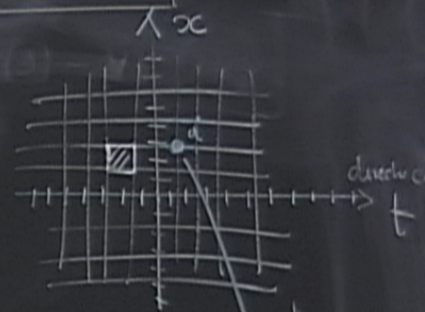
$$\begin{array}{c} | \\ \hline b \quad t_1 \quad t_2 \quad t \\ \hline \end{array} \rightarrow = \langle q_{F,t} | T [A_1(t_1), A_2(t_2)] | q_{I,0} \rangle \quad 0 < t_1, t_2 < t$$

Path-Integral Quantization \Rightarrow Functional integral over Field-configuration

$$Z = \int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

How to define this properly?

$$Z = \lim_{\epsilon \rightarrow 0} \int \prod_{\vec{a} \in \mathbb{Z}^d} d\phi_{\vec{a}} \underbrace{\left(\frac{2\pi\hbar}{\epsilon^{d-2}} \right)^{-1/2}}_{D[\phi]} \exp\left(\frac{i}{\hbar} S_{\epsilon}[\phi]\right)$$



discretize time
" space
 $\Delta t = \Delta x = \epsilon$
" space-time

$$S_{\epsilon}[\phi] = \sum_{\vec{a} \in \mathbb{Z}^d} \epsilon^d \left[\left(\frac{\phi_{\vec{a}} - \phi_{\vec{a} + \vec{e}_0}}{\epsilon} \right)^2 \frac{1}{2} - \dots \right]$$