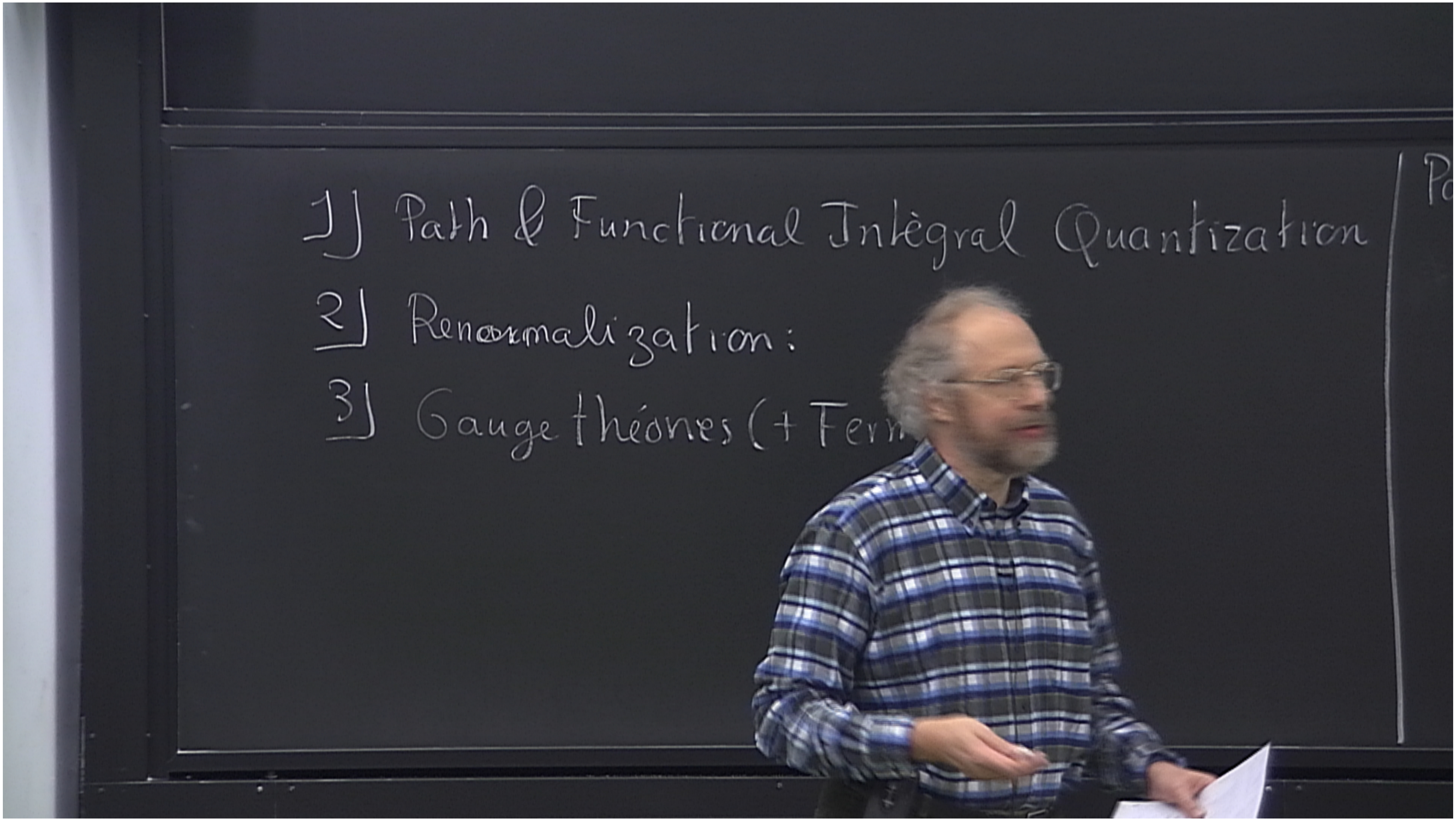
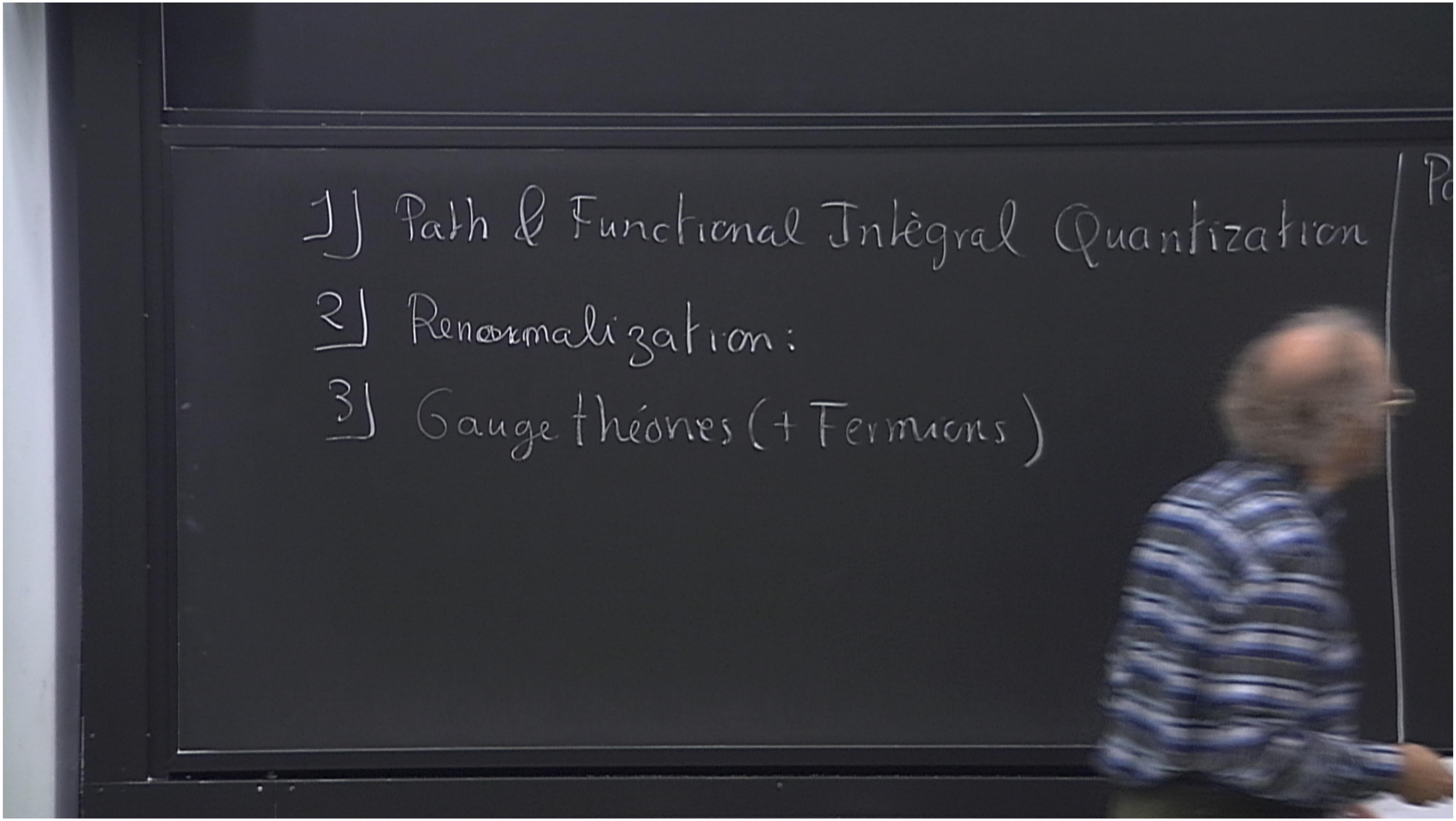
Title: 14/15 PSI - Quantum Field Theory II - Lecture 1

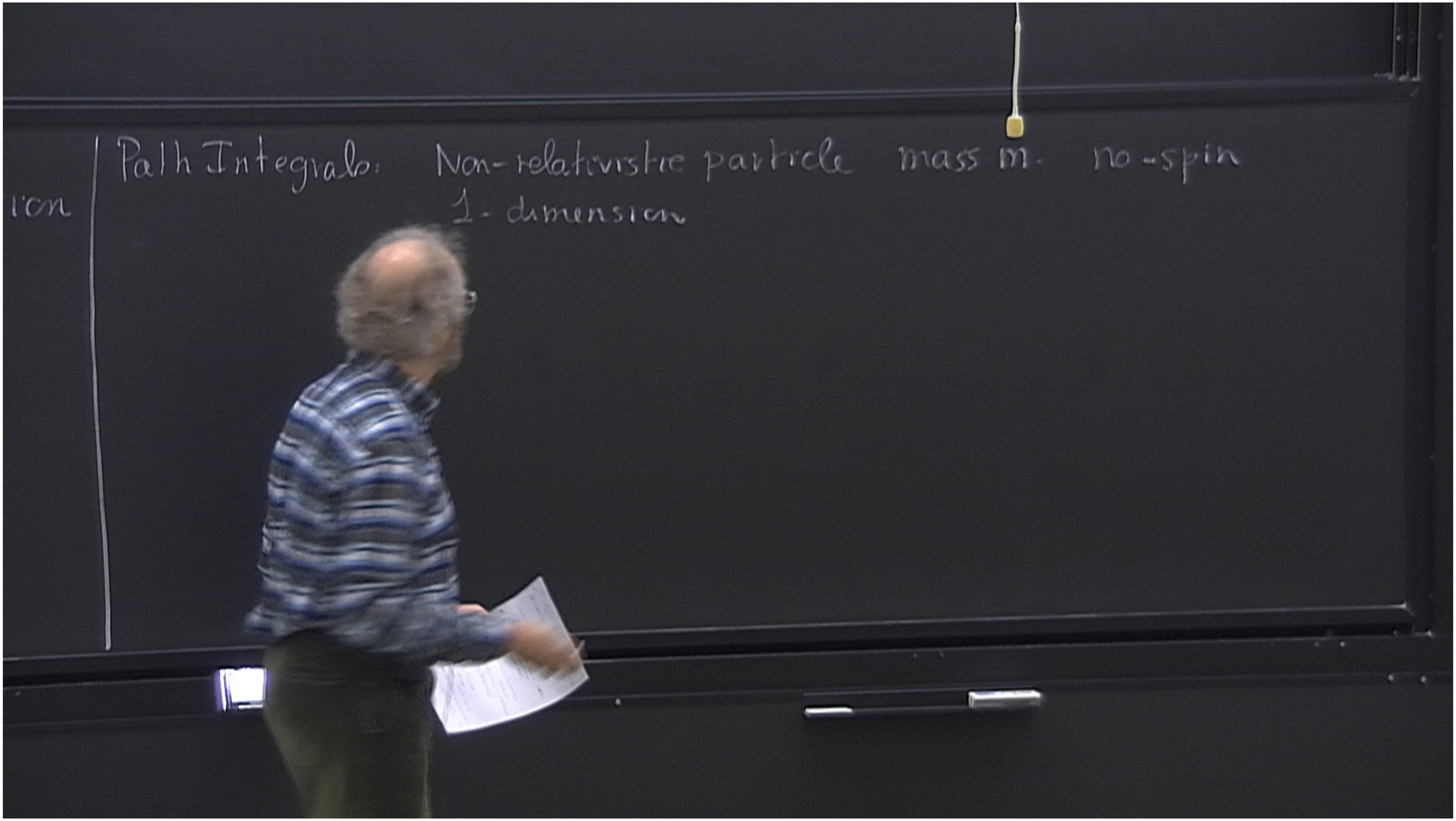
Date: Nov 10, 2014 09:00 AM

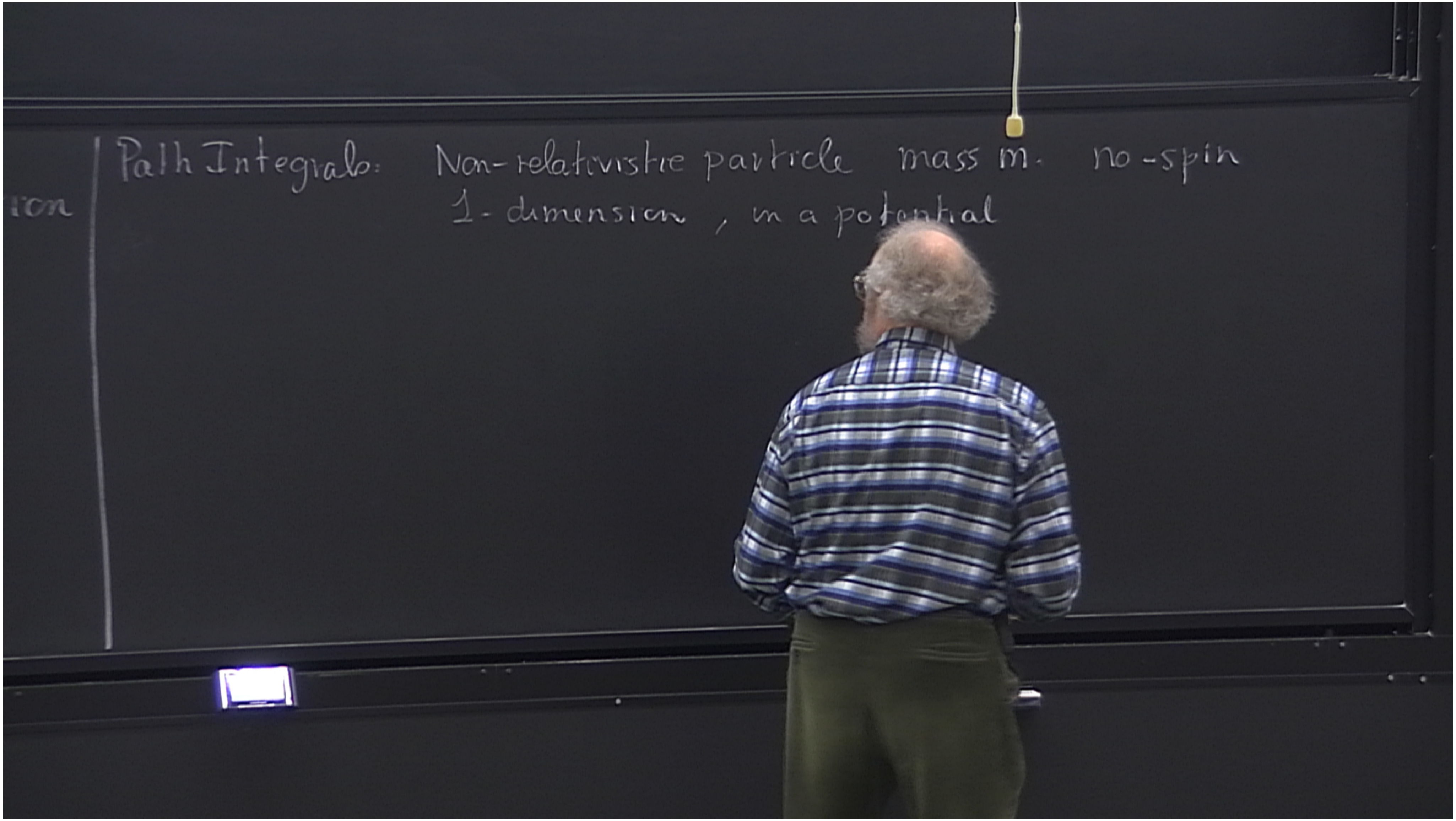
URL: http://pirsa.org/14110006

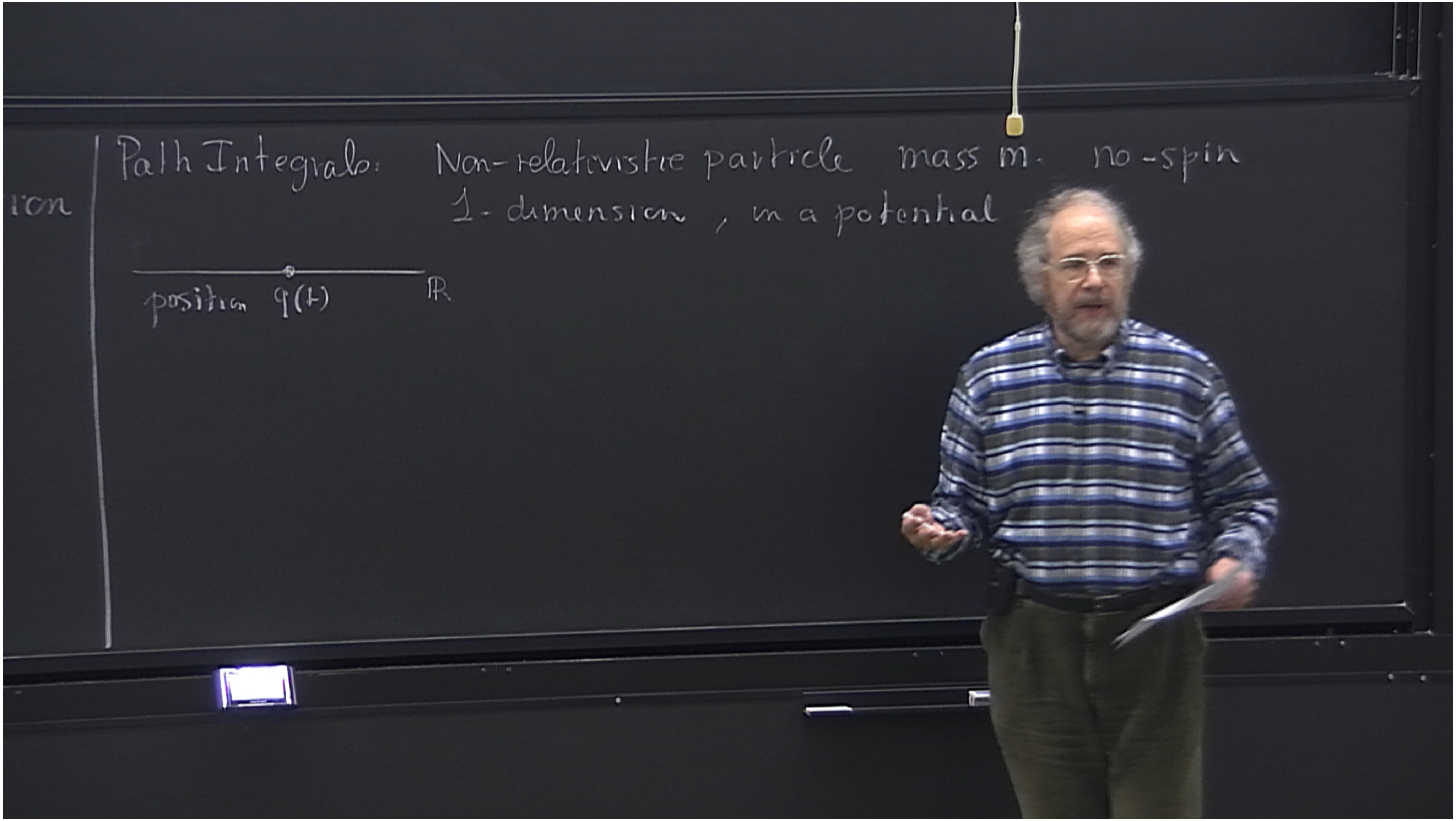
Abstract:


1] Path & Functional Intègral Quantization
2] Renormalization:
3] Gauge théories (+ Ferm

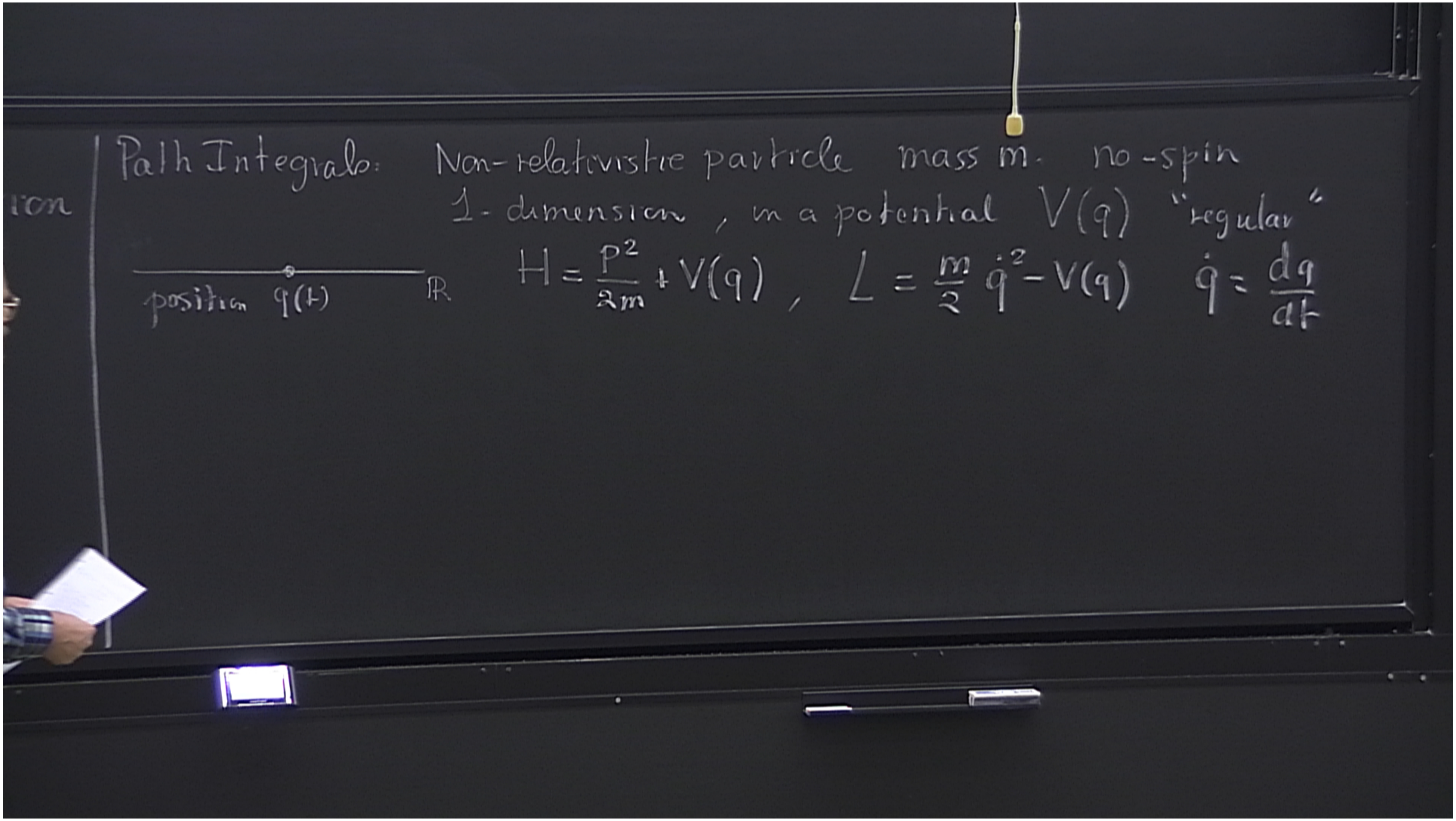

1) Path & Functional Intègral Quantization
2) Renormalization:
3) Gauge théories (+ Fermions)


Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension

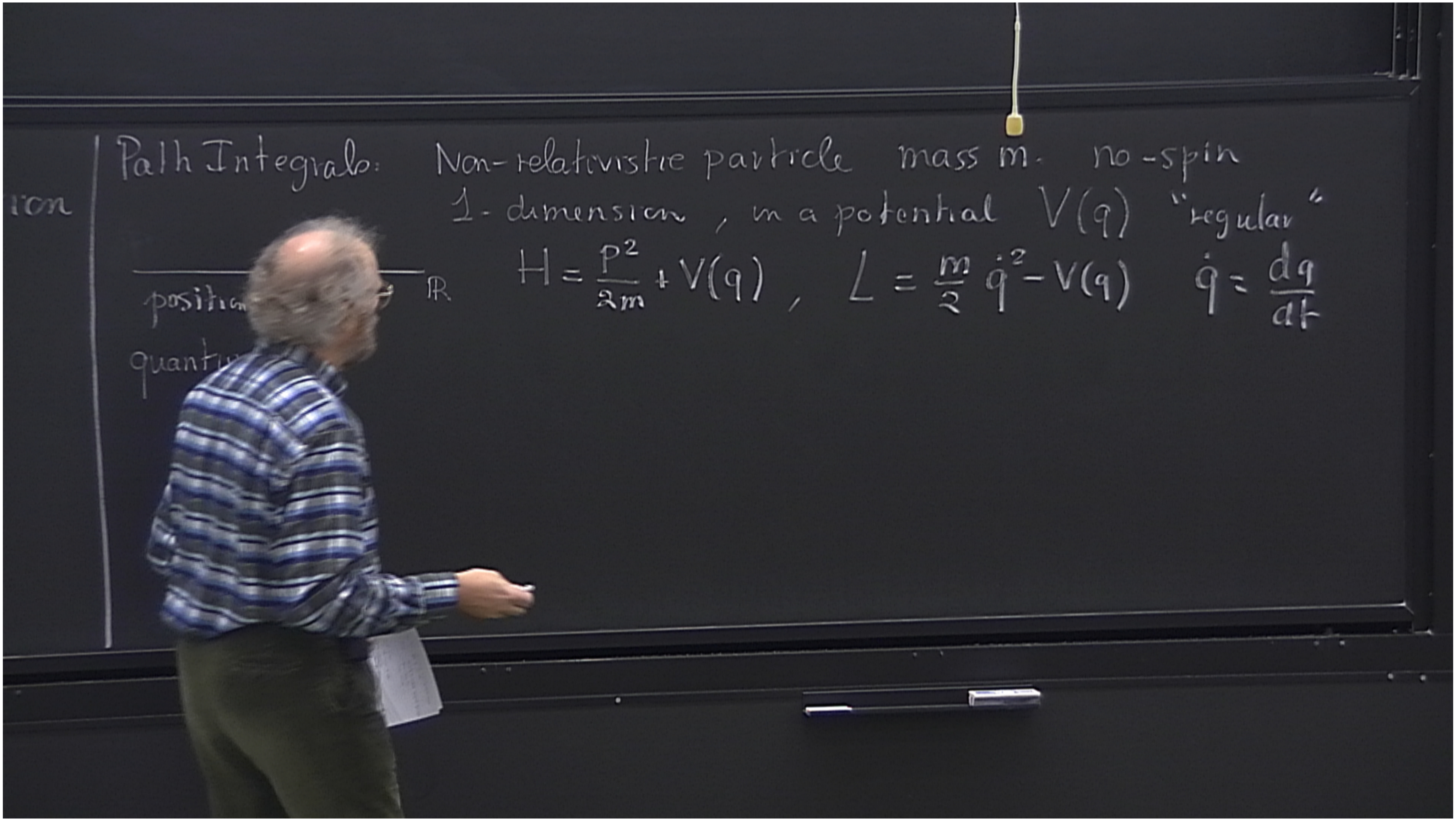

Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential

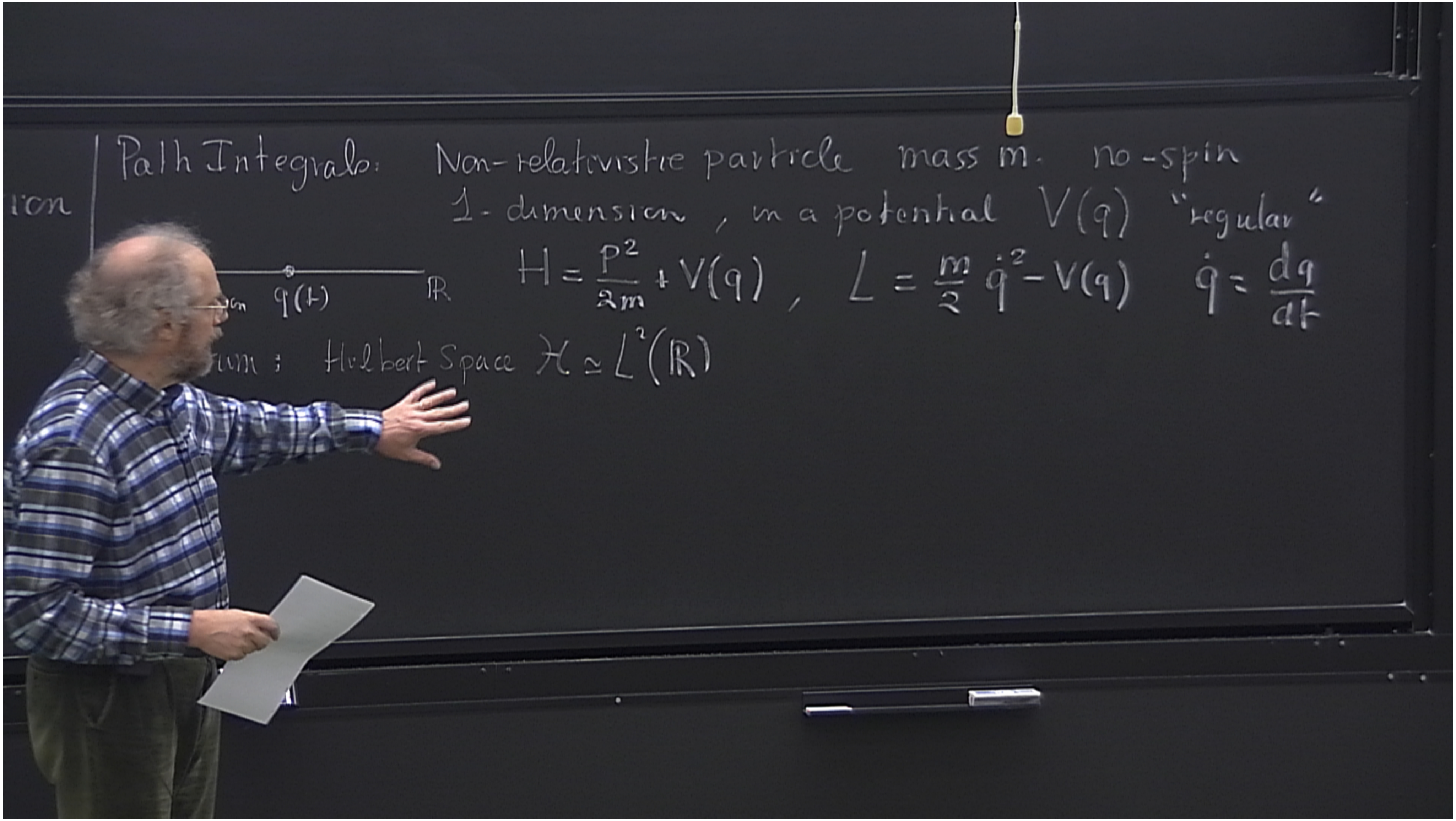


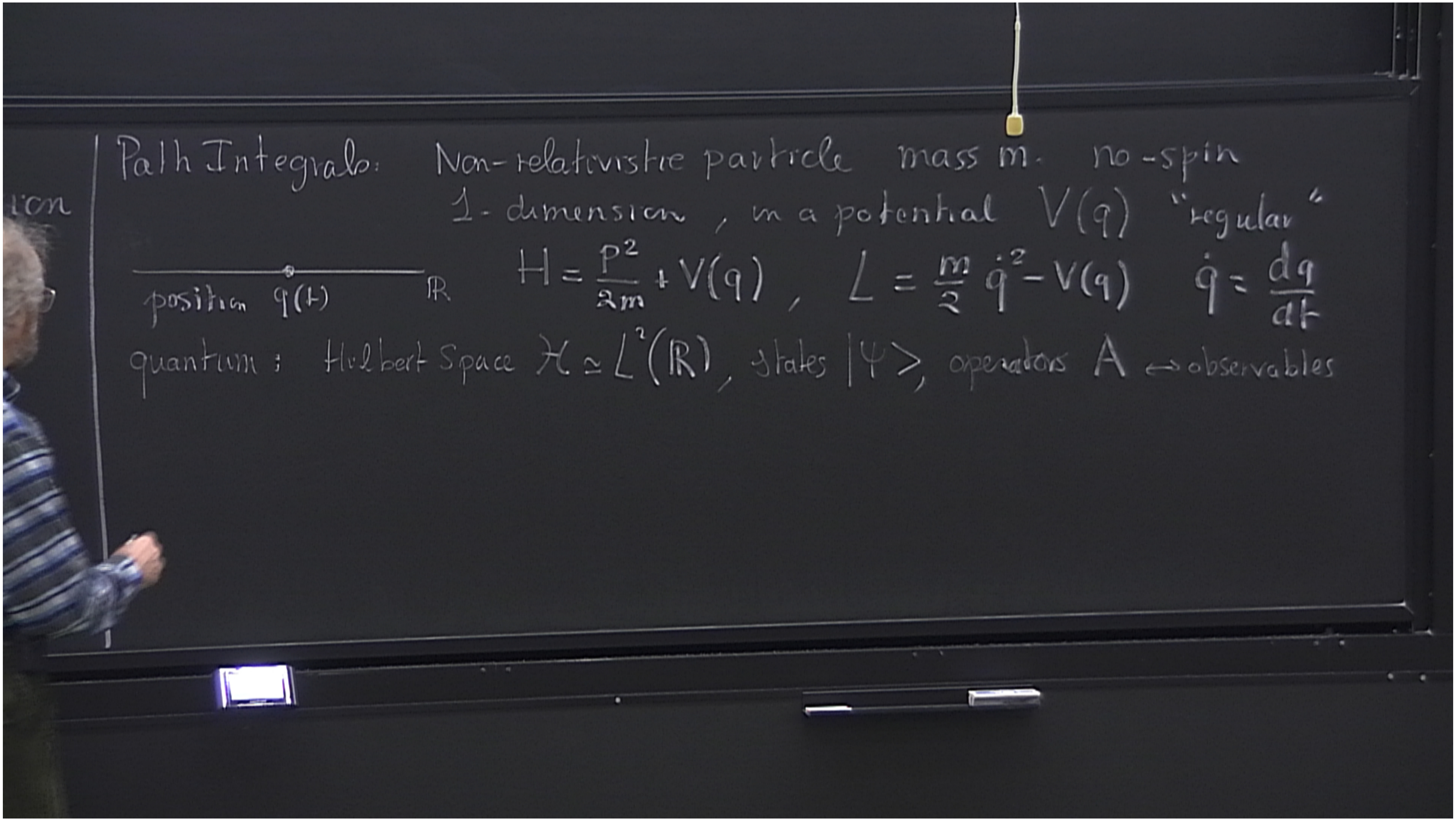
Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension , in a potential
position q(t)
ℝ

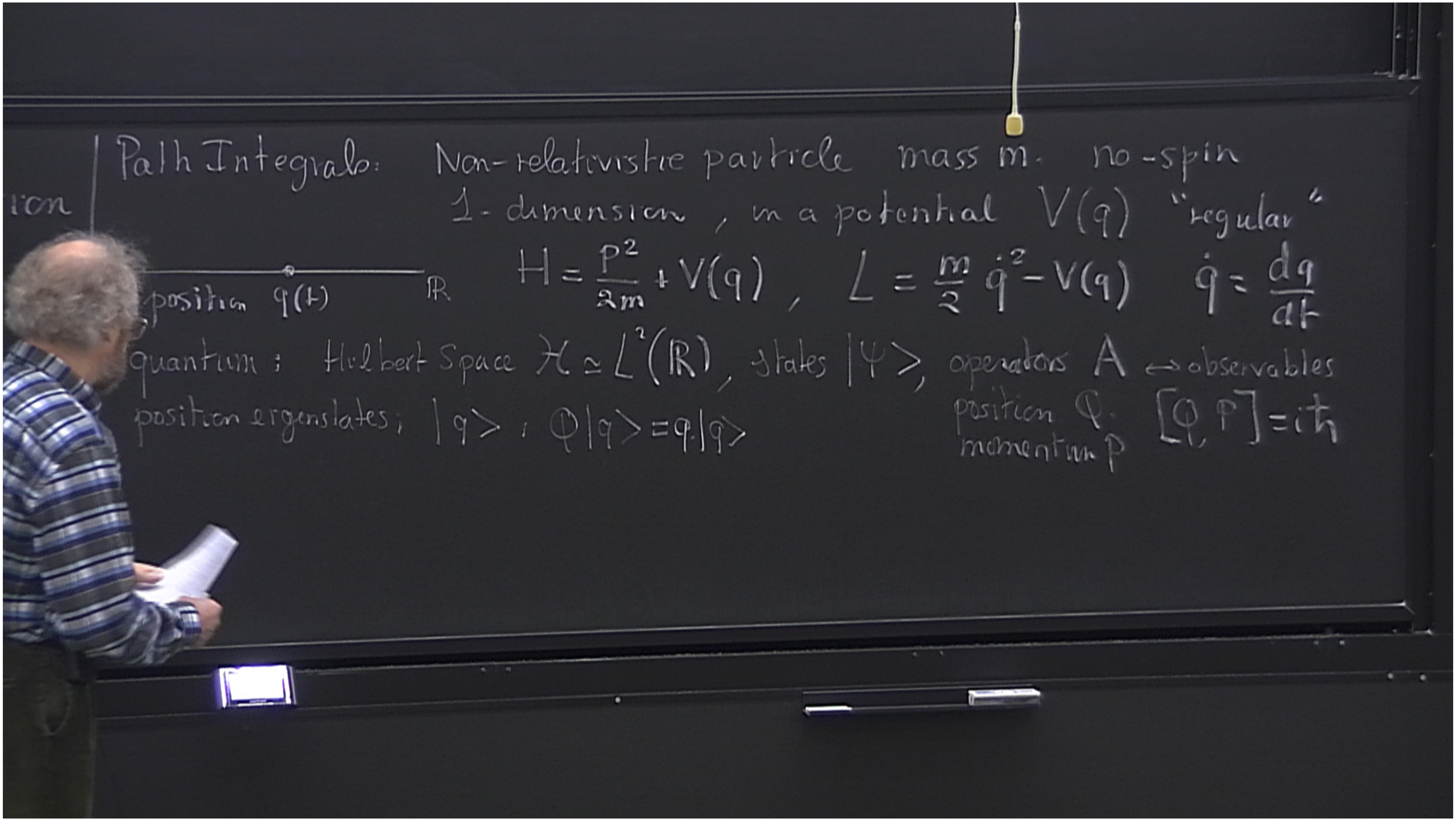


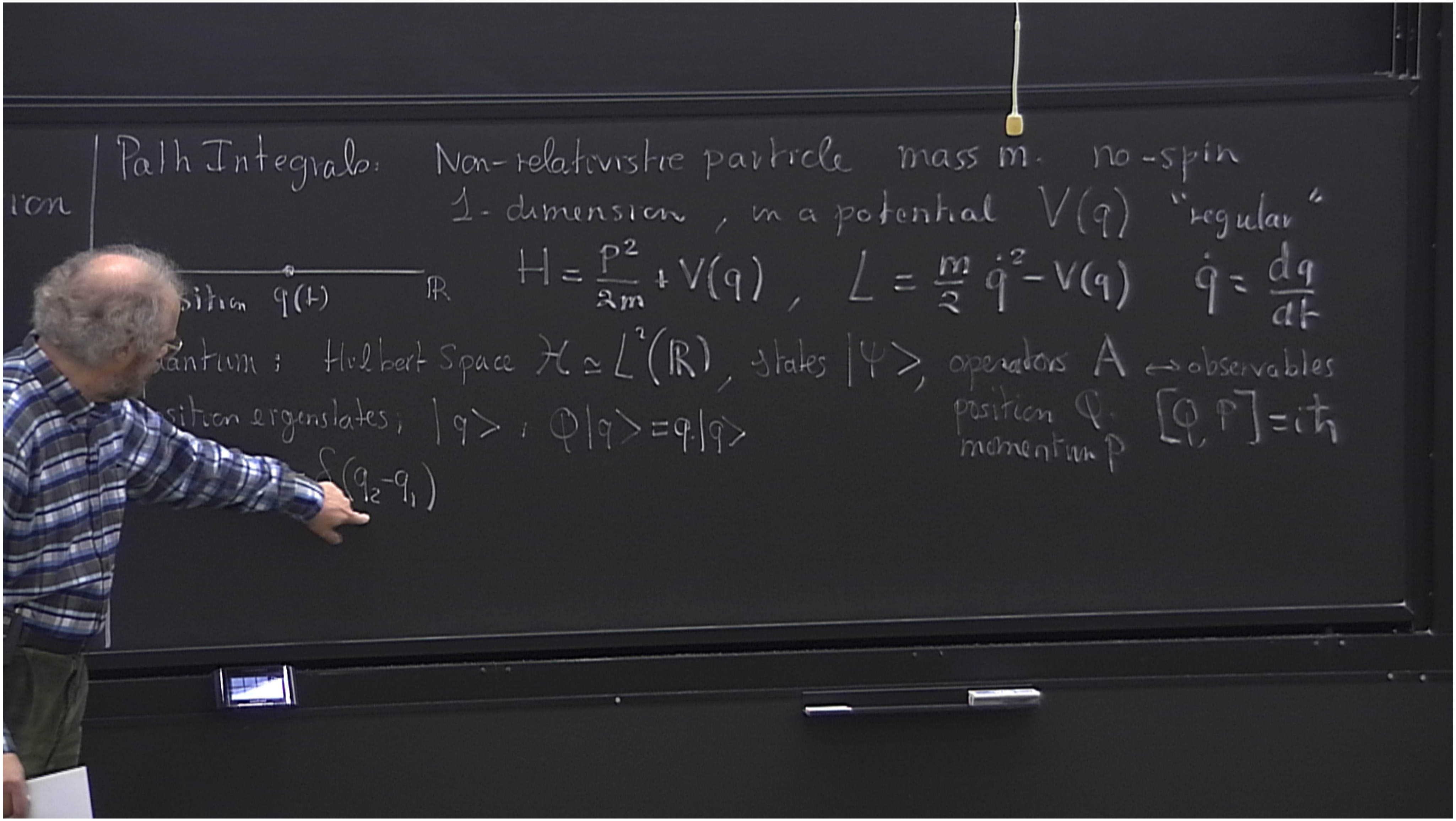
Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
$H = \frac{p^2}{2m} + V(q)$, $L = \frac{m}{2}\dot{q}^2 - V(q)$ $\dot{q} = \frac{dq}{dt}$
position q(t)
$\mathbb{R}$


Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
$H = \frac{p^2}{2m} + V(q)$, $L = \frac{m}{2}\dot{q}^2 - V(q)$ $\dot{q} = \frac{dq}{dt}$
ℝ

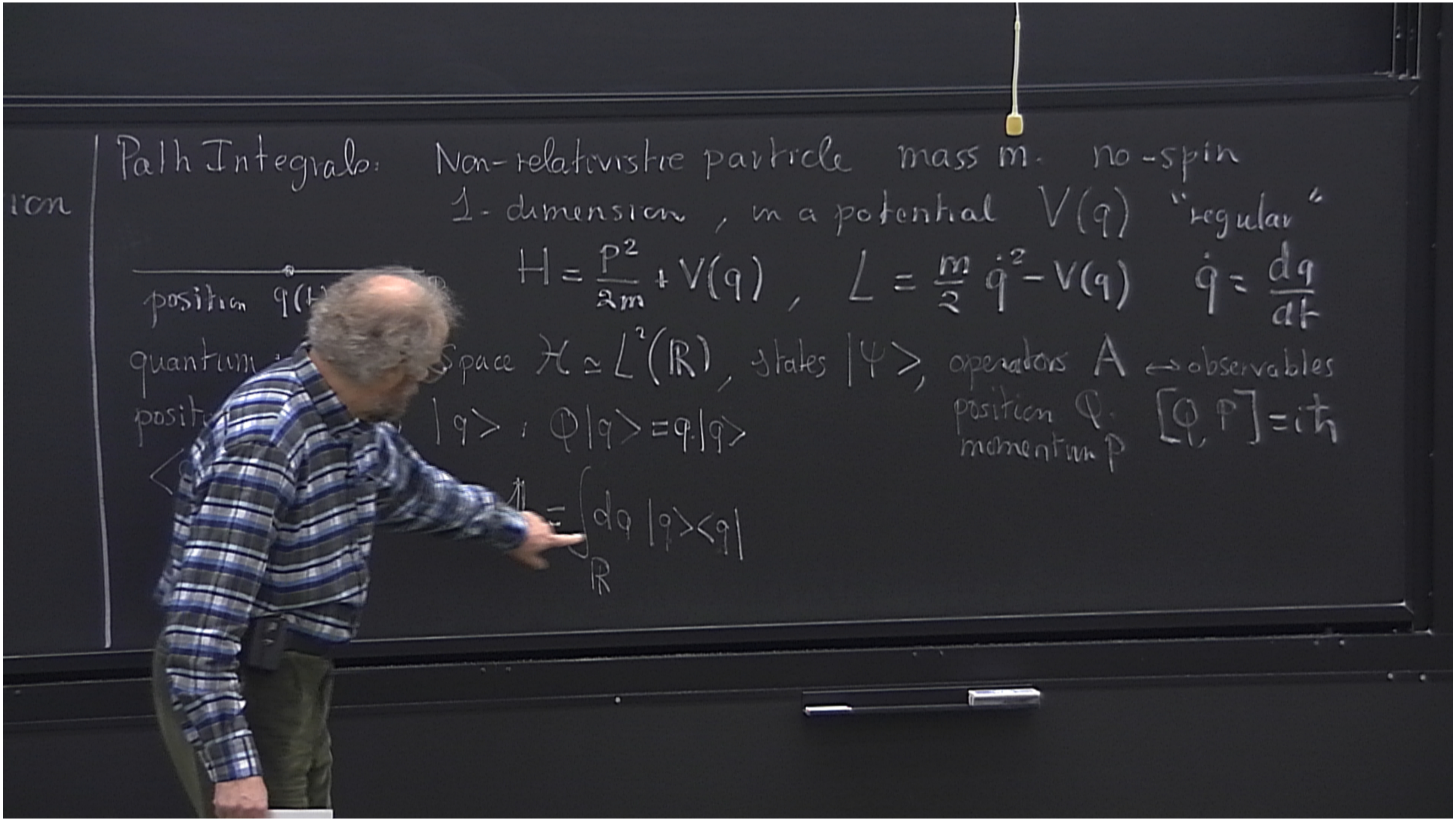

Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
H = p²/2m + V(q), L = m/2 q̇² − V(q) q̇ = dq/dt
q(t) ℝ
Hilbert Space ℋ ≃ L²(ℝ)


Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
position q(t) ℝ
H = p²/2m + V(q), L = m/2 q̇² − V(q) q̇ = dq/dt
quantum: Hilbert Space ℋ ≃ L²(ℝ), states |Ψ>, operators A ↔ observables


ion
Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
position q(t) ℝ
H = p²/2m + V(q), L = m/2 q̇² − V(q) q̇ = dq/dt
quantum: Hilbert Space ℋ ≃ L²(ℝ), states |ψ>, operators A ↔ observables
position eigenstates: |q>, Q|q> = q|q>
position Q
momentum P
[Q, P] = iħ



Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
position q(t) ℝ
$H = \frac{p^2}{2m} + V(q)$, $L = \frac{m}{2}\dot{q}^2 - V(q)$ $\dot{q} = \frac{dq}{dt}$
Quantum: Hilbert Space $\mathcal{H} \simeq L^2(\mathbb{R})$, states $|\psi\rangle$, operators A ↔ observables
position eigenstates: $|q\rangle$, $Q|q\rangle = q|q\rangle$
position Q
momentum P
$[Q, P] = i\hbar$
$\delta(q_2 - q_1)$



Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
position q(t)
$H = \frac{p^2}{2m} + V(q)$, $L = \frac{m}{2}\dot{q}^2 - V(q)$ $\dot{q} = \frac{dq}{dt}$
quantum
space $\mathcal{H} \simeq L^2(\mathbb{R})$, states $|\psi>$, operators A ↔ observables
position Q. momentum P $[Q, P] = i\hbar$
$|q>$; $Q|q> = q|q>$
$\mathbb{1} = \int_{\mathbb{R}} dq\, |q><q|$

ion

Path Integrals: Non-relativistic particle mass m. no-spin

1-dimension, in a potential $V(q)$ "regular"

position $q(t)$ $\mathbb{R}$

$$H = \frac{p^2}{2m} + V(q), \quad L = \frac{m}{2}\dot{q}^2 - V(q) \quad \dot{q} = \frac{dq}{dt}$$

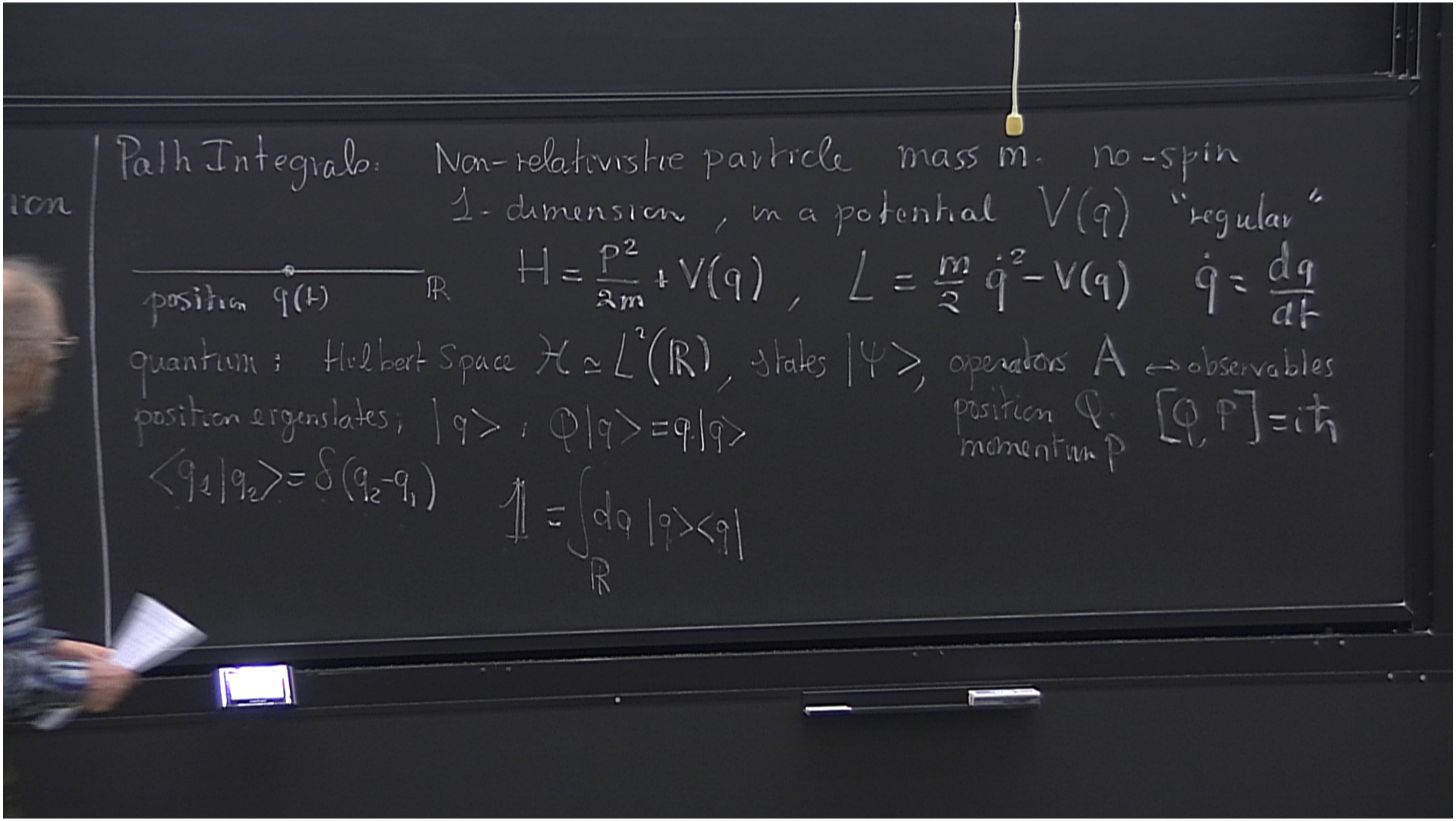
quantum: Hilbert Space $\mathcal{H} \simeq L^2(\mathbb{R})$, states $|\psi\rangle$, operators $A \leftrightarrow$ observables

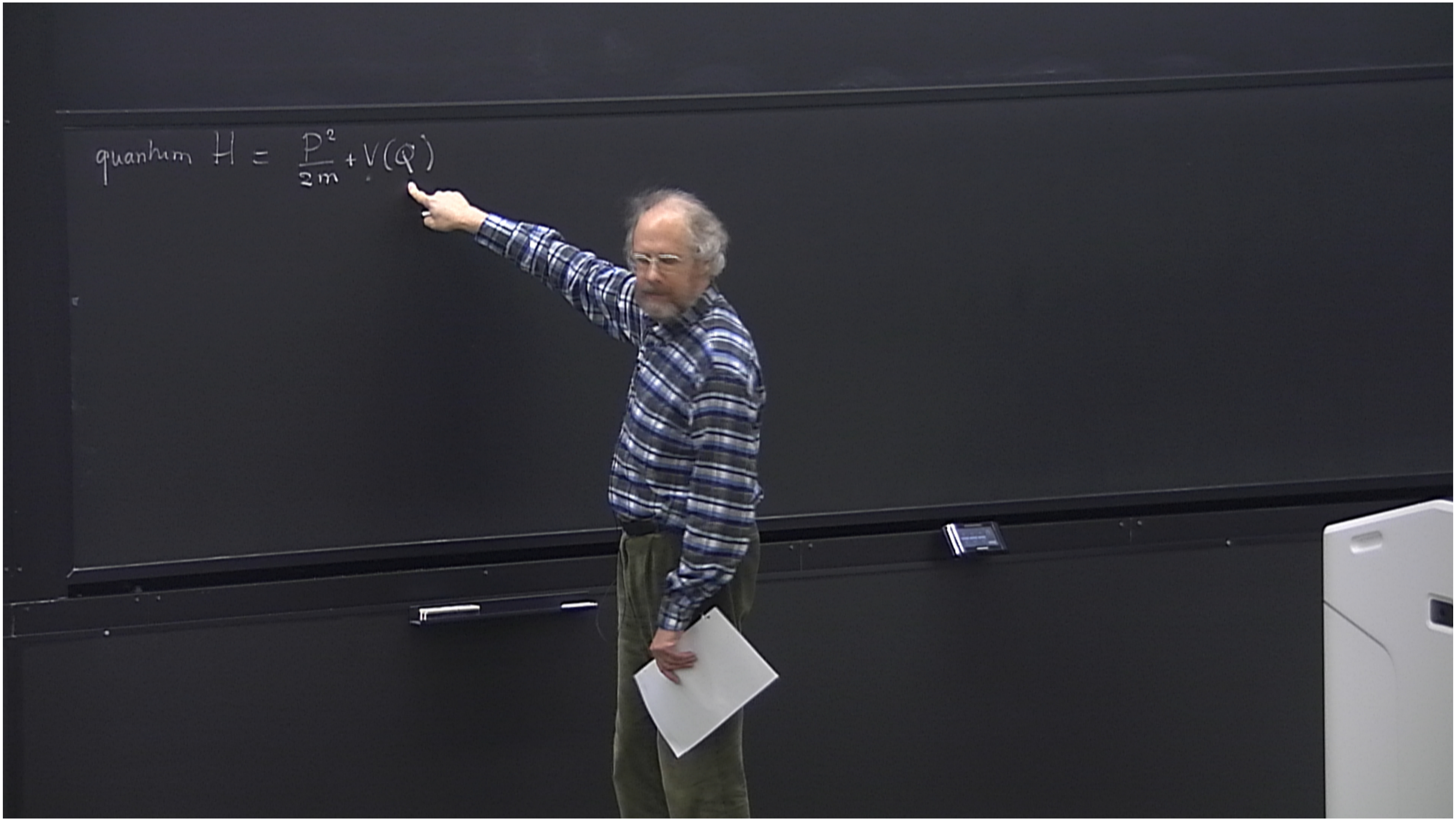
position eigenstates: $|q\rangle$; $Q|q\rangle = q|q\rangle$

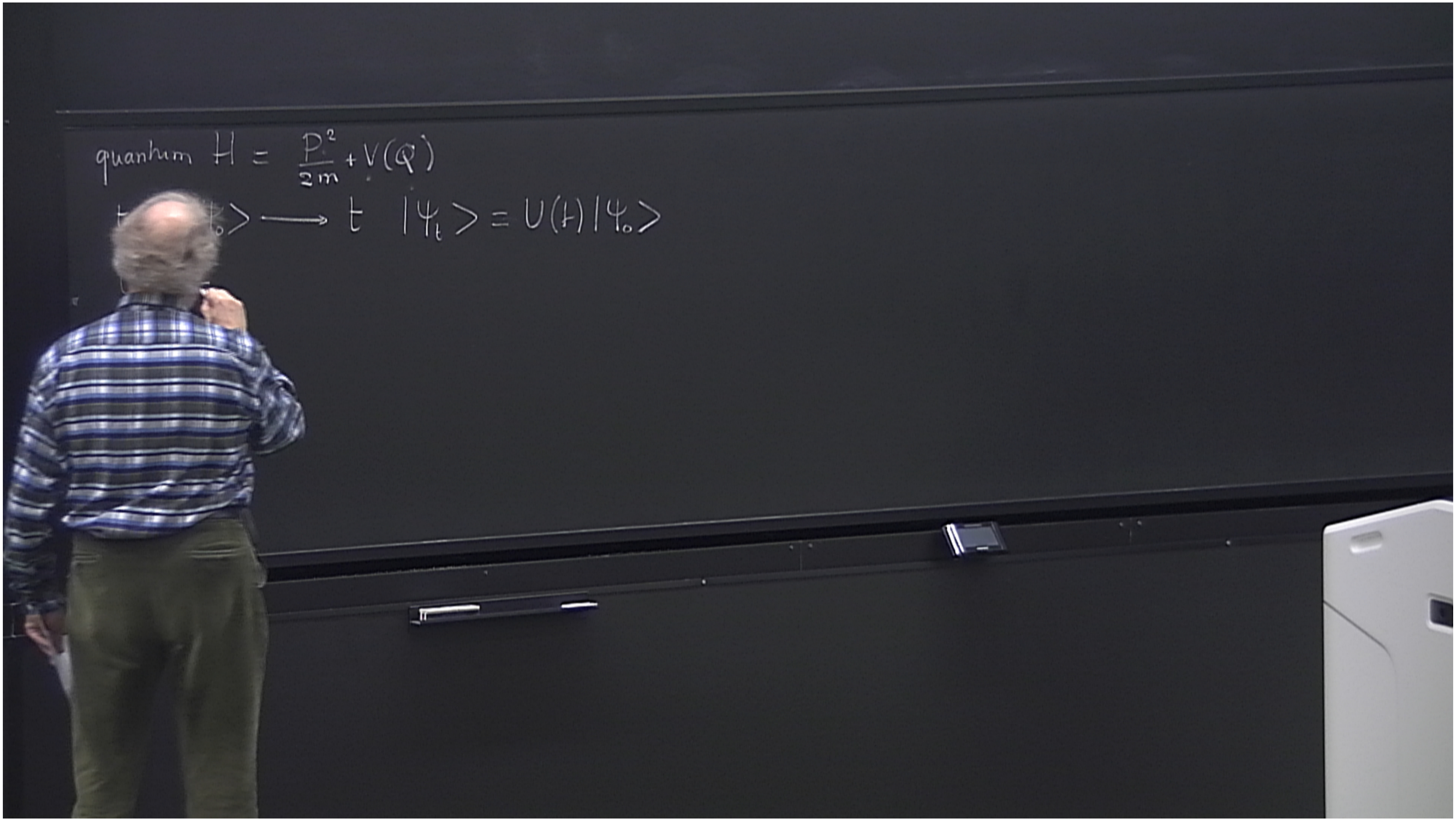
position Q, momentum P: $[Q, P] = i\hbar$

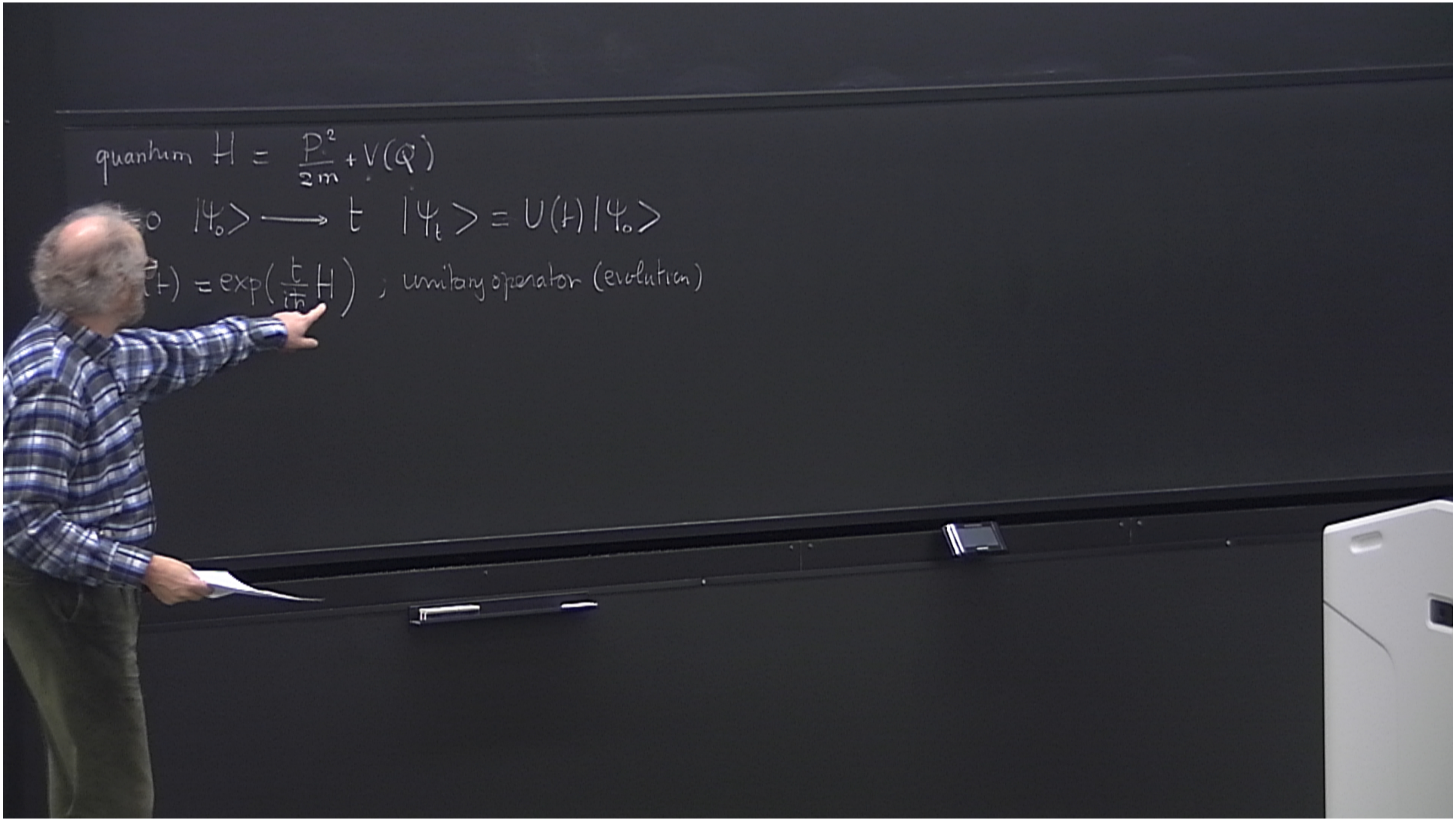
$$\langle q_1 | q_2 \rangle = \delta(q_2 - q_1)$$

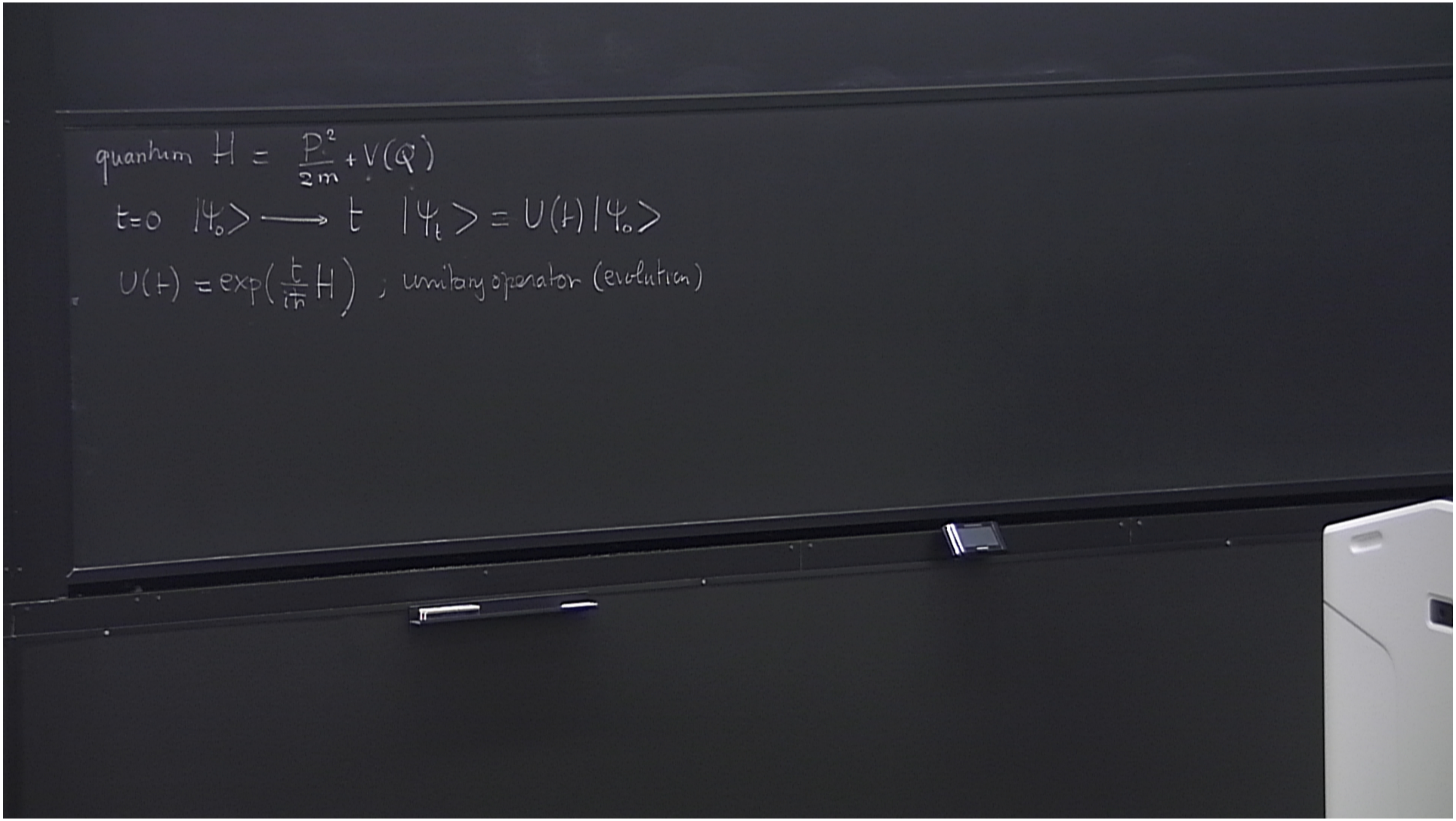
$$\mathbb{1} = \int_{\mathbb{R}} dq\, |q\rangle\langle q|$$

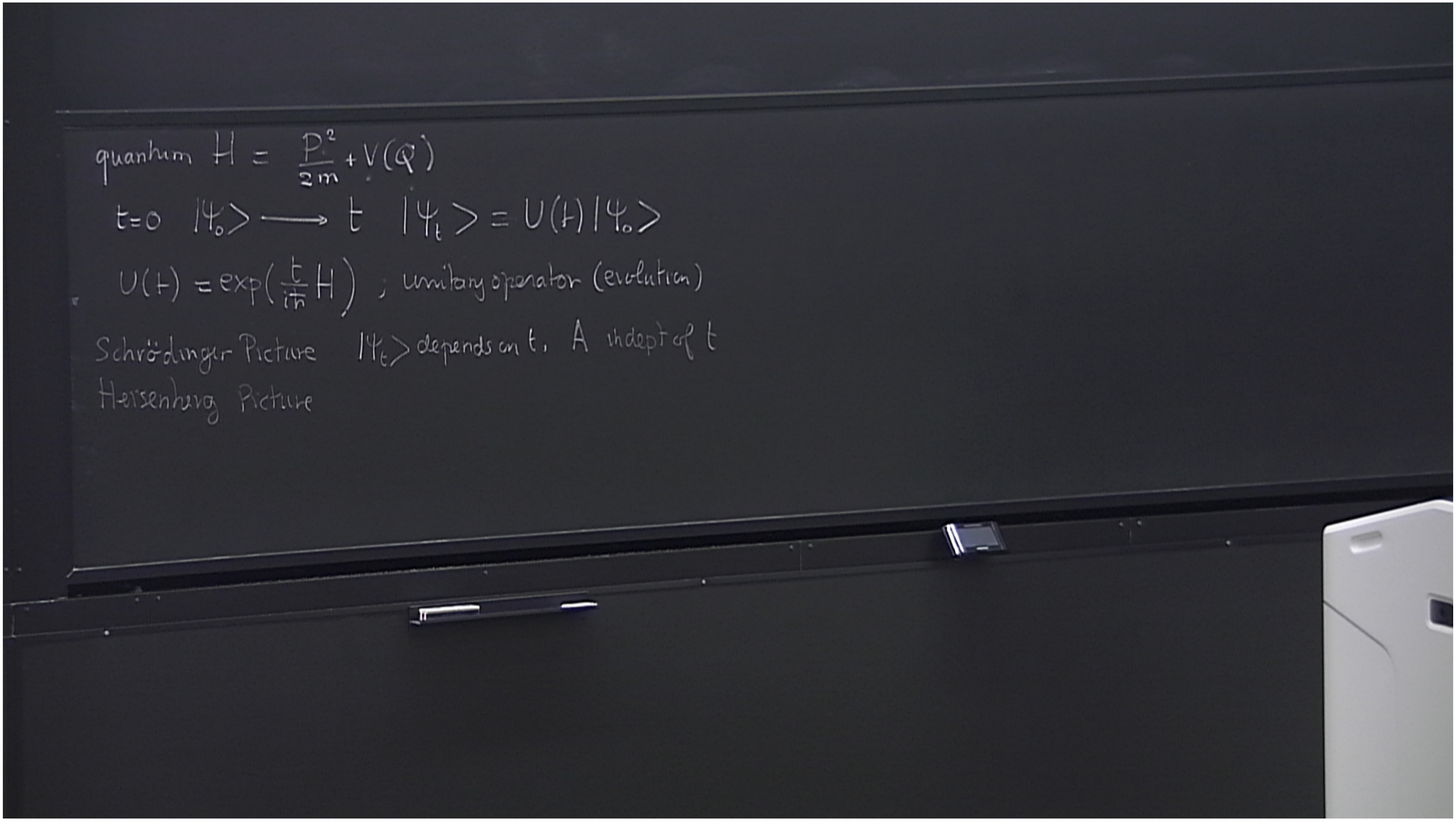

ion
Path Integrals: Non-relativistic particle mass m. no-spin
1-dimension, in a potential V(q) "regular"
position q(t) ℝ
H = p²/2m + V(q), L = m/2 q̇² − V(q)   q̇ = dq/dt
quantum: Hilbert Space ℋ ≃ L²(ℝ), states |ψ>, operators A ↔ observables
position eigenstates; |q> ; Q|q> = q|q>
position Q, momentum P   [Q, P] = iℏ
<q₁|q₂> = δ(q₂ − q₁)
𝟙 = ∫ℝ dq |q><q|

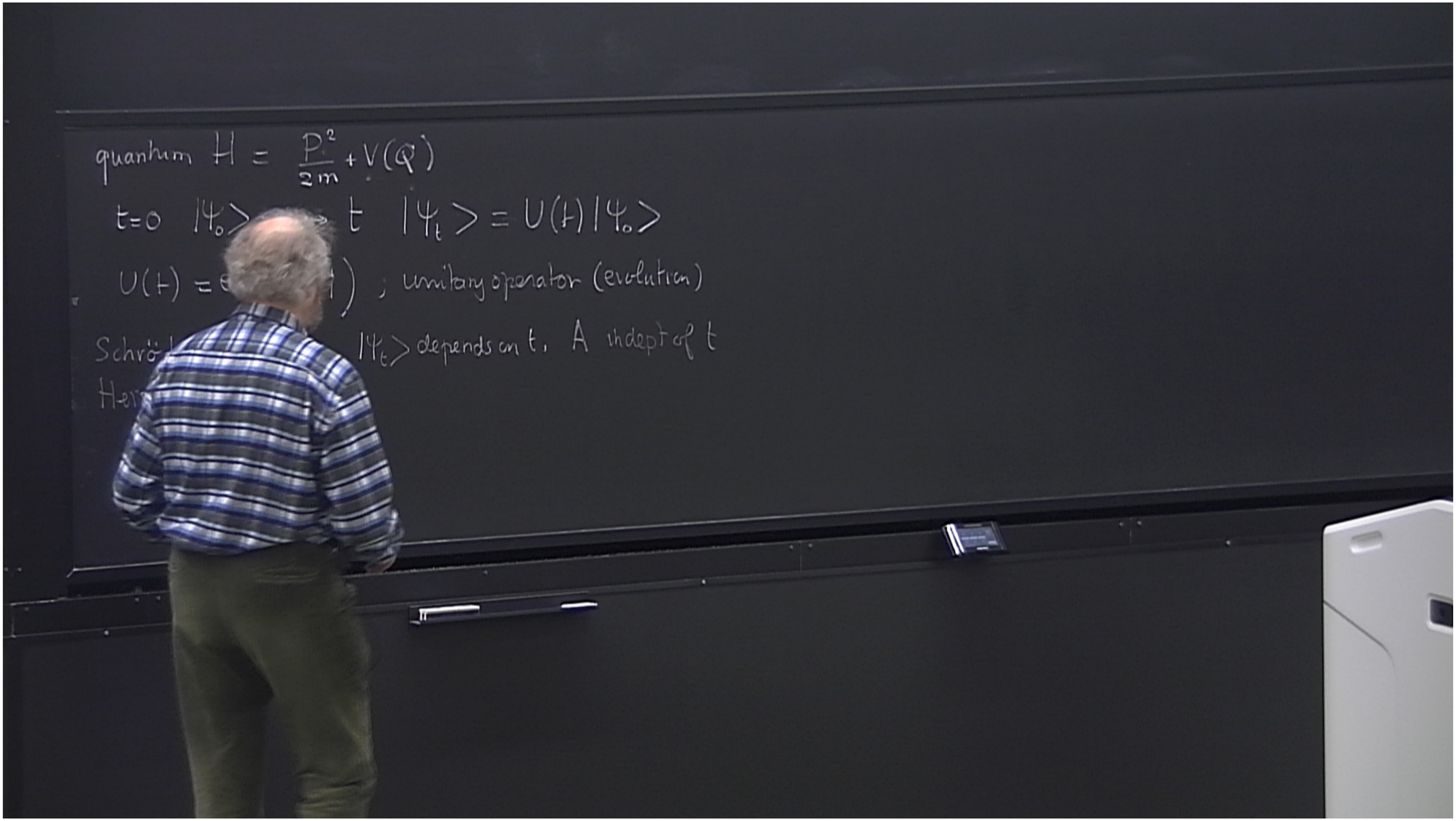

quantum $H = \frac{P^2}{2m} + V(Q)$

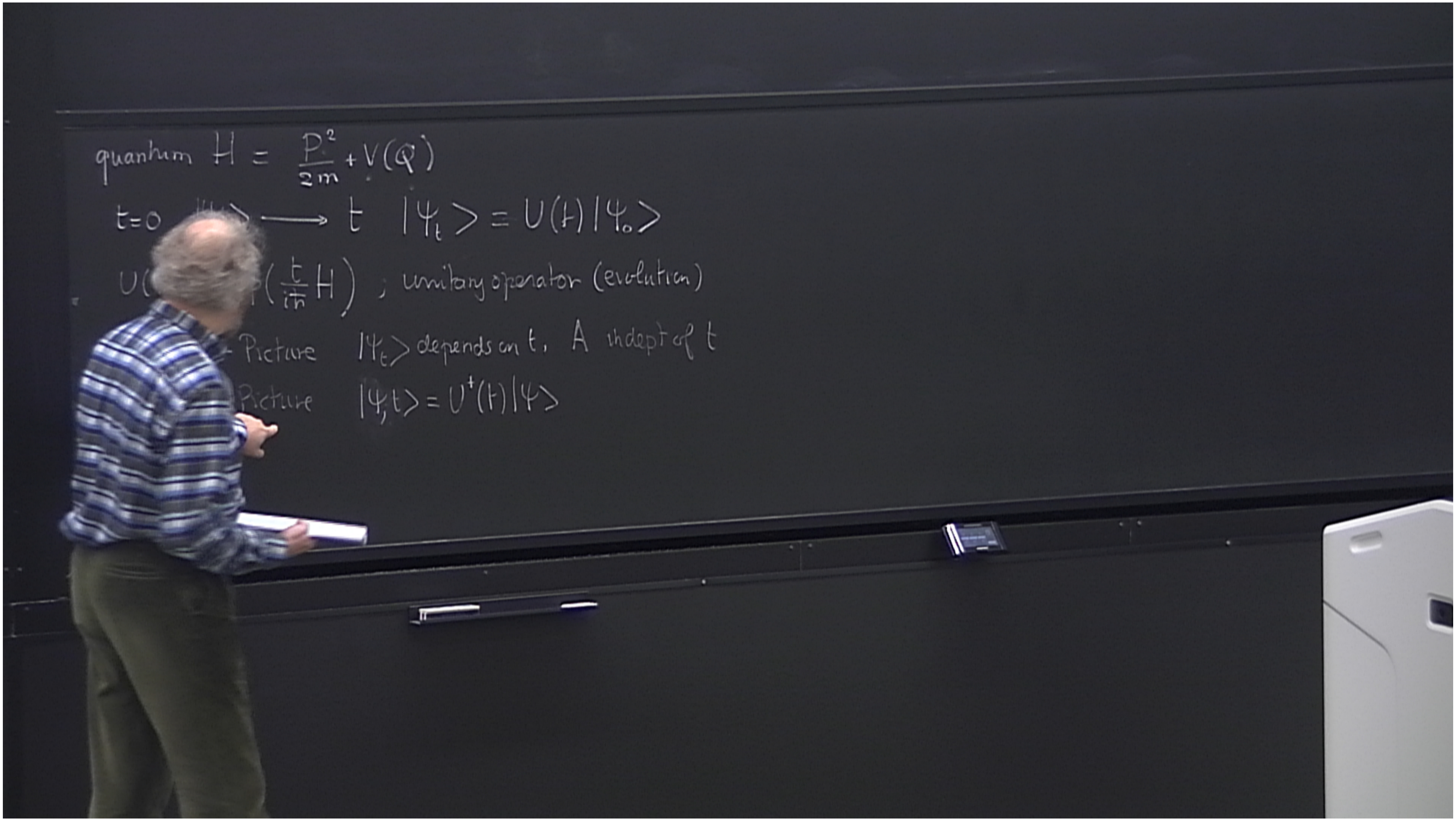

quantum H = P²/2m + V(Q)
> ⟶ t |ψ_t > = U(t) |ψ_0 >

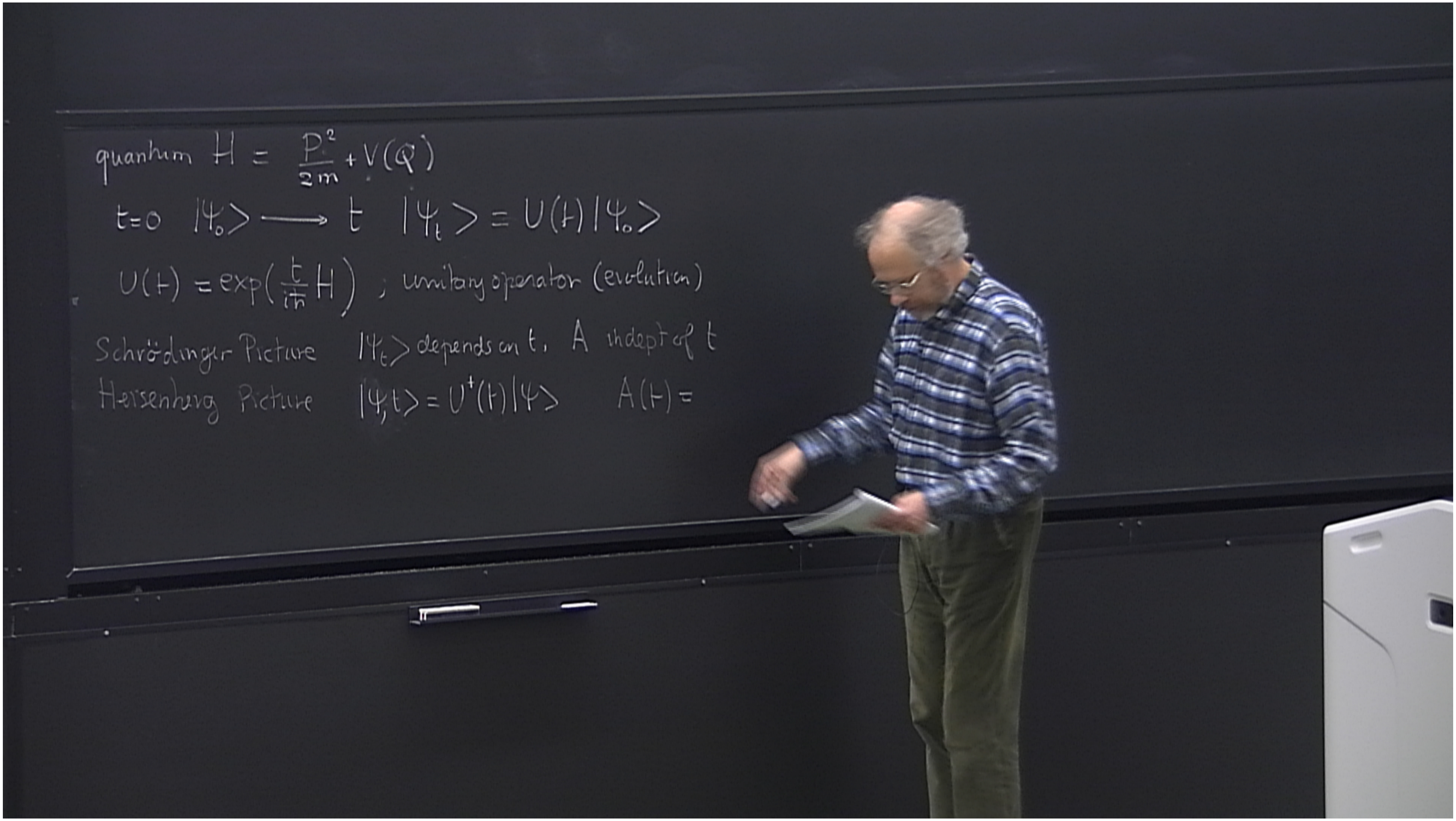

quantum $H = \frac{P^2}{2m} + V(Q)$
$|\Psi_0\rangle \longrightarrow t \quad |\Psi_t\rangle = U(t)|\Psi_0\rangle$
$= \exp(\frac{t}{i\hbar} H)$; unitary operator (evolution)

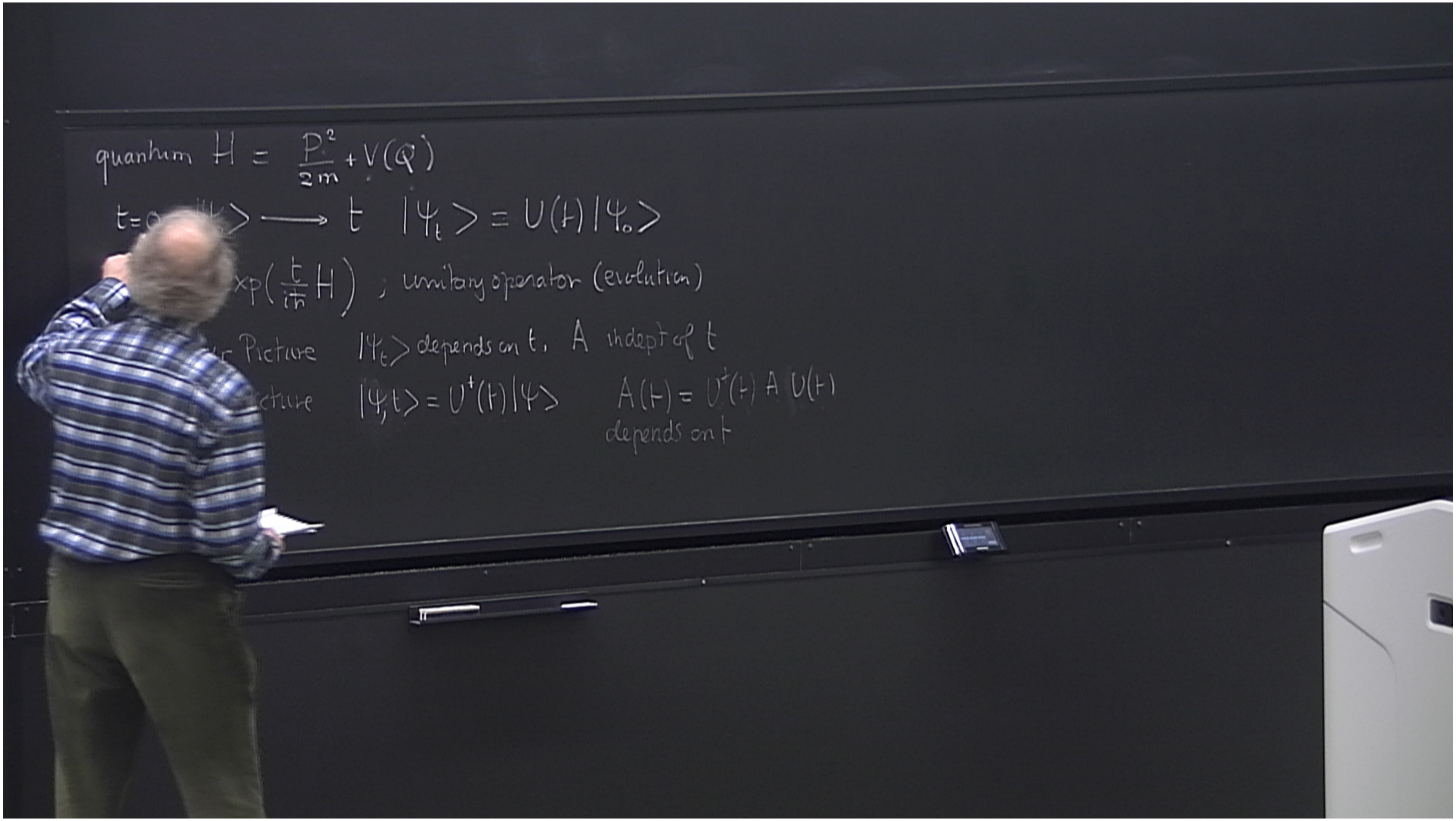

quantum $H = \frac{P^2}{2m} + V(Q)$
$t=0 \quad |\psi_0\rangle \longrightarrow t \quad |\psi_t\rangle = U(t)|\psi_0\rangle$
$U(t) = \exp\left(\frac{t}{i\hbar} H\right)$; unitary operator (evolution)

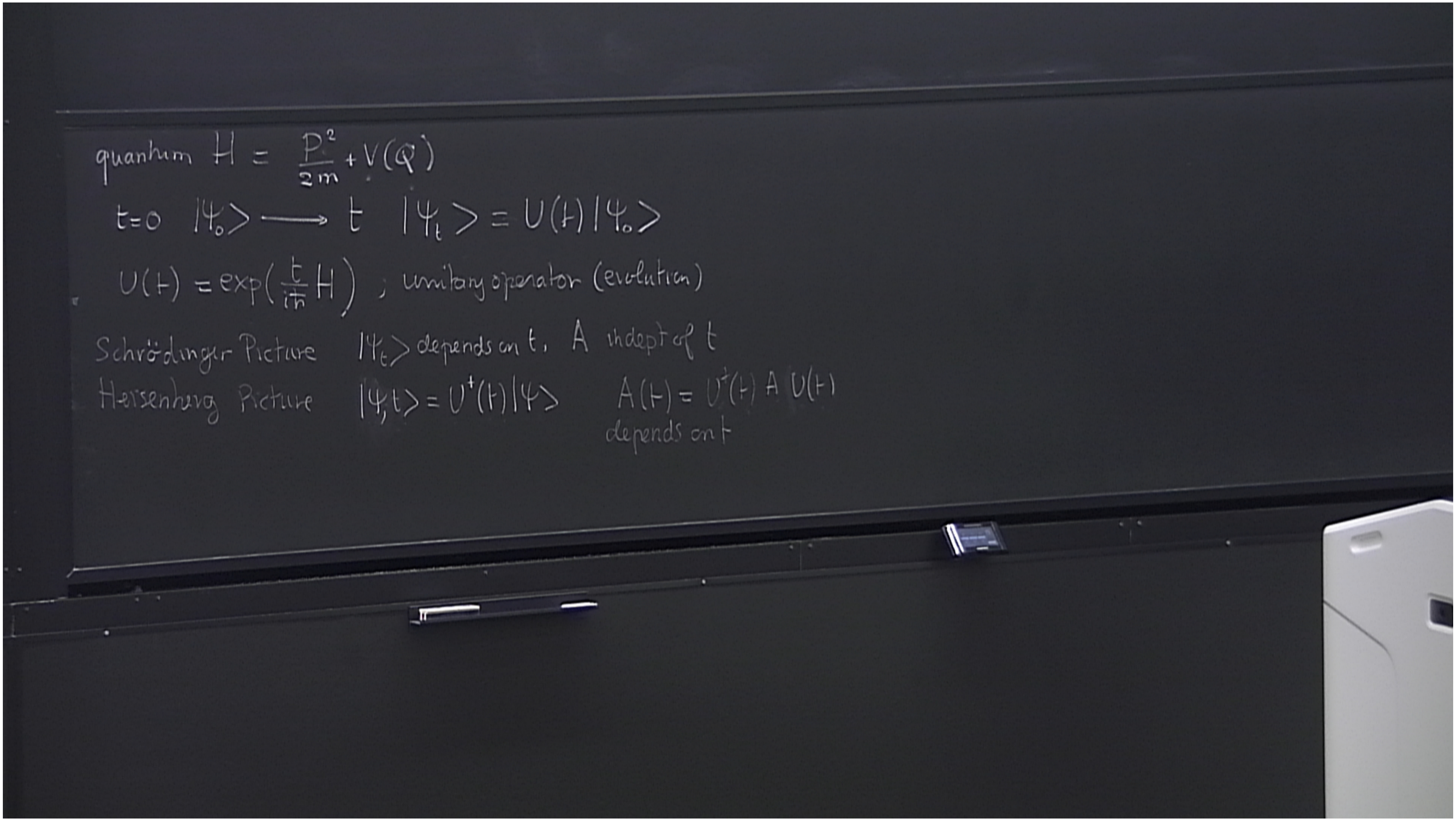

quantum H = P²/2m + V(Q)
t=0 |ψ₀> ⟶ t |ψₜ> = U(t)|ψ₀>
U(t) = exp(t/iħ H) ; unitary operator (evolution)
Schrödinger Picture |ψₜ> depends on t, A indept of t
Heisenberg Picture


quantum $H = \frac{P^2}{2m} + V(Q)$
$t=0$ $|\psi_0\rangle$
t $|\psi_t\rangle = U(t)|\psi_0\rangle$
$U(t) =$
; unitary operator (evolution)
Schrö
$|\psi_t\rangle$ depends on t, A indept of t

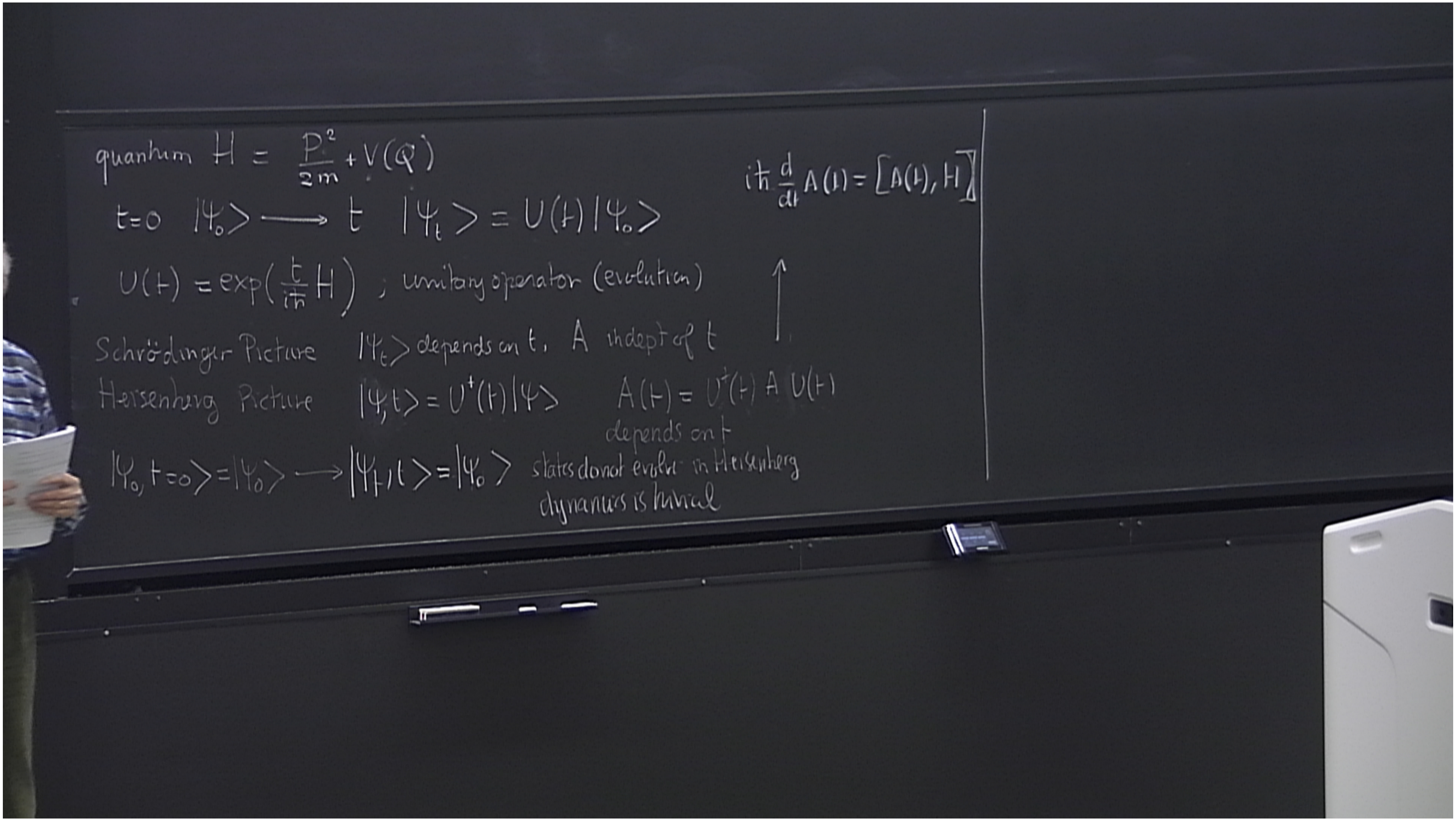

quantum H = P²/2m + V(Q)
t=0 → t |Ψt> = U(t)|Ψ0>
(t/iħ H) ; unitary operator (evolution)
Picture |Ψt> depends on t, A indep't of t
Picture |Ψ,t> = U†(t)|Ψ>

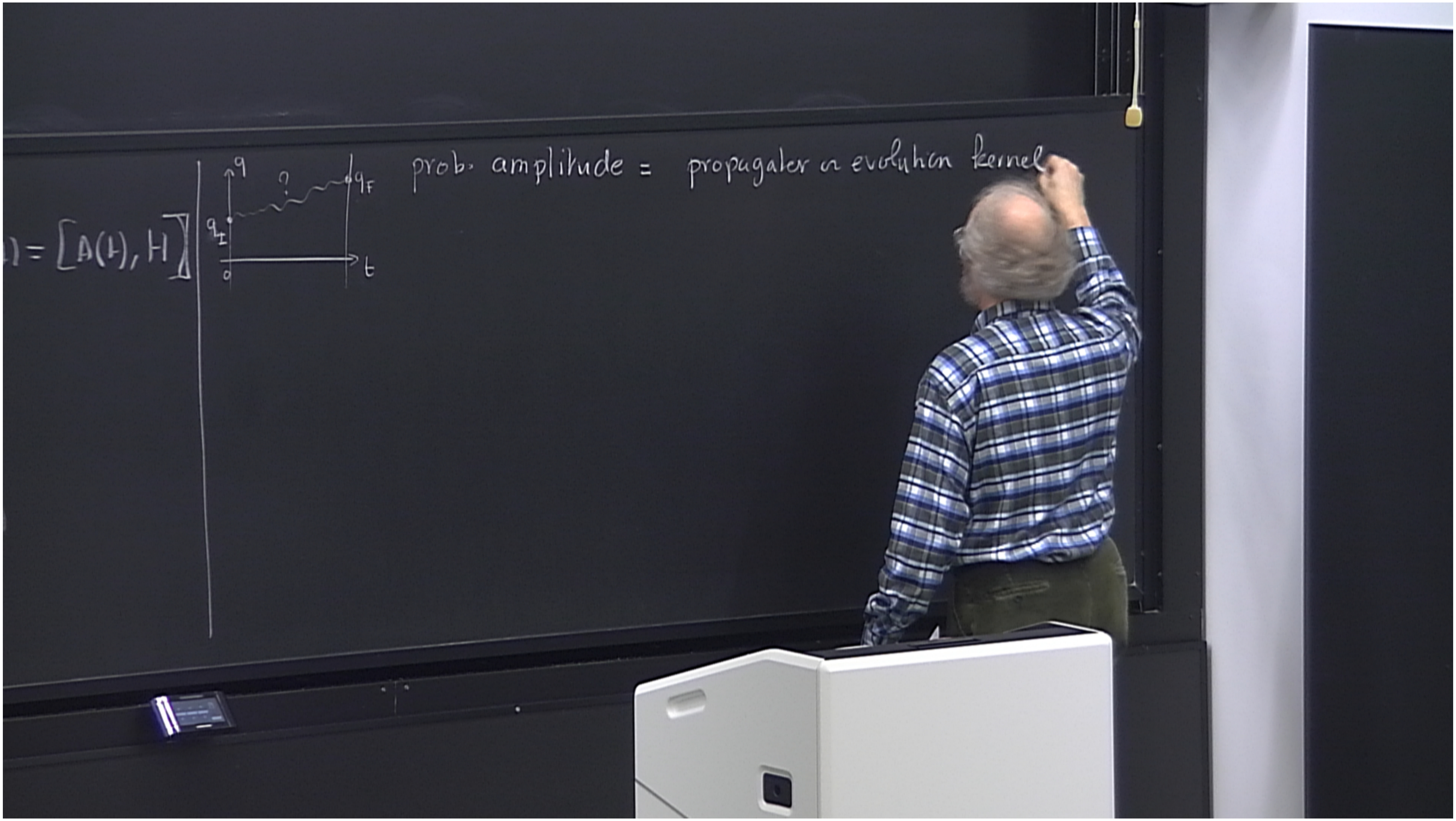

quantum H = P²/2m + V(Q)
t=0 |ψ₀> ⟶ t |ψ_t> = U(t)|ψ₀>
U(t) = exp(t/iħ H) ; unitary operator (evolution)
Schrödinger Picture |ψ_t> depends on t, A indept of t
Heisenberg Picture |ψ,t> = U†(t)|ψ> A(t) =

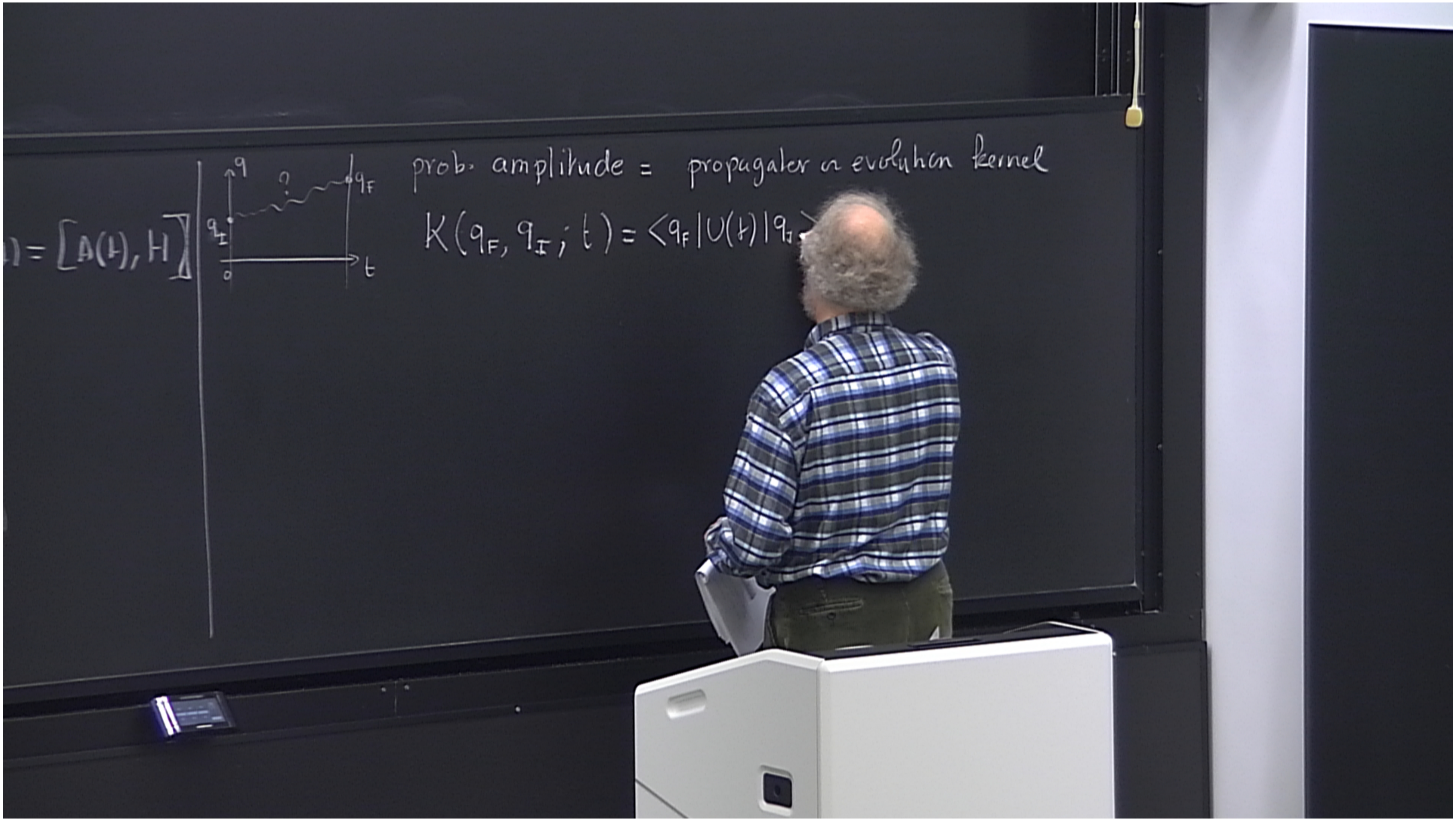

quantum H = P²/2m + V(Q)
t=0 |ψ> ⟶ t |ψ_t> = U(t)|ψ_0>
xp(t/iħ H) ; unitary operator (evolution)
Picture |ψ_t> depends on t, A indep't of t
cture |ψ,t> = U†(t)|ψ> A(t) = U†(t) A U(t)
depends on t

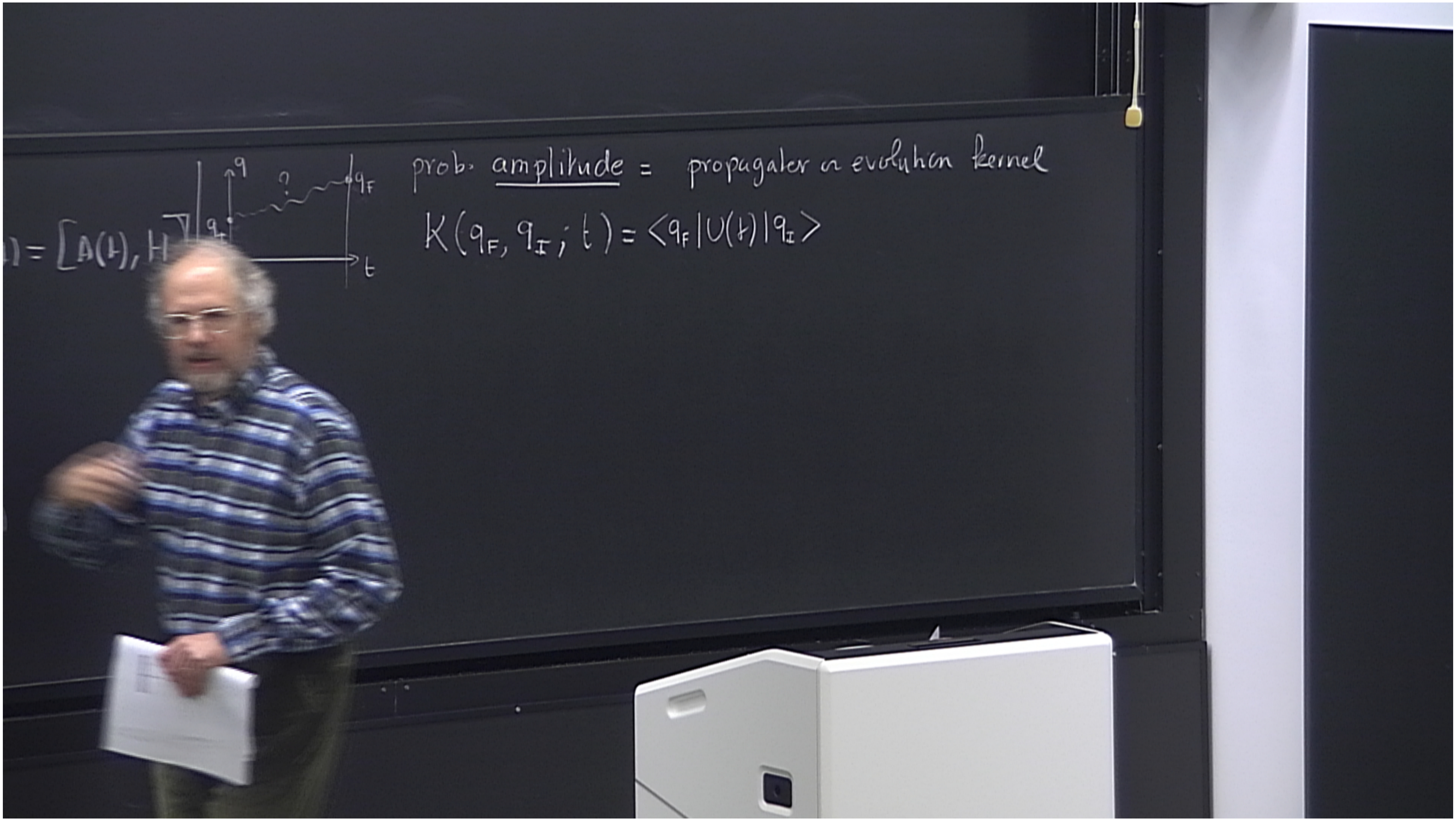


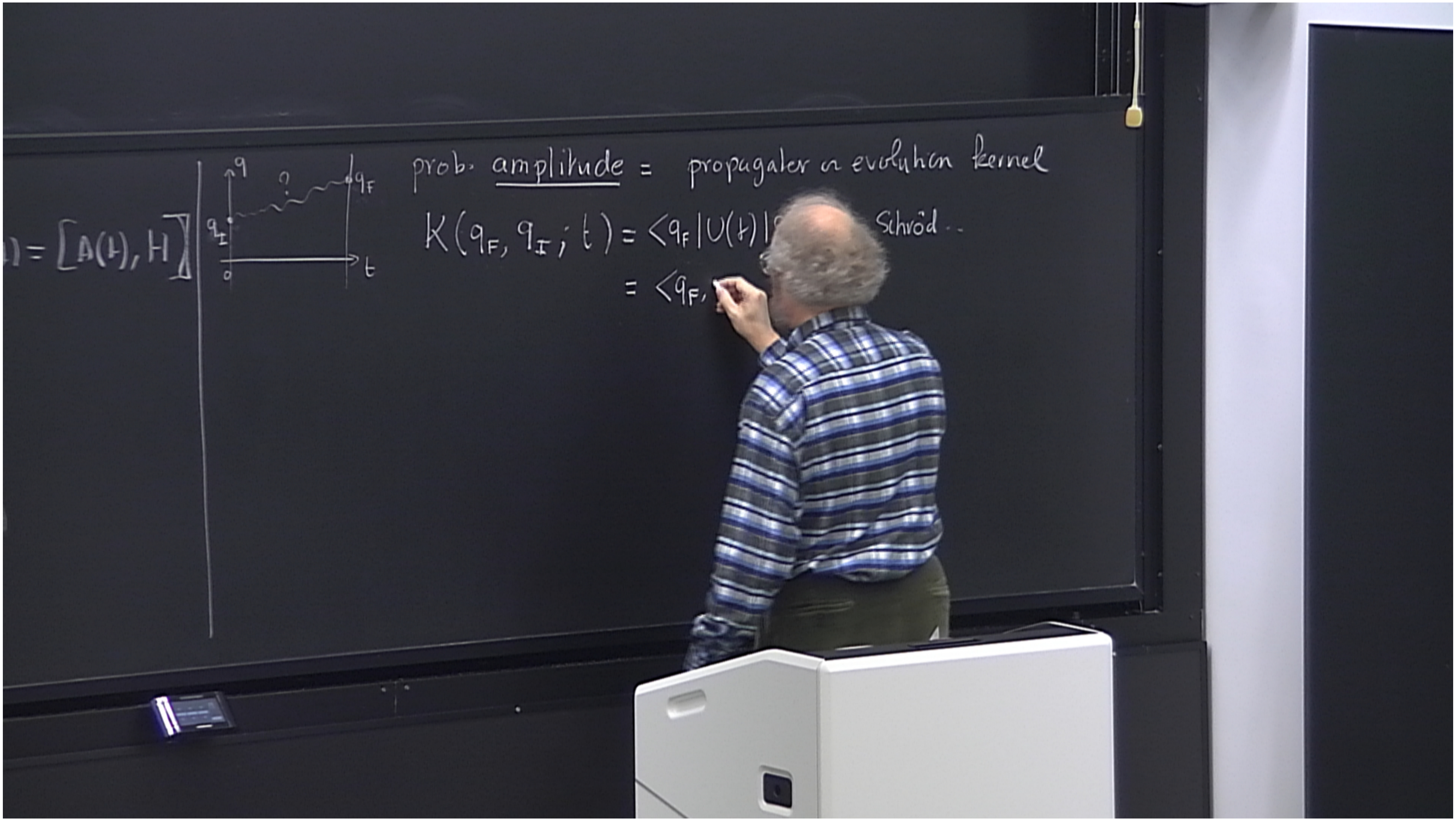
quantum $H = \frac{P^2}{2m} + V(Q)$
$t=0 \quad |\psi_0\rangle \longrightarrow t \quad |\psi_t\rangle = U(t)|\psi_0\rangle$
$U(t) = \exp\left(\frac{t}{i\hbar} H\right)$; unitary operator (evolution)
Schrödinger Picture $|\psi_t\rangle$ depends on t, A indep't of t
Heisenberg Picture $|\psi, t\rangle = U^{\dagger}(t)|\psi\rangle$ $A(t) = U^{\dagger}(t)\, A\, U(t)$
depends on t

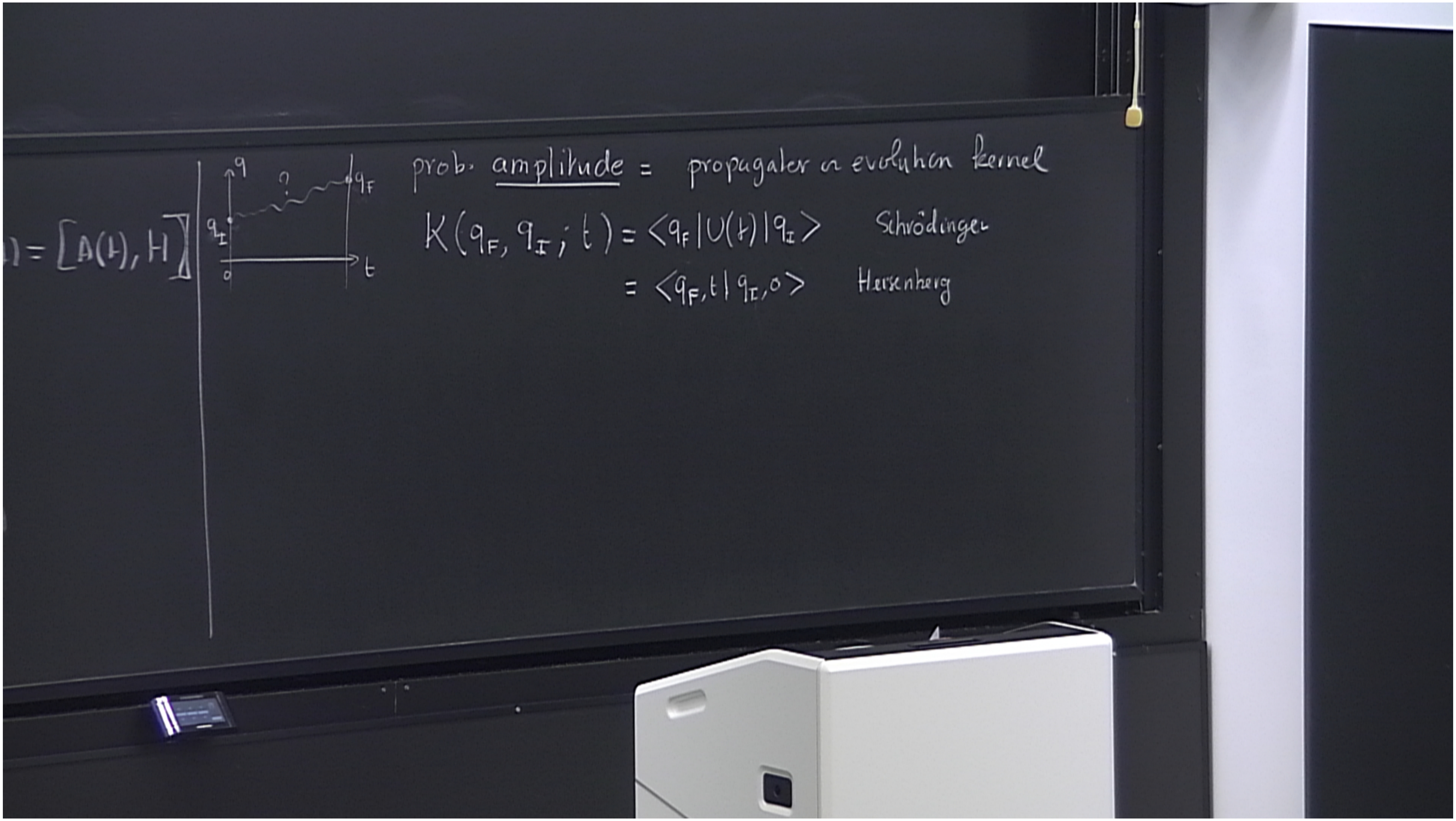

quantum $H = \frac{P^2}{2m} + V(Q)$
$i\hbar \frac{d}{dt} A(t) = [A(t), H]$
$t=0 \quad |\psi_0\rangle \longrightarrow t \quad |\psi_t\rangle = U(t)|\psi_0\rangle$
$U(t) = \exp(\frac{t}{i\hbar} H)$; unitary operator (evolution)
Schrödinger Picture $|\psi_t\rangle$ depends on t, A indep t of t
Heisenberg Picture $|\psi, t\rangle = U^\dagger(t)|\psi\rangle$ $A(t) = U^\dagger(t) A U(t)$
depends on t
$|\psi_0, t=0\rangle = |\psi_0\rangle \longrightarrow |\psi_t, t\rangle = |\psi_0\rangle$ states do not evolve in Heisenberg
dynamics is trivial

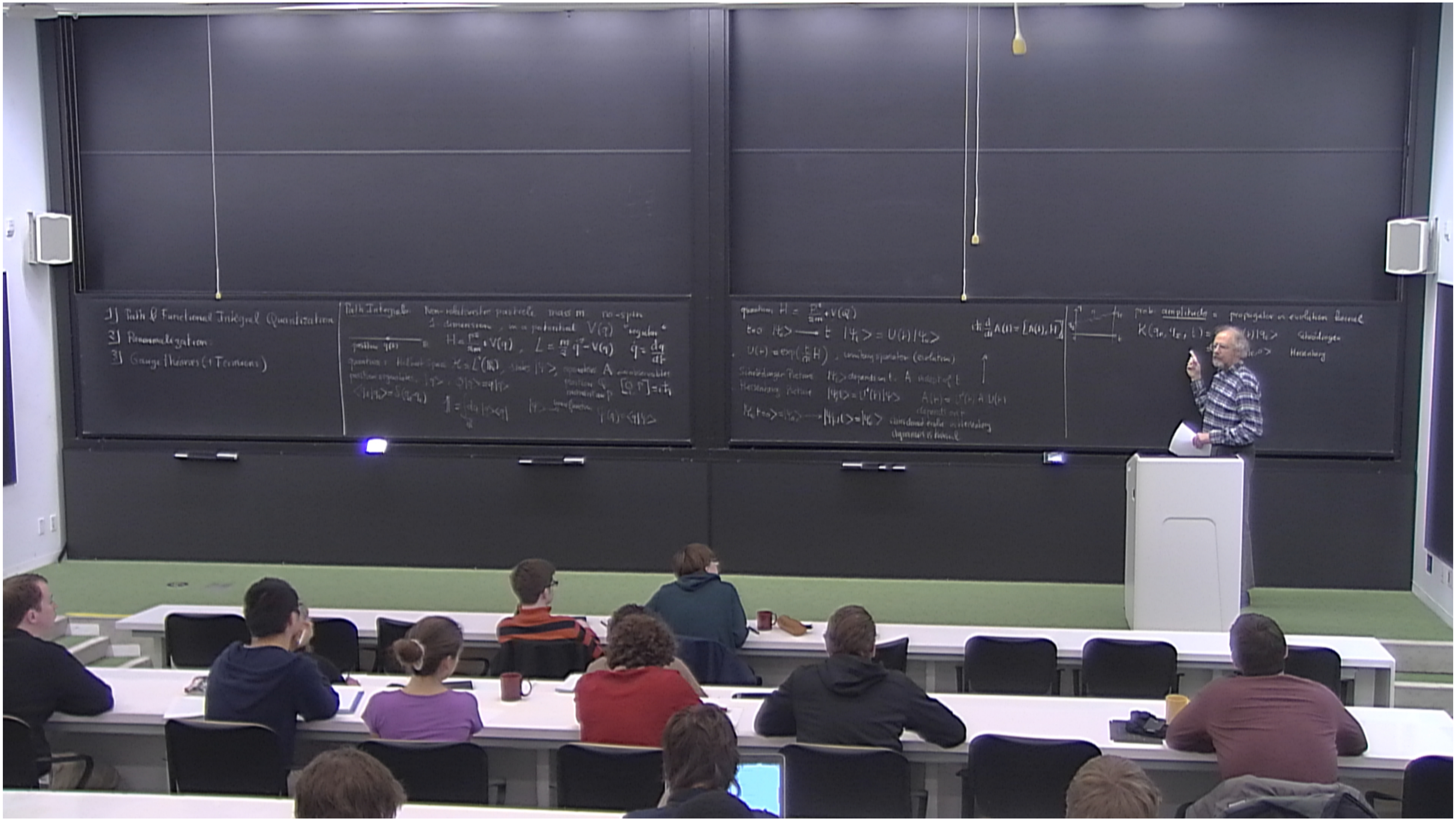

= [A(t), H]
q
?
q_F
q_I
0
t
prob. amplitude = propagator or evolution kernel

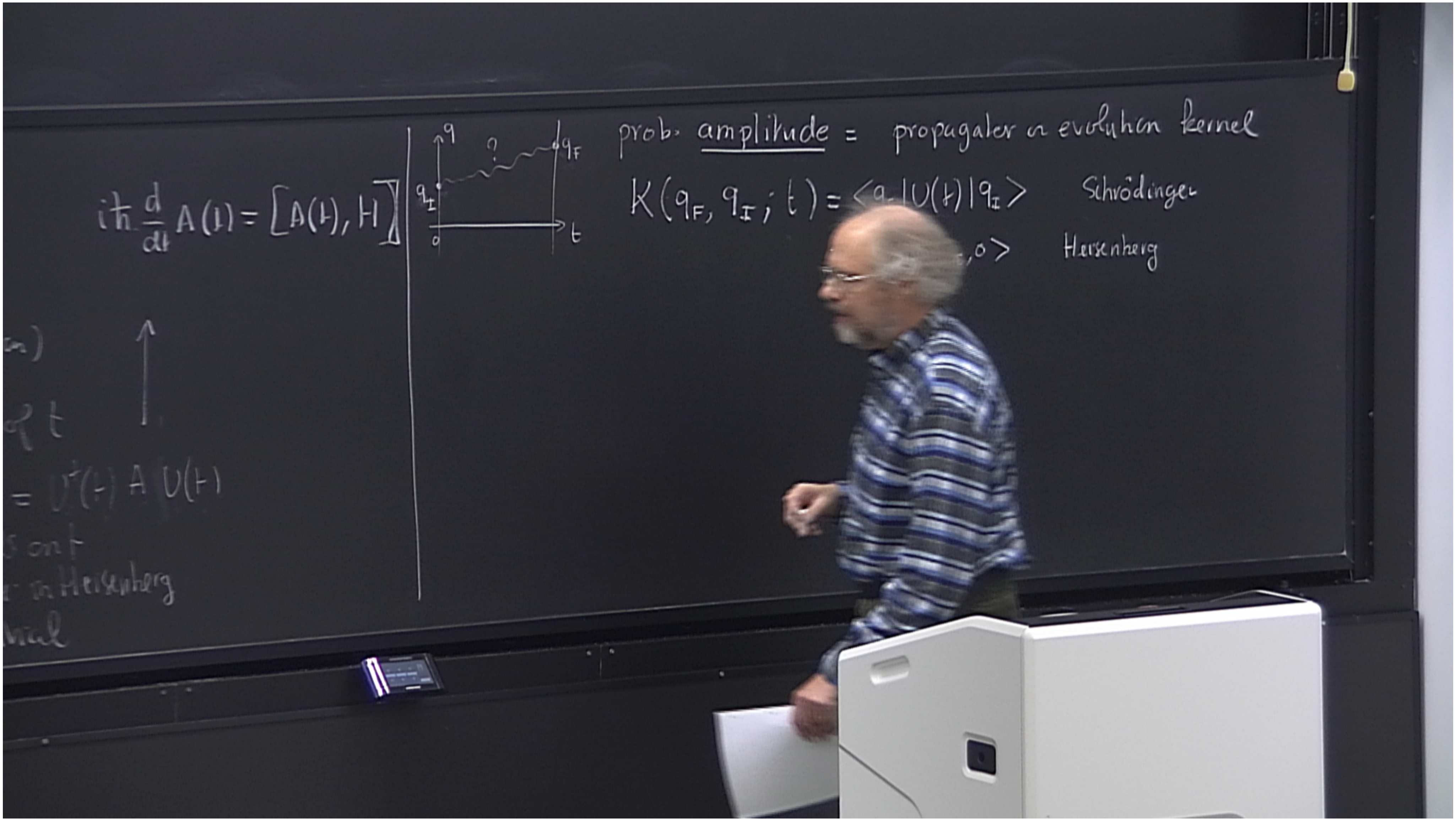

= [A(t), H]
q
?
q_F
q_I
0
t
prob. amplitude = propagator or evolution kernel
K(q_F, q_I ; t) = <q_F | U(t) | q_I

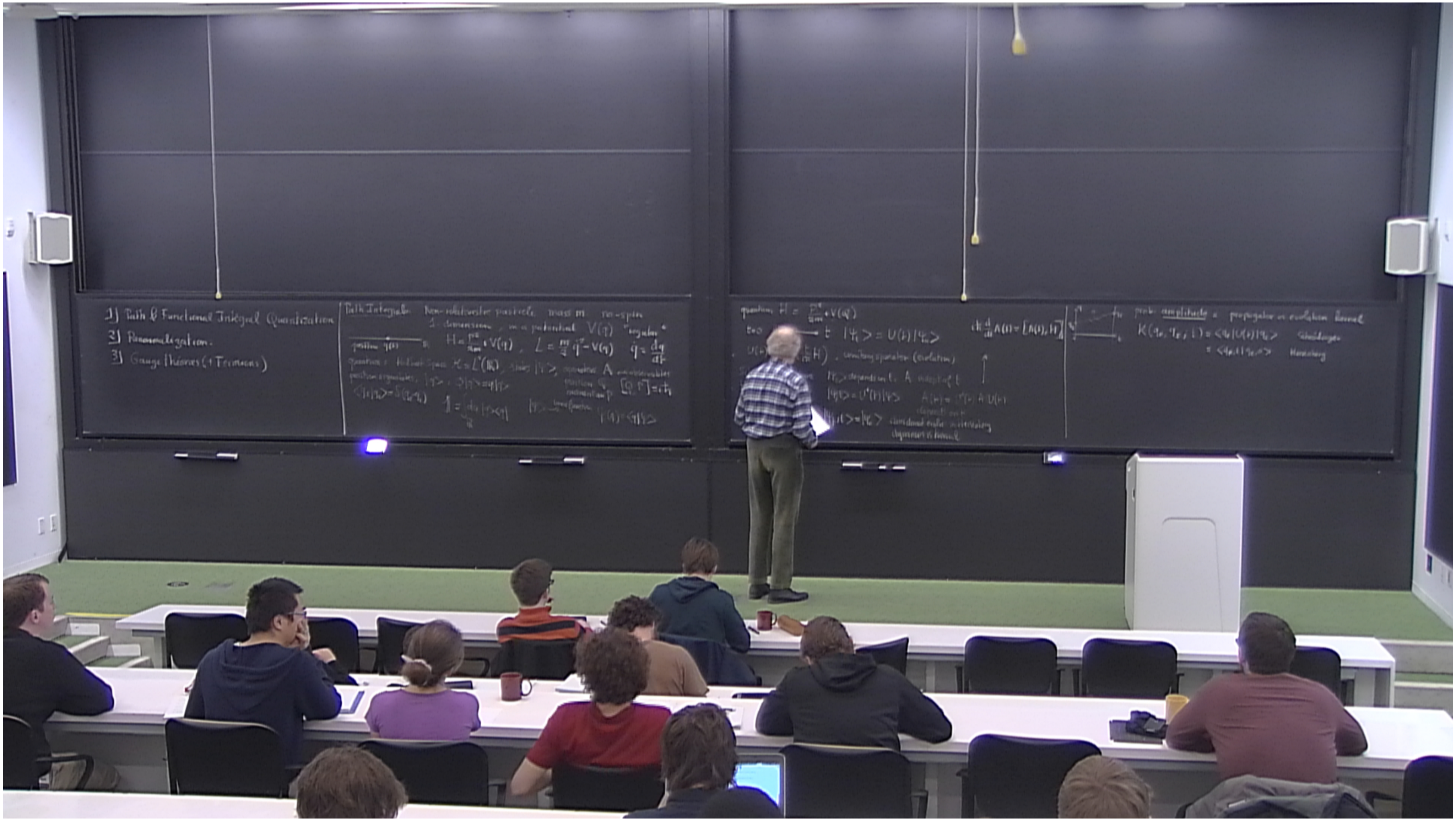

prob. amplitude = propagator or evolution kernel
K(qF, qI; t) = <qF|U(t)|qI>
= [A(t), H
q
qF
?
t


prob. amplitude = propagator or evolution kernel
K(qF, qI ; t) = <qF|U(t)|
Schröd..
= <qF,
= [A(t), H]
q
qF
qI
t

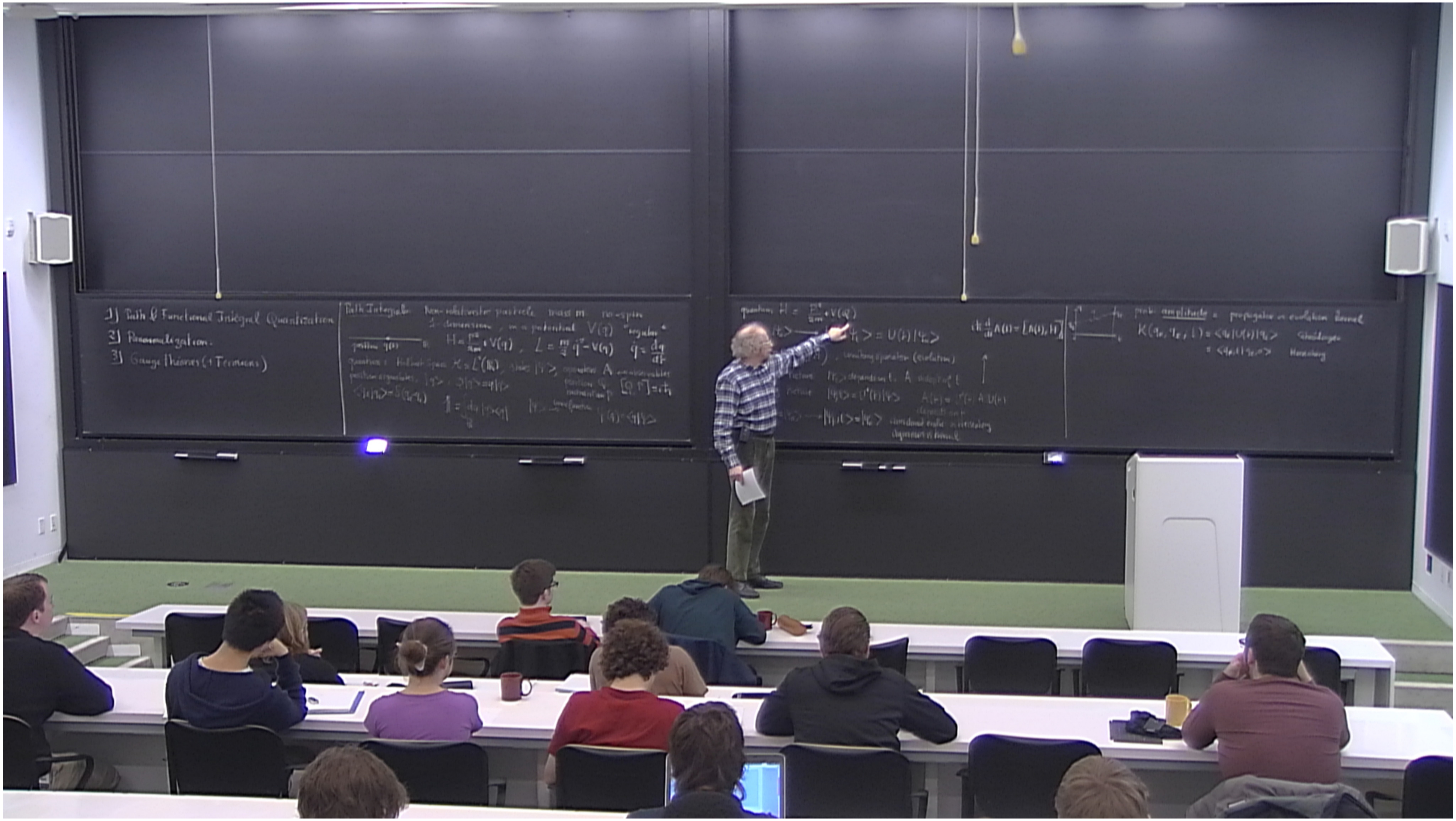

= [A(t), H]
q
?
q_F
q_I
0
t
prob. amplitude = propagator or evolution kernel
K(q_F, q_I ; t) = <q_F|U(t)|q_I>
Schrödinger
= <q_F,t|q_I,0>
Heisenberg

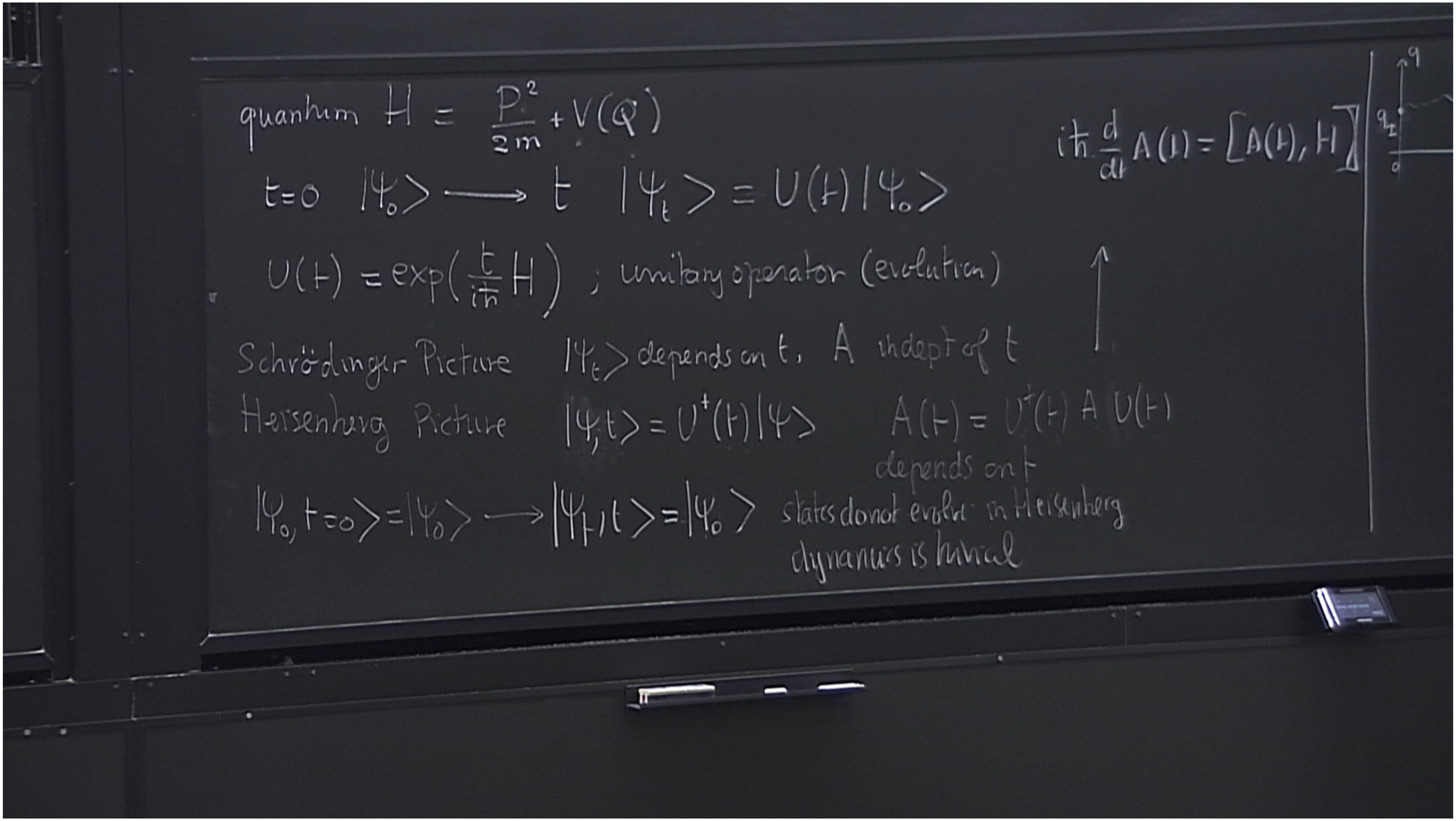

1) Path & Functional Integral Quantization
2) Renormalization
3) Gauge Theories (+ Fermions)
Path Integrals
Schrödinger Picture
Heisenberg Picture
prob. amplitude = propagator or evolution kernel

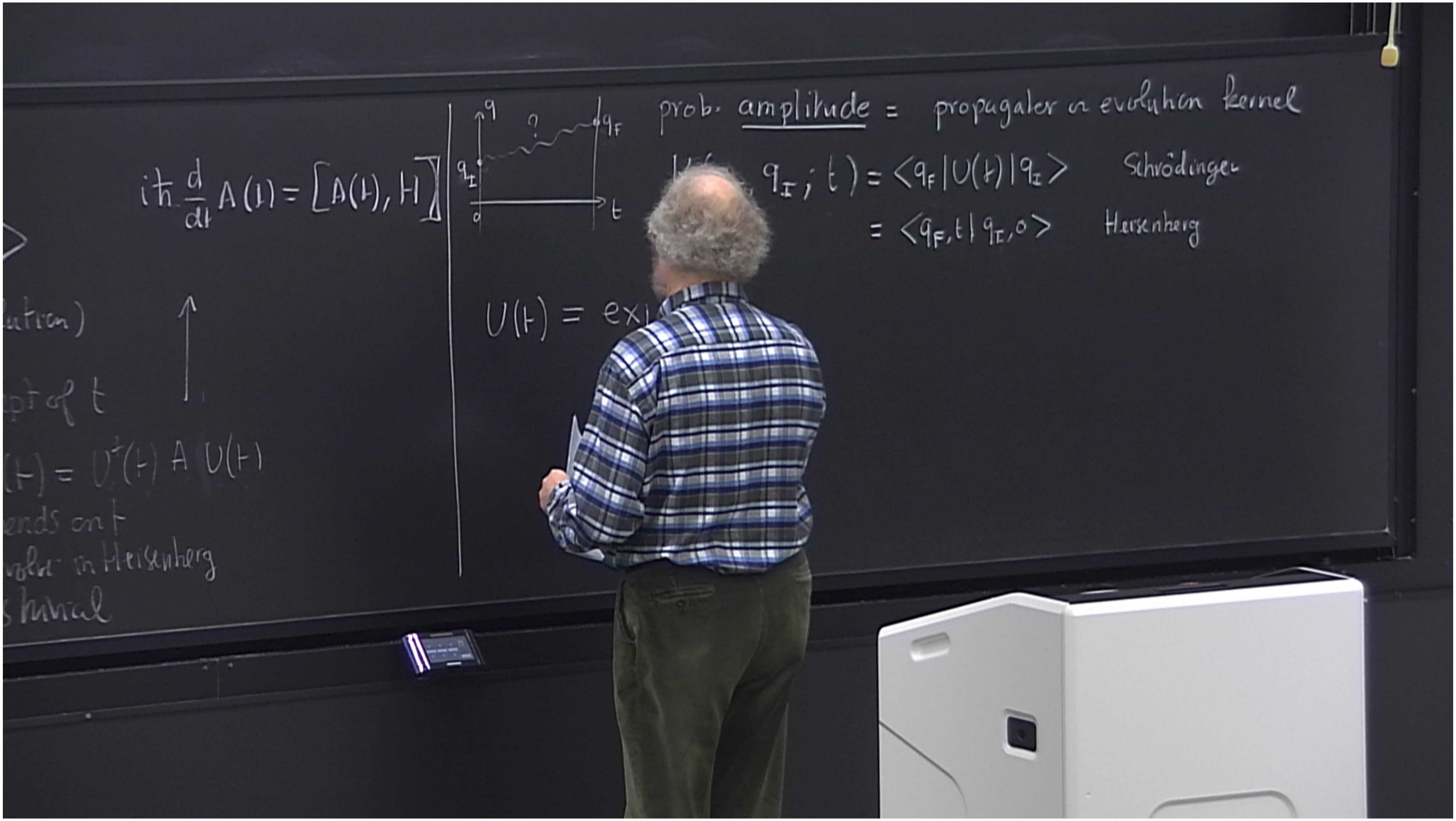

prob. amplitude = propagator or evolution kernel
K(qF, qI ; t) =
Schrödinger
Heisenberg
iħ d/dt A(t) = [A(t), H]
= U†(t) A U(t)
in Heisenberg

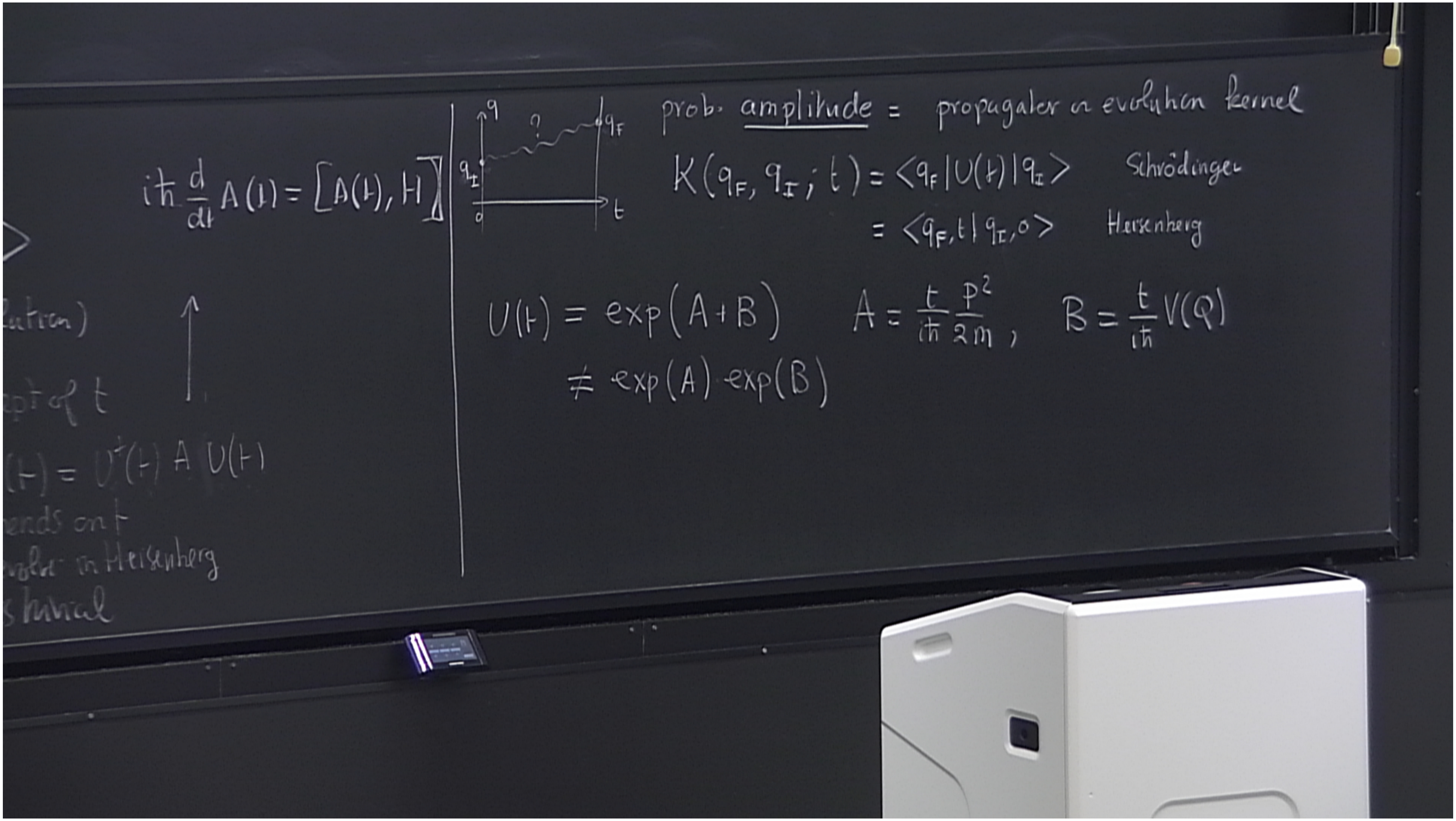


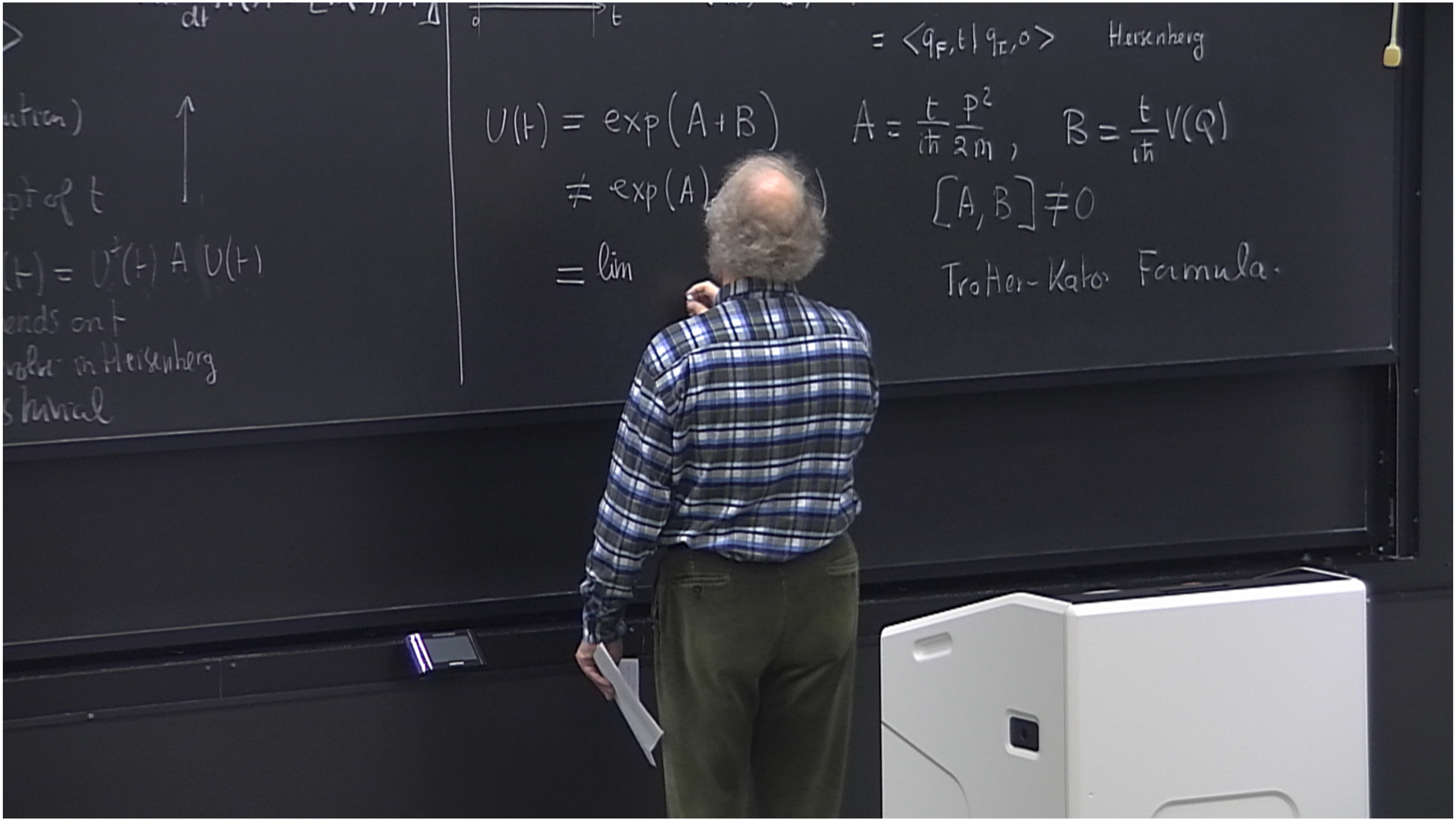
1) Path & Functional Integral Quantization
2) Renormalization.
3) Gauge Theories (+ Fermions)
Path Integrals
mass m
no-spin

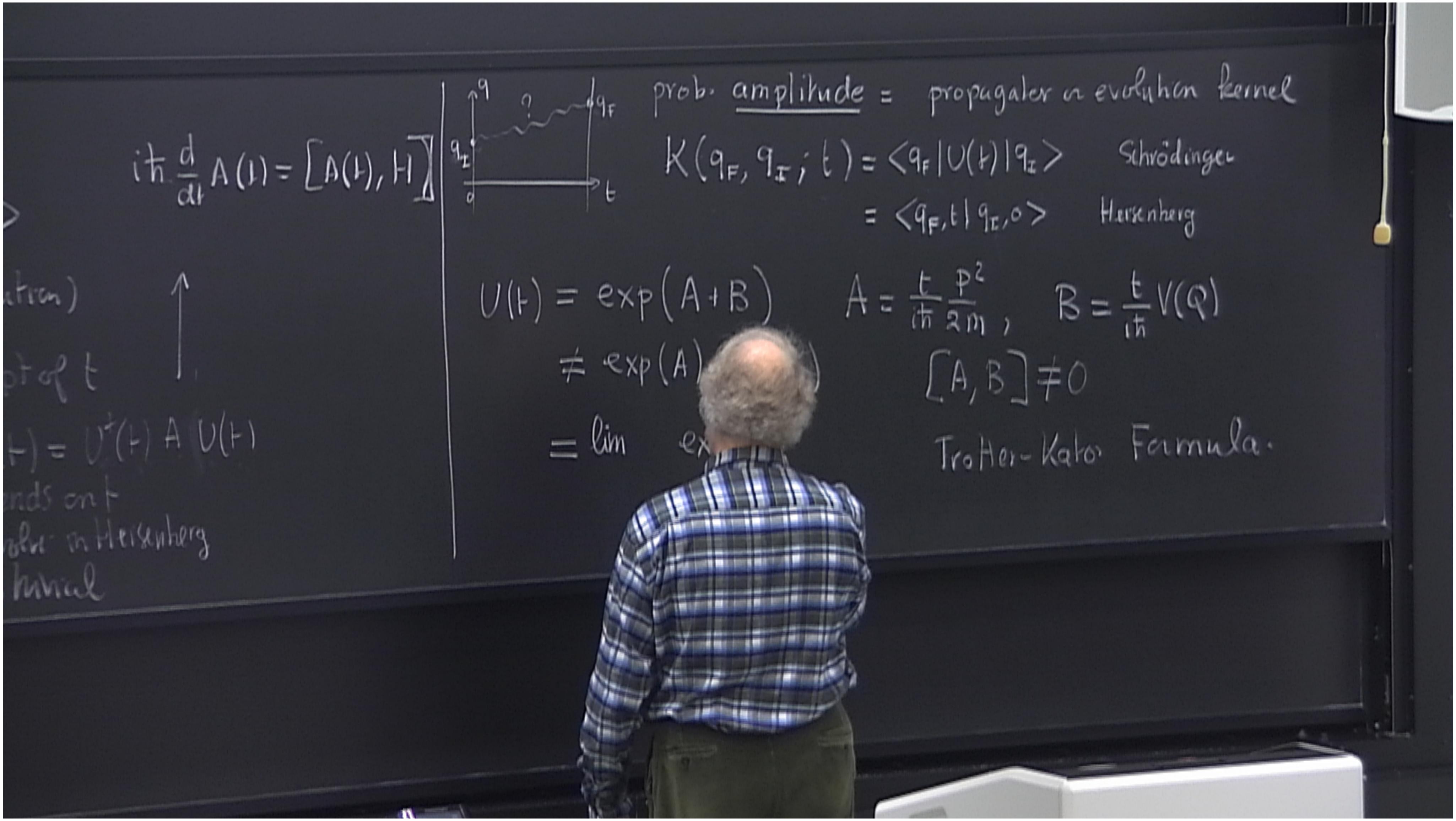

1) Path & Functional Integral Quantization
3) Gauge Theories (+ Fermions)
Path Integrals

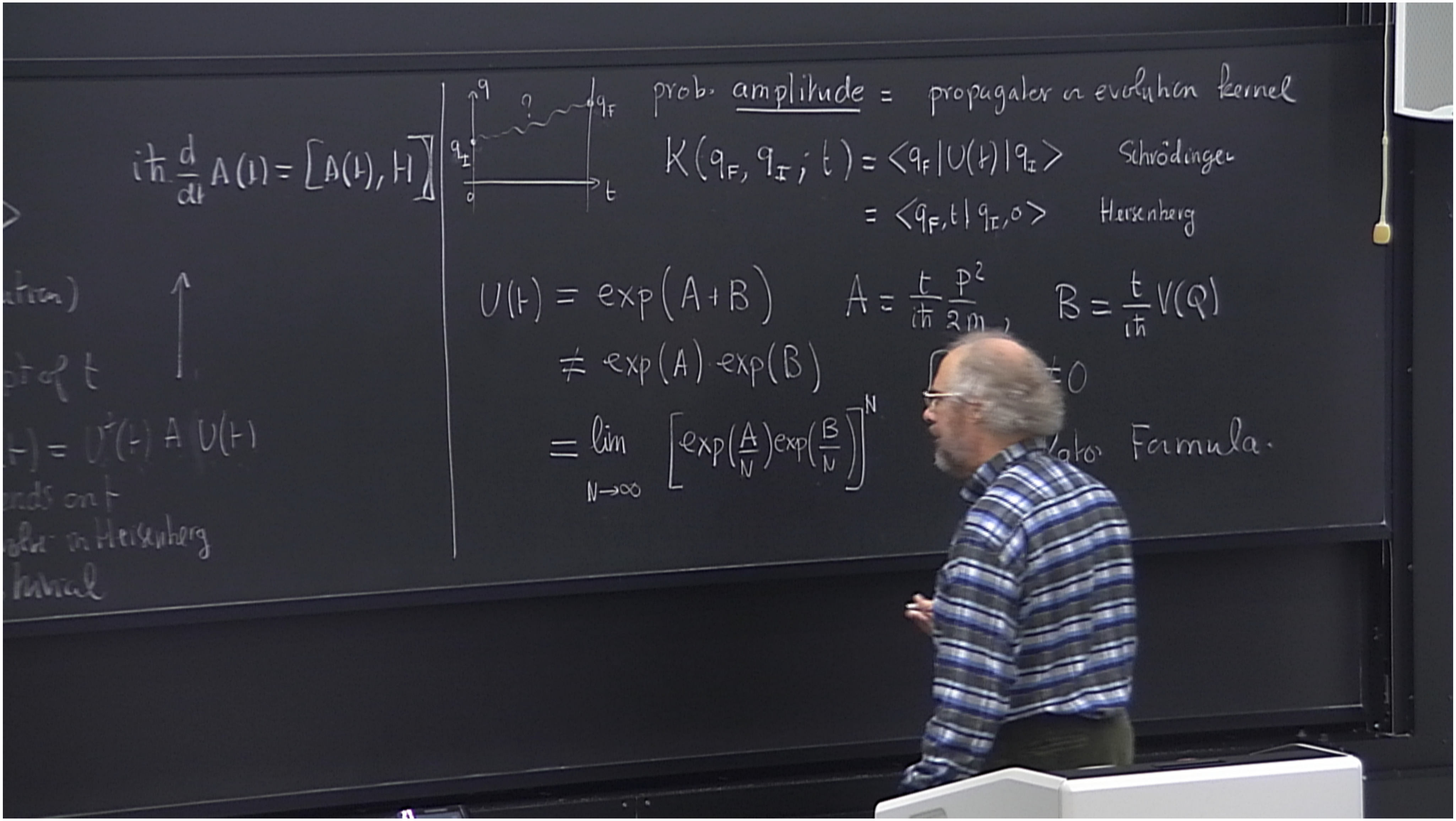

quantum H = P²/2m + V(Q)
iħ d/dt A(t) = [A(t), H]
t=0 |ψ₀> ⟶ t |ψ_t> = U(t)|ψ₀>
U(t) = exp(t/iħ H) ; unitary operator (evolution)
Schrödinger Picture |ψ_t> depends on t, A indept of t
Heisenberg Picture |ψ, t> = U⁺(t)|ψ> A(t) = U⁺(t) A U(t)
depends on t
|ψ₀, t=0> = |ψ₀> ⟶ |ψ_t, t> = |ψ₀> states do not evolve in Heisenberg
dynamics is trivial


prob. amplitude = propagaler or evolution kernel
Schrödinger
Heisenberg

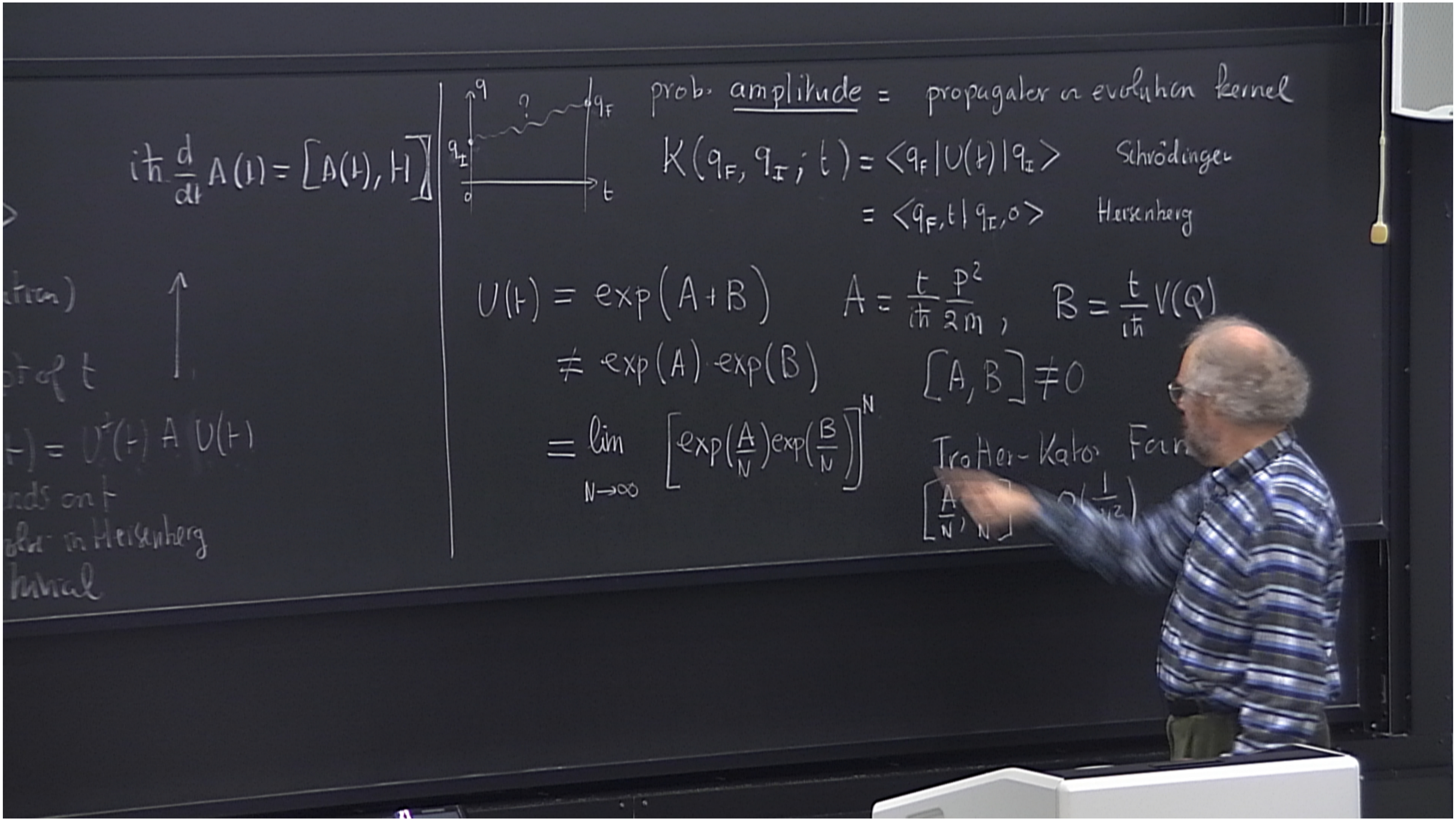

$i\hbar \frac{d}{dt} A(t) = [A(t), H]$
prob. amplitude = propagator or evolution kernel
$K(q_F, q_I; t) = \langle q_F | U(t) | q_I \rangle$ Schrödinger
$= \langle q_F, t | q_I, 0 \rangle$ Heisenberg
$U(t) = \exp(A+B)$ $A = \frac{t}{i\hbar}\frac{P^2}{2m}$, $B = \frac{t}{i\hbar}V(Q)$
$\neq \exp(A)\exp(B)$
q
q_F
q_I
t
0
?

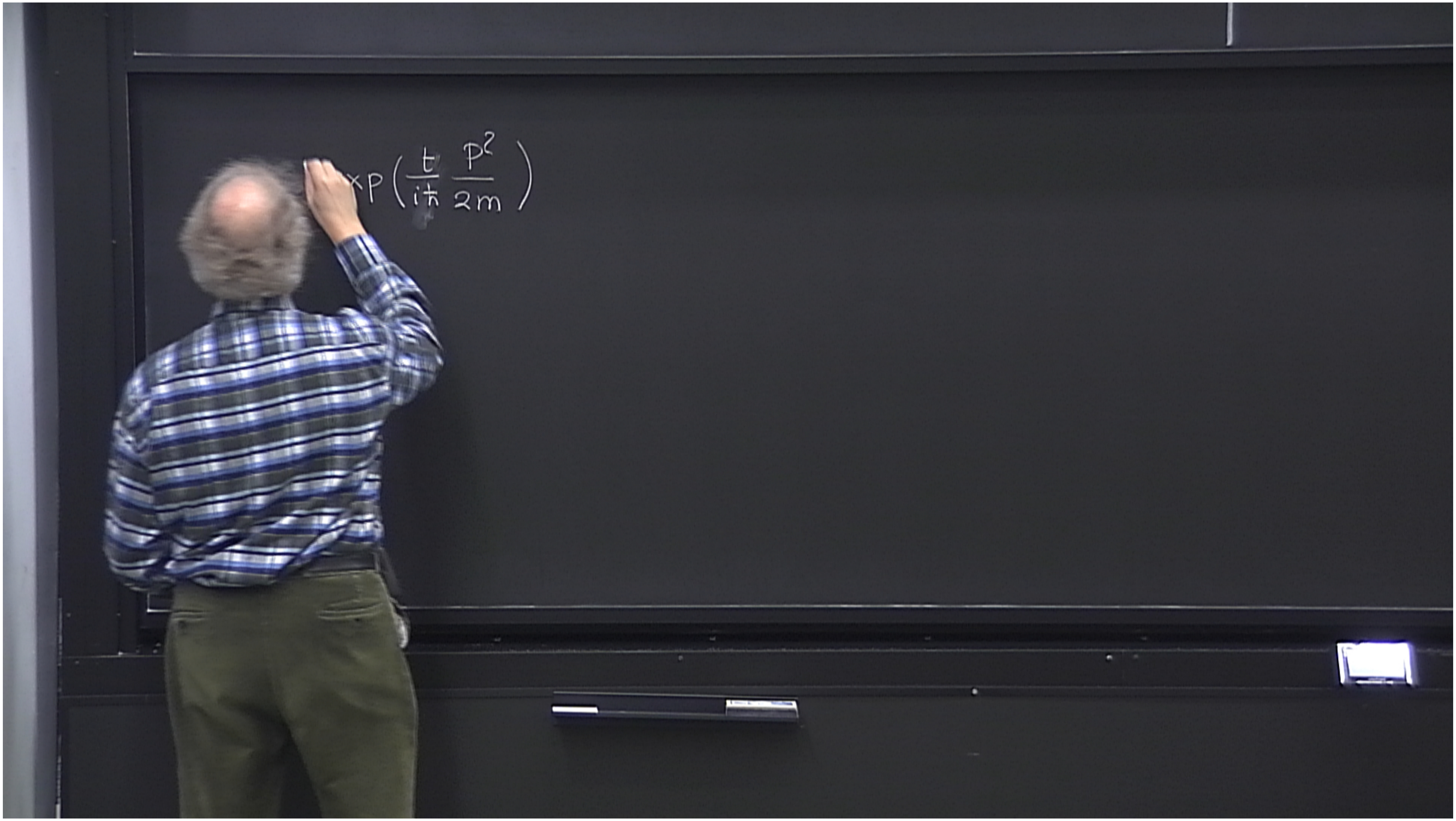

= ⟨q_F, t | q_I, 0⟩ Heisenberg
U(t) = exp(A + B)
A = t/iħ p²/2m , B = t/iħ V(Q)
≠ exp(A)
[A, B] ≠ 0
= lim
Trotter-Kato Formula.

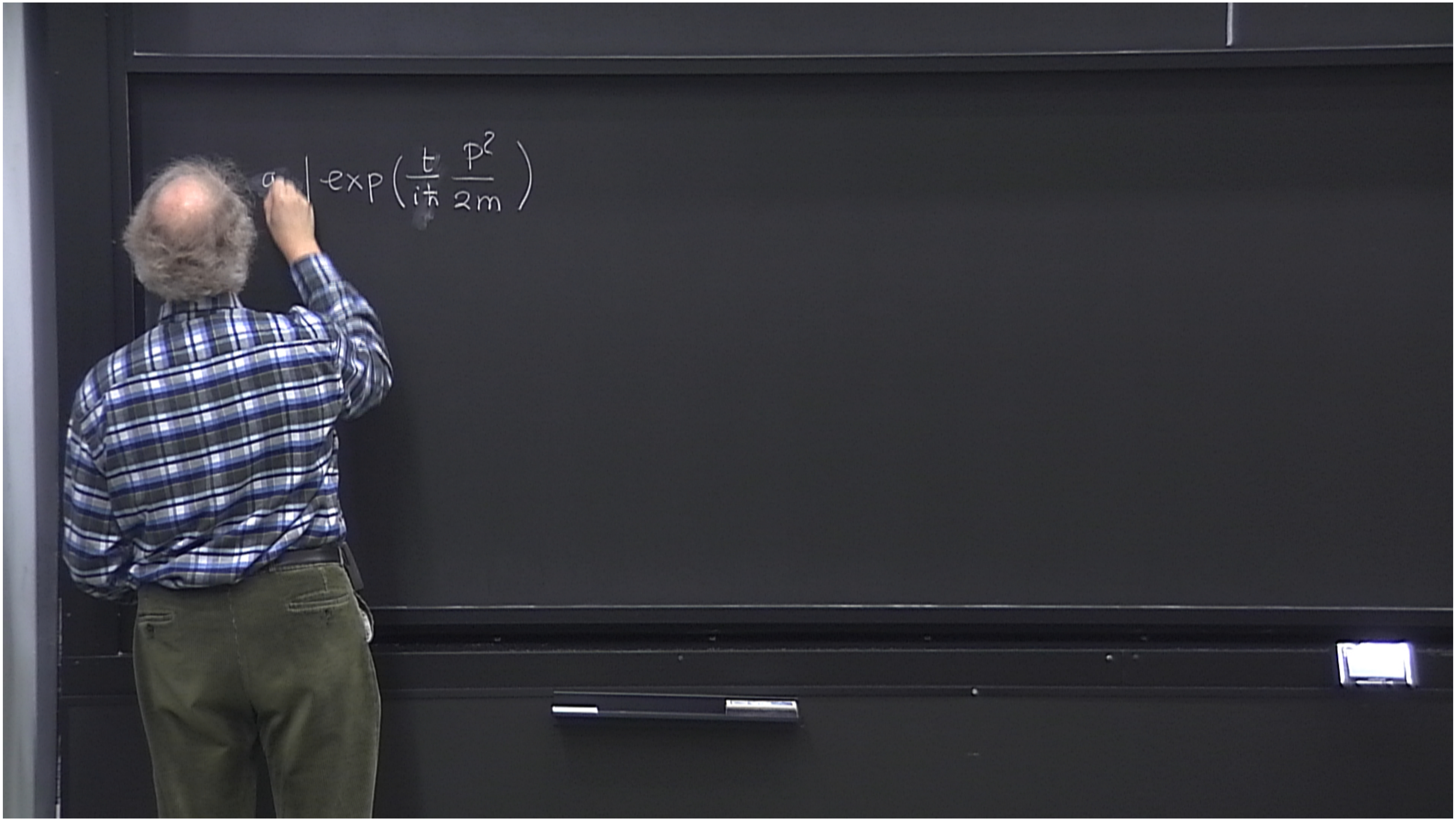

prob. amplitude = propagator or evolution kernel
K(q_F, q_I ; t) = <q_F|U(t)|q_I> Schrödinger
= <q_F, t|q_I, 0> Heisenberg
iħ d/dt A(t) = [A(t), H]
U(t) = exp(A+B)
A = t/iħ p²/2m , B = t/iħ V(Q)
[A, B] ≠ 0
Trotter-Kato Formula.

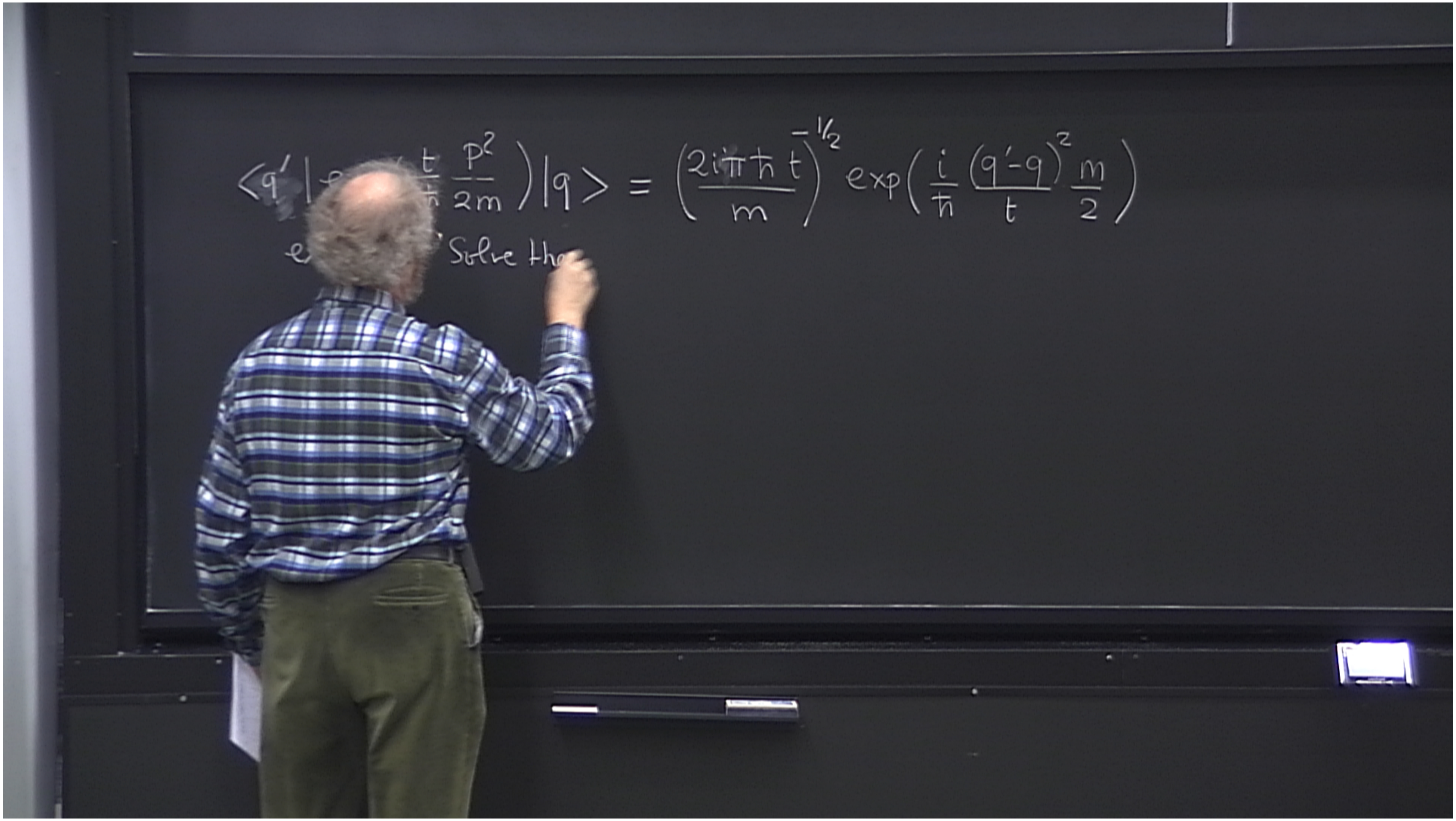


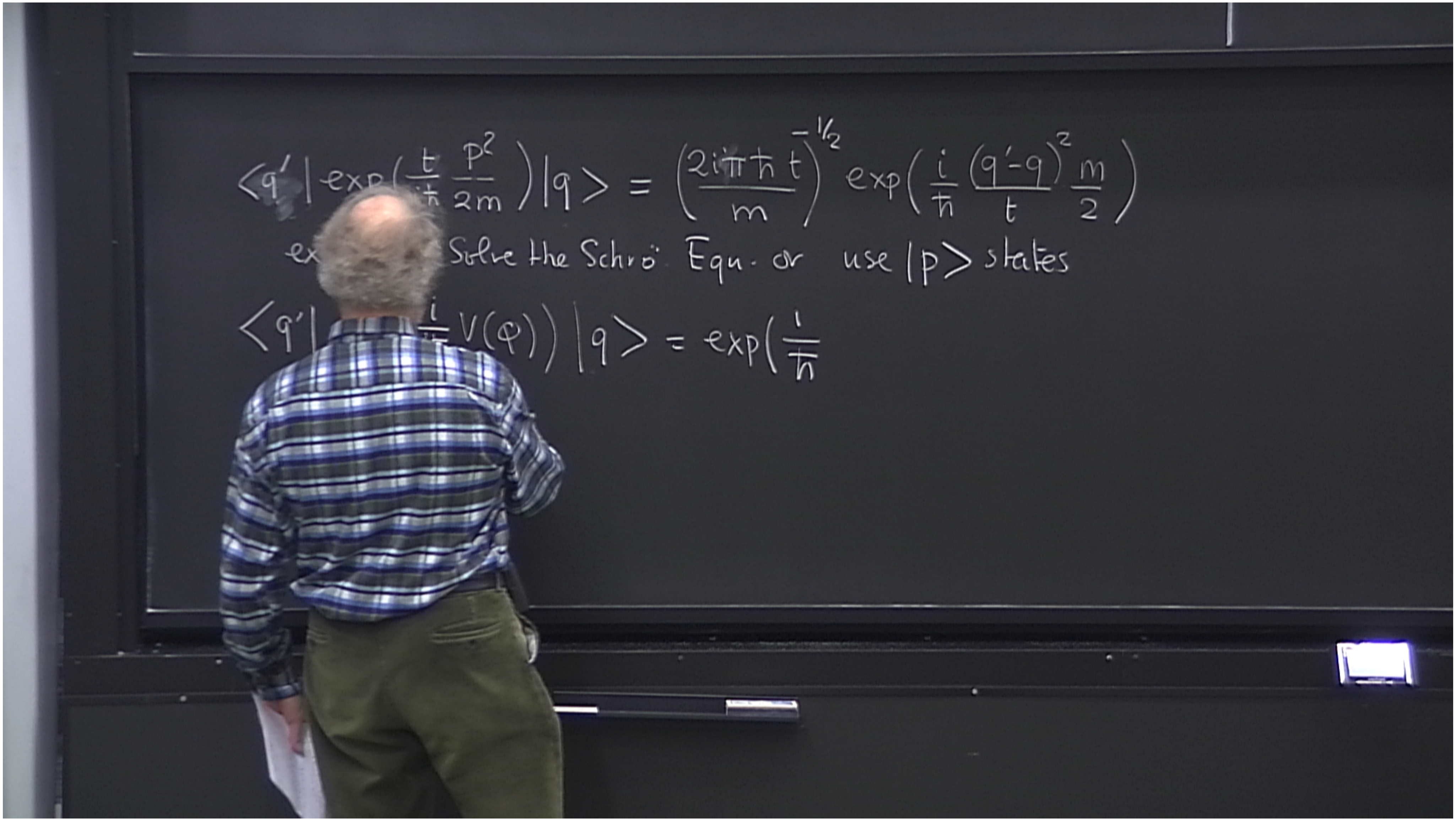
prob. amplitude = propagator or evolution kernel
K(q_F, q_I ; t) = <q_F | U(t) | q_I> Schrödinger
= <q_F, t | q_I, 0> Heisenberg
iħ d/dt A(t) = [A(t), H]
U(t) = exp(A + B)
A = t/iħ p²/2m , B = t/iħ V(Q)
≠ exp(A) exp(B)
= lim N→∞ [exp(A/N) exp(B/N)]^N
Formula.
ends on t
in Heisenberg


prob. amplitude = propagator or evolution kernel
K(q_F, q_I ; t) = <q_F|U(t)|q_I> Schrödinger
= <q_F, t | q_I, 0> Heisenberg
iħ d/dt A(t) = [A(t), H]
U(t) = exp(A+B) A = t/iħ p²/2m , B = t/iħ V(Q)
≠ exp(A) exp(B) [A, B] ≠ 0
= lim N→∞ [exp(A/N) exp(B/N)]^N
Trotter-Kato


xp (t/iħ p²/2m)


q | exp (t/iħ p²/2m)


$\left(\frac{2i\pi\hbar t}{m}\right)^{-1/2} \exp\left(\frac{i}{\hbar}\frac{(q'-q)^2}{t}\frac{m}{2}\right)$
Solve the



⟨q'| exp(t/iħ p²/2m)|q⟩ = (2iπħt/m)^{-1/2} exp(i/ħ (q'-q)²/t m/2)
Solve the Schrö. Equ. or use |p⟩ states
⟨q'| i/ħ V(φ)) |q⟩ = exp(i/ħ

$$\langle q'|\exp\left(\frac{t}{i\hbar}\frac{p^2}{2m}\right)|q\rangle = \left(\frac{2i\pi\hbar t}{m}\right)^{-1/2}\exp\left(\frac{i}{\hbar}\frac{(q'-q)^2}{t}\frac{m}{2}\right)$$

exercise: Solve the Schrö. Equ. or use $|p\rangle$ states

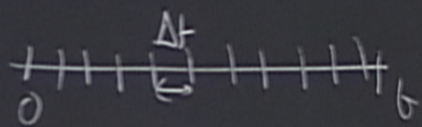
$$\langle q'|\exp\left(\frac{t}{i\hbar}V(q)\right)|q\rangle = \exp\left(\frac{t}{i\hbar}V(q)\right)\delta(q'-q)$$

$$\langle q'| \exp\left(\frac{t}{i\hbar}\frac{p^2}{2m}\right)|q\rangle = \left(\frac{2i\pi\hbar t}{m}\right)^{-1/2} \exp\left(\frac{i}{\hbar}\frac{(q'-q)^2}{t}\frac{m}{2}\right)$$

exercise: Solve the Schrö. Equ. or use $|p\rangle$ states

$$\langle q'|\exp\left(\frac{t}{i\hbar}V(\varphi)\right)|q\rangle = \exp\left(\frac{t}{i\hbar}V(q)\right)\delta(q'-q)$$

decompose the interval time $[0, t]$ into N small intervals



$$\Delta t = \frac{t}{N}$$

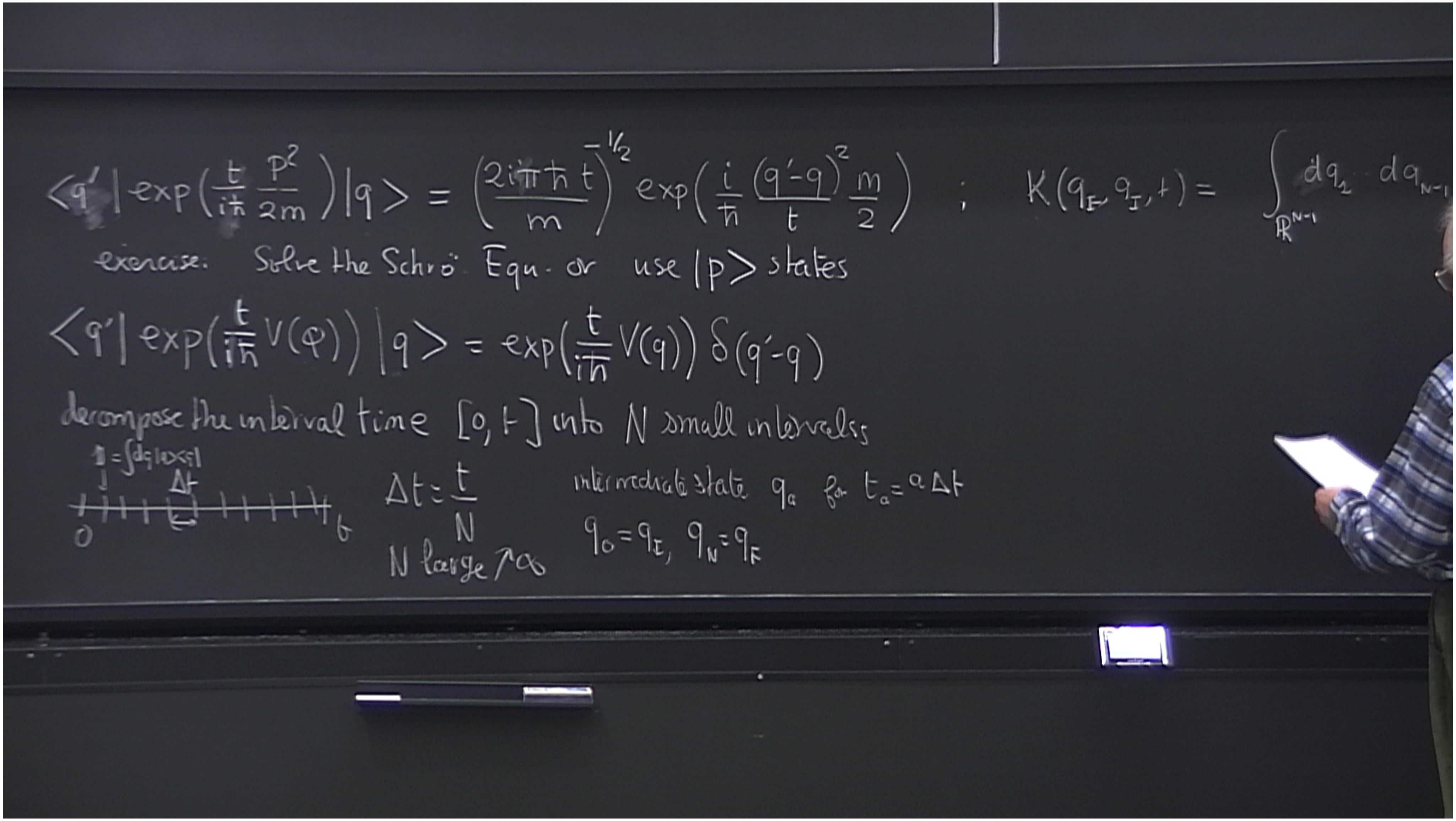
N large $\nearrow \infty$

$$\frac{q'-q)^2}{t}\frac{m}{2}\Big) \quad ; \quad K(q_F, q_I, t) = \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1}$$

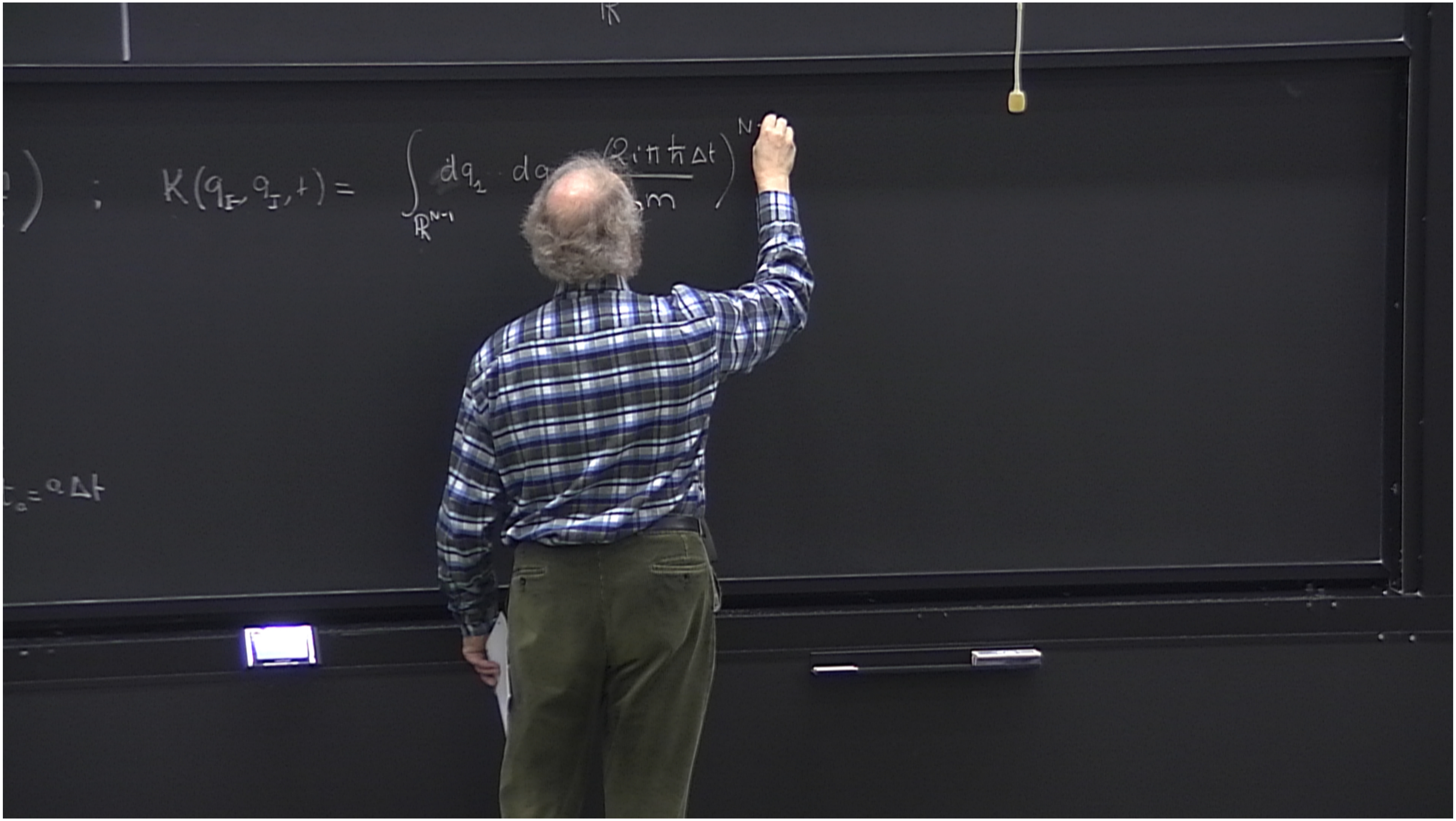
states

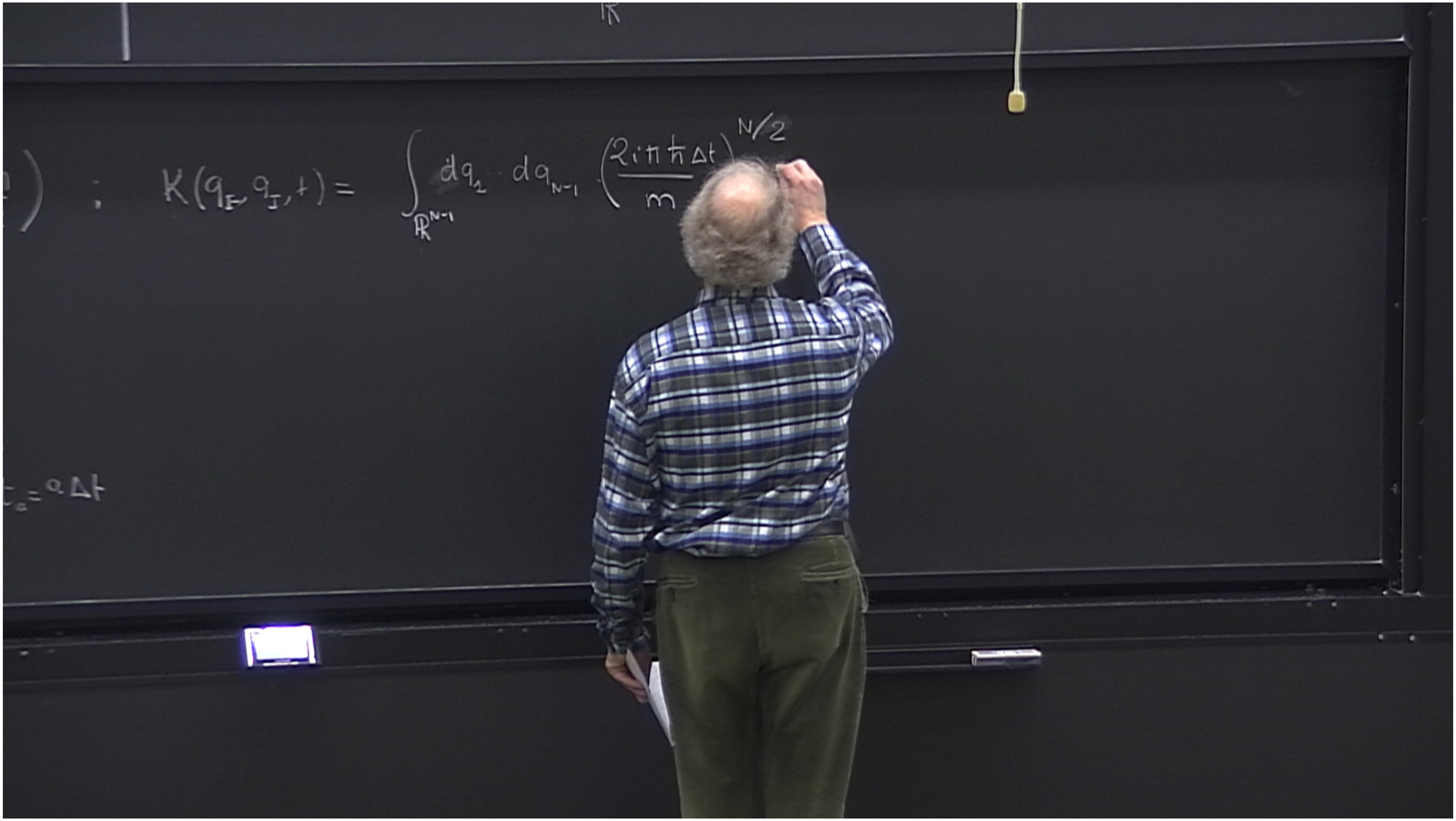
$'-q)$

ntervals



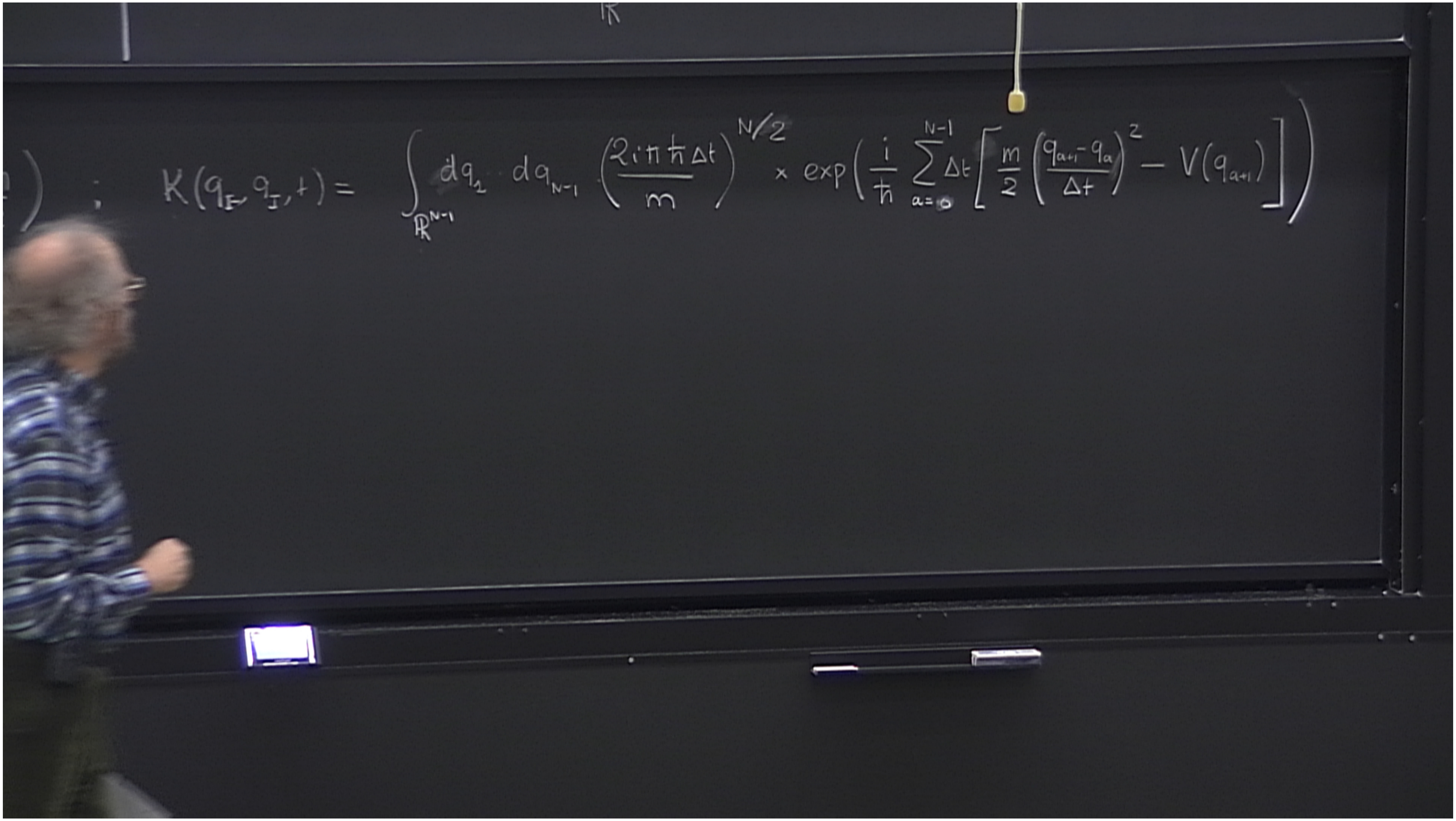
$\langle q'|\exp\left(\frac{t}{i\hbar}\frac{p^2}{2m}\right)|q\rangle = \left(\frac{2i\pi\hbar t}{m}\right)^{-1/2}\exp\left(\frac{i}{\hbar}\frac{(q'-q)^2}{t}\frac{m}{2}\right)$;
$K(q_F, q_I, t) = \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1}$
exercise: Solve the Schrö. Equ. or use $|p\rangle$ states
$\langle q'|\exp\left(\frac{t}{i\hbar}V(q)\right)|q\rangle = \exp\left(\frac{t}{i\hbar}V(q)\right)\delta(q'-q)$
decompose the interval time $[0, t]$ into N small intervals
$1 = \int dq\, |q\rangle\langle q|$
Δt
0
t
$\Delta t = \frac{t}{N}$
N large $\nearrow \infty$
intermediate state q_a for $t_a = a\Delta t$
$q_0 = q_I$, $q_N = q_F$

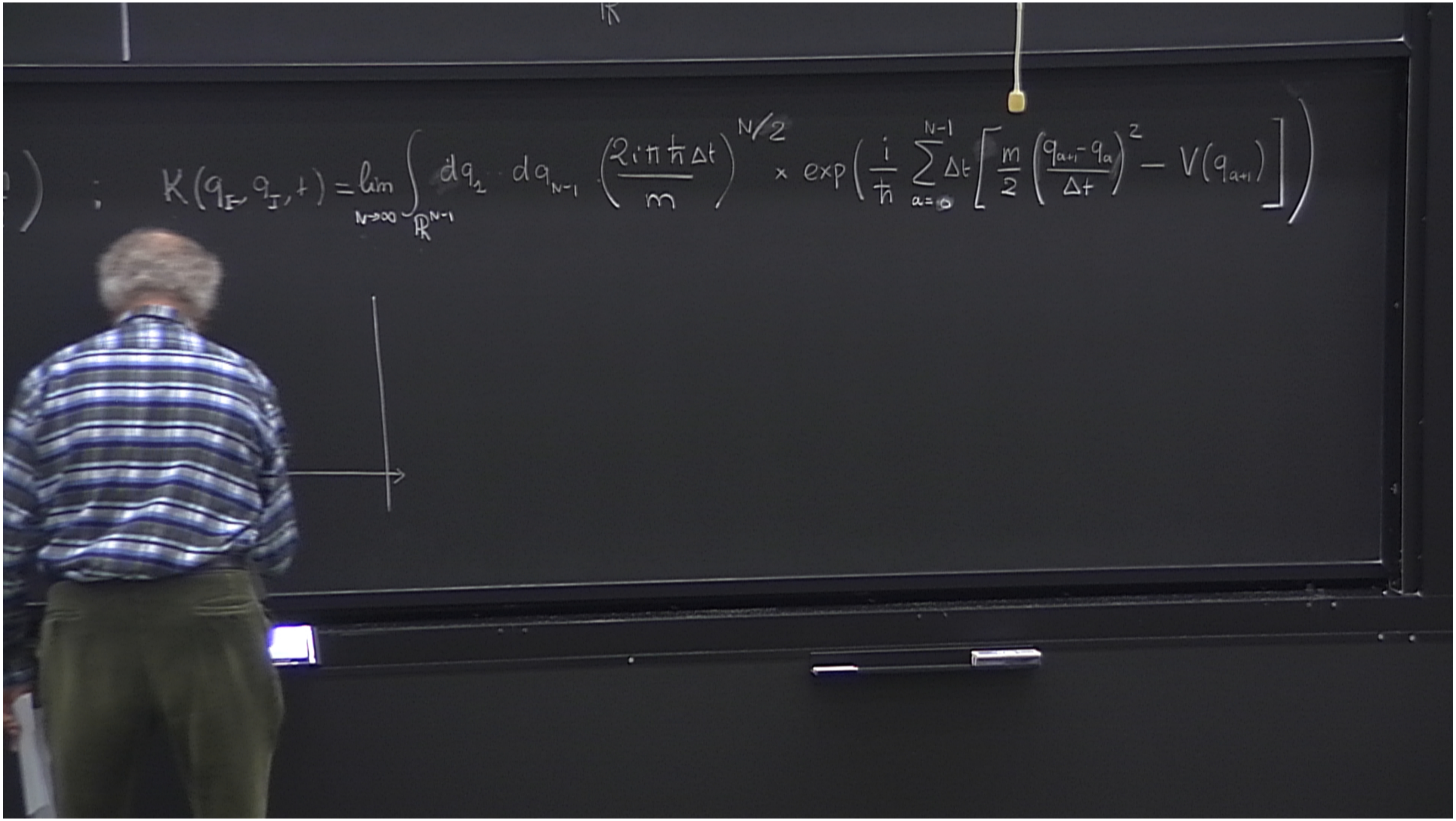

K(q_F, q_I, t) =

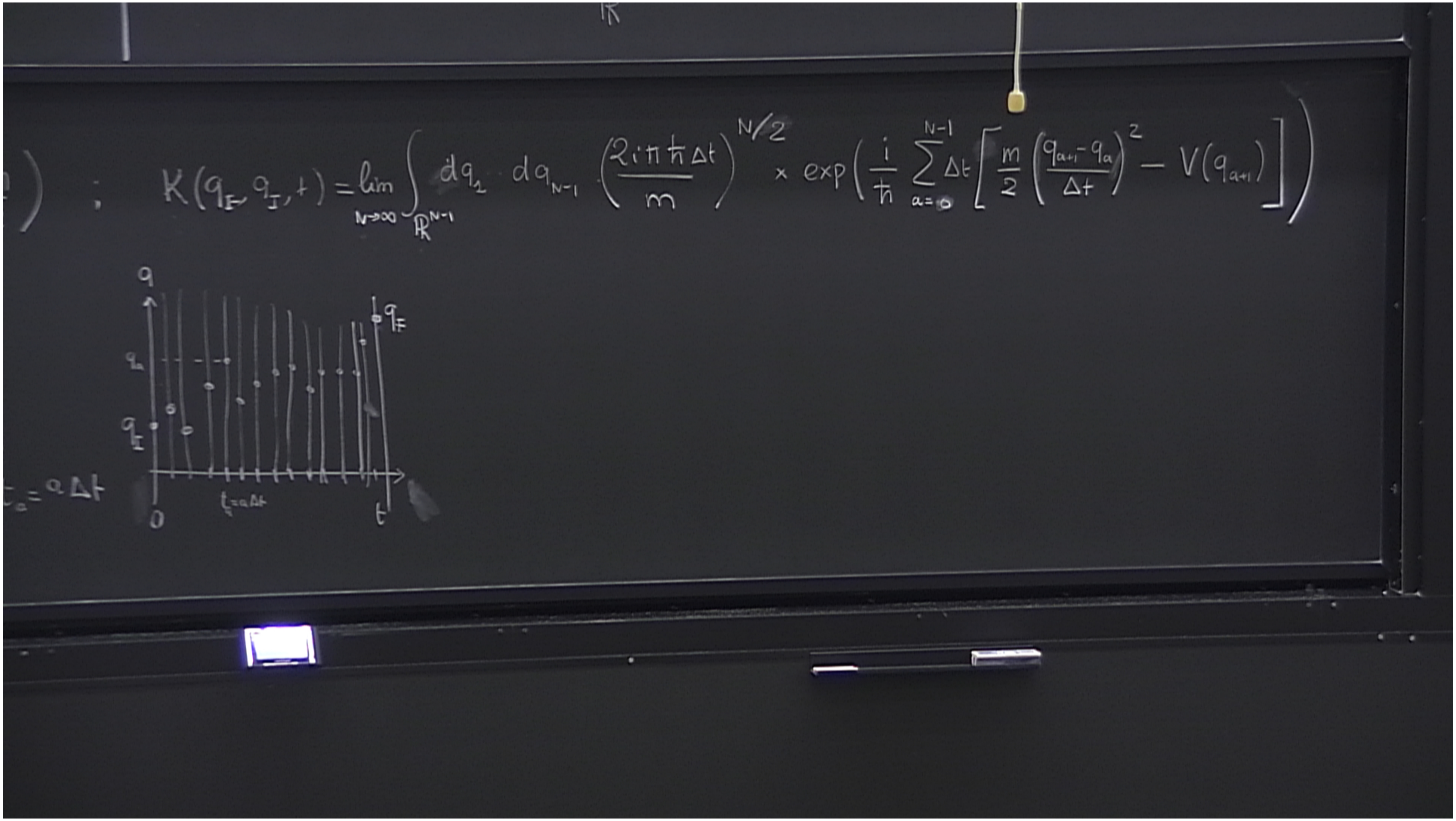

K(q_F, q_I, t) = ∫_{R^{N-1}} dq_1 · dq_{N-1} (2iπħΔt / m)^{N/2}
t_a = aΔt

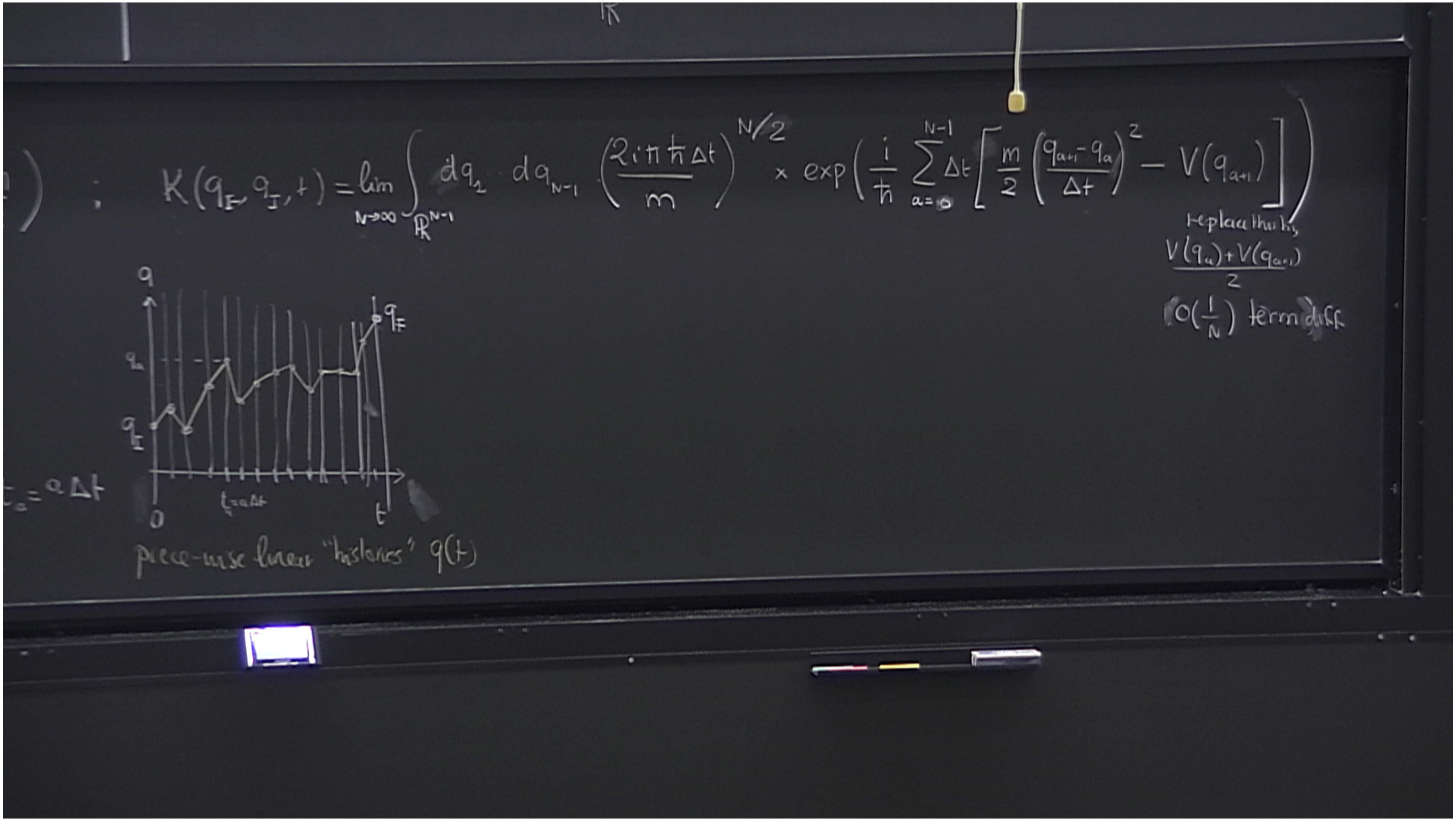
$$K(q_F, q_I, t) = \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1} \left(\frac{2 i \pi \hbar \Delta t}{m}\right)^{N/2} \times \exp\left(\frac{i}{\hbar} \sum_{a=0}^{N-1} \Delta t \, \frac{m}{2} \left(\frac{q_{a+1} - q_a}{\Delta t}\right)^2\right)$$

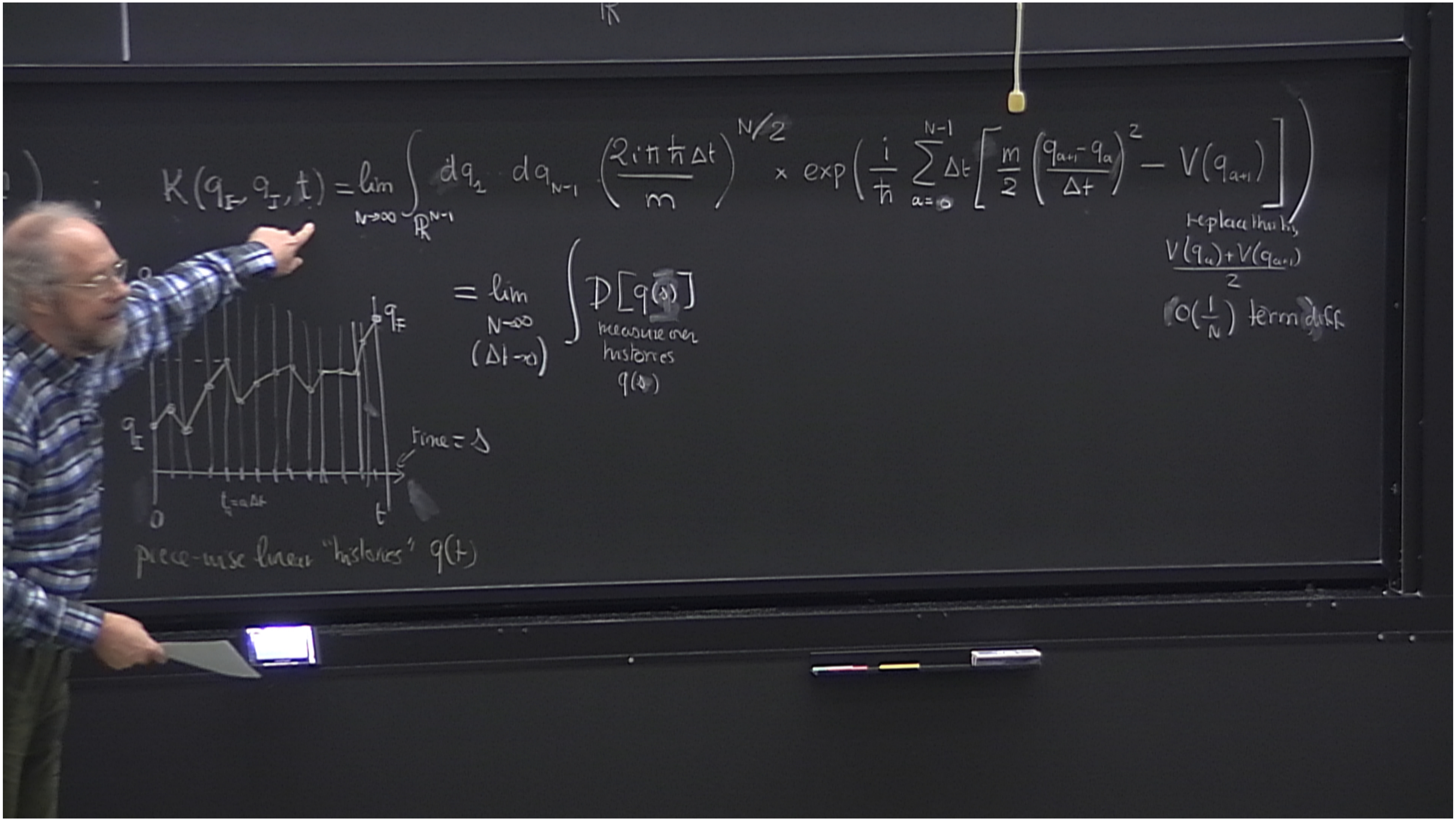
$t_a = a \Delta t$


$K(q_F, q_I, t) = \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1} \left(\frac{2 i \pi \hbar \Delta t}{m}\right)^{N/2} \times \exp\left(\frac{i}{\hbar} \sum_{a=0}^{N-1} \Delta t \left[\frac{m}{2}\left(\frac{q_{a+1}-q_a}{\Delta t}\right)^2 - V(q_{a+1})\right]\right)$

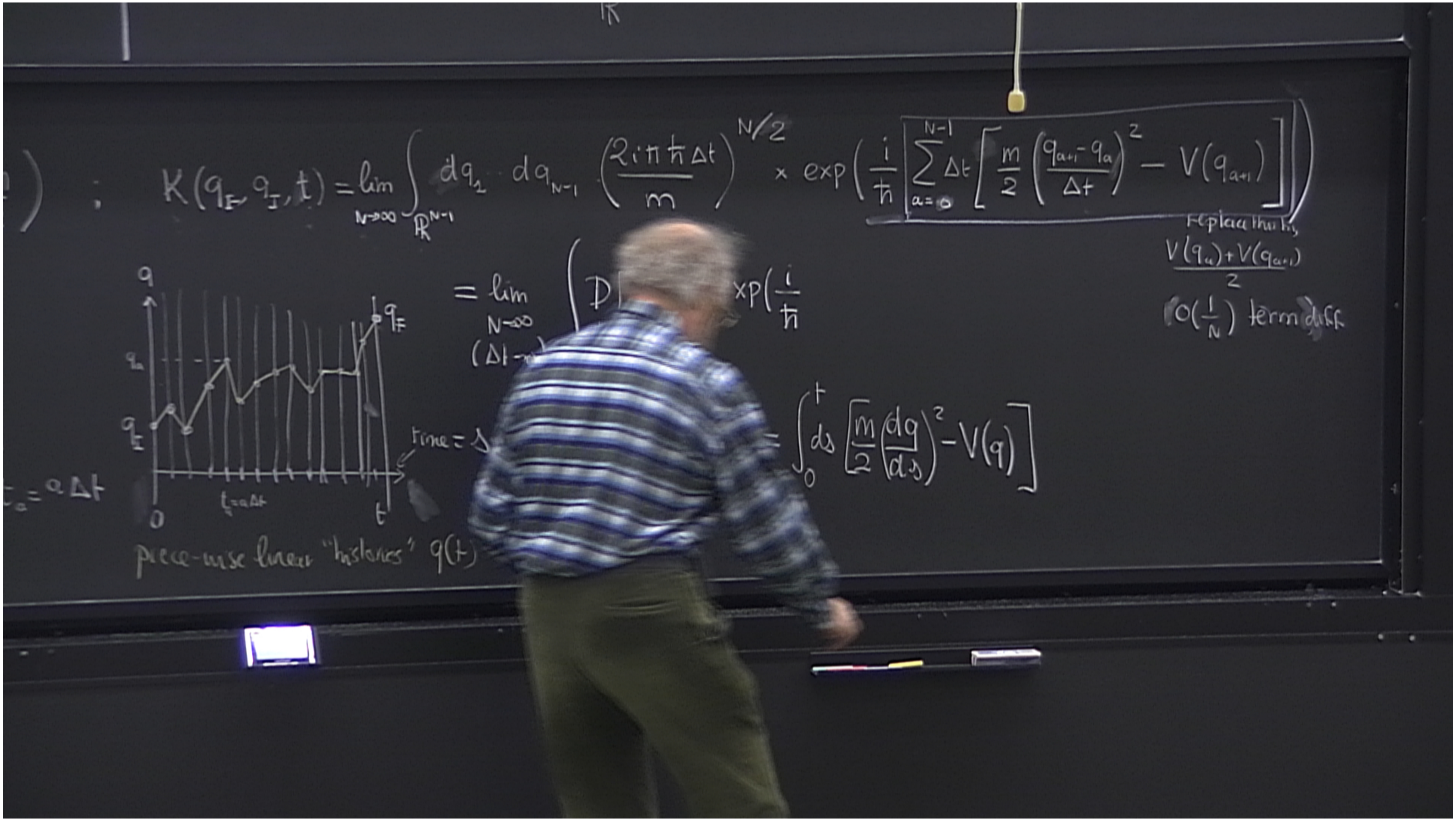

$K(q_F, q_I, t) = \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1} \left(\frac{2i\pi\hbar\Delta t}{m}\right)^{N/2} \times \exp\left(\frac{i}{\hbar}\sum_{a=0}^{N-1}\Delta t\left[\frac{m}{2}\left(\frac{q_{a+1}-q_a}{\Delta t}\right)^2 - V(q_{a+1})\right]\right)$

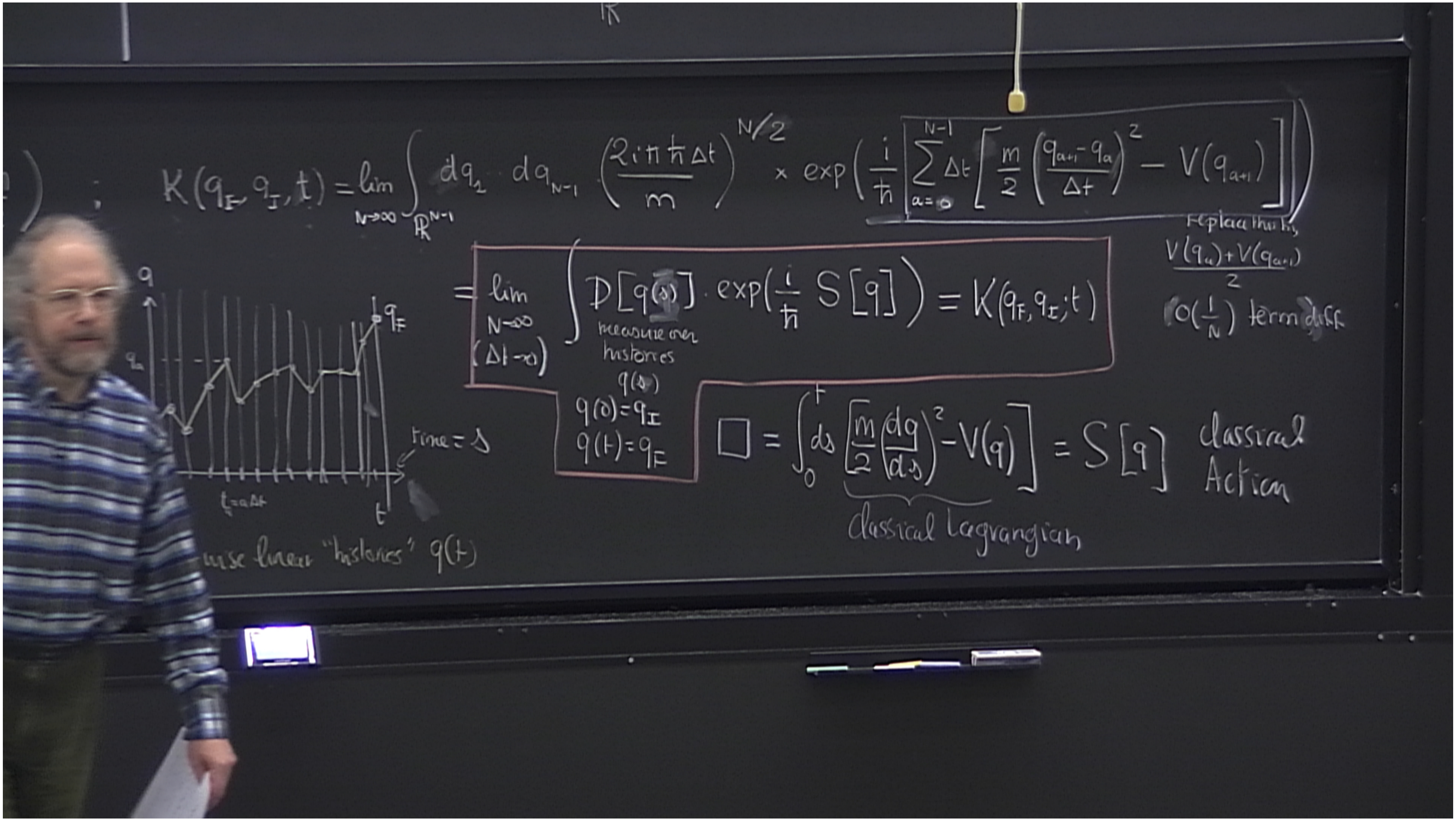

$K(q_F, q_I, t) = \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} dq_1 \, dq_{N-1} \left(\frac{2i\pi\hbar\Delta t}{m}\right)^{N/2} \times \exp\left(\frac{i}{\hbar} \sum_{a=0}^{N-1} \Delta t \left[\frac{m}{2}\left(\frac{q_{a+1}-q_a}{\Delta t}\right)^2 - V(q_{a+1})\right]\right)$
q
q_F
q_a
q_I
$t_a = a\Delta t$
0
t

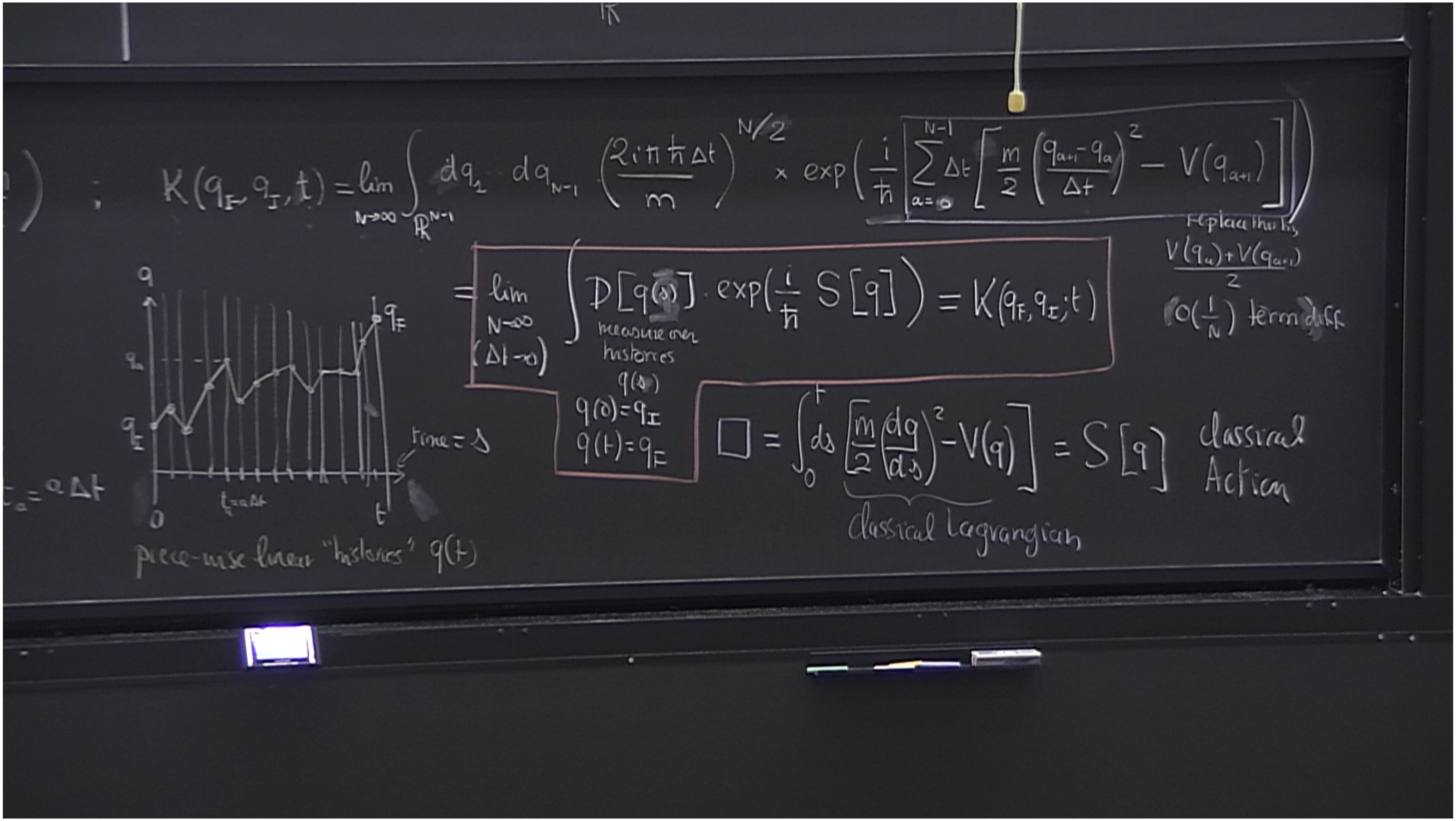

K(q_F, q_I, t) = lim_{N→∞} ∫_{R^{N-1}} dq_1 … dq_{N-1} (2iπħΔt / m)^{N/2} × exp(i/ħ Σ_{a=0}^{N-1} Δt [m/2 ((q_{a+1} − q_a)/Δt)^2 − V(q_{a+1})])
replace this by
(V(q_a) + V(q_{a+1})) / 2
O(1/N) term diff
t_a = a Δt
q
q_a
q_I
q_F
0
t_a = a Δt
t
piece-wise linear "histories" q(t)

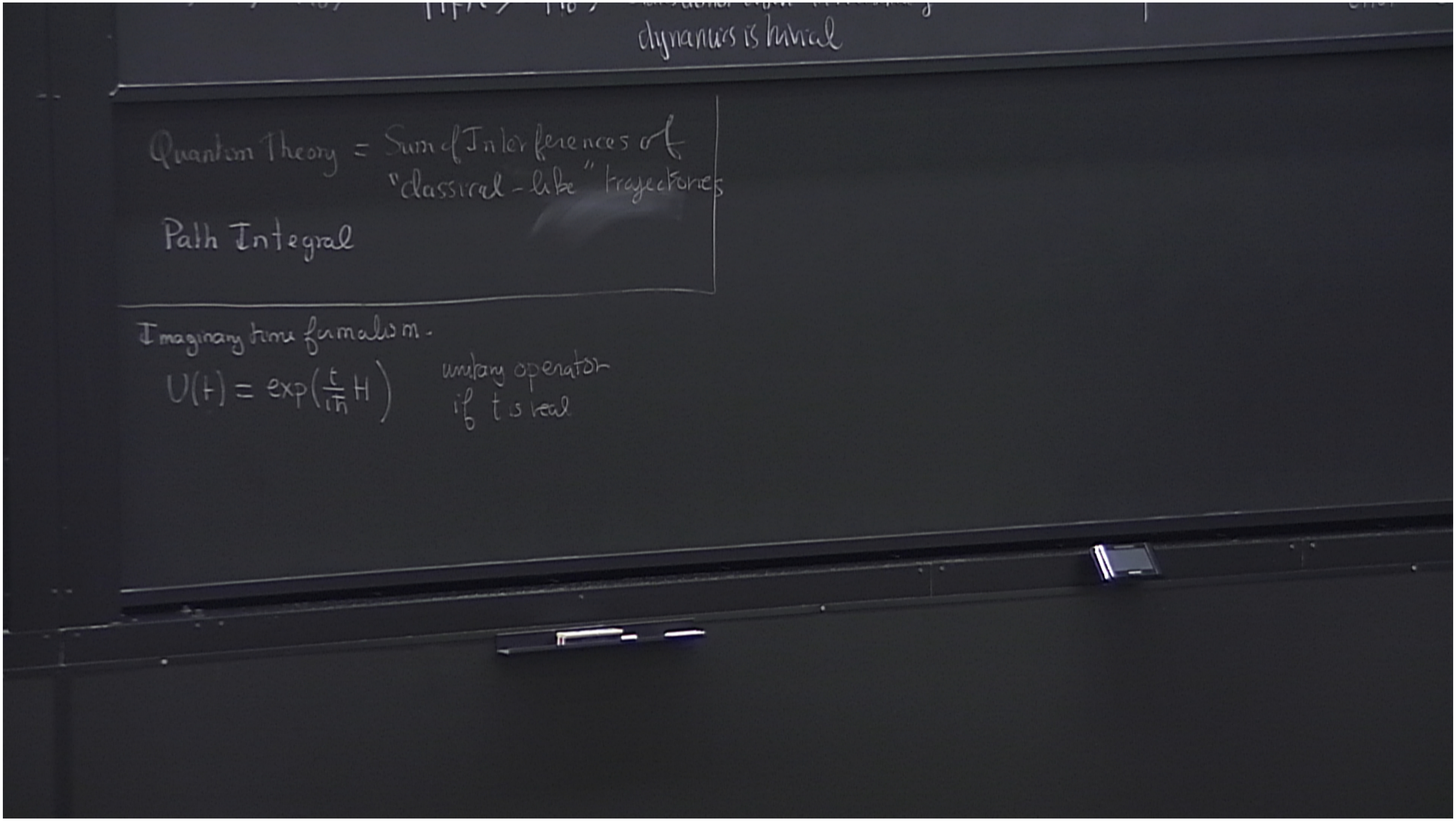


$K(q_F, q_I, t) = \lim_{N\to\infty} \int_{\mathbb{R}^{N-1}} dq_1 \cdots dq_{N-1} \left(\frac{2 i \pi \hbar \Delta t}{m}\right)^{N/2} \times \exp\left(\frac{i}{\hbar} \sum_{a=0}^{N-1} \Delta t \left[\frac{m}{2}\left(\frac{q_{a+1} - q_a}{\Delta t}\right)^2 - V(q_{a+1})\right]\right)$
replace this by
$\frac{V(q_a) + V(q_{a+1})}{2}$
$O(\frac{1}{N})$ term diff
$= \lim_{N\to\infty, (\Delta t \to 0)} \int D[q(s)]$
measure over histories $q(s)$
q_F
q_I
time = s
$t_a = a\Delta t$
0
t
piece-wise linear "histories" $q(t)$

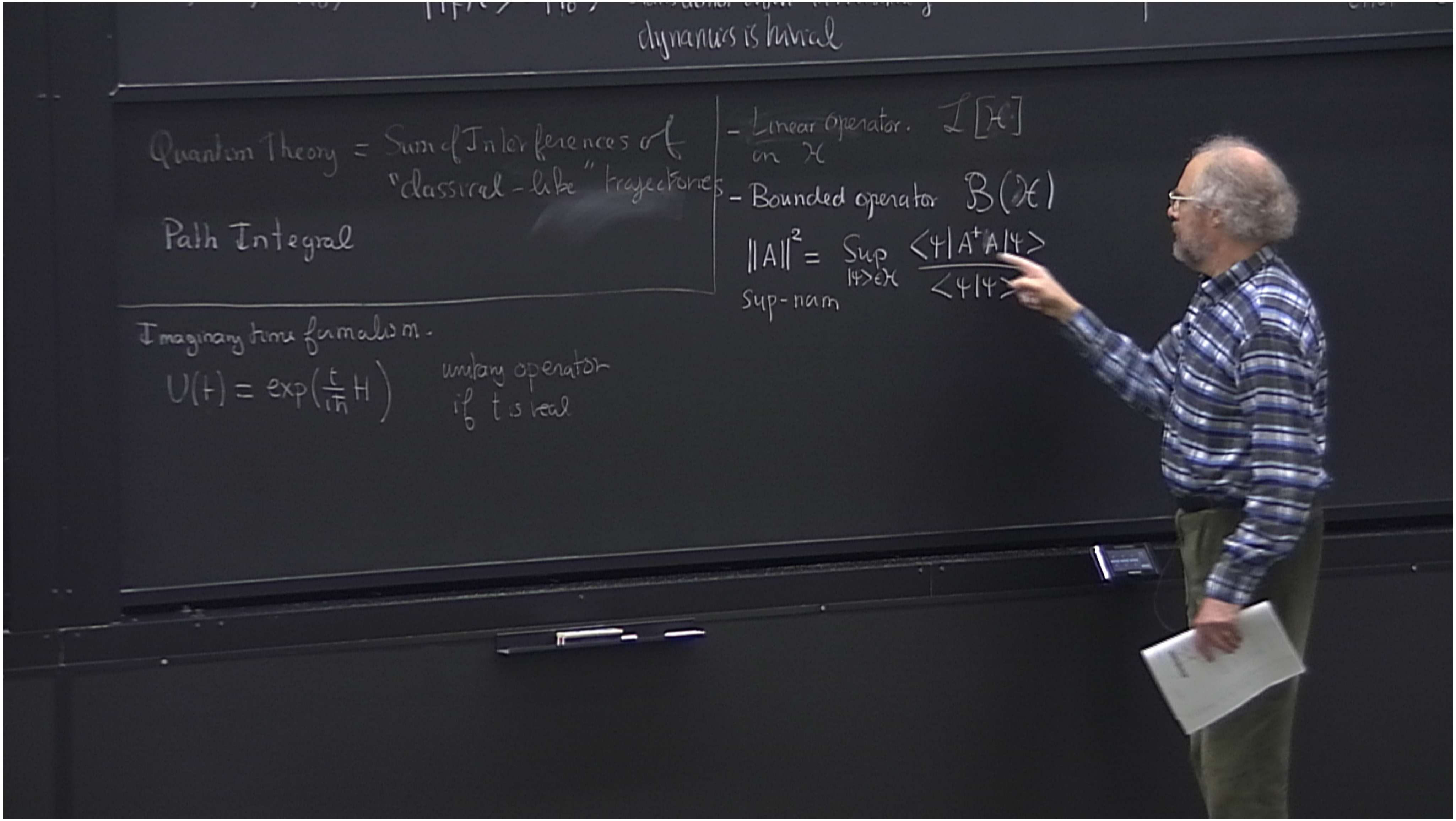

K(q_F, q_I, t) = lim ∫ dq_1 ... dq_{N-1} (2iπħΔt / m)^{N/2} × exp(i/ħ Σ_{a=0}^{N-1} Δt [m/2 ((q_{a+1} − q_a)/Δt)^2 − V(q_{a+1})])
replace this by
V(q_a)+V(q_{a+1}) / 2
O(1/N) term diff
time = s
t_a = aΔt
piece-wise linear "histories" q(t)

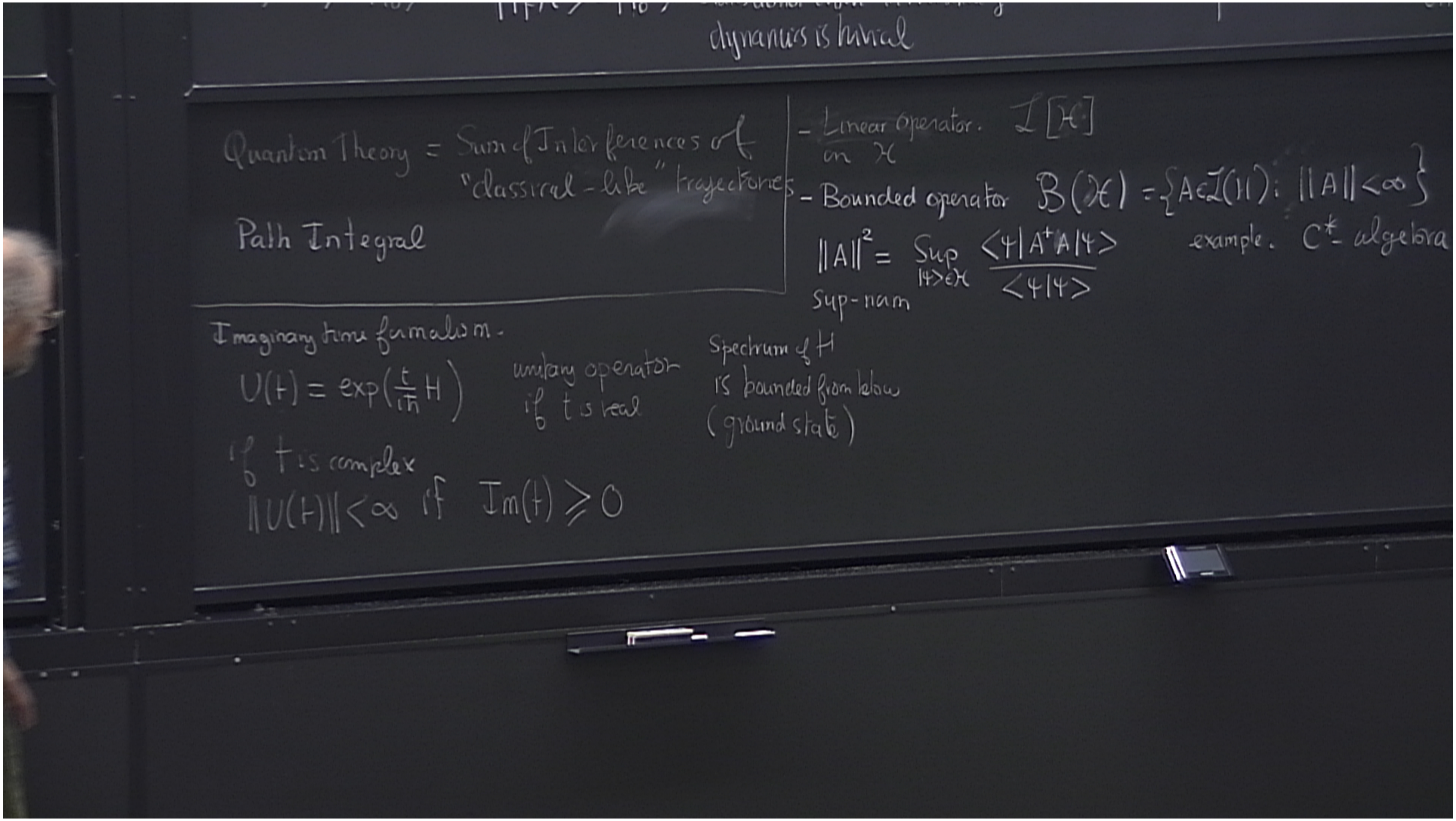

K(q_F, q_I, t) = lim N→∞ ∫ dq_1 ... dq_N-1 (2iπ ħ Δt / m)^N/2 × exp(i/ħ Σ_a=0^N-1 Δt [m/2 ((q_a+1 − q_a)/Δt)^2 − V(q_a+1)])
replace this by
V(q_a)+V(q_a+1) / 2
O(1/N) term diff
= lim N→∞ (Δt→0) ∫ D[q(s)] · exp(i/ħ S[q]) = K(q_F, q_I; t)
measure over histories q(s)
q(0) = q_I
q(t) = q_F
time = s
□ = ∫_0^t ds [m/2 (dq/ds)^2 − V(q)] = S[q]
classical Action
classical Lagrangian
"histories" q(t)

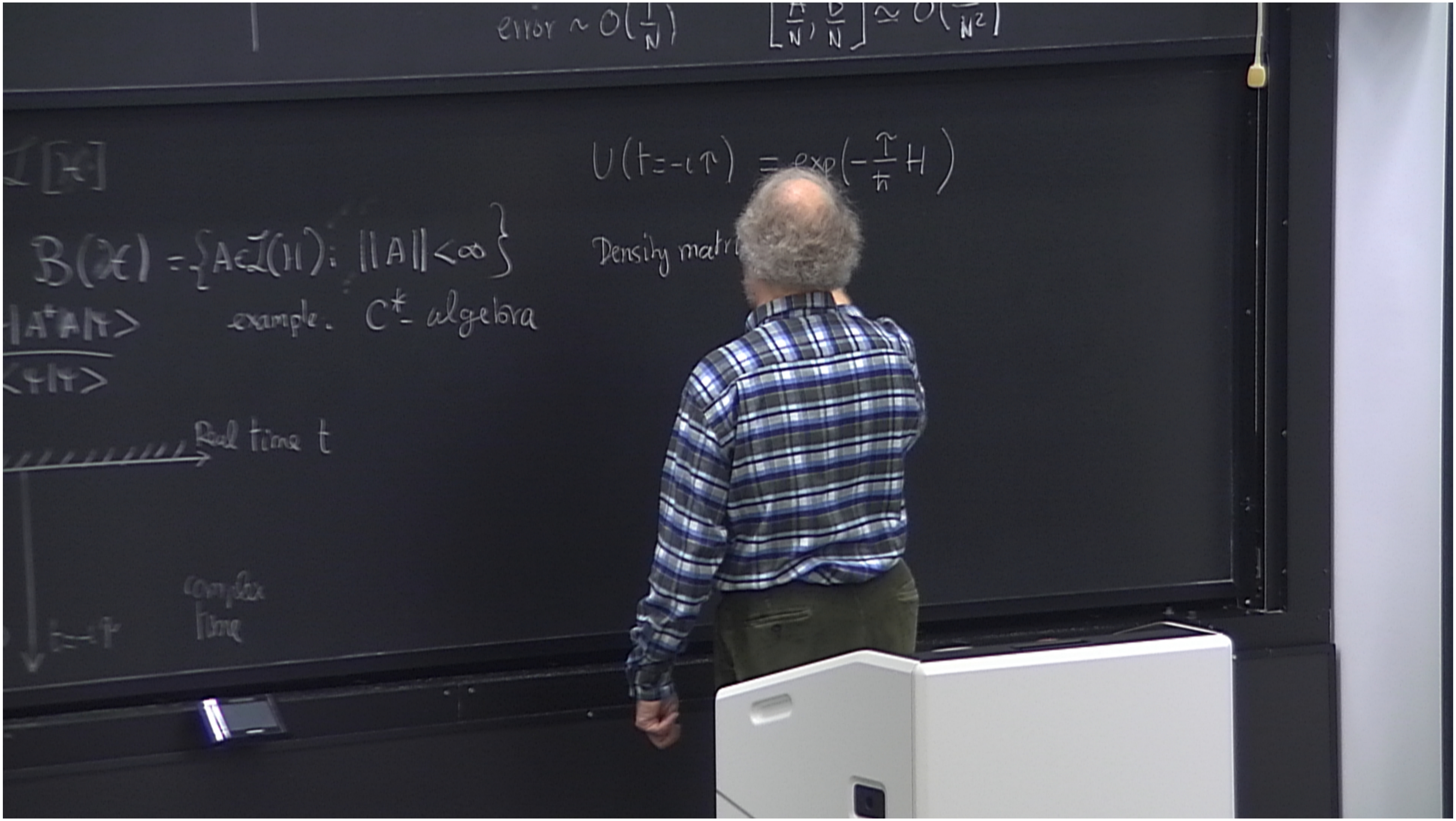

K(q_F, q_I, t) = lim_{N→∞} ∫_{R^{N-1}} dq_1 · dq_{N-1} (2iπħΔt / m)^{N/2} × exp(i/ħ Σ_{a=0}^{N-1} Δt [m/2 ((q_{a+1} − q_a)/Δt)^2 − V(q_{a+1})])
replace this by
V(q_a)+V(q_{a+1}) / 2
O(1/N) term diff
= lim N→∞ (Δt→0) ∫ D[q(s)] · exp(i/ħ S[q]) = K(q_F, q_I; t)
measure over histories q(s)
q(0) = q_I
q(t) = q_F
□ = ∫_0^t ds [m/2 (dq/ds)^2 − V(q)] = S[q]
classical Action
classical Lagrangian
time = s
t_a = aΔt
piece-wise linear "histories" q(t)

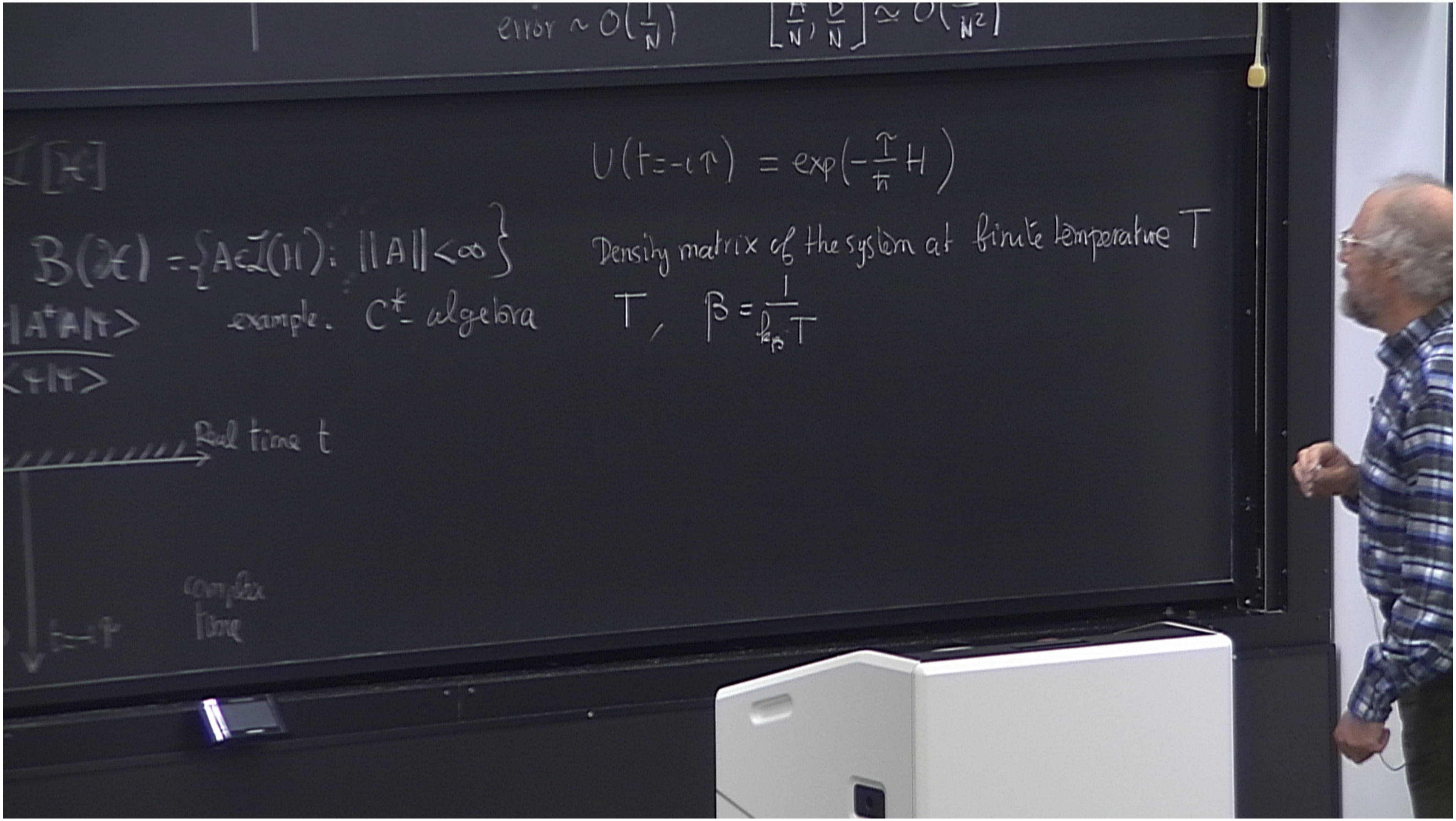


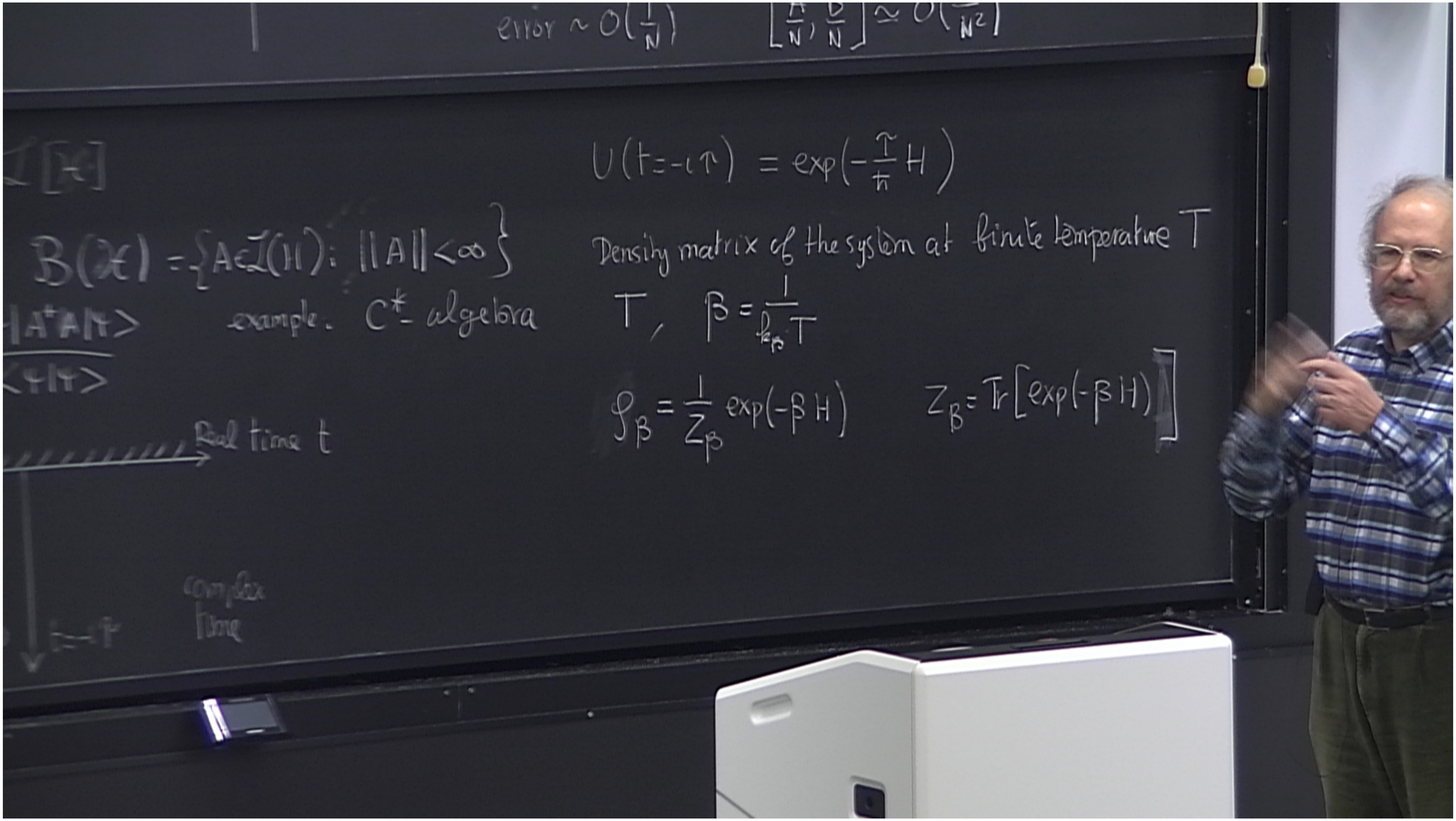
dynamics is trivial
Quantum Theory = Sum of Interferences of "classical-like" trajectories
Path Integral
Imaginary time formalism.
U(t) = exp(t/iħ H)
unitary operator
if t is real

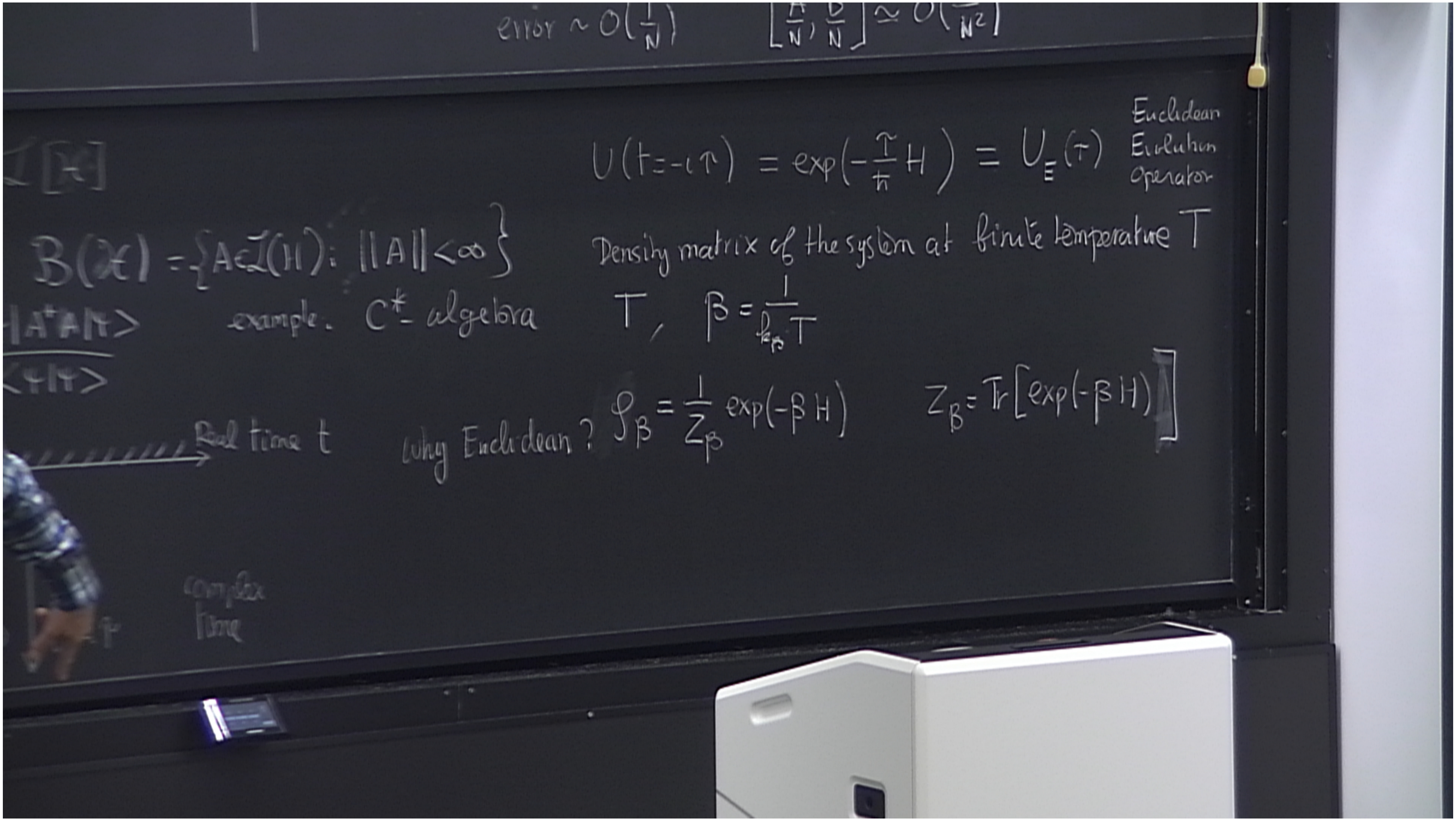

dynamics is trivial
Quantum Theory = Sum of Interferences of "classical-like" trajectories
Path Integral
- Linear Operator. on $\mathcal{H}$ $\mathcal{L}[\mathcal{H}]$
- Bounded operator $\mathcal{B}(\mathcal{H})$
$\|A\|^2 = \sup_{|\psi\rangle \in \mathcal{H}} \frac{\langle\psi|A^+A|\psi\rangle}{\langle\psi|\psi\rangle}$
sup-nam
Imaginary time fumalism.
$U(t) = \exp\left(\frac{t}{i\hbar} H\right)$
unitary operator if t is real

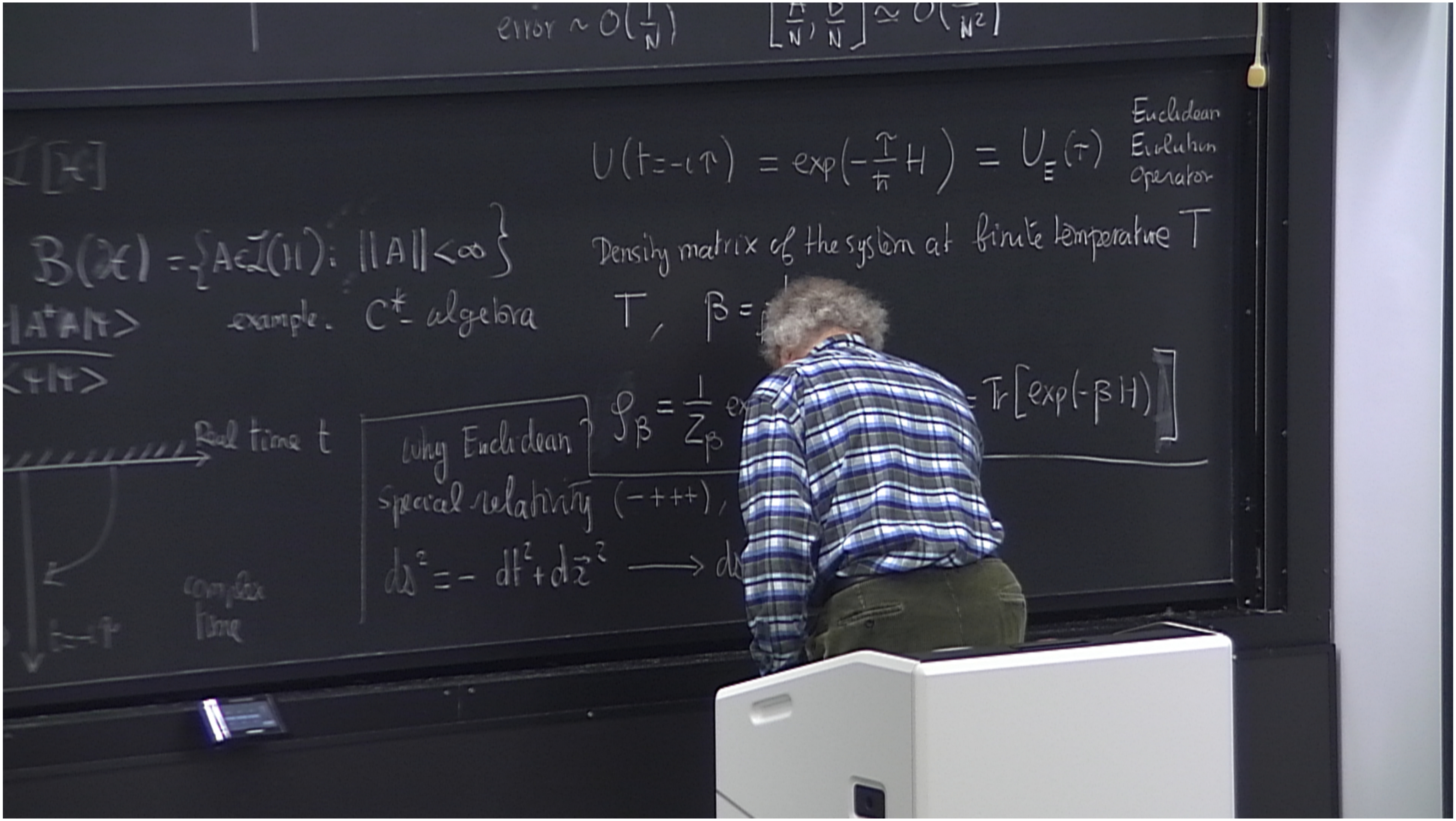

dynamics is trivial
Quantum Theory = Sum of Interferences of "classical-like" trajectories
Path Integral
- Linear operator. on H L[H]
- Bounded operator B(H) = {A ∈ L(H) ; ||A|| < ∞}
||A||² = Sup |ψ>∈H <ψ|A†A|ψ> / <ψ|ψ>
Sup-norm
example. C*- algebra
Imaginary time formalism.
U(t) = exp(t/iħ H)
unitary operator if t is real
Spectrum of H is bounded from below (ground state)
if t is complex
||U(t)|| < ∞ if Im(t) ≥ 0

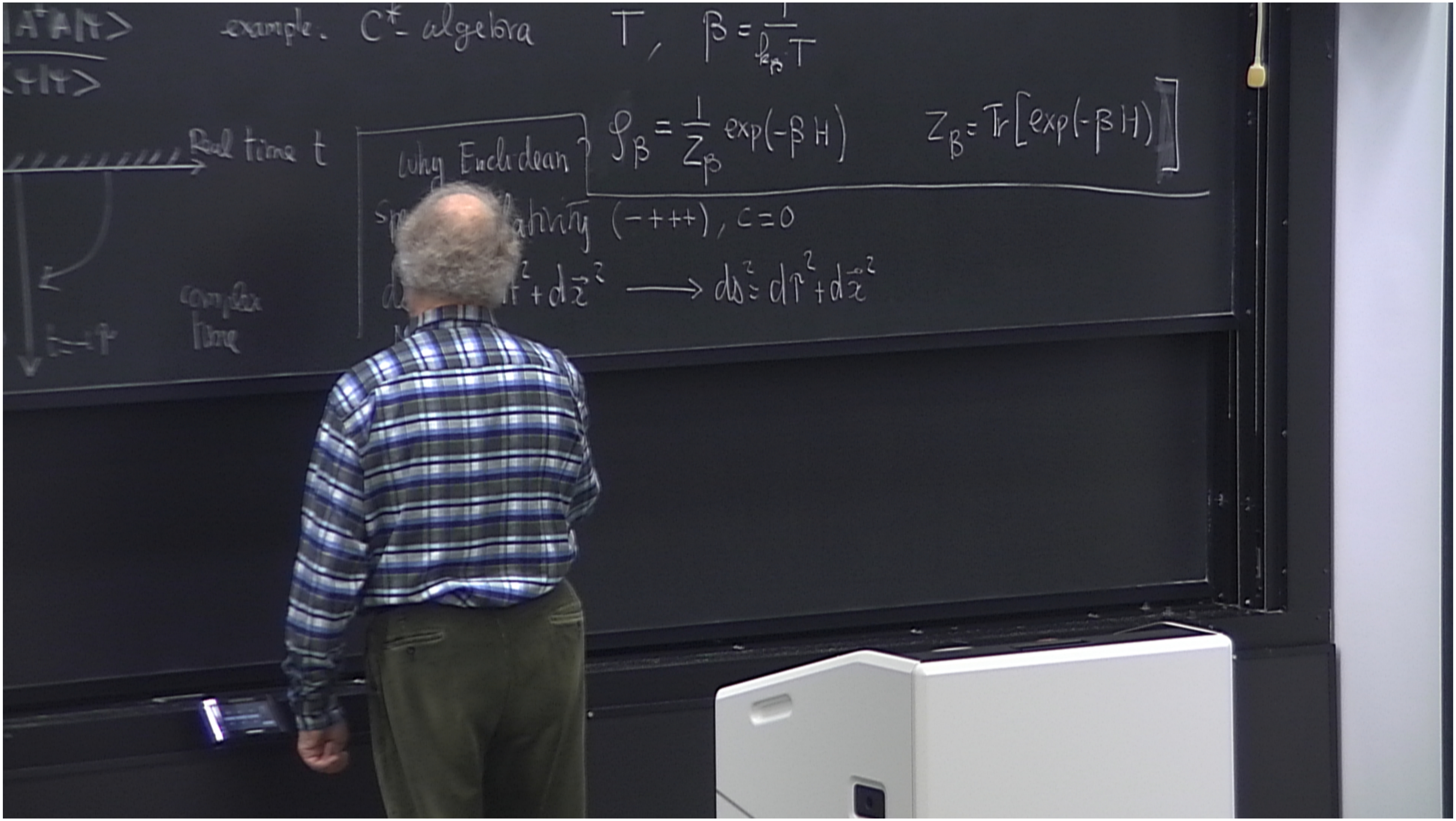

error ~ O(1/N)
U(t=-iτ) = exp(-τ/ħ H)
B(H) = {A∈L(H): ||A||<∞}
Density matri
example. C*- algebra
Real time t
complex
time

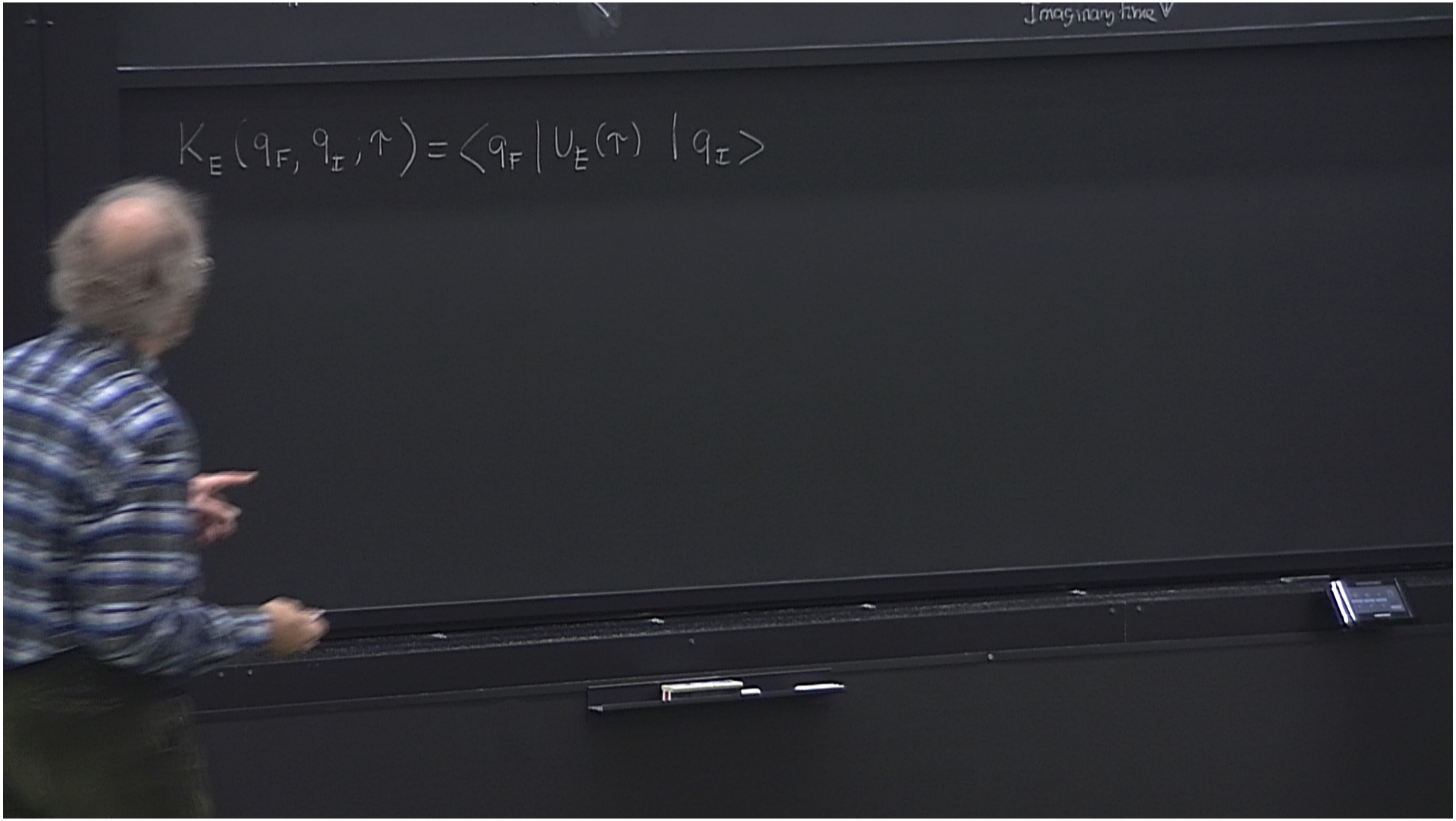

error ~ O(1/N)
U(t=-iτ) = exp(-τ/ħ H)
B(H) = {A∈L(H): ||A||<∞}
Density matrix of the system at finite temperature T
example. C*- algebra
T, β = 1/(k_B T)
Real time t
complex time

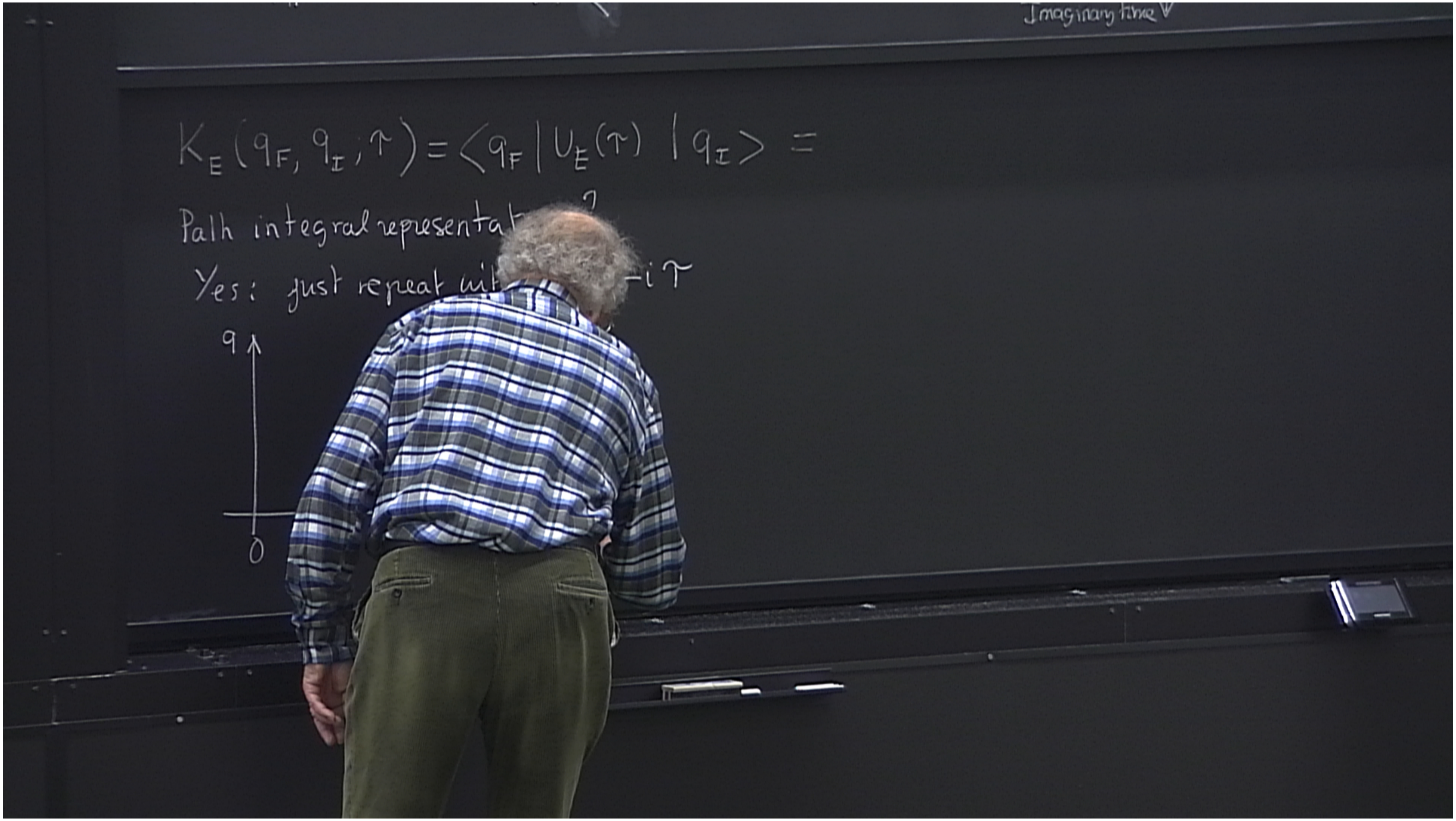

error ~ O(1/N)
U(t=-iτ) = exp(-τ/ħ H)
Density matrix of the system at finite temperature T
T, β = 1/(k_B T)
ρ_β = 1/Z_β exp(-β H)
Z_β = Tr[exp(-β H)]
B(H) = {A∈L(H): ||A||<∞}
example. C*- algebra
Real time t
complex time

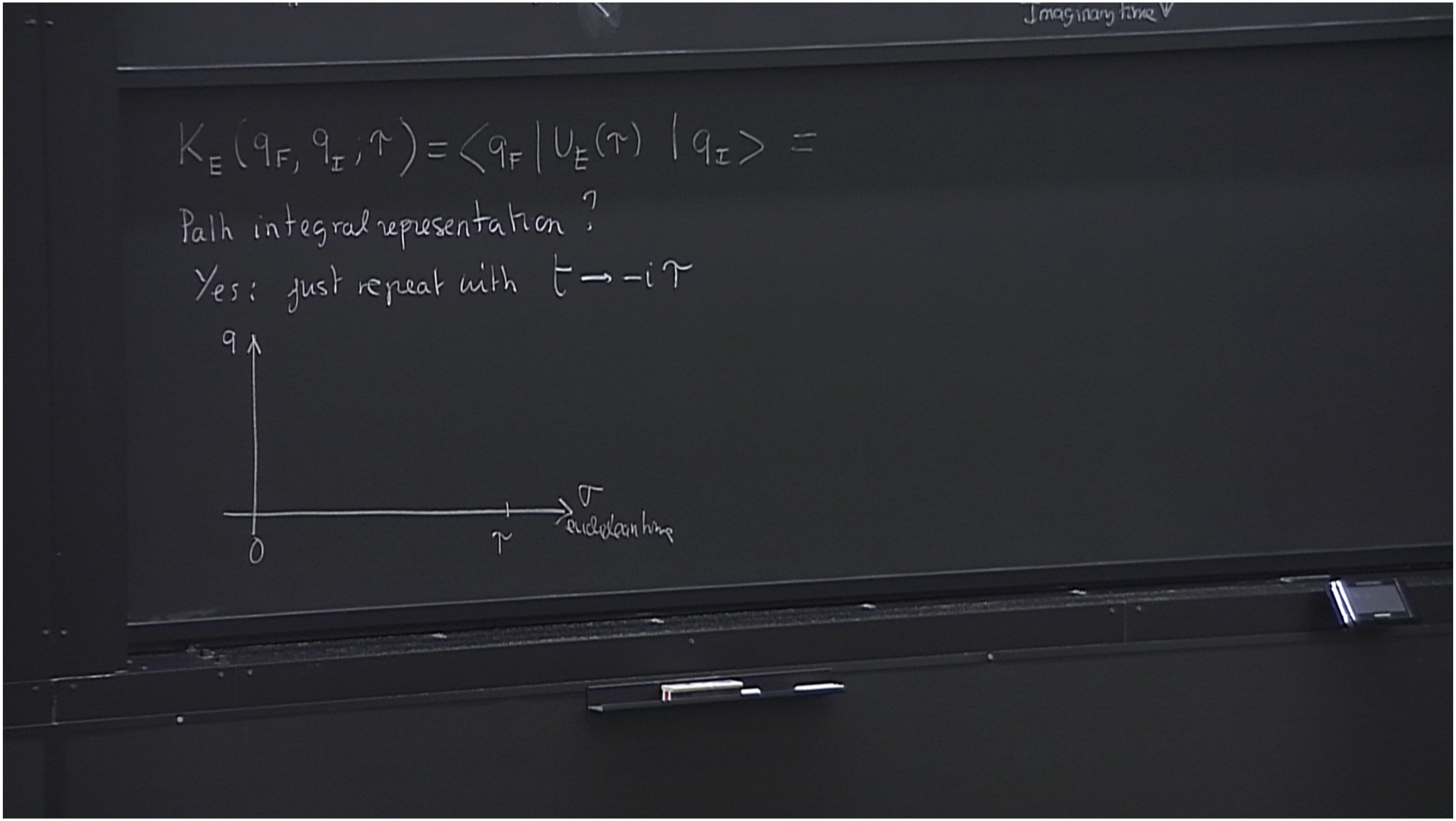

error ~ O(1/N)
U(t=-iτ) = exp(-τ/ħ H) = U_E(τ)
Euclidean Evolution Operator
B(H) = {A∈L(H): ||A||<∞}
example. C*-algebra
Density matrix of the system at finite temperature T
T, β = 1/(k_B T)
ρ_β = 1/Z_β exp(-βH)
Z_β = Tr[exp(-βH)]
Why Euclidean?
Real time t
complex time

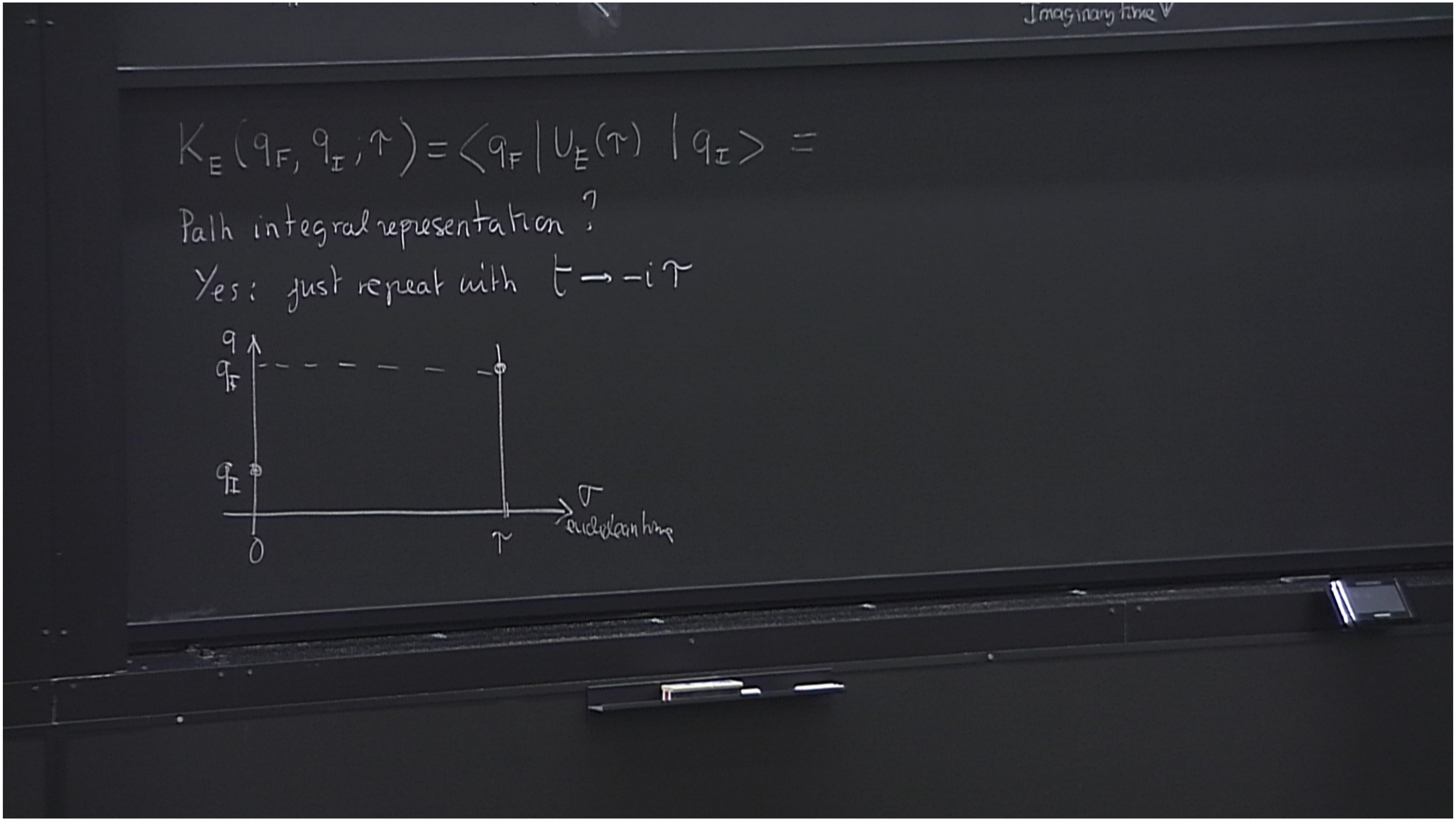

error ~ O(1/N)
U(t=-iτ) = exp(-τ/ħ H) = U_E(τ)
Euclidean Evolution Operator
B(H) = {A∈L(H): ||A||<∞}
example. C*-algebra
Density matrix of the system at finite temperature T
T, β =
ρ_β = 1/Z_β
= Tr[exp(-βH)]
Real time t
Why Euclidean?
Special relativity (-+++),
ds² = - dt² + dx²
complex time

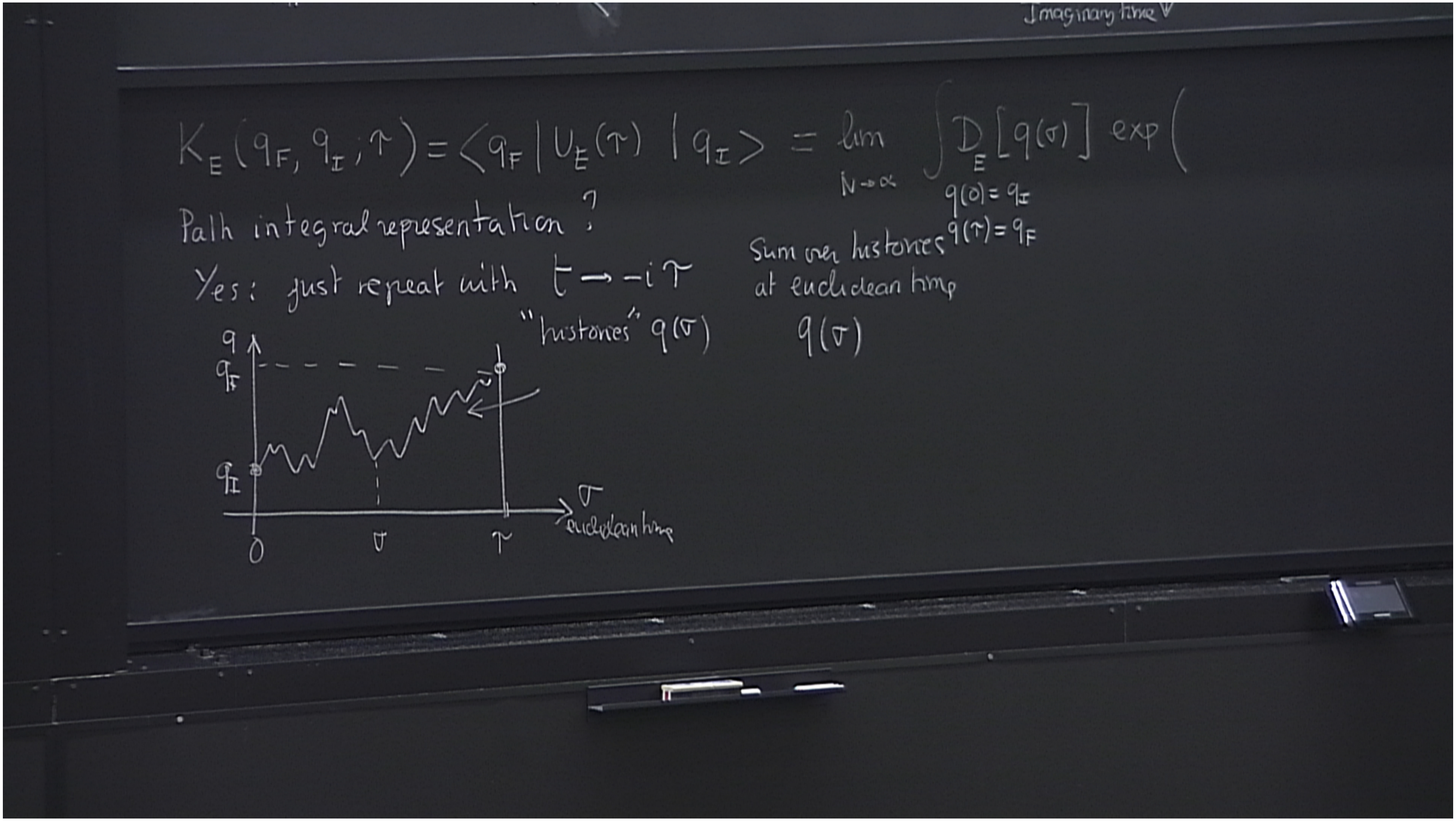

example. C*- algebra
Real time t
Why Euclidean?
complex time
(−+++), c=0

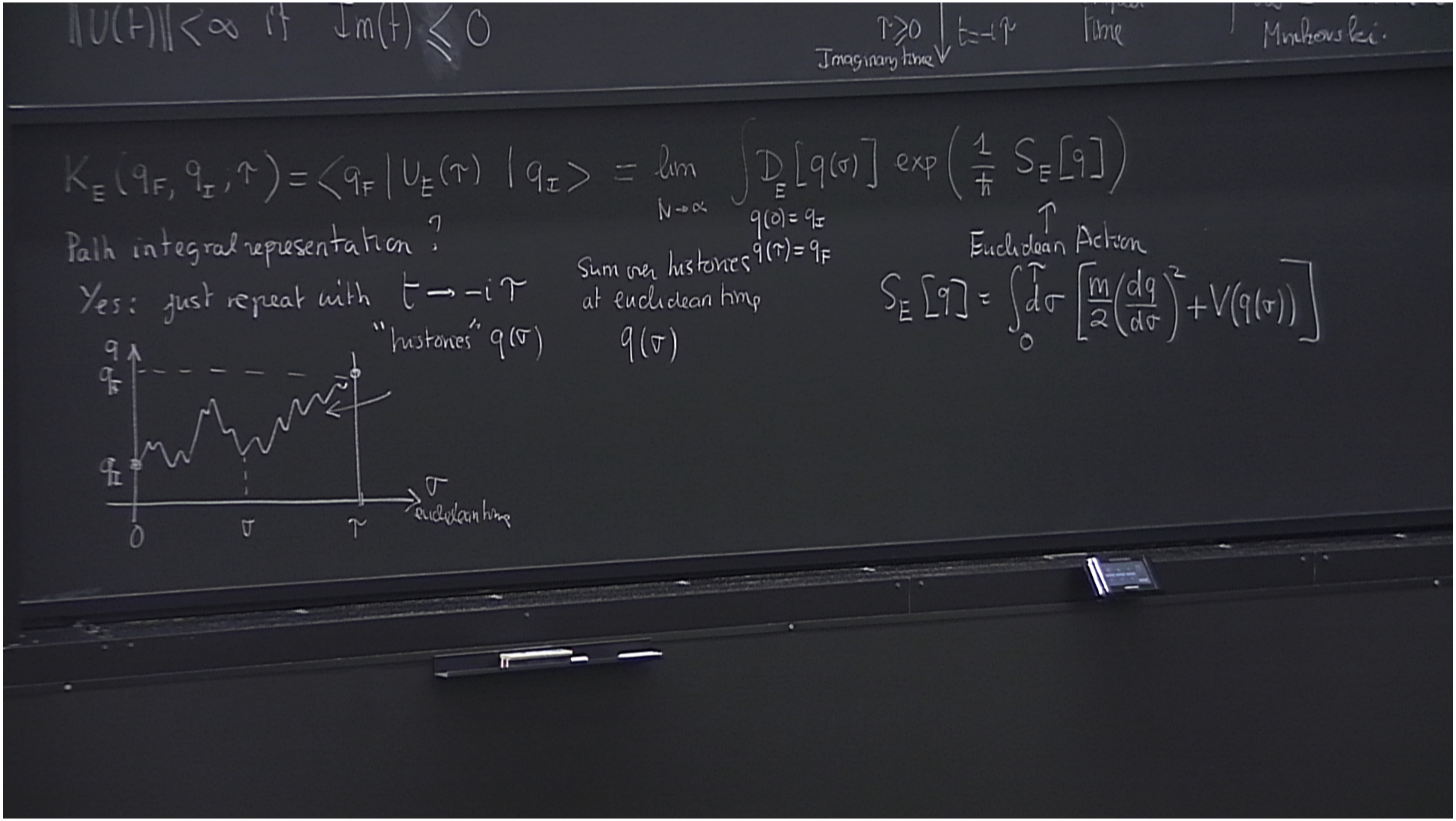

Imaginary time
$K_E(q_F, q_I; \tau) = \langle q_F | U_E(\tau) | q_I \rangle$


Imaginary time
$K_E(q_F, q_I; \tau) = \langle q_F | U_E(\tau) | q_I \rangle =$
Path integral representat
Yes: just repeat wi
$-i\tau$
q
0

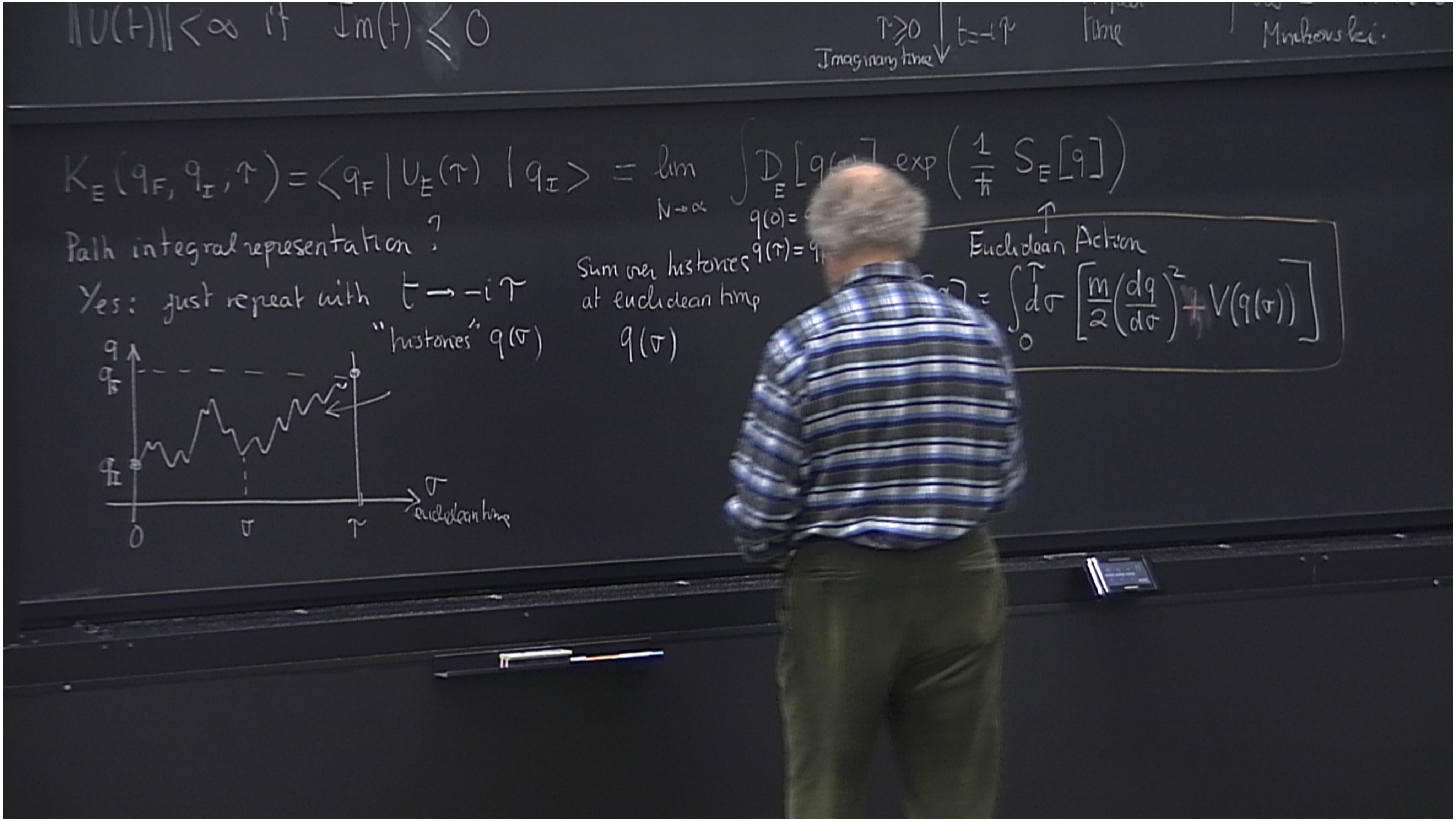

Imaginary time
K_E(q_F, q_I; τ) = <q_F | U_E(τ) | q_I> =
Path integral representation ?
Yes: just repeat with t → −iτ
q
0
τ
τ
euclidean time

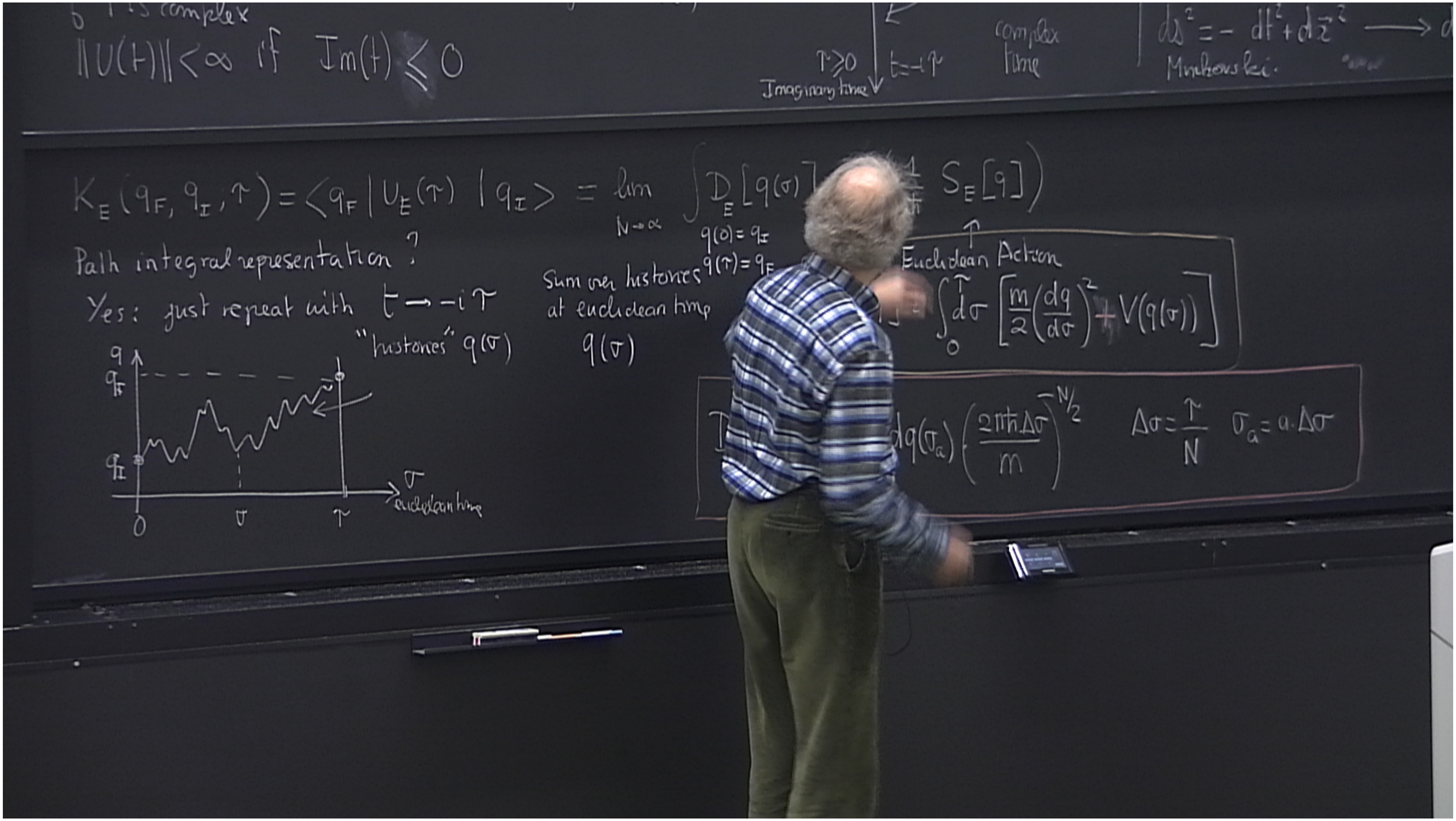

Imaginary time
$K_E(q_F, q_I; \tau) = \langle q_F | U_E(\tau) | q_I \rangle =$
Path integral representation ?
Yes: just repeat with $t \to -i\tau$
q
q_F
q_I
τ
euclidean time
0
τ


Imaginary time
$K_E(q_F, q_I; \tau) = \langle q_F | U_E(\tau) | q_I \rangle = \lim_{N\to\infty} \int D_E[q(\sigma)] \exp($
$q(0) = q_I$
$q(\tau) = q_F$
Path integral representation ?
Yes: just repeat with $t \to -i\tau$
Sum over histories at euclidean time
$q(\sigma)$
"histories" $q(\sigma)$
q
q_F
q_I
0
σ
τ
σ
euclidean time

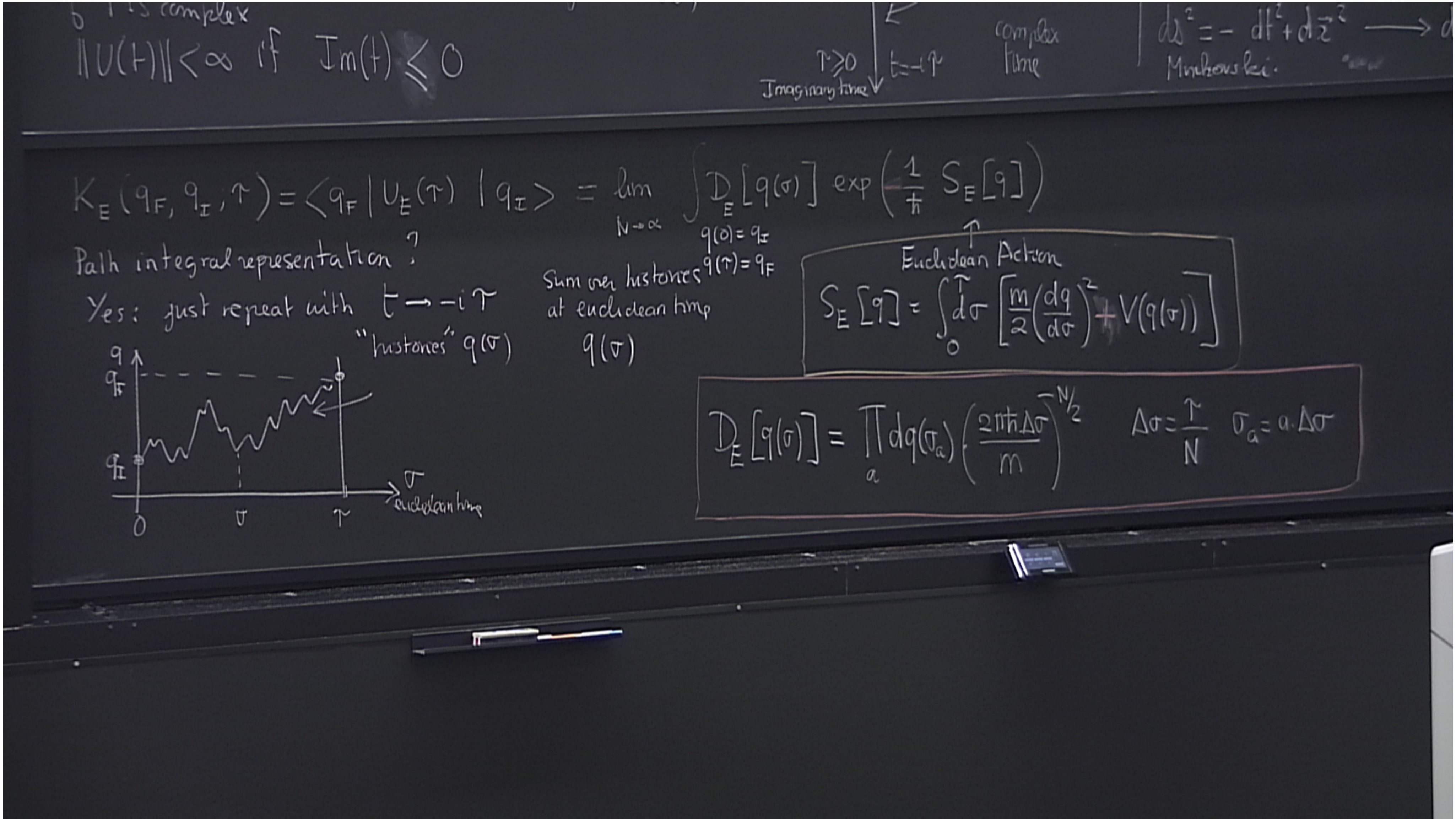


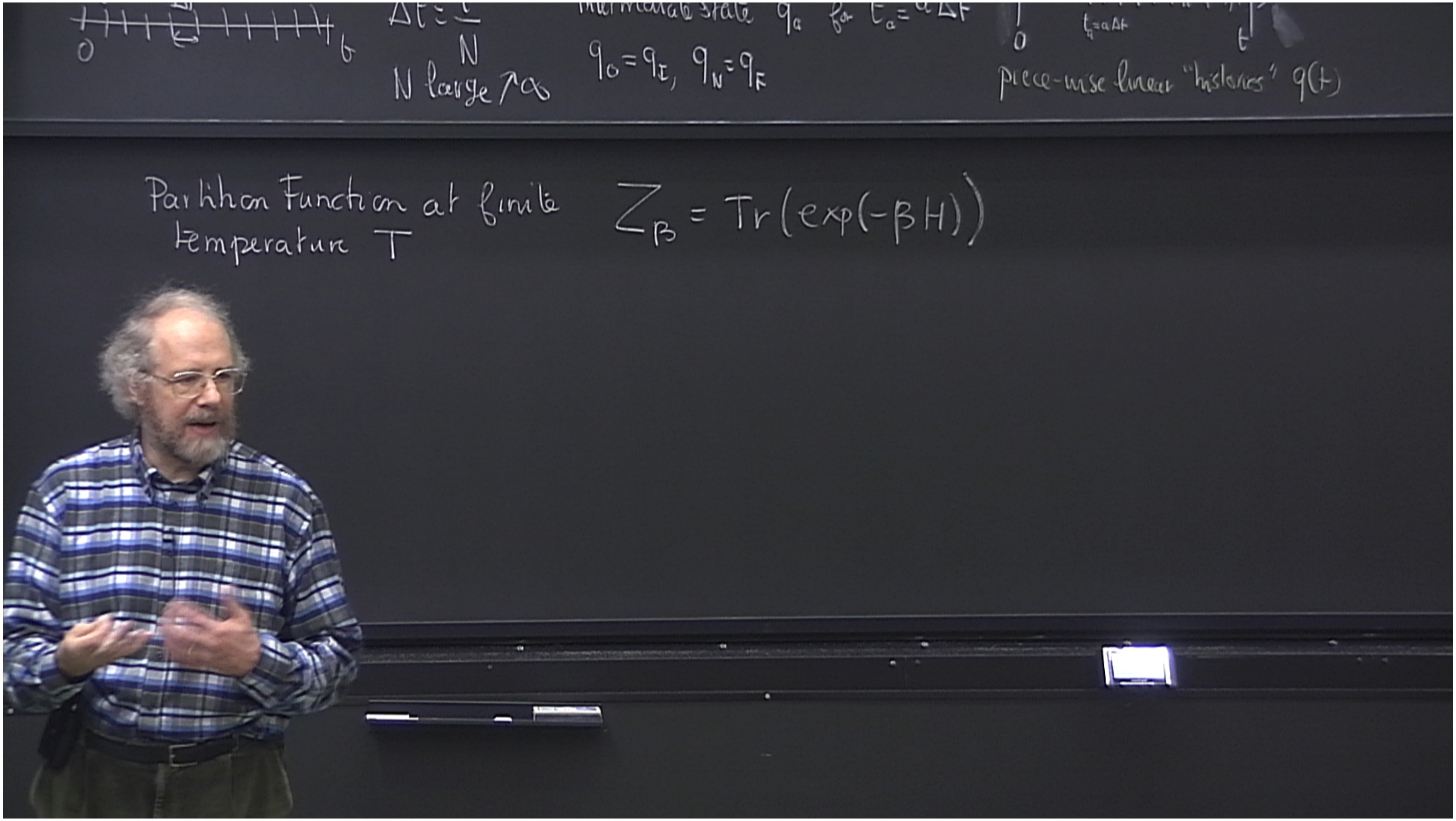
$\|U(t)\| < \infty$ if $\mathrm{Im}(t) \leq 0$
$\tau \geq 0$ $t = -i\tau$
Imaginary time
time
Minkowski.
$K_E(q_F, q_I; \tau) = \langle q_F | U_E(\tau) | q_I \rangle = \lim_{N \to \infty} \int D_E[q(\sigma)] \exp\left(\frac{1}{\hbar} S_E[q]\right)$
$q(0) = q_I$
$q(\tau) = q_F$
Euclidean Action
Path integral representation ?
Yes: just repeat with $t \to -i\tau$
Sum over histories at euclidean time
"histories" $q(\sigma)$
$q(\sigma)$
$S_E[q] = \int_0^\tau d\sigma \left[\frac{m}{2}\left(\frac{dq}{d\sigma}\right)^2 + V(q(\sigma))\right]$
q
q_F
q_I
σ
euclidean time
0
σ
τ

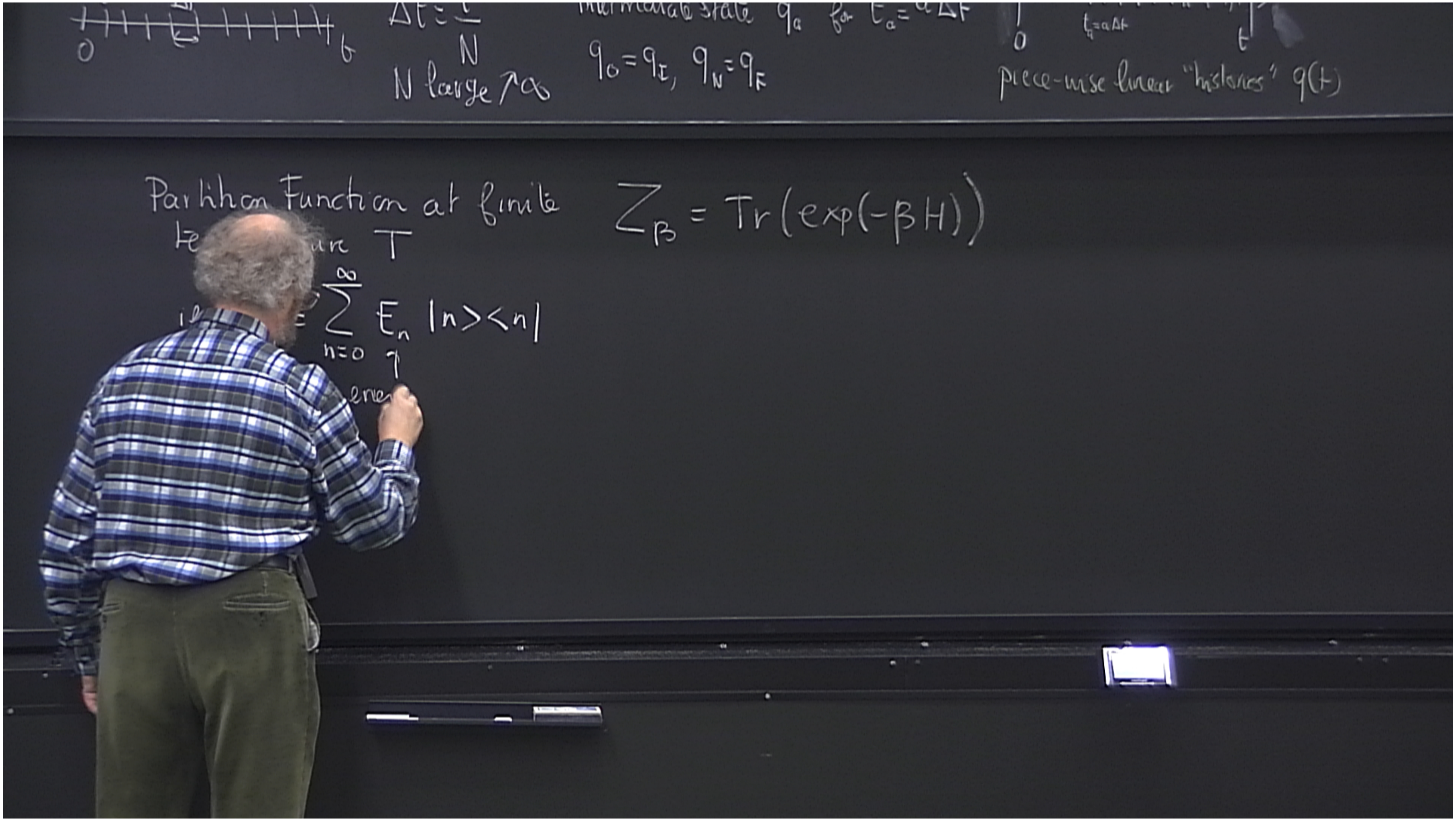

||U(t)|| < ∞ if Im(t) ≤ 0
τ≥0 t=-iτ
Imaginary time
time
Minkowski.
K_E(q_F, q_I, τ) = <q_F | U_E(τ) | q_I> = lim N→∞ ∫ D_E[q] exp(1/ħ S_E[q])
Path integral representation ?
Yes: just repeat with t → -iτ
"histories" q(σ)
Sum over histories at euclidean time
q(σ)
Euclidean Action
= ∫_0^τ dσ [m/2 (dq/dσ)^2 + V(q(σ))]
euclidean time

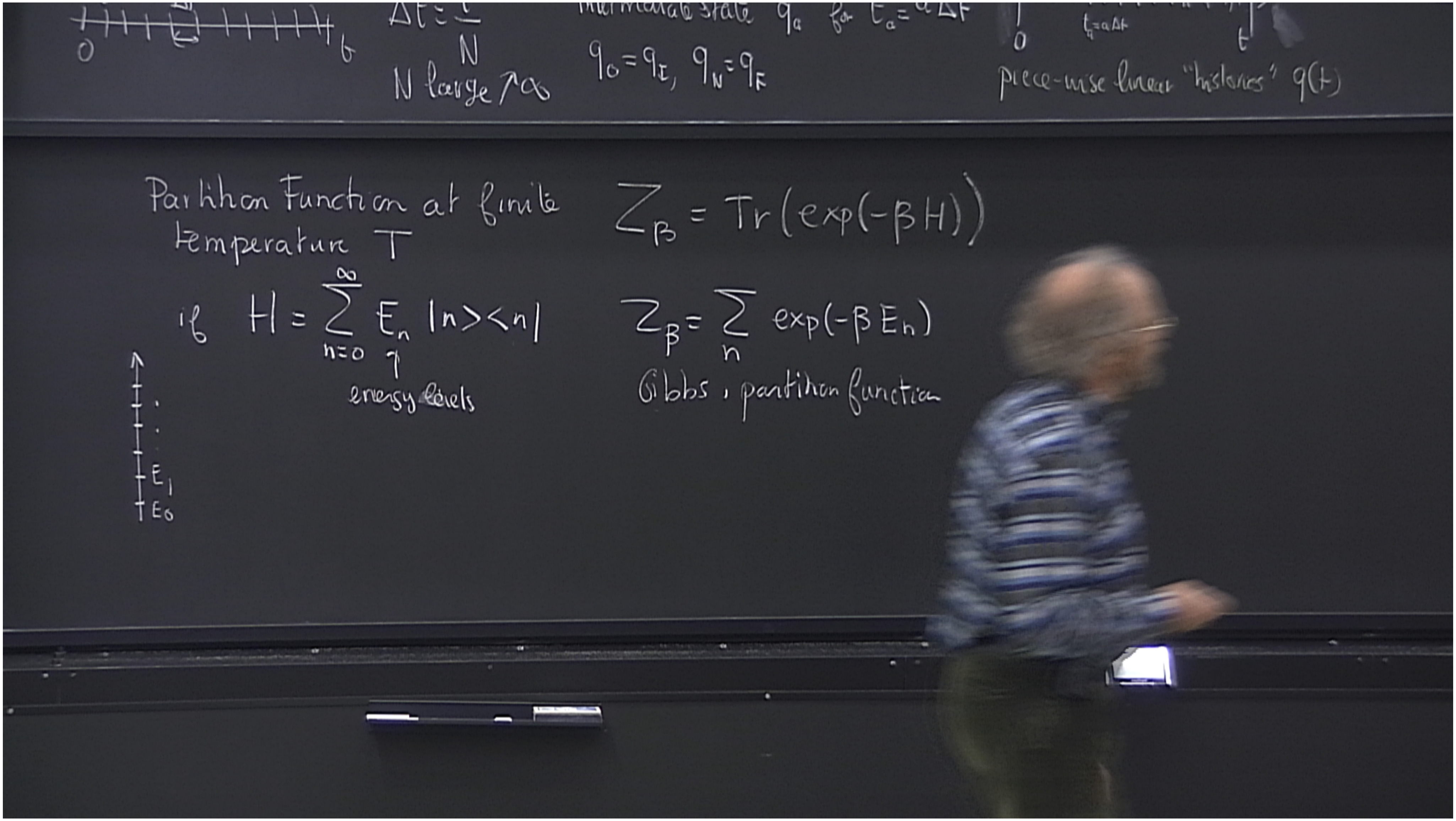


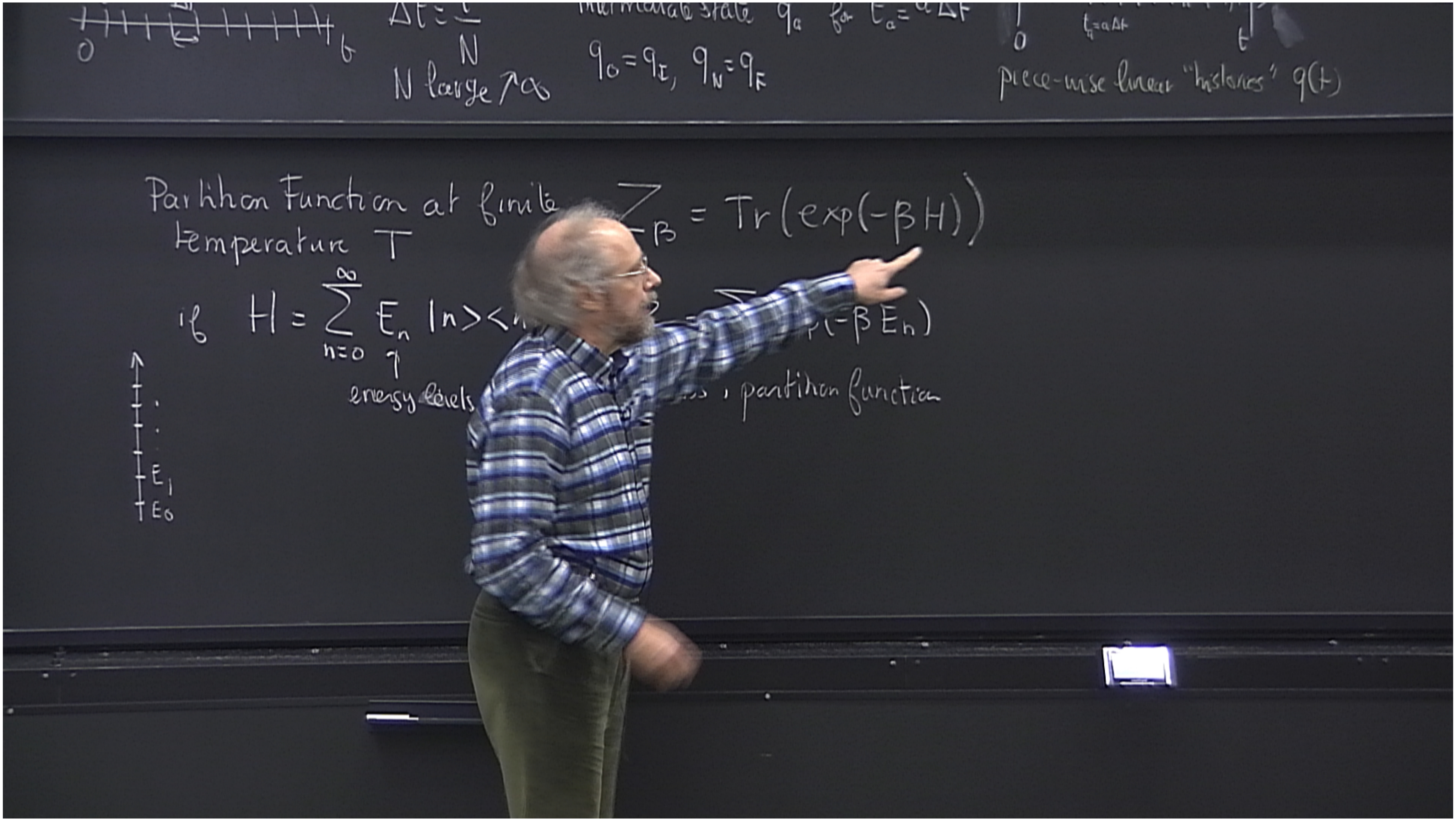
||U(t)|| < ∞ if Im(t) ≤ 0
τ ≥ 0
Imaginary time
t = -iτ
complex time
ds² = - dt² + dx²
Minkowski
K_E(q_F, q_I, τ) = <q_F | U_E(τ) | q_I> = lim N→∞ ∫ D_E[q(σ)]
S_E[q]
q(0) = q_I
q(τ) = q_F
Path integral representation?
Yes: just repeat with t → -iτ
Sum over histories at euclidean time
"histories" q(σ)
q(σ)
Euclidean Action
Δσ = τ/N
σ_a = a·Δσ
euclidean time


complex
||U(t)|| < ∞ if Im(t) ≤ 0
τ≥0 t=-iτ
Imaginary time
complex time
ds² = - dt² + dx²
Minkowski
K_E(q_F, q_I; τ) = ⟨q_F | U_E(τ) | q_I⟩ = lim N→∞ ∫ D_E[q(σ)] exp(-1/ħ S_E[q])
q(0) = q_I
q(τ) = q_F
Path integral representation ?
Yes: just repeat with t → -iτ
Sum over histories at euclidean time
"histories" q(σ)
q(σ)
Euclidean Action
S_E[q] = ∫₀^τ dσ [m/2 (dq/dσ)² + V(q(σ))]
D_E[q(σ)] = ∏_a dq(σ_a) (2πħΔσ/m)^(-N/2)
Δσ = τ/N
σ_a = a·Δσ
q
q_F
q_I
σ
euclidean time
0
σ
τ

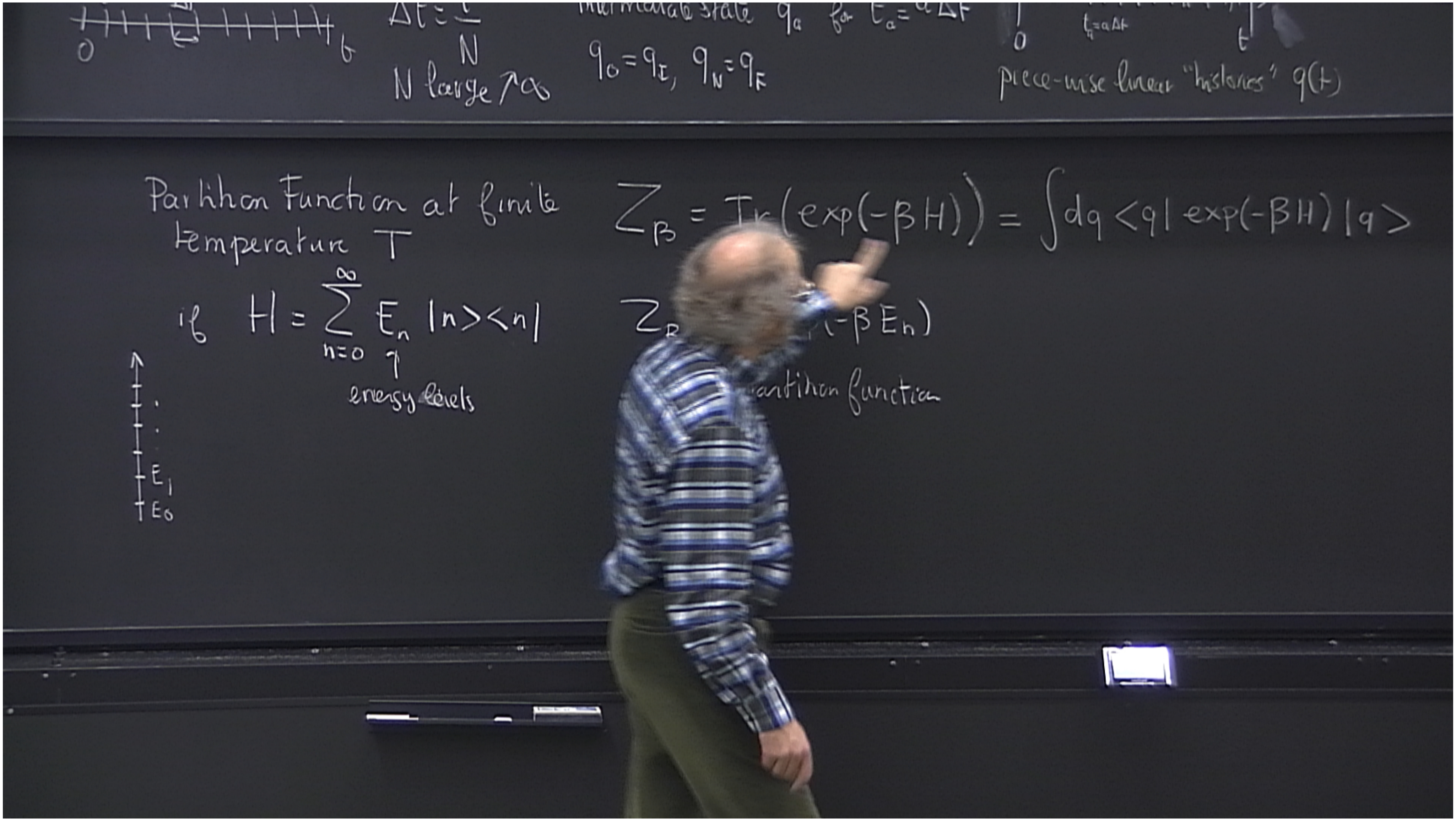

N large ↗ ∞
q_0 = q_I, q_N = q_F
piece-wise linear "histories" q(t)
Partition Function at finite temperature T
Z_β = Tr(exp(−βH))


N large ↗ ∞
q_0 = q_I, q_N = q_F
piece-wise linear "histories" q(t)
Partition Function at finite
T
Z_β = Tr(exp(−βH))
E_n |n><n|
n=0


N large ↗ ∞
$q_0 = q_I, \; q_N = q_F$
piece-wise linear "histories" q(t)
Partition Function at finite temperature T
$Z_\beta = Tr(\exp(-\beta H))$
if $H = \sum_{n=0}^{\infty} E_n |n\rangle\langle n|$
energy levels
$Z_\beta = \sum_n \exp(-\beta E_n)$
Gibbs, partition function
E_1
E_0



N large ↗∞
q_0 = q_I, q_N = q_F
piece-wise linear "histories" q(t)
Partition Function at finite temperature T
Z_β = Tr(exp(-βH))
if H = Σ_{n=0}^∞ E_n |n><n|
energy levels
partition function
E_1
E_0

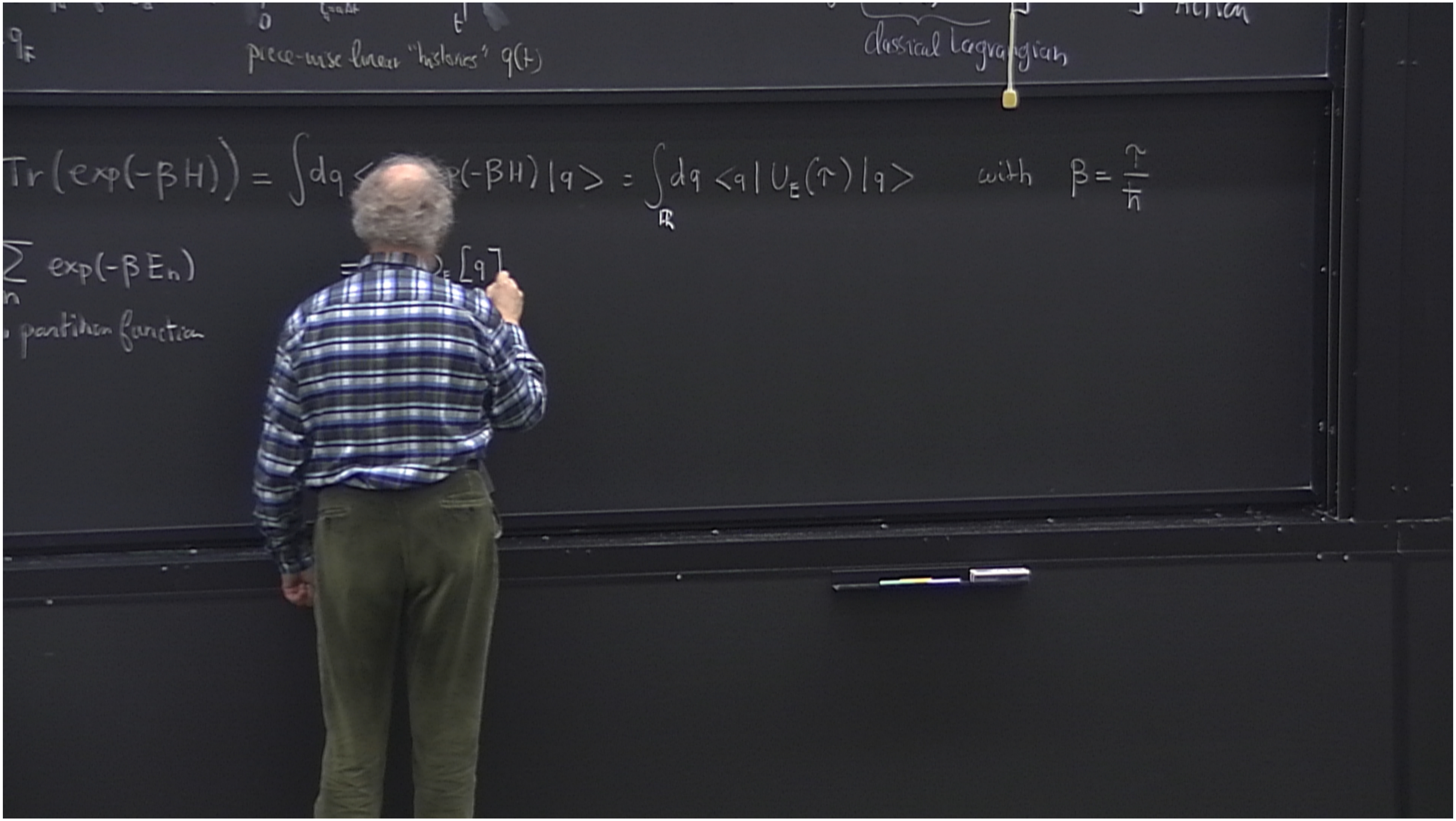

N large ↗ ∞
q_0 = q_I , q_N = q_F
piece-wise linear "histories" q(t)
Partition Function at finite temperature T
Z_β = Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q>
if H = Σ_{n=0}^{∞} E_n |n><n|
energy levels
E_1
E_0
partition function

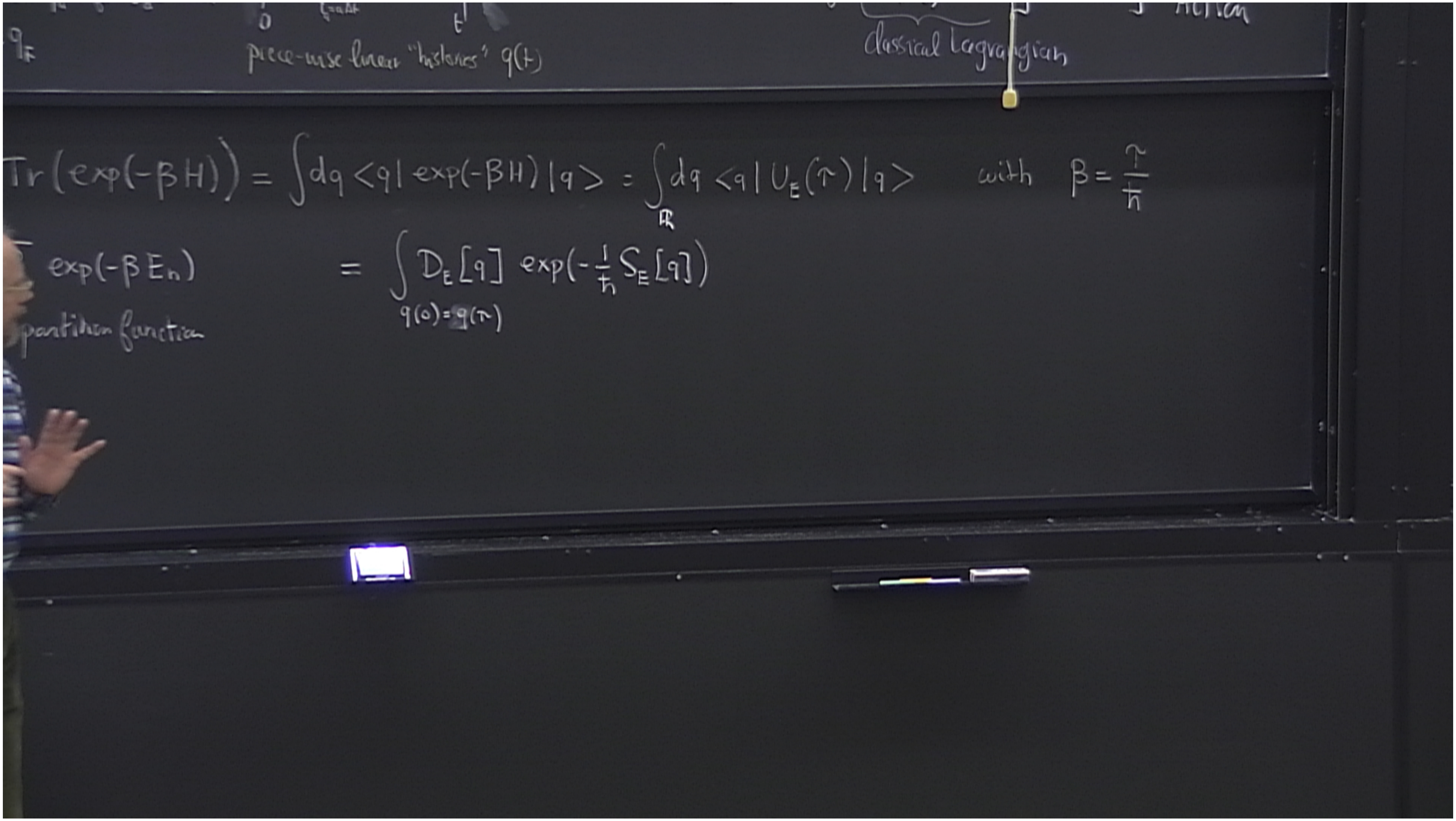
$q_0 = q_I, \; q_N = q_F$ piece-wise linear "histories" $q(t)$ Classical Lagrangian

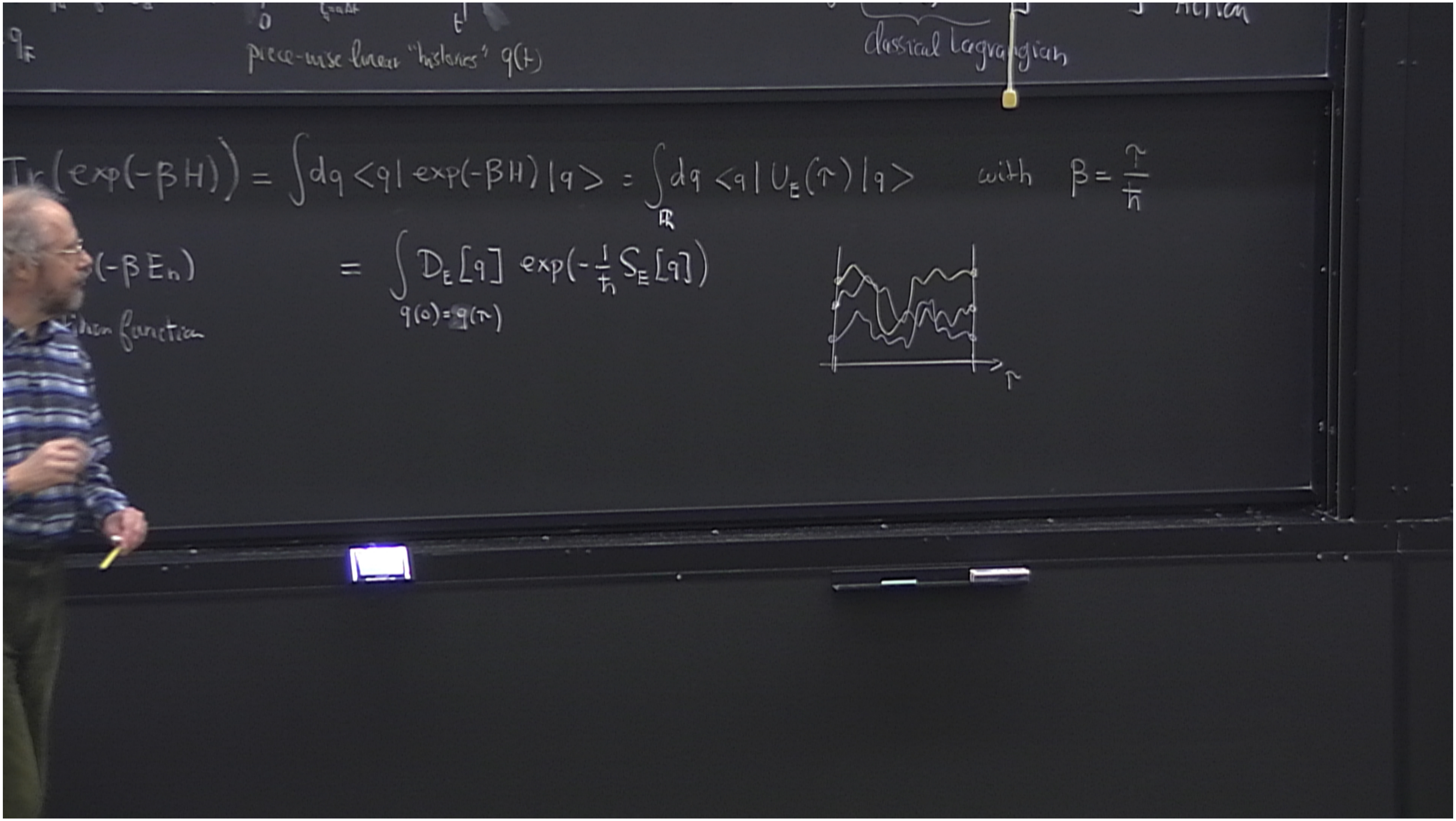
$$Z_\beta = \mathrm{Tr}\left(\exp(-\beta H)\right) = \int dq \, \langle q | \exp(-\beta H) | q \rangle = \int_{\mathbb{R}} dq \, \langle q | U_E(\tau) | q \rangle \quad \text{with}$$

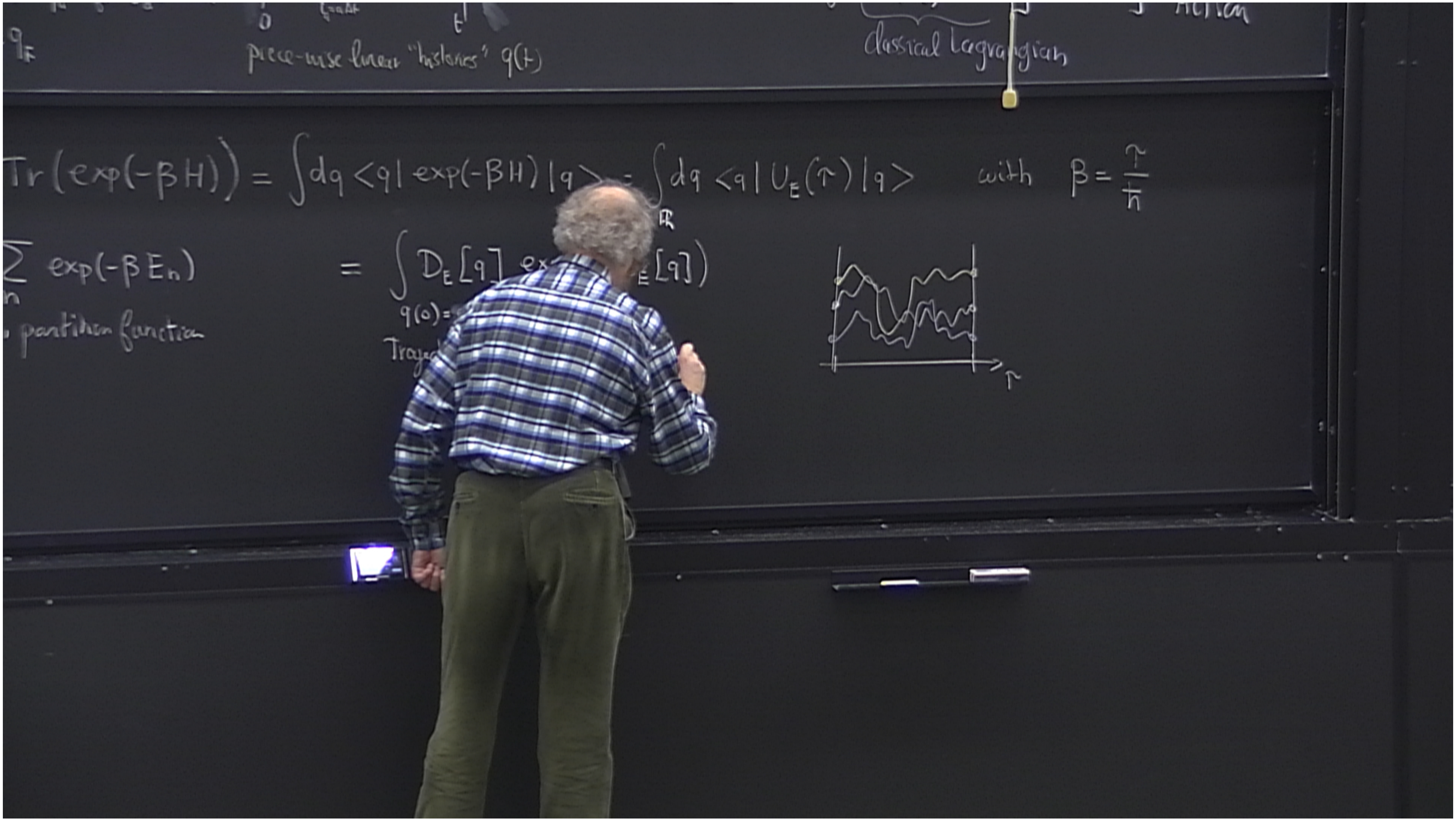
$$Z_\beta = \sum_n \exp(-\beta E_n)$$

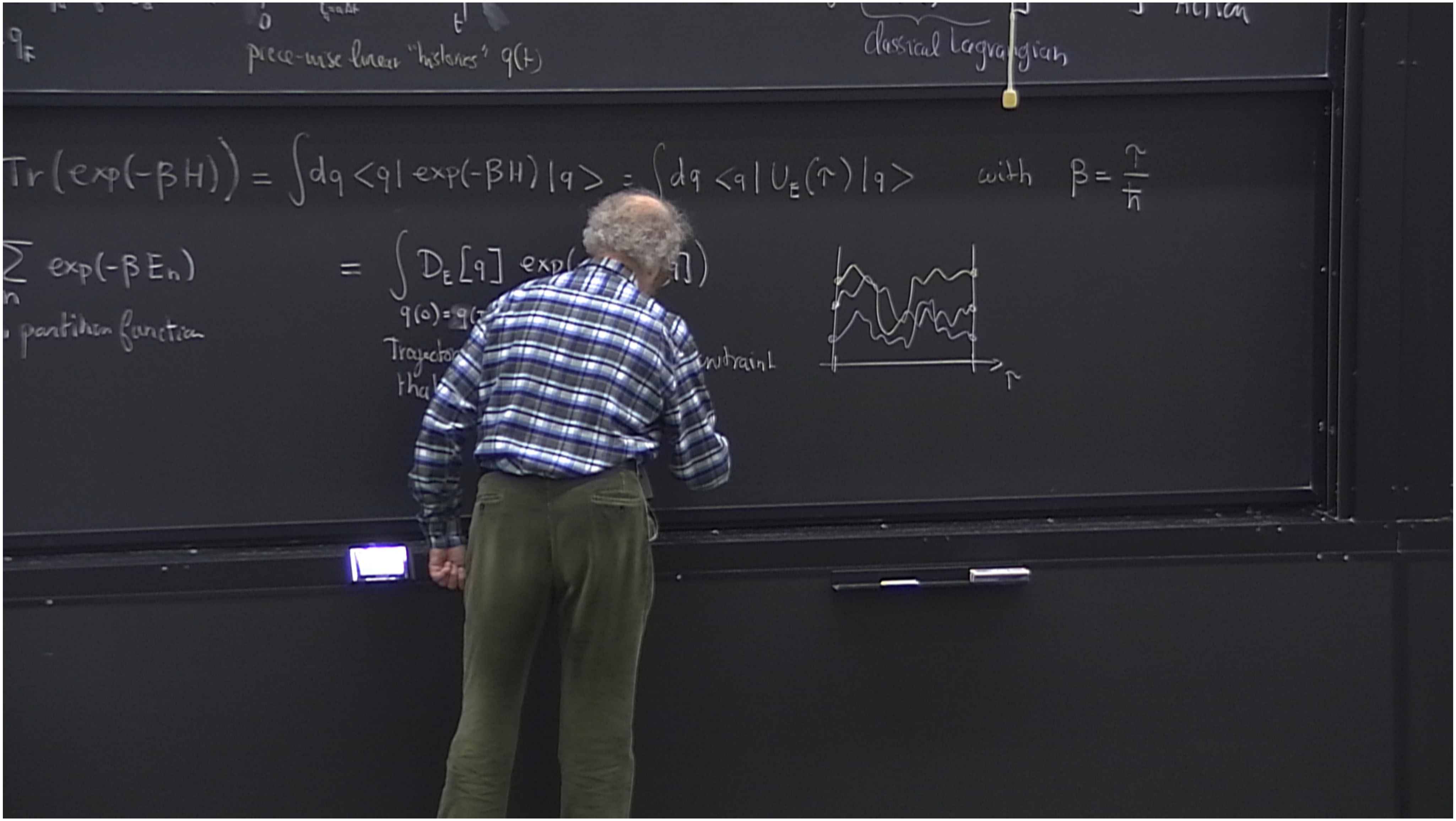
Gibbs, partition function

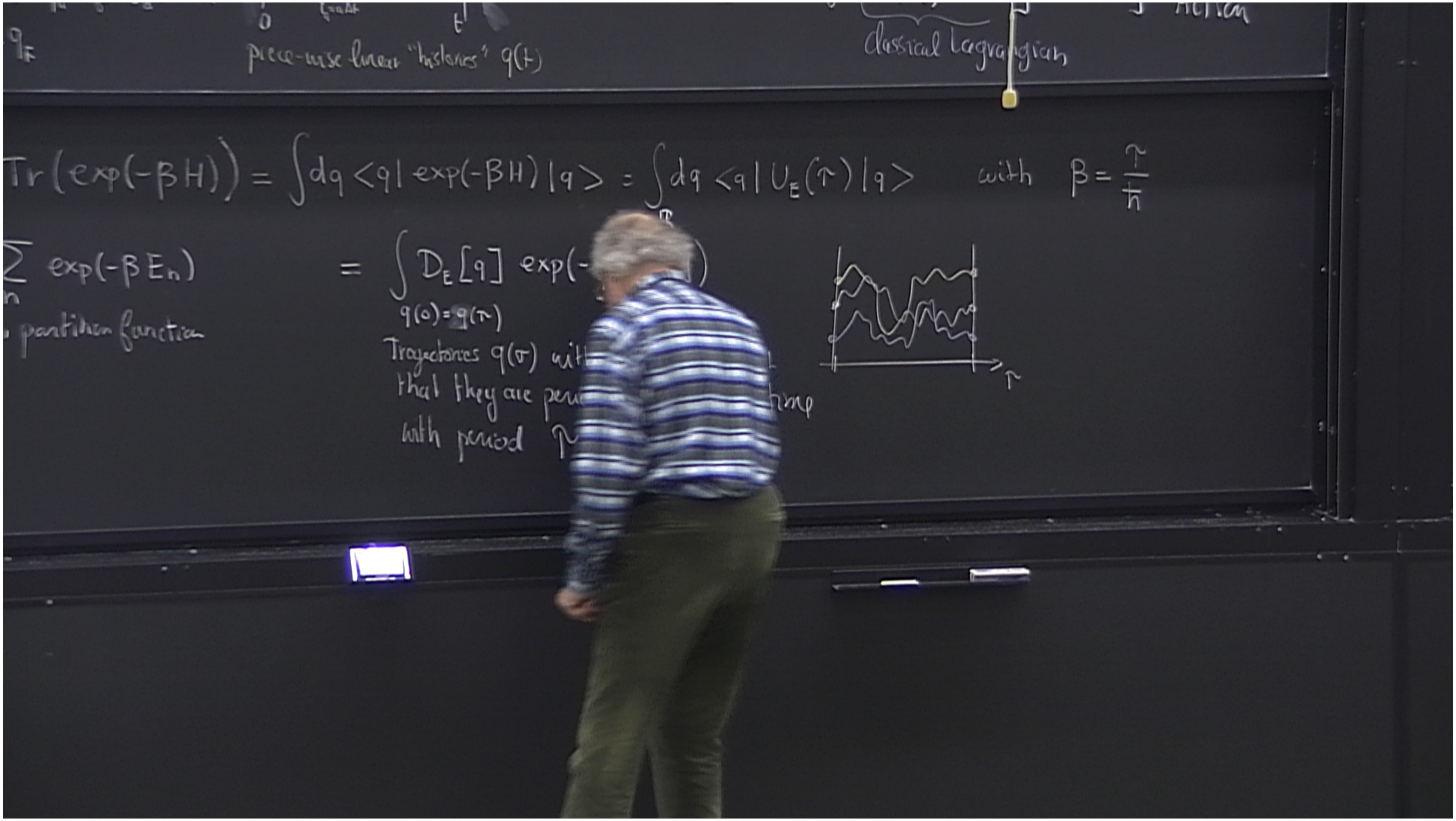

piece-wise linear "histories" q(t)
classical Lagrangian
with
partition function

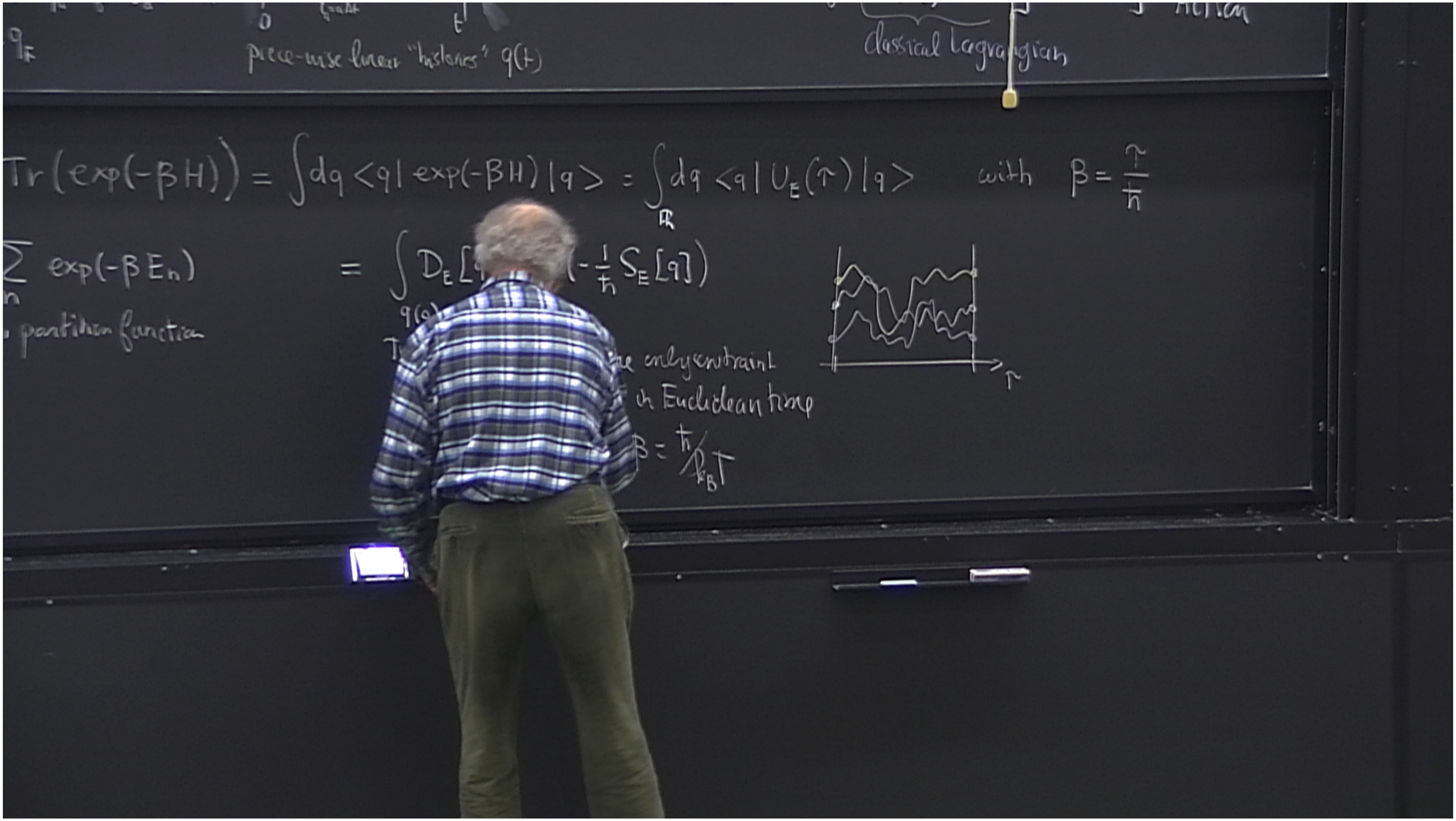

piece-wise linear "histories" q(t)
classical Lagrangian
Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q> = ∫dq <q| U_E(τ) |q> with β = τ/ħ
exp(-βE_n) = ∫D_E[q] exp(-1/ħ S_E[q])
q(0)=q(τ)
partition function


piece-wise linear "histories" q(t)
classical Lagrangian
Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q> = ∫dq <q| U_E(τ) |q> with β = τ/ħ
(-βE_n) = ∫ D_E[q] exp(-1/ħ S_E[q])
q(0)=q(τ)
function

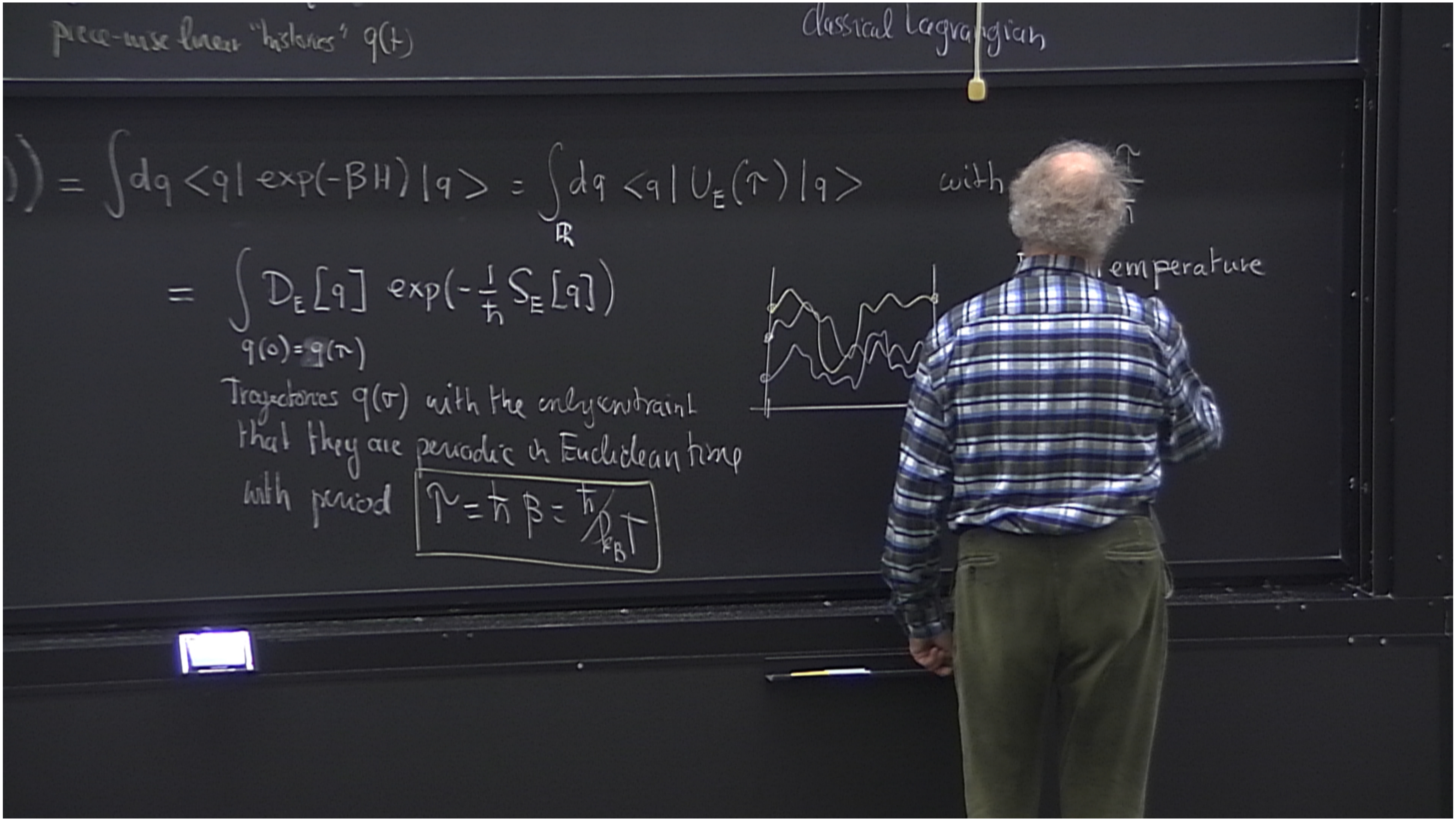

piece-wise linear "histories" q(t)
classical Lagrangian
Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q> = ∫dq <q| U_E(τ) |q>
with β = τ/ħ
∑ exp(-βE_n)
partition function


piece-wise linear "histories" q(t)
classical Lagrangian
Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q> = ∫dq <q| U_E(τ) |q>   with β = τ/ħ
Σ exp(-βE_n)
n
partition function
= ∫ D_E[q] exp(
q(0) = q(
Trajector
that
nstraint
τ

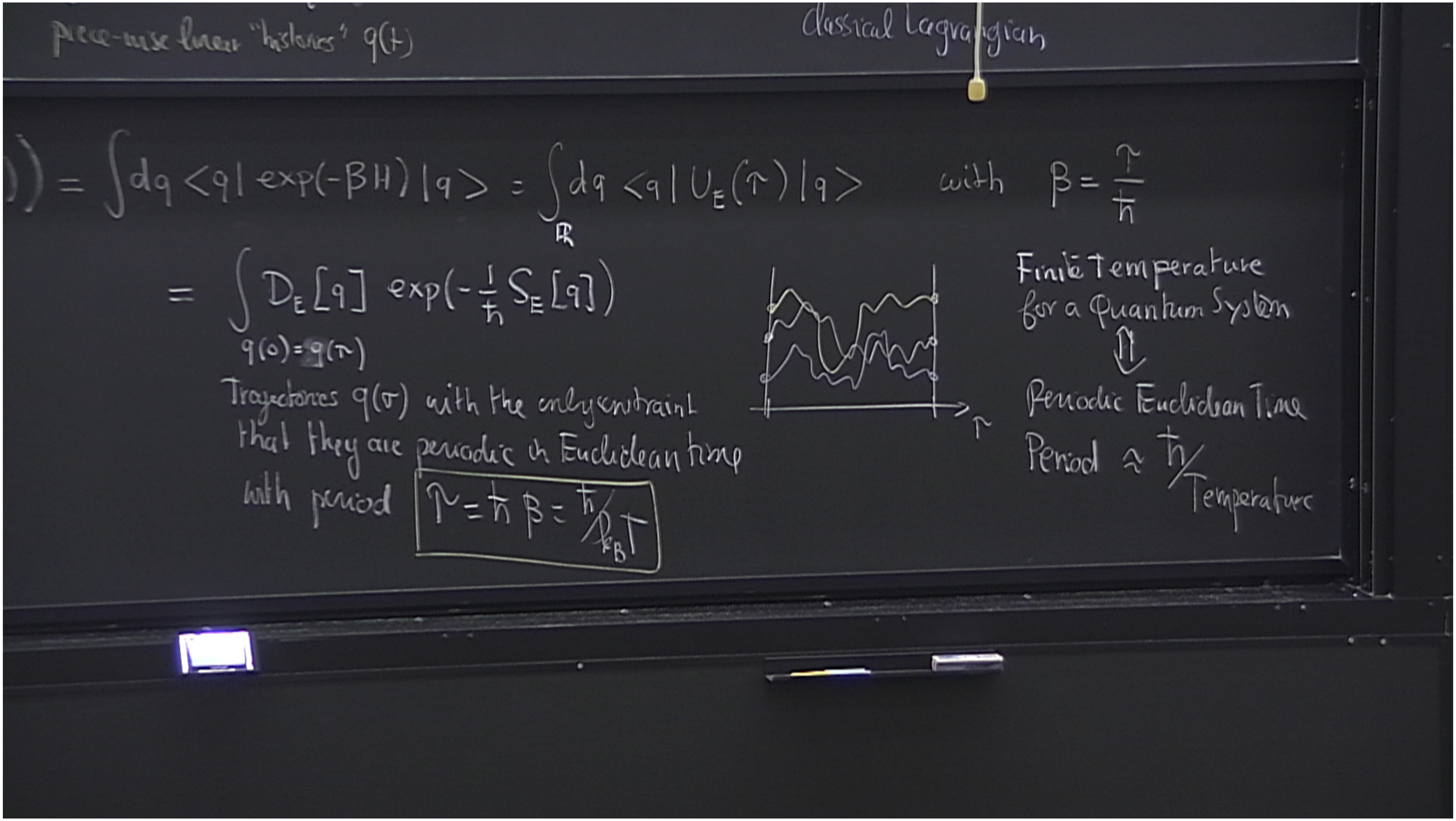

piece-wise linear "histories" q(t)
classical Lagrangian
Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q> = ∫dq <q| U_E(τ) |q> with β = τ/ħ
∑ exp(-βE_n)
partition function
= ∫ D_E[q] exp(-
q(0)=q(τ)
Trajectories q(σ)
that they are
with period τ

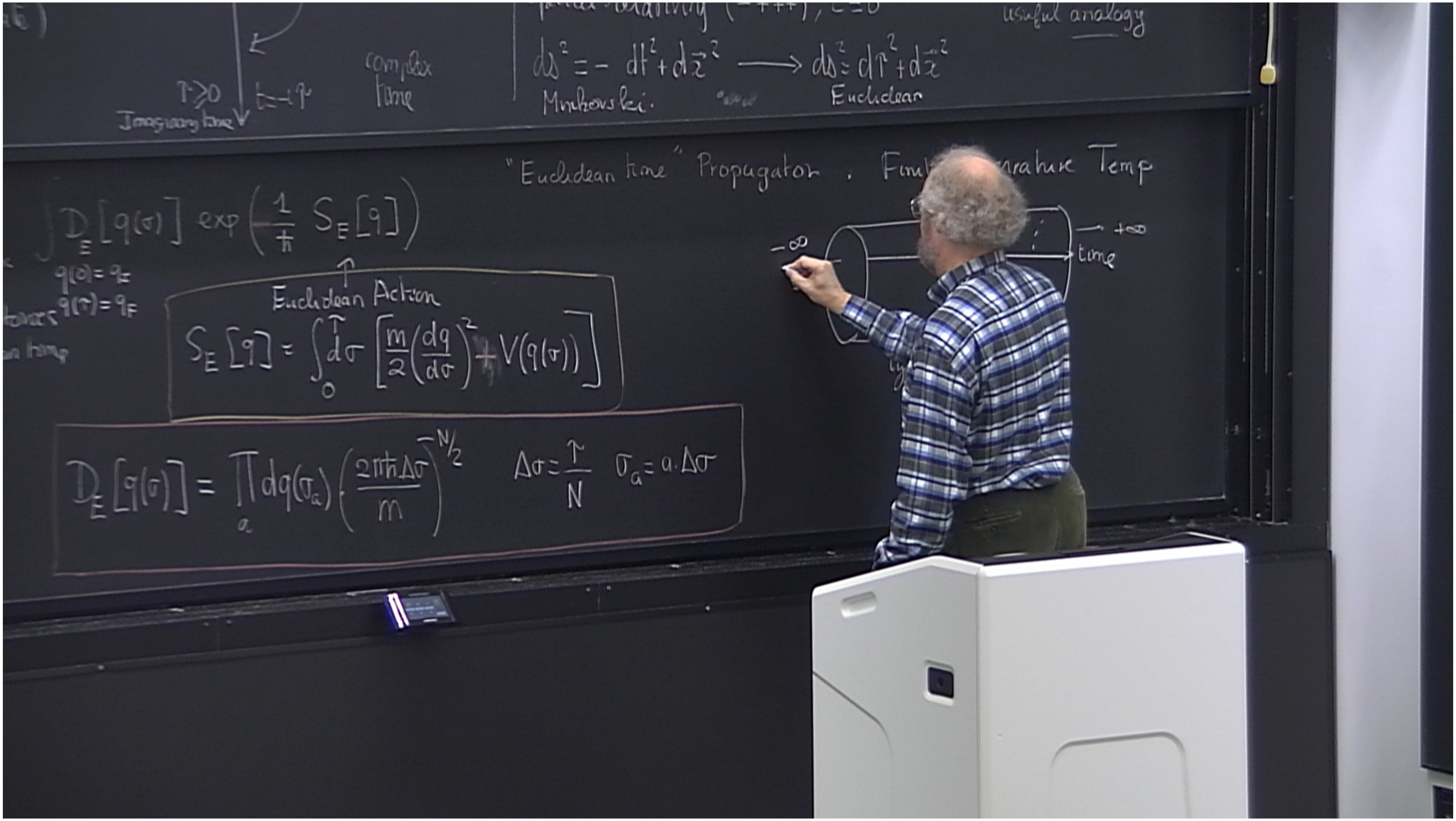


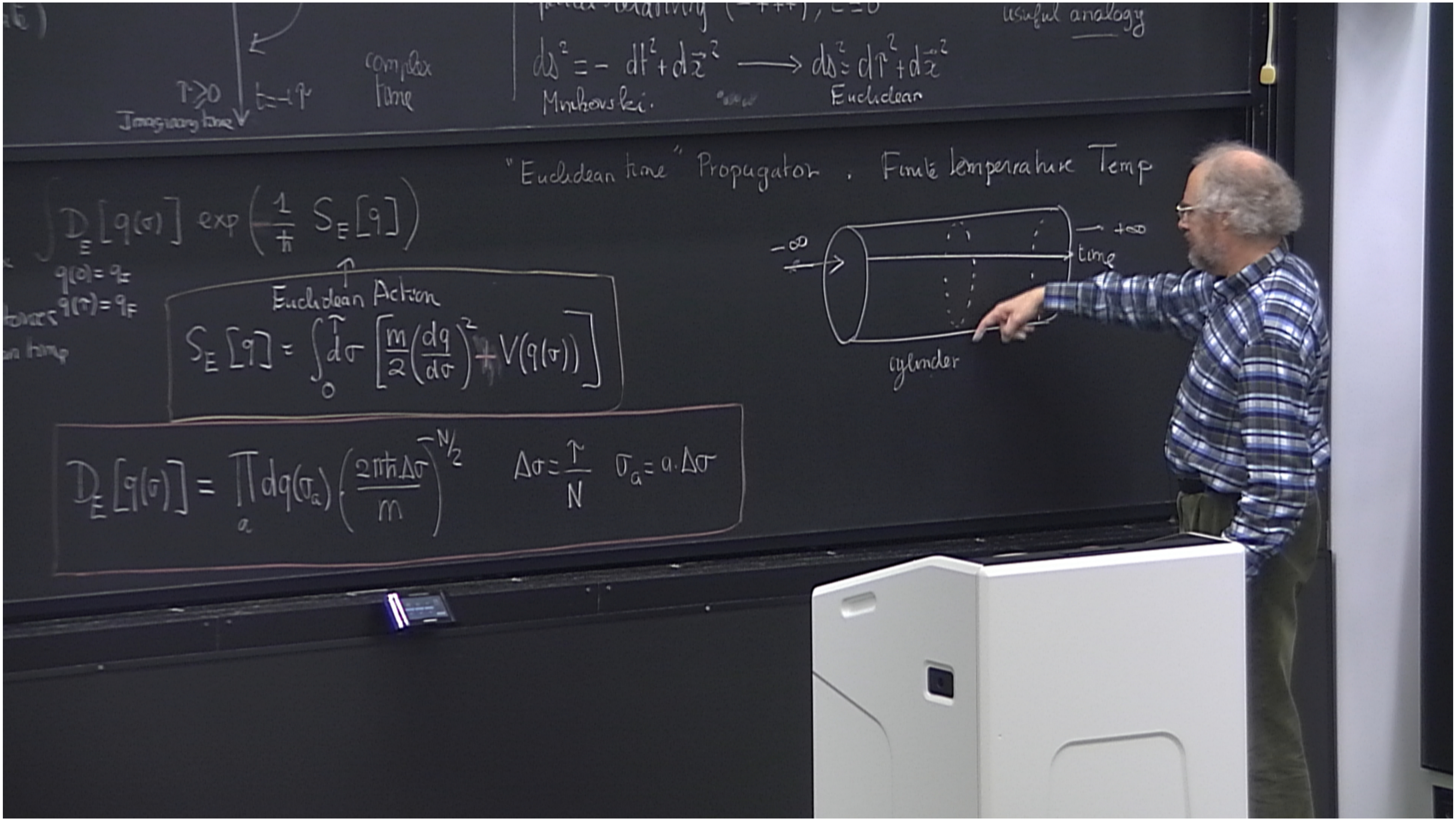
piece-wise linear "histories" q(t)
classical Lagrangian
with
partition function
in Euclidean time

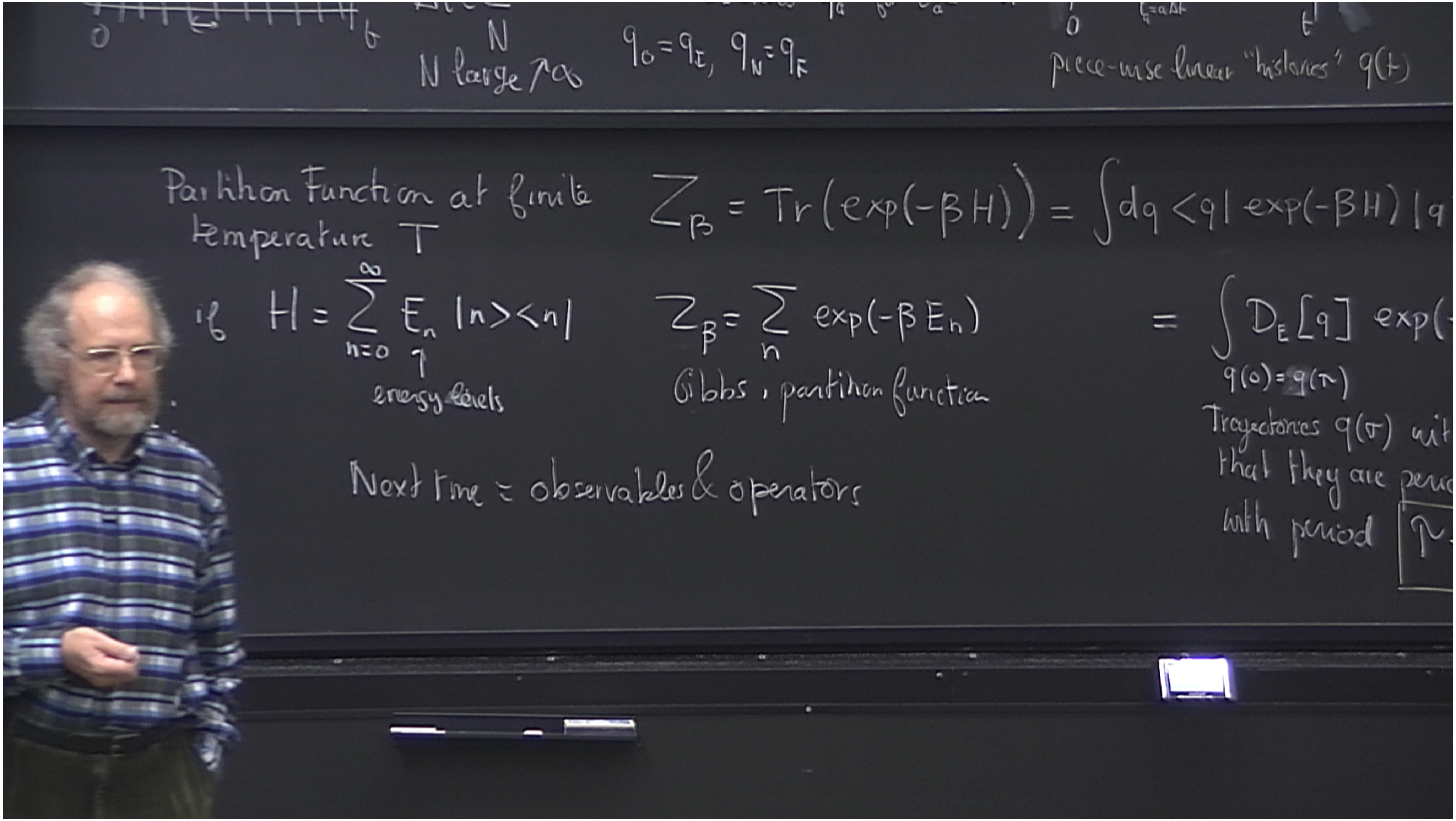


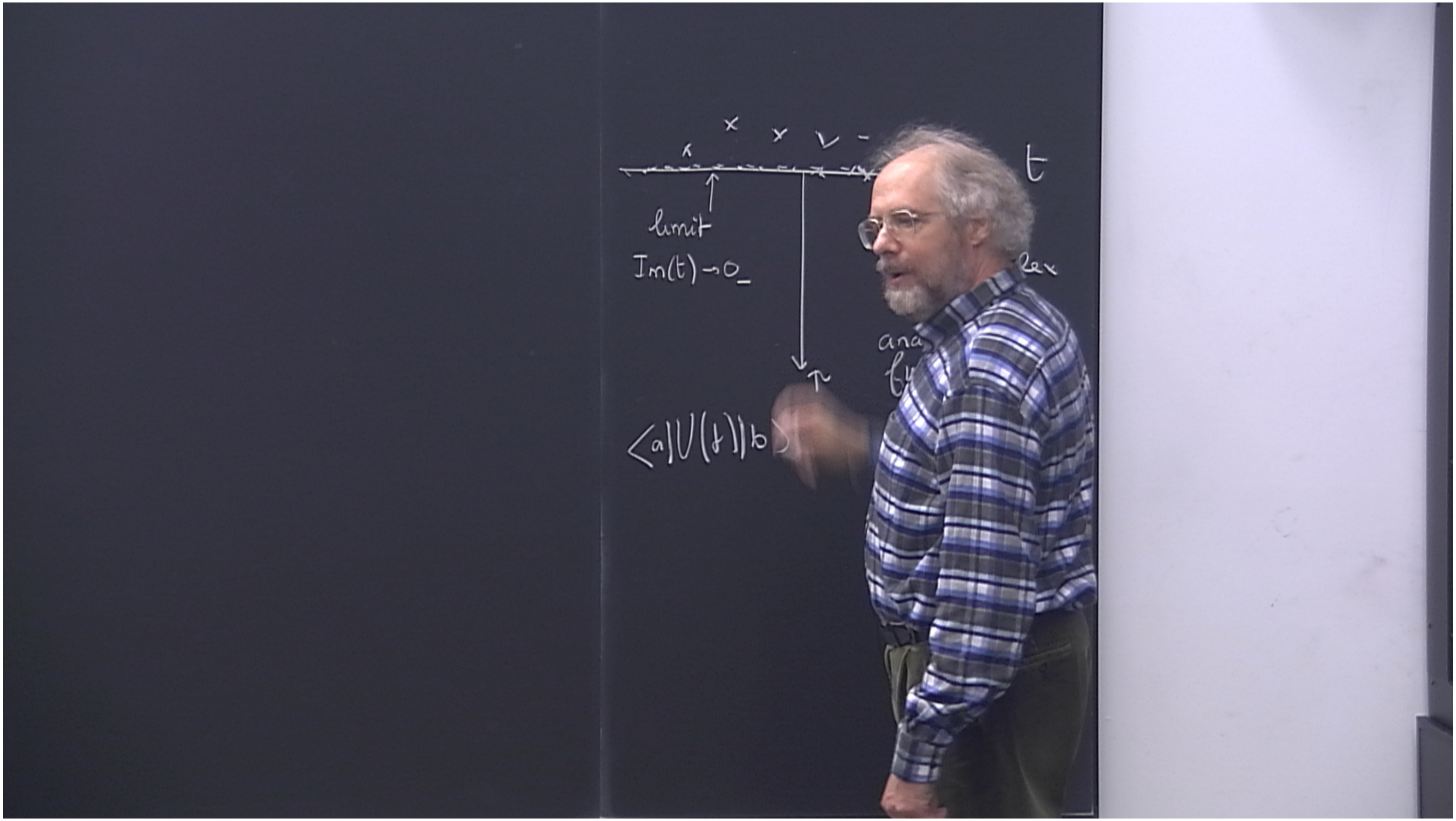
piece-wise linear "histories" q(t)
Classical Lagrangian
)) = ∫dq <q| exp(-βH) |q> = ∫_R dq <q| U_E(τ) |q>
with
= ∫ D_E[q] exp(-1/ħ S_E[q])
q(0) = q(τ)
Trajectories q(τ) with the only constraint
that they are periodic in Euclidean time
with period τ = ħβ = ħ/k_B T
emperature

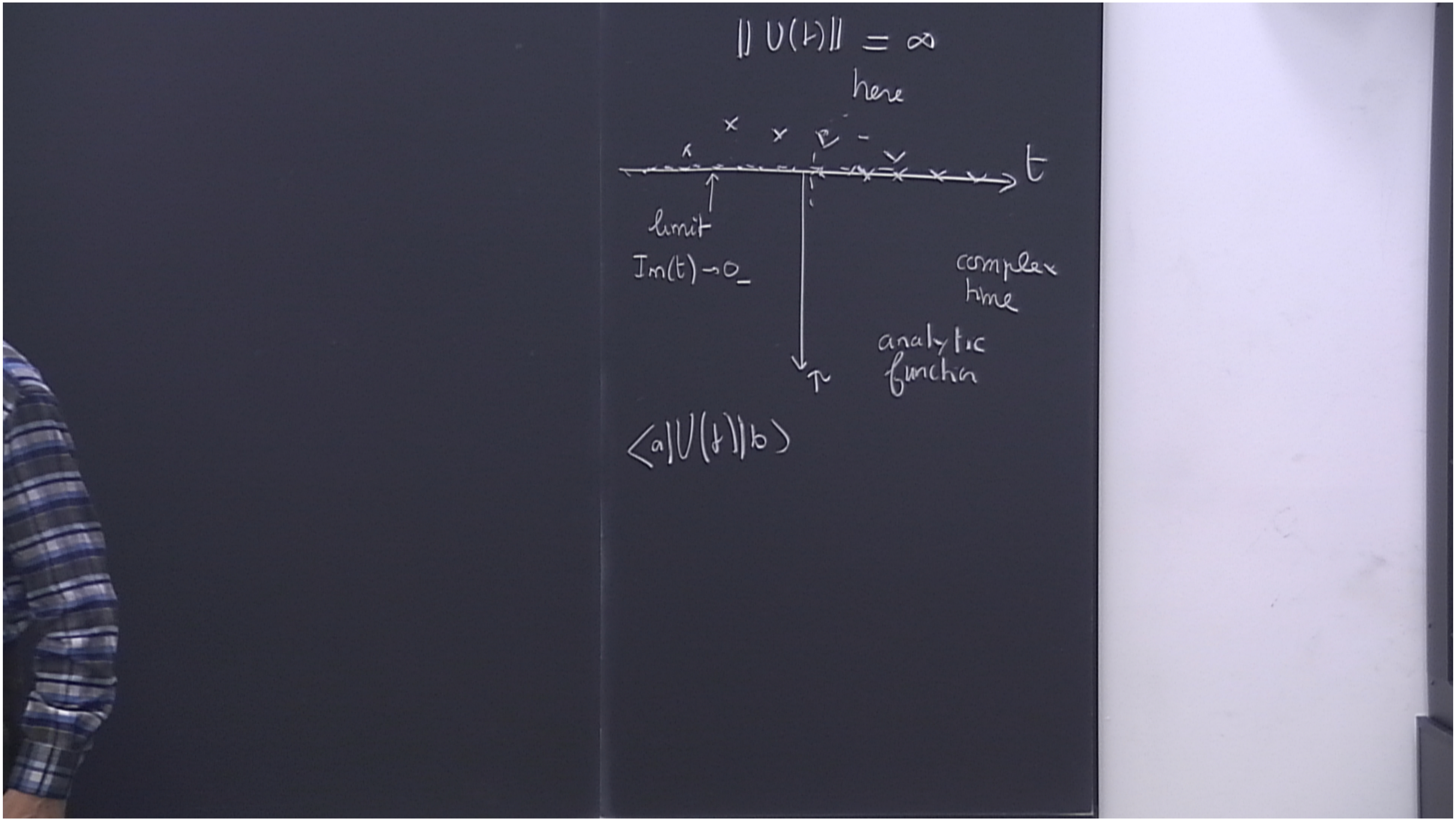

piece-wise linear "histories" q(t)
Classical Lagrangian
)) = ∫dq <q| exp(-βH) |q> = ∫ dq <q| U_E(τ) |q> with β = τ/ħ
= ∫ D_E[q] exp(-1/ħ S_E[q])
q(0) = q(τ)
Trajectories q(τ) with the only constraint
that they are periodic in Euclidean time
with period τ = ħ β = ħ/k_B T
τ
Finite Temperature
for a Quantum System
⇕
Periodic Euclidean Time
Period ≈ ħ/Temperature

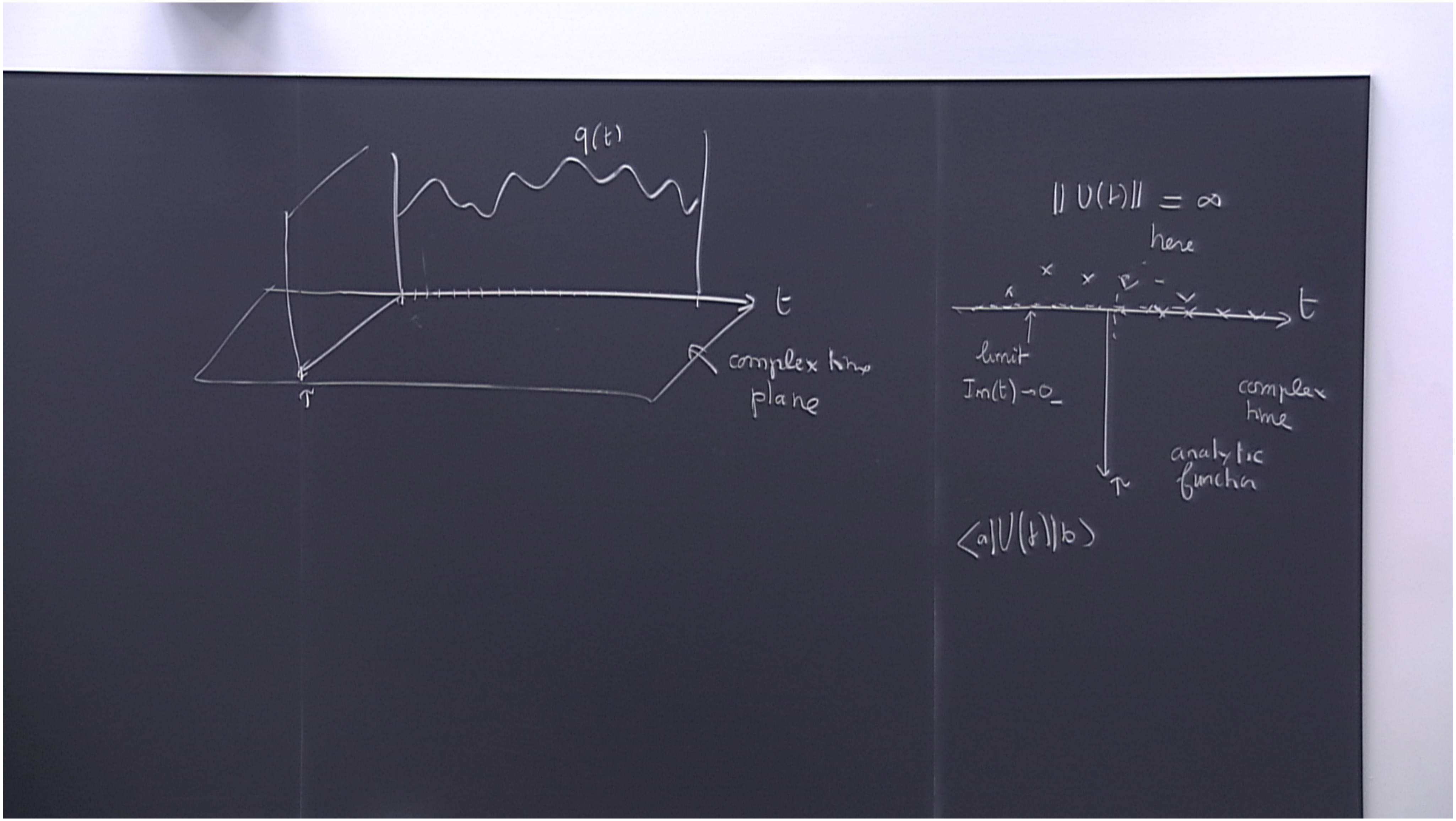

useful analogy
complex time
Imaginary time
Minkowski
Euclidean
"Euclidean time" Propagator
time
Euclidean Action

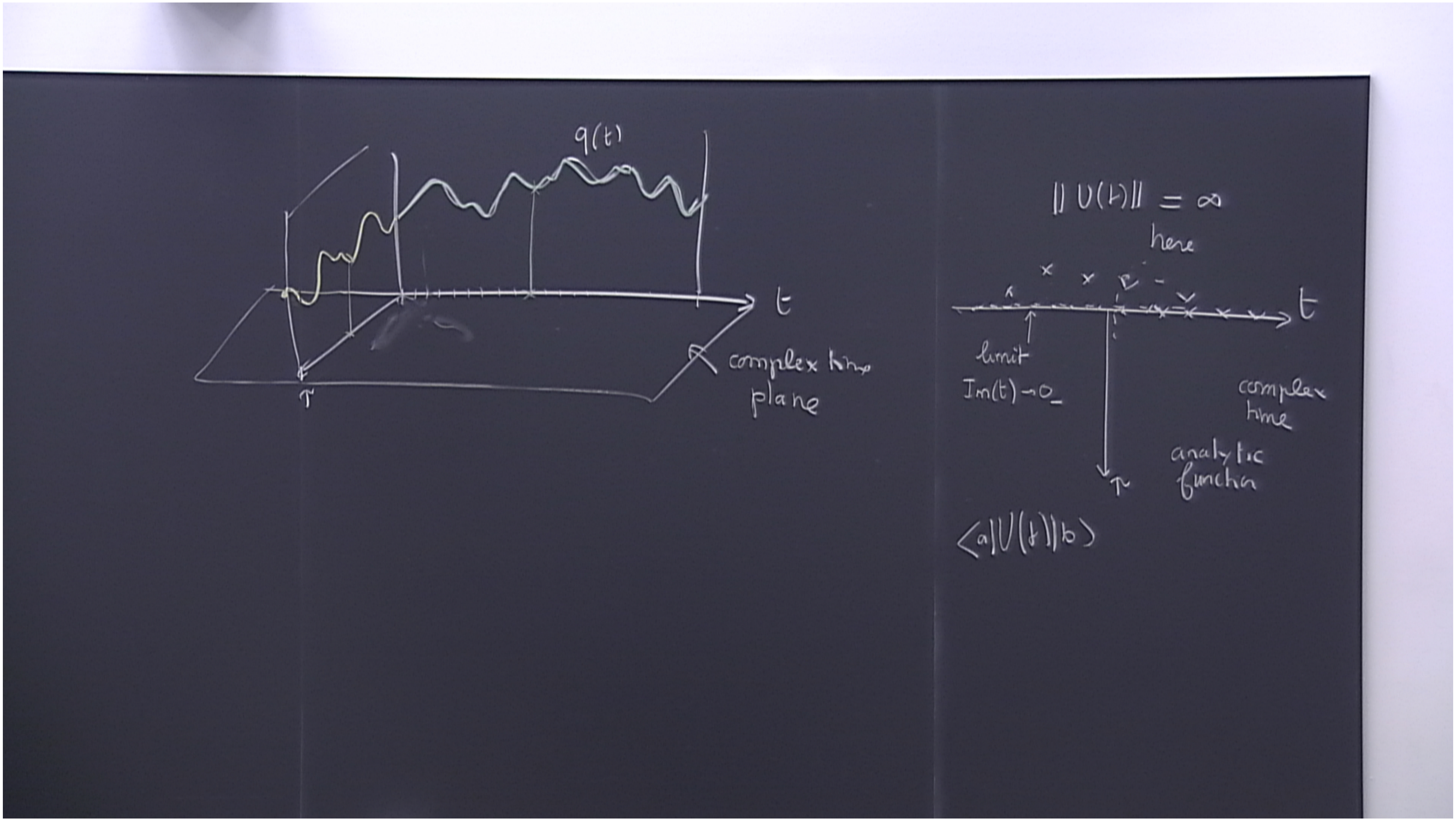

Complex time
Imaginary time
Minkowski
Euclidean
useful analogy
"Euclidean time" Propagator . Finite temperature Temp
Euclidean Action
cylinder
time
$-\infty$
$+\infty$
$ds^2 = -dt^2 + d\vec{x}^2 \longrightarrow ds^2 = d\tau^2 + d\vec{x}^2$
$\int D_E[q(\sigma)] \exp\left(-\frac{1}{\hbar} S_E[q]\right)$
$q(0) = q_I$
$q(\tau) = q_F$
$S_E[q] = \int_0^\tau d\sigma \left[\frac{m}{2}\left(\frac{dq}{d\sigma}\right)^2 + V(q(\sigma))\right]$
$D_E[q(\sigma)] = \prod_a dq(\sigma_a) \left(\frac{2\pi\hbar\Delta\sigma}{m}\right)^{-N/2}$
$\Delta\sigma = \frac{\tau}{N}$
$\sigma_a = a \cdot \Delta\sigma$

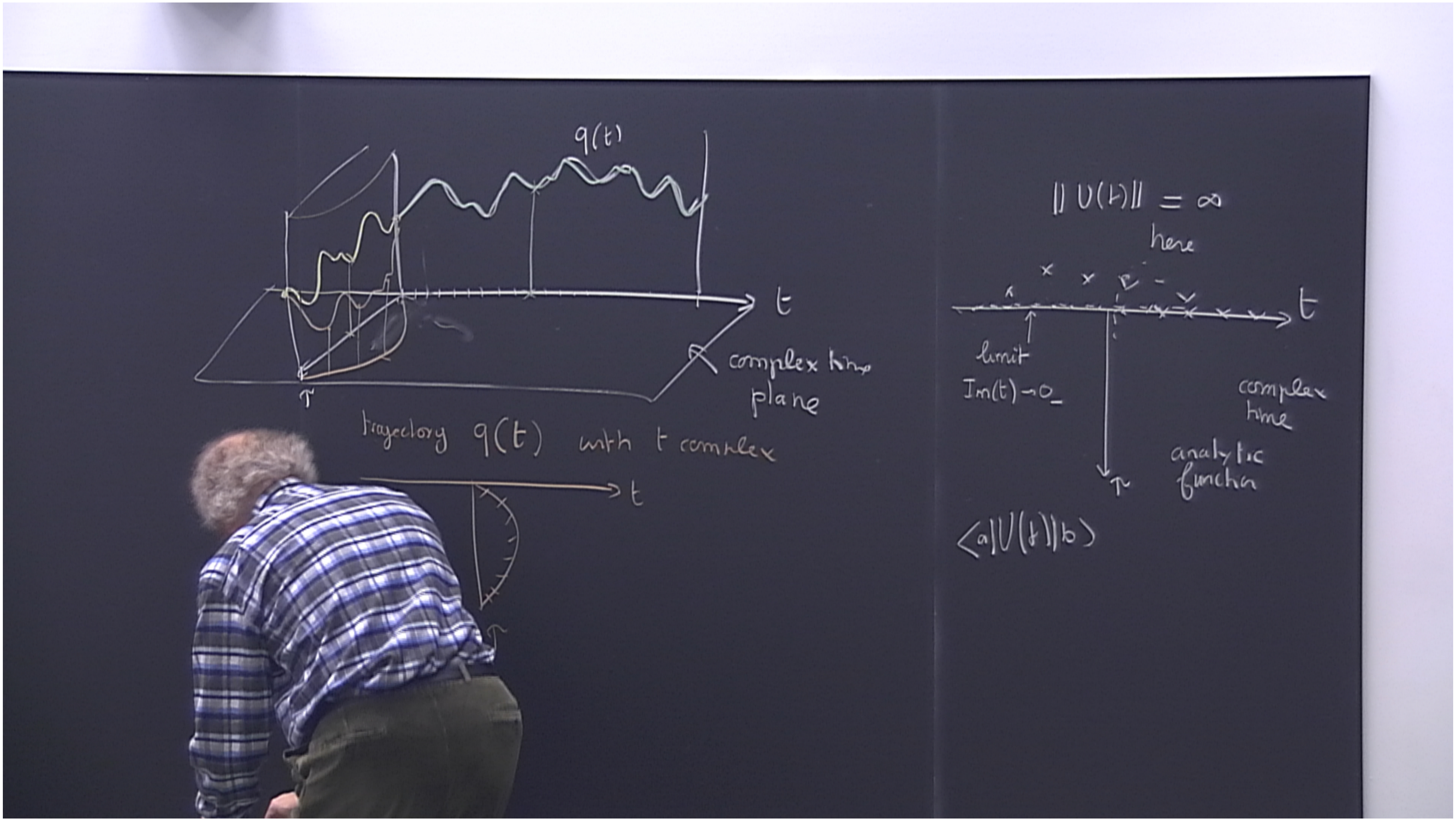

N
N large ↗ ∞
q0 = qI, qN = qF
piece-wise linear "histories" q(t)
Partition Function at finite temperature T
Zβ = Tr(exp(-βH)) = ∫dq <q| exp(-βH) |q
if H = Σ_{n=0}^{∞} En |n><n|
energy levels
Zβ = Σ_n exp(-βEn)
Gibbs, partition function
= ∫ DE[q] exp(
q(0) = q(τ)
Trajectories q(τ) wit
that they are peri
with period τ
Next time: observables & operators


t
limit
Im(t) → 0_
<a|U(t)|b>


||U(t)|| = ∞
here
t
limit
Im(t) → 0_
complex time
analytic function
τ
<a|U(t)|b>


q(t)
t
τ
complex time plane
||U(t)|| = ∞
here
t
limit
Im(t)→0_
complex time
analytic function
τ
<a|U(t)|b>


q(t)
t
complex time plane
τ
||U(t)|| = ∞
here
t
limit
Im(t) → 0_
complex time
τ
analytic function
⟨a|U(t)|b⟩


q(t)
t
complex time plane
trajectory q(t) with t complex
t
|| U(t) || = ∞
here
t
limit
Im(t) → 0_
complex time
analytic function
<a|U(t)|b>