Title: Numerical Study of a Bosonic Topological Insulator in Three Dimensions

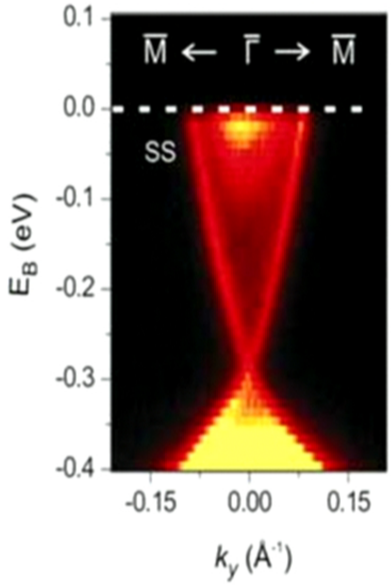
Date: Oct 21, 2014 03:30 PM

URL: http://pirsa.org/14100113

Abstract: We construct a model which realizes a (3+1)-dimensional symmetry-protected topological phase of bosons with U(1) charge conservation and time reversal symmetry, envisioned by A. Vishwanath and T. Senthil [PRX 4 011016]. Our model works by introducing an additional spin degree of freedom, and binding its hedgehogs to a species of charged bosons. We study the model using Monte Carlo and determine its bulk phase diagram; the phase where the bound states of hedgehogs and charges condense is the topological phase, and we demonstrate this by observing a Witten effect. We also study the surface phase diagram on a (2+1)-dimensional boundary between the topological and trivial insulators. We find a number of exotic phases on the surface, including exotic superfluids, a phase with a Hall conductivity quantized to half the value possible in 2D, and a phase with intrinsic topological order. We also find a new bulk phase with intrinsic topological order.

Effects of Interactions

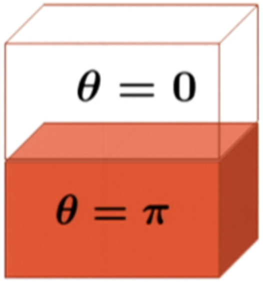
- Experimentally realized TIs made up of free fermions
- Can be studied using band theory
- How does the physics change when (strong) interactions are added?
- Are new phases possible?
- How can they be realized?
- Band theory doesn't work, new techniques needed



Properties of TIs (independent of band theory)

- Write low-energy effective field theory (independent of microscopic details)
- Analogy with Quantum Hall Effect, where effective theory has Chern-Simons term
- Topological physics (e.g. Hall conductivity, gapless edge transport) comes from this term
- Topological term for TI is E · B term:

$$S = \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu \partial_\rho A_\lambda$$



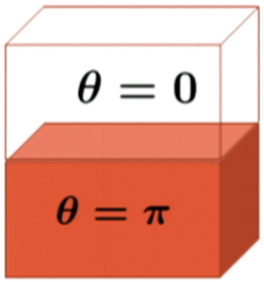
- E · B is a total derivative, so no bulk physics
- Need to study surface instead

Surface Hall conductivity

- Spatially varying θ leads to Chern-Simons term on the surface

$$S = \frac{\theta(r)}{8\pi^2}\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu \partial_\rho A_\lambda = \left(\frac{\partial_\mu \theta(r)}{8\pi^2}\right)\epsilon^{\nu\mu\rho\lambda}A_\nu \partial_\rho A_\lambda$$

$$= \left(\frac{\sigma_{xy}}{4\pi}\right)\epsilon^{\nu\mu\rho\lambda}A_\nu \partial_\rho A_\lambda \to \sigma_{xy} = \frac{\Delta\theta}{2\pi} = \frac{1}{2}$$



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- Therefore we are on the surface of a topological phase

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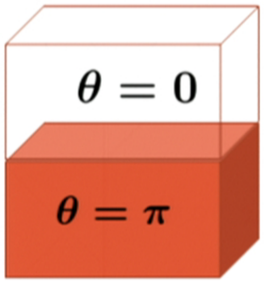

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Bulk response from topological term

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- Total charge in a given volume:

$$\rho = \frac{\delta S}{\delta A_0} = \frac{\theta}{8\pi^2}\nabla \cdot B$$

Rosenberg and Franz (2010)

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Topological Phases of Interacting Bosons

- Many different (SRE) topological phases can be constructed from interacting bosons[1]
- With bosons, interactions must be important or the system would bose-condense
- SRE phases: 'minimal' way to study this physics
- These phases can host exotic new phenomena
- Much work to be done
  - Understand properties of phases
  - How to do experiments with them
- Interactions make the problem hard—many of our favourite tools don't apply

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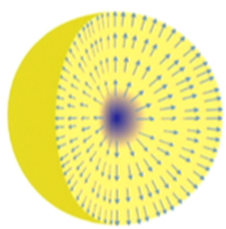
1: Chen, Liu, Gu, Wen, Science 338 1604 (2012)

How to make a model of a bosonic topological phase

- In 1D, point topological defect is a kink (discrete symmetry)
- Can bind these to charges and proliferate bound states
- Get $Z_N x Z_N$ versions of Haldane Phase[1]

- In 2D, point topological defect is a vortex (U(1) symmetry)
- Proliferate vortex+boson bound states
- Bosonic quantum Hall effect' in 2D[2]



- In 3D, point topological defect is a hedgehog
- Proliferating bound states of hedgehogs+bosons will give us a topological phase[3]

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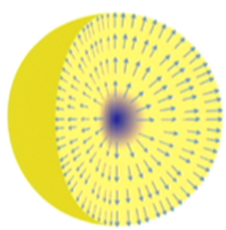

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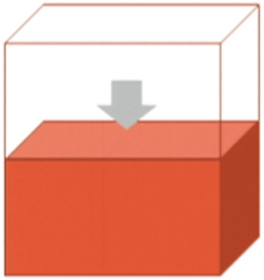
"Bosonic Topological Insulator"

- A 3D Phase with charge conservation and time reversal symmetry
- Vishwanath and Senthil: Postulated field theory for the surface of the system
- Showed the surface has exotic properties
- This physics can't exist in purely 2D system



Vishwanath and Senthil, PRX 3 011016 (2013)

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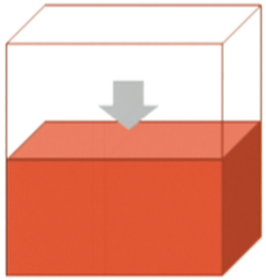


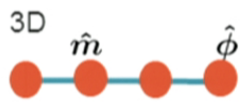
Vishwanath and Senthil, PRX 3 011016 (2013)

Our model

- Define concrete model of bosons on a lattice which realizes the bosonic topological insulator
- Model binds bosons with ‘hedgehogs’, proliferates bound states
- Study in Monte Carlo
- Measure bulk and surface properties to demonstrate that we have the bosonic TI

Constructing the model 1: Bosons

- Start with 3D Hamiltonian of bosons hopping on lattice, Trotter decompose to 4D classical action of conserved currents



$$\langle m|\phi\rangle = e^{im\phi}$$

$$m \in \mathbb{Z}, \quad \phi \in [0, 2\pi)$$

$$H = U\sum_r \hat{m}(r)^2 - t\sum_{r,\mu=x,y,z} \cos[\nabla_\mu \hat{\phi}(r)]$$

$$Z = \langle\Phi|e^{-\beta H}|\Phi\rangle = \sum_{\{m\}}\prod_\tau \langle m(\tau)|e^{-\delta\tau H}|m(\tau+\delta\tau)\rangle$$

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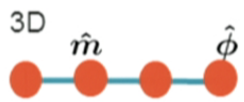

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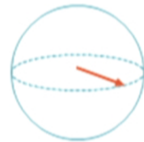
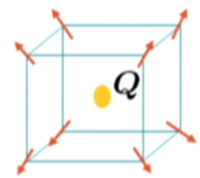
Constructing the model 2: Hedgehogs

- O(3) quantum rotor



- Hamiltonian contains ferromagnetic interaction, and kinetic energy
- After Trotter decomposition, get 4D classical model of an O(3) magnet

$$S = -\beta \sum_{r,\mu=x,y,z,\tau} \vec{n}(r) \cdot \vec{n}(r+\mu)$$



- Take a cube out of lattice, can define hedgehog number at the center
- In 4D, hedgehogs become line objects
- So we can bind them to boson currents

Monte Carlo setup

$$S = -\beta \sum_{r,\mu=x,y,z,\tau} \vec{n}(r) \cdot \vec{n}(r+\mu) + \frac{\lambda}{2} \sum_{r,\mu} [J_\mu(r) - Q_\mu(r)]^2$$

- *Classical* Monte Carlo on this 4D action gives partition sum
- Monte Carlo variables are $\vec{n}$ and J
- Pick an $\vec{n}$ or small loop of J, change it, decide whether to keep the update based on usual Metropolis criterion

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Phase Diagram

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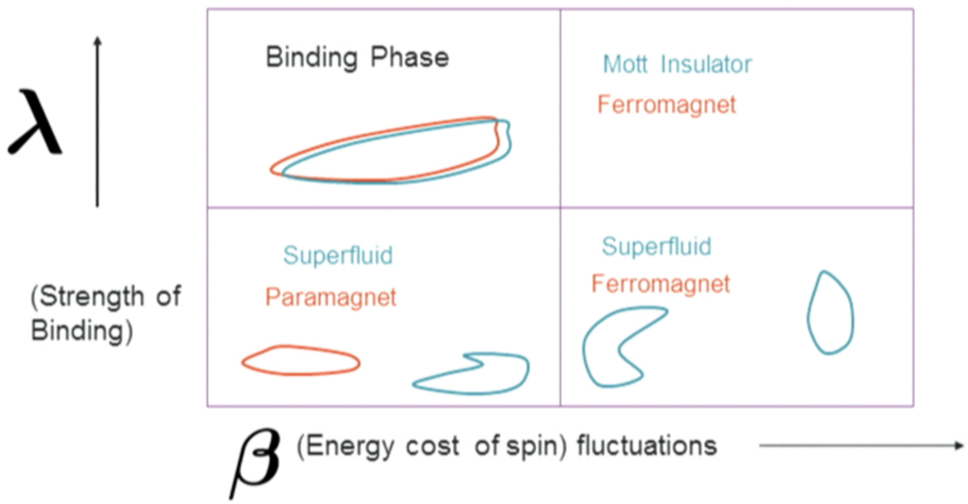

- Detect transitions by measuring specific heat
- Order parameters:
  - Magnetization
  - Superfluid stiffness

Boson current: J Hedgehog current: Q

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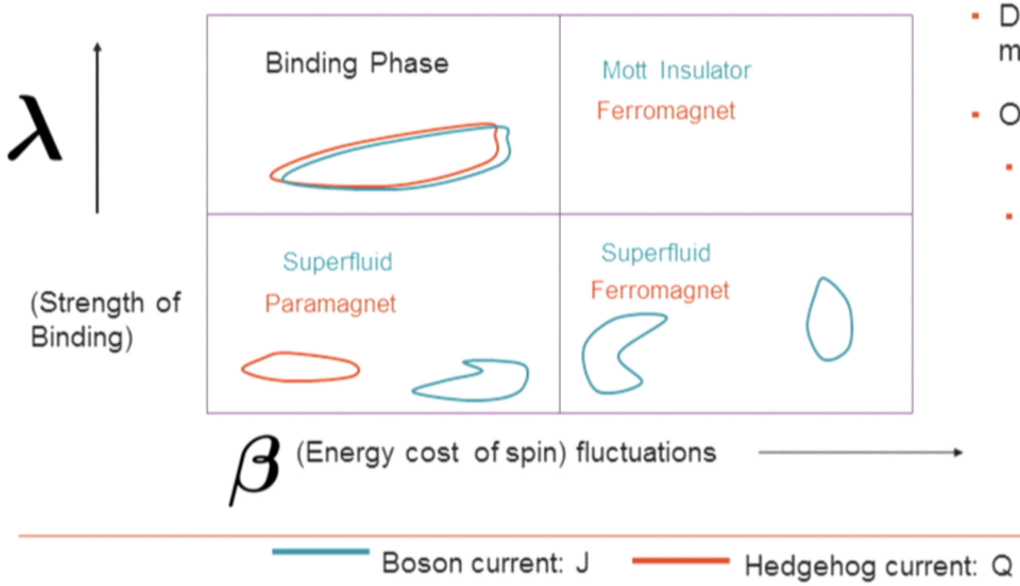

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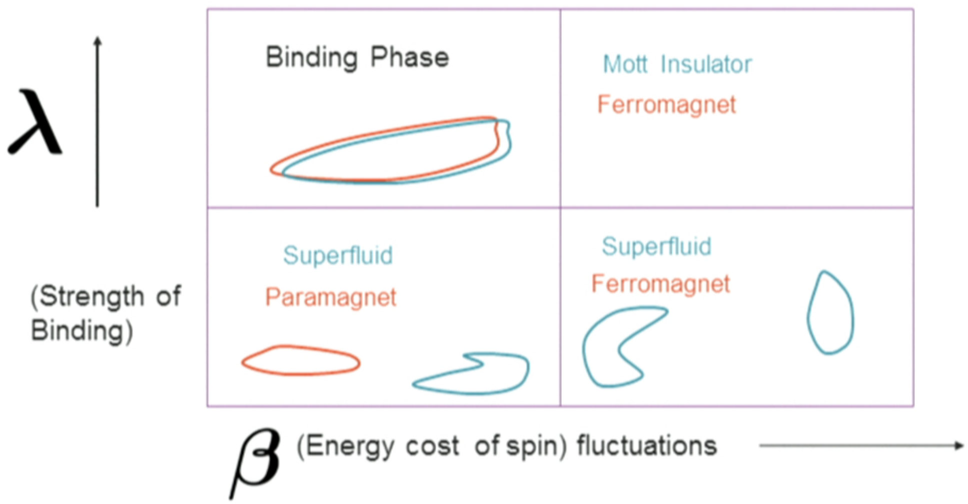

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How to actually construct hedgehogs

- Previous setup doesn't allow the measurements we need to make
- Can instead represent the spins by an easy-plane CP1 model
- This has U(1) symmetry instead of O(3)—but it still has hedgehogs!
- This model can also be thought of as an action for bosons
- **So we are studying a model of two species of bosons**

Constructing the model 2: Hedgehogs

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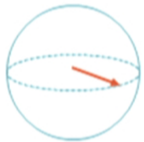

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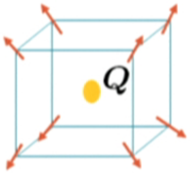

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Measuring the Witten effect

- Two species of bosons leads to two external fields. E · B term:

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- One electric charge binds to each monopole
- Technically we have two species of boson (one from currents, one from spins)
- Imagine adding tunnelling between the two types

Ye and Wen (2014), Ye and Wang (2013), Metlitski, Kane, Fisher (2013)

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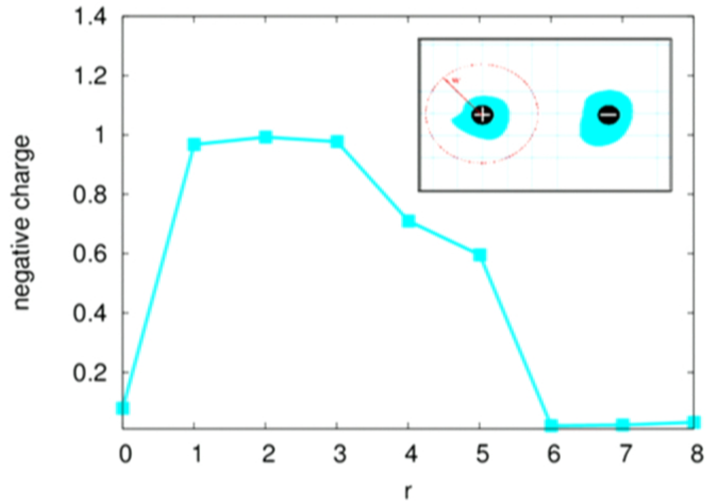
Witten effect—Numerical Results



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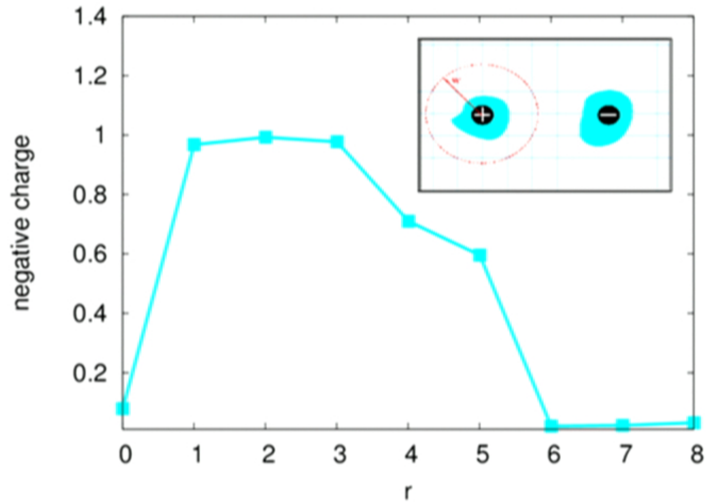

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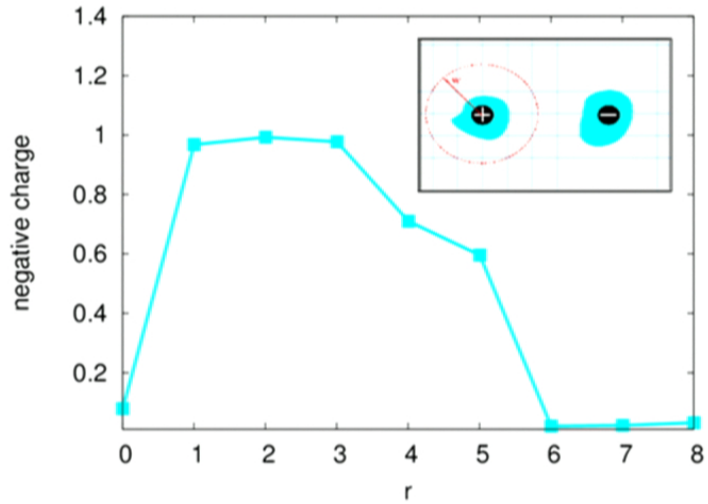

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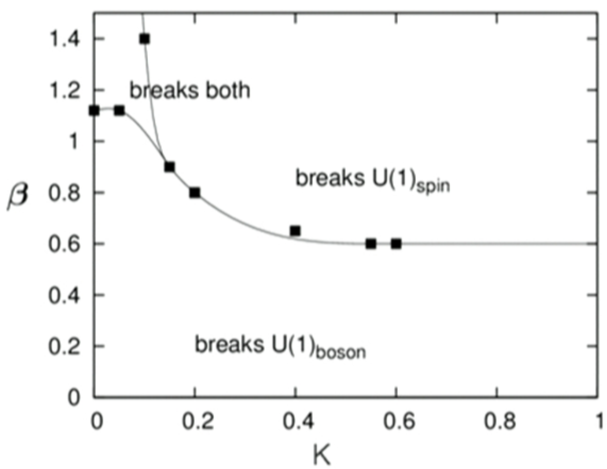
Surfaces of SRE Topological Phases: Basics

- Cannot continuously connect a topological phase to a trivial phase
- Need to either undergo a phase transition or break a symmetry
- The same holds on the edge:
  - Long ranged entanglement (gaplessness or topological order) OR
  - Broken Symmetry
- Our system has U(1) symmetry from bosons, U(1) from spins (also bosons) and time-reversal

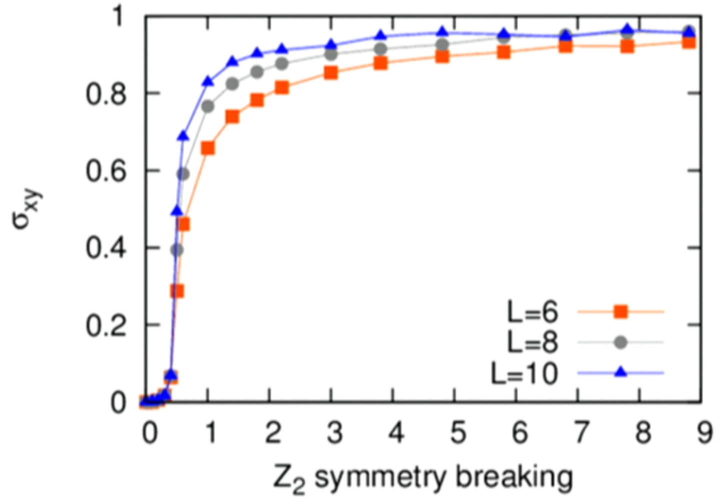
Surface Phase Diagram

- Tune the parameters on the surface to access different phases
- Measure order parameters: magnetization and superfluid stiffness
- In our model, we always break a U(1) symmetry



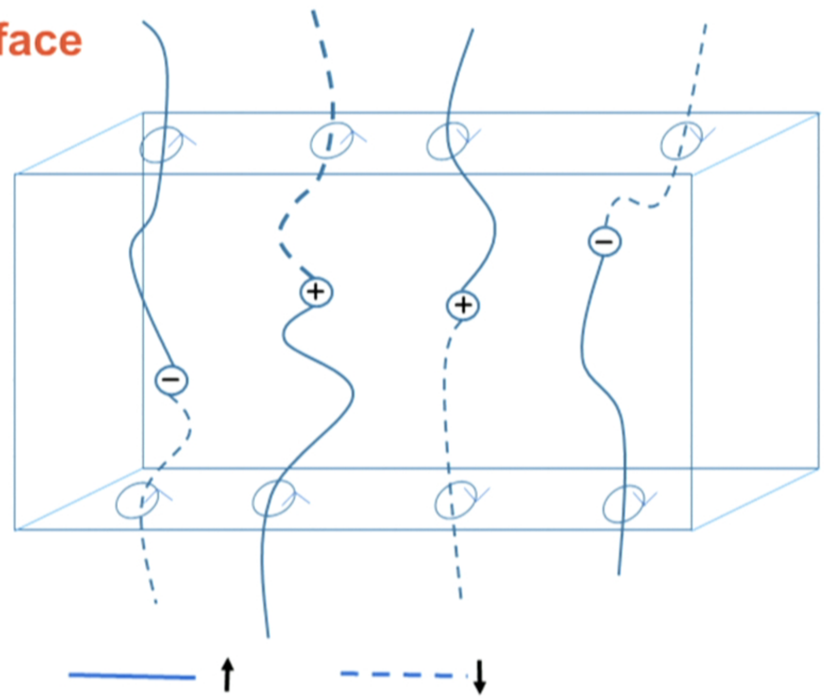
- Break $U(1)_{boson}$: superfluid
- Break $U(1)_{spin}$: ferromagnet, but can also think of as a superfluid
- Direct transition between superfluids

Numerical results



Understanding Surface

- Sign of hedgehog and boson determined by orientation of type of vortex lines
- Proliferate vortices: preserve spin symmetry but break boson symmetry
- Forbid bosons: preserve boson symmetry but break spin symmetry
- Forbid one type of vortex line: breaks time reversal, leads to binding between boson charge and vorticity



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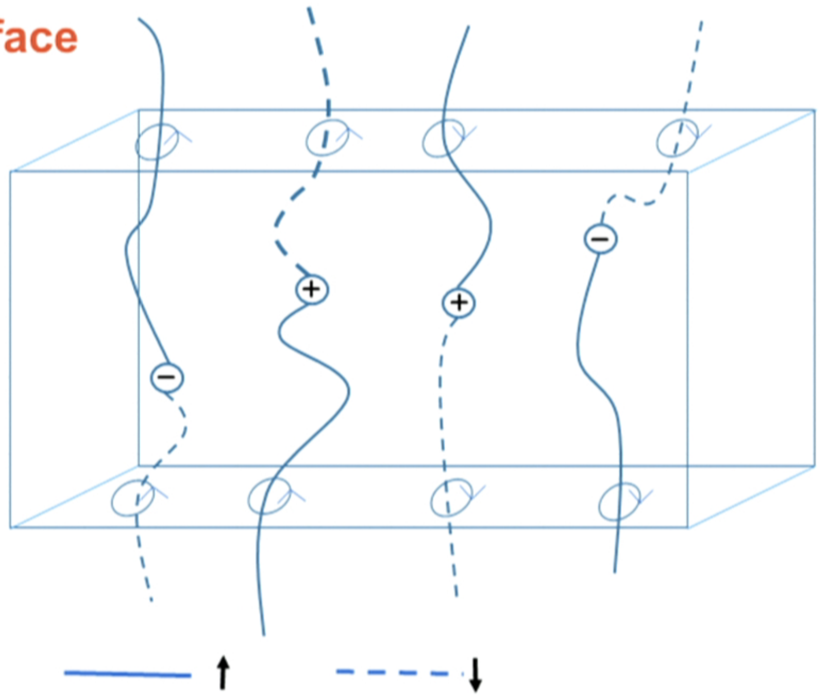
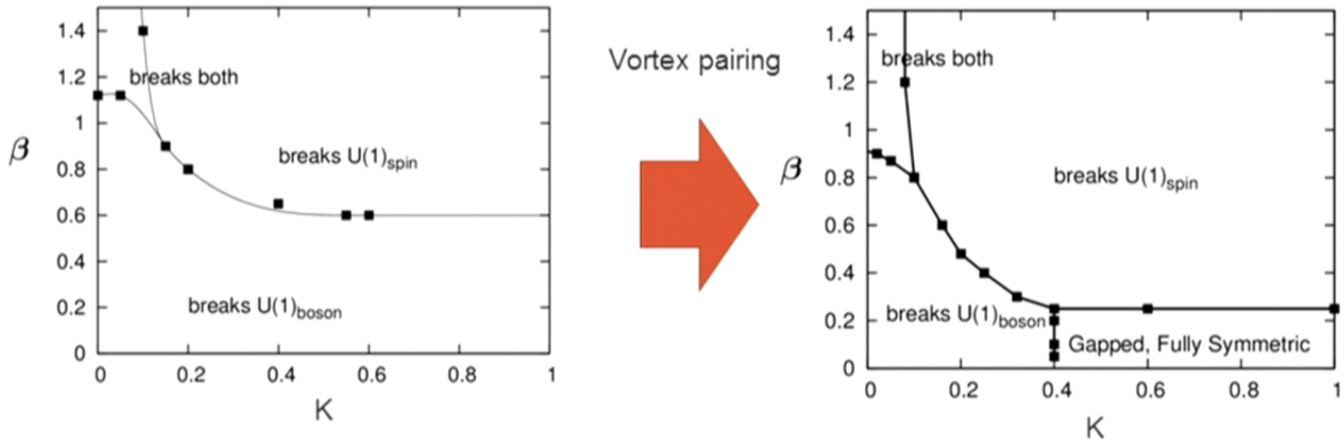


Another option: surface topological order

- Can have a surface phase which is gapped, and breaks no symmetries, but is topologically ordered



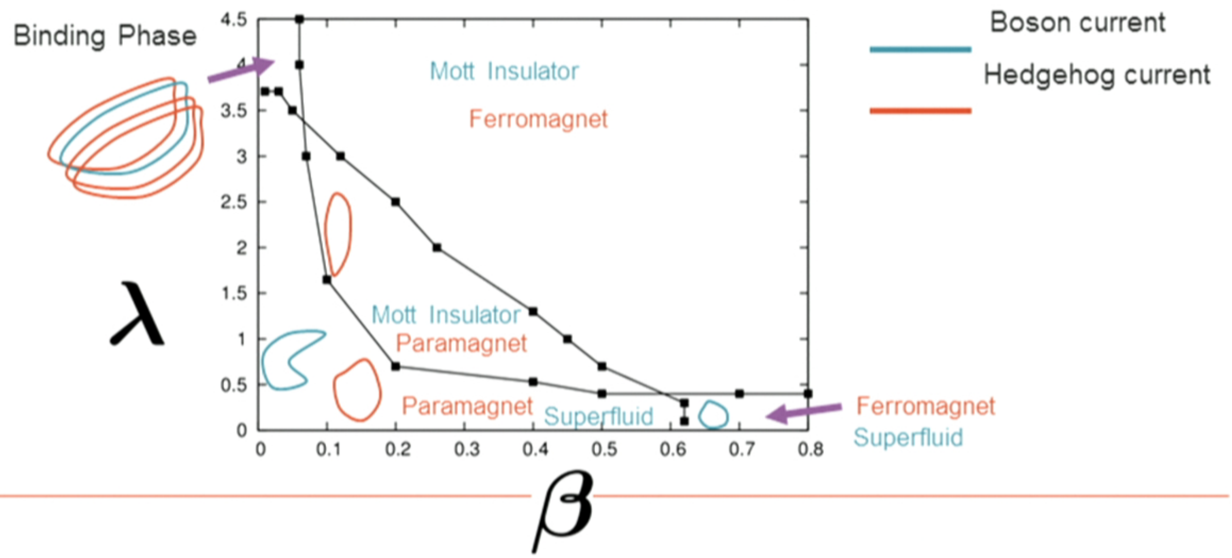
Binding multiple defects: topological order

- Everything so far is about SRE phases analogous to free fermion TI, field theory described by Vishwanath and Senthil
- We bound one topological defect to each charge
- We can also think about binding multiple defects
- Get long-ranged entangled phase
- 'Fractionalized Bosonic Topological Insulator'

Binding multiple defects: topological order

- Bind *d* hedgehogs to *c* charges
- New binding phase
- Another new phase with breaks no symmetries

$$\frac{\lambda}{2}\sum_{r,\mu}[dJ_{\mu}(r) - cQ_{\mu}(r)]^2$$



Gapped excitations in fractionalized phase

- Idea: start with an action in terms of physical variables
- Make 'change of variables' to express action in terms of gapped quasiparticles
- Properties of quasiparticles can be read off of this action

$$
\begin{aligned}
S &= -\beta \sum_{r,\mu} \cos[\nabla_\mu \phi_1(r) - a_\mu(r)] - \beta \sum_{r,\mu} \cos[\nabla_\mu \phi_2(r) - a_\mu(r)] \\
&+ \frac{K}{2} \sum_{r,\mu<\nu} [\nabla_\mu a_\nu(r) - 2\pi B_{\mu\nu}(r)]^2 + \frac{\lambda}{2} \sum_{r,\mu} [dJ_\mu(r) - cQ_\mu(r)]^2 \\
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Dirac Strings-> Electric field lines

- The P variables are integer-valued, we can explicitly write this constraint as:

$$\sum_{\{M\}} e^{2\pi i M_{\mu\nu} B^P_{\mu\nu}}$$

- We then integrate out the P variables, the M variables are 'Faraday line' excitations
- If this was done on the original (Q) variables in the absence of charge, the M would be closed lines which the magnetic monopoles see as 2π flux
- If charges are allowed, the charges are sources and sinks of the M's
- This case: Integrating out gauge field 'a' shows that a boson is a source for an M line of strength *d*
- M contains 'ordinary' part, and fractionalized, divergenceless part

Action in terms of only gapped excitations

Terms which gap G and M

$$S = \frac{1}{2Kd^2}\sum_{r,\mu,\nu} M_{\mu\nu}^2 + \frac{\lambda}{2}\sum_{r,\mu} G_\mu^2$$
$$- i\sum_{r\rho}\frac{1}{d}G_\mu(r)A_\mu^{\text{ext}}(r) + i\sum_{R,\mu<\nu}\frac{2\pi b}{d}M_{\mu\nu}(R)B_{\mu\nu}^{G}(R)$$

Particle excitations carrying 1/d of a unit of charge

Gives a phase when a particle goes around a line, like 3D toric code

Future directions

- Build more models to represent other phases
  - ‘Kramers’ bosonic topological phase?
  - Binding particles to intersections of domain walls? (J. Wang, Gu & Wen 2014)
- Use the models already built to do things
  - Critical properties
  - Lattice defects
  - Microscopic Hamiltonians → finding them and making them more realistic
  - Measuring quantities in these models to compare to more realistic systems

Input: DVI - 1024x768p@59.82Hz
Output: SDI - 1920x1080i@60Hz

Conclusions

- Interacting Boson TI can be realized in a lattice model
- Physics is to bind topological defects to charges
- Can measure the predicted properties in Monte Carlo
- Also find new phase with long-ranged entanglement

Reading List

- This work: arXiv:1408.1096
- 1D model: arXiv 1410.1580
- 2D model: Annals of Physics 334, 288 (2013)
- Topological phase of lattice gauge theory:
  arXiv 1408.3146
- SL(2,Z) change of variables: PRB 85, 245121 (2012)

Acknowledgments

Olexei Motrunich
Max Metlitski
Matthew Fisher

Conclusions

- Interacting Boson TI can be realized in a lattice model
- Physics is to bind topological defects to charges
- Can measure the predicted properties in Monte Carlo
- Also find new phase with long-ranged entanglement

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Binding multiple defects: topological order

- Everything so far is about SRE phases analogous to free fermion TI, field theory described by Vishwanath and Senthil
- We bound one topological defect to each charge
- We can also think about binding multiple defects
- Get long-ranged entangled phase
- 'Fractionalized Bosonic Topological Insulator'

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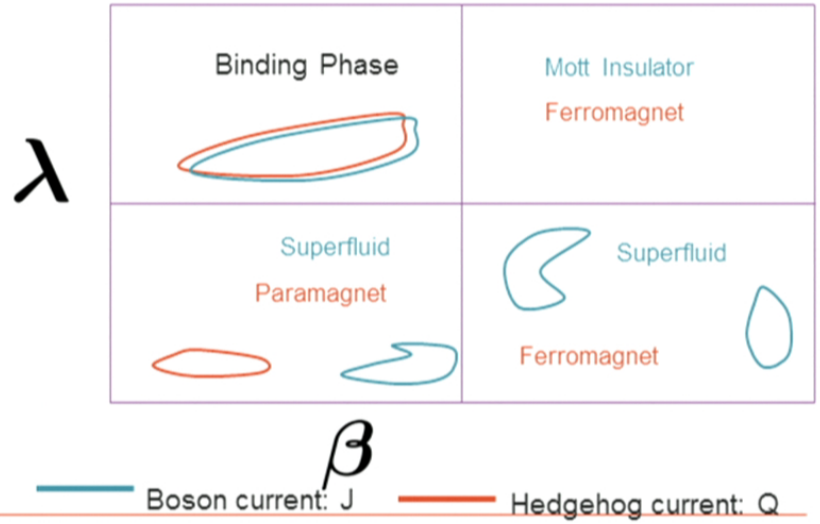

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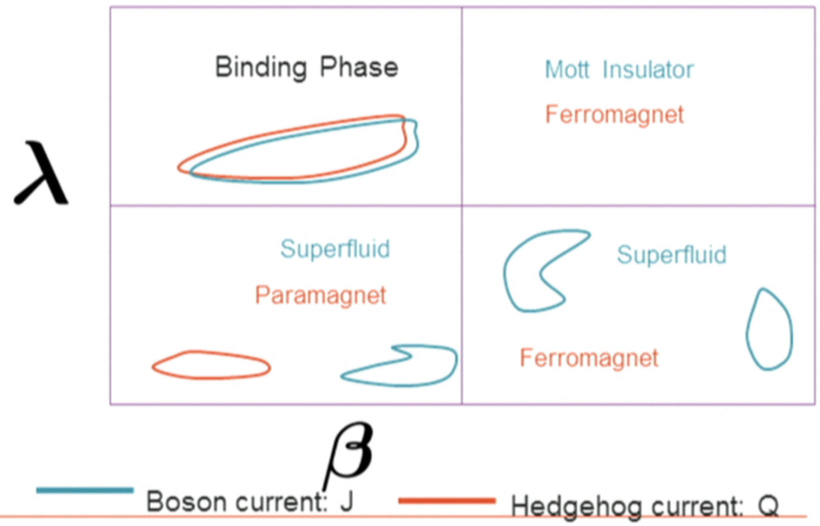
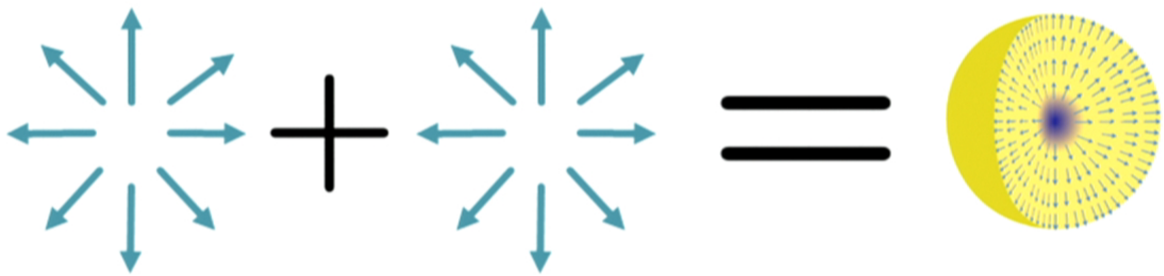
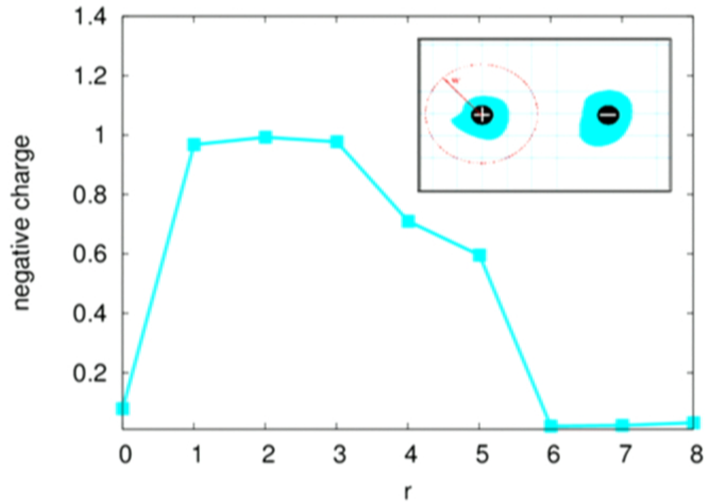


Vortex lines picture

- In the spin sector, there are vortex line topological defects
- Two types of vortex lines
- Where the types of vortex lines meet, that is a hedgehog



Witten effect—Numerical Results



- Can introduce monopole into Monte Carlo
- Measure total charge in a sphere of radius r around one monopole

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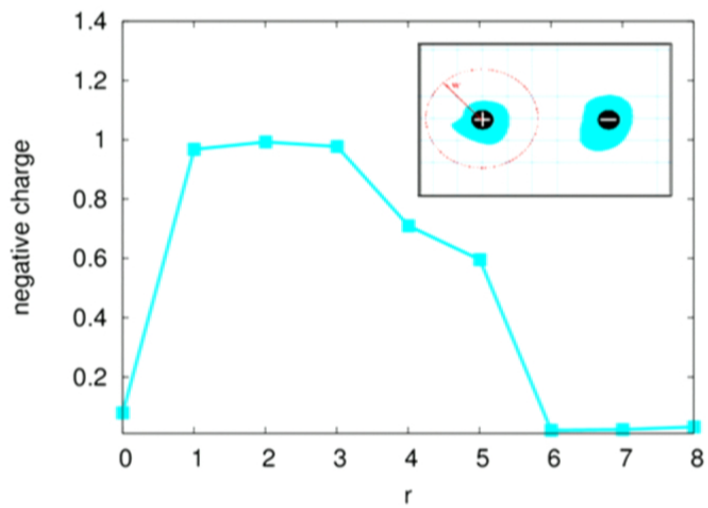

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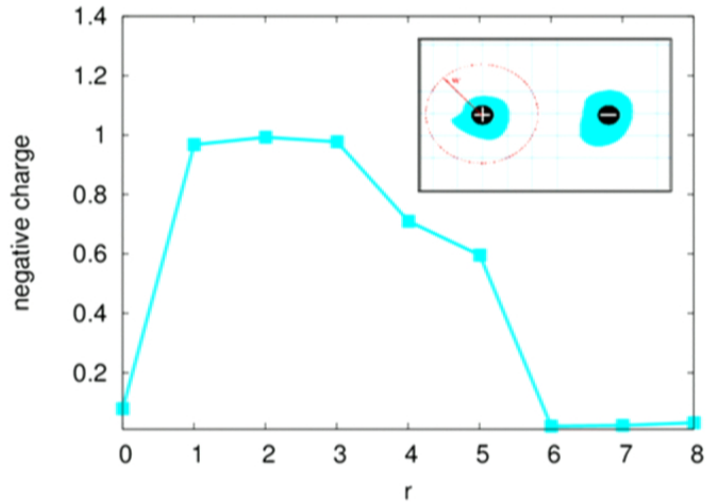

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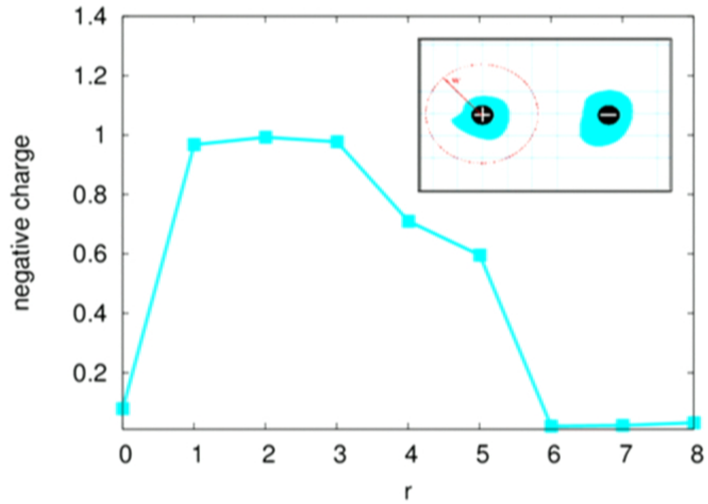

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How to actually construct hedgehogs

- Previous setup doesn't allow the measurements we need to make
- Can instead represent the spins by an easy-plane CP1 model
- This has U(1) symmetry instead of O(3)—but it still has hedgehogs!
- This model can also be thought of as an action for bosons
- **So we are studying a model of two species of bosons**

$$S = -\beta \sum_{r,\mu} \cos[\nabla_\mu \phi_1(r) - a_\mu(r)] - \beta \sum_{r,\mu} \cos[\nabla_\mu \phi_2(r) - a_\mu(r)]$$

$$+ \frac{K}{2} \sum_{r,\mu<\nu} [\nabla_\mu a_\nu(r) - 2\pi B_{\mu\nu}(r)]^2 + \frac{\lambda}{2} \sum_{r,\mu} [J_\mu(r) - Q_\mu(r)]^2$$

$$Q_\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \nabla_\nu B_{\rho\sigma}$$